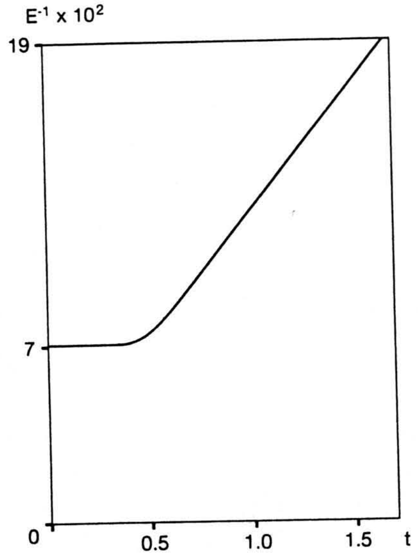

Fig. 4.9. Different stages of the evolution are depicted

5. Physical Applications

... a time to gather stones together

Ecclesiastes

Now it is time for our theory to bear fruit and for us to harvest. In this chapter we shall recollect the results concerning specific physical systems and shall obtain some new results by specializing general approaches from the previous chapters. We shall try to describe most hitherto known facts with regard to the spectra of developed weak turbulence. The general model of wave turbulence adopted in this book is based on the detailed consideration of the wave-wave interaction. The interaction with the external environment was described by the function $\Gamma(\boldsymbol{k})$ specifiying the decrement of wave attenuation or the growth-rate of wave instability. Nature is naturally more complex (for example, the interaction of water waves with wind and currents or wave-particle interactions in plasmas). Nevertheless, we believe the following fomulas and interpretations to provide a good basis for the further development of the theory of wave turbulence in various systems each of which might require a separate monograph for an adequate detailed description.

5.1 Weak Acoustic Turbulence

In this section we shall discuss wave turbulence with a near-sound dispersion law. Plenty of physical systems belong to this type. In spatially homogeneous media according to the Goldsone theorem, the wave frequency $\omega(k)$ should vanish together with the wave number k. In most cases the frequency expansion at small k starts from the first (i.e., linear) term. So the large-scale perturbations produce acoustic waves in solids, fluids, gases, and plasmas. The dispersion is supposed to be weak but sufficiently large to justify applicability of the kinetic equation and to be greater than dissipation. The magnitude of the nonlinear interaction coefficient and the sign of the dispersive frequency addition are different for various media. As repeatedly mentioned above, the properties of acoustic turbulence strongly depend on the sign of dispersion and the dimension of the space. Indeed, for positive dispersion three-wave interactions are allowed, while for negative dispersion one should take four-wave processes into account. Different kinetic equations are to be used in these cases. As far as the space dimension is concerned, we shall mention here only one fact to demonstrate the large difference between two- and three-dimensional cases. Considering acoustic turbulence to be close to scale invariant turbulence and supposing the index of the Kolmogorov spectrum to be close to the general formula $s_0 = m + d$, we obtain the correct value for the three-dimensional case only (see Sect. 3.2). For the two-dimensional case it is not possible to use the scale-invariant approximation of the dispersion law $\omega(k) = ck^{1+\epsilon}$ instead of the physically better justified expression (1.2.22)

$$\omega(k) = ck(1 + a^2k^2) . \tag{5.1.1}$$

Even for the index of the Komogorov solution one gets different answers. Two-dimensional sound turbulence may be referred to as "nonanalytical" with regard to the dispersion parameter, i.e., the solution depends on the way which the dispersion law tends to a linear one. We shall treat the case with positive dispersion in Sect. 5.1.1 ($d = 3$) and Sect. 5.1.2 ($d = 2$) and the one with negative dispersion in the first part of the Sect. 5.1.3.

It should be noted that the dispersion law can be close to a linear one not only for long waves. There exist short acoustic-like waves in some systems. For example, the dispersion law of spin waves in an antiferromagnet (1.4.18) coincides with that of relativistic particle and has the form

$$\omega^2(k) = \omega_0^2 + (ck)^2 . \tag{5.1.2a}$$

In the "ultrarelativistic" limit $vk \gg \omega_0$ the frequency is approximately equal to

$$\omega(k) = ck + \omega_0^2/(2vk) . \tag{5.1.2b}$$

The second term on the right-hand-side describes small dispersion. With the help of a figure like Fig. 1.1 it is easy to understand that both dispersion laws (5.1.2a, b) are of the nondecay type. Therefore in the case of short waves, the positive additional term leads to the four-wave kinetic equation as opposed to the long wave case. Generally speaking, the decay criterion for the almost acoustic dispersion law

$$\omega(k) = ck + \Omega(k) , \qquad \Omega(k) \ll ck \tag{5.1.3}$$

can be formulated as follows

$$\operatorname{sign}\left[\Omega(k)\frac{\partial}{\partial k}\frac{\Omega(k)}{ck}\right] > 0 . \tag{5.1.4}$$

The dispersion law (5.1.2) is also valid for atmospheric inertio-gravity waves with lengths (≃100–1000 km) shorter than the Rossby radius. Such waves are two-dimensional. The second part of Sect. 5.1.3 deals with short-wave acoustic turbulence.

5.1.1 Three-Dimensional Acoustics with Positive Dispersion: Magnetic Sound and Phonons in Helium

This subsection is devoted to long three-dimensional sound waves with positive dispersion thus allowing for three-wave interactions. In magnetized media, a positive dispersion term proportional to powers of k that are greater than unity is frequently oberved. The interaction between acoustic and spin-wave subsystems in crystals gives rise to the both dispersion and nonlinear interaction of sound, since the spin subsystem is usually strongly dispersive and nonlinear unlike the acoustic one. This interaction is especially strong in antiferromagnets [5.1]. Magnetic sound in plasmas possesses also positive dispersion [5.2]. In the presence of a magnetic field the dispersion law is usually nonisotropic. For example, in plasmas, the dispersive addition to the linear term depends on the angle and even changes sign at some angles. However, in the case of small dispersion only waves with close propagation directions interact with each other. Thus, the angular dependence can often be ignored. Considering locally isotropic and slightly anisotropic distributions seems thus reasonable. Within some interval of the values of the pressure, phonons in helium provide another example for systems with positive dispersion [5.3].

Thus, we consider three-dimensional waves with the dispersion law (5.1.1). The coefficient of the three-wave interaction can be represented in the extremely simple form

$$|V(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)|^2 = bkk_1k_2 , \tag{5.1.5}$$

defined by a single dimensional constant b. Indeed, the scaling index m of the interaction coefficient of long acoustic waves can be obtained from a dimensional analysis: $m = 3/2$ (see Sect. 1.1.4). The powers of k_1 and k_2 should be equal because of the symmetry properties of the system. The power of k can be obtained by the following simple argument. Let us consider the case $k \ll k_1, k_2$ with $k_1 \approx k_2$. The long k-wave can be regarded as giving rise to density variations thus varying also the frequency. Therefore, the interaction Hamiltonian

$$\int V(k, k_1, k_2)a(k)a(k_1)a^*(k_1)\, d\boldsymbol{k}d\boldsymbol{k}_1d\boldsymbol{k}_2$$

may be written in the form

$$\int \delta\omega(k_1)|a(k_1)|^2\, d\boldsymbol{k}_1 .$$

Consequently, $V(k, k_1, k_1)$ varies with k_1 like

$$\lim_{k\ll k_1} V(k, k_1, k_1) \propto k_1 k^{1/2} . \tag{5.1.6}$$

If we add the requirement that the interaction coefficient has to vanish with every wave number k, k_1, or k_2 (since there should be no interaction with a homogeneous shift) then we arrive at (5.1.5) with an accuracy of a dimensionless function of angles, see also (1.1.38). But the angular dependence of the interaction coefficient can be neglected, because the space-time synchronization condition $\omega(\boldsymbol{k}) = \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k} - \boldsymbol{k}_1)$ allows in the case of weak dispersion for interactions between waves with almost collinear wave vectors, see (3.2.2).

Thus, we can write the kinetic equation for three-dimensional sound in the form

$$\begin{aligned}\frac{\partial n(\boldsymbol{k},t)}{b\partial t} = \int & dk_1 d\theta_1 d\varphi_1 k k_1^3 (k-k_1)\Big\{\delta\big(3ca^2 k k_1(k-k_1) - \theta_1^2 k k_1/2\big)\\ &\times \big[n(\boldsymbol{k}_1)n(\boldsymbol{k}-\boldsymbol{k}_1) - n(\boldsymbol{k})n(\boldsymbol{k}_1) - n(\boldsymbol{k})n(\boldsymbol{k}-\boldsymbol{k}_1)\big]\\ &- 2\delta\big(3ca^2 k k_1(k_1-k) - \theta_1^2 k k_1/2\big)\\ &\times \big[n(\boldsymbol{k})n(\boldsymbol{k}_1-k) - n(\boldsymbol{k}_1)n(\boldsymbol{k}) - n(\boldsymbol{k}_1)n(\boldsymbol{k}_1-\boldsymbol{k})\big]\Big\}\end{aligned} \tag{5.1.7}$$

where we retained the small dispersion parameter a only in the arguments of the δ-functions. The applicability criterion for this equation will be derived in the second volume of this book and is given by

$$bk^2 N \ll c\ln(ak)^{-2} .$$

For isotropic distributions, (5.1.7) can be integrated over the angles to arrive at (3.2.3):

$$\begin{aligned}\frac{\partial n(k,t)}{4\pi b\partial t} = &\int_0^k k_1^2(k-k_1)^2\big\{n(k_1)n(k-k_1)\\ &- n(k)\big[n(k_1)+n(k-k_1)\big]\big\}\,dk_1\\ &- \int_k^\infty k_1^2(k_1-k)^2\big\{n(k)n(k_1-k)\\ &- n(k_1)\big[n(k)+n(k_1-k)\big]\big\}\,dk_1 .\end{aligned} \tag{5.1.8}$$

Thus, the small dispersion parameter a is eliminated from the isotropic kinetic equation of the zeroth order.

There exists only one isotropic Kolmogorov solution (2.4.5)

$$n(k) = \lambda P^{1/2} k^{-9/2} , \qquad \lambda \approx (5/b)^{-1/2}(4\pi)^{-1} \tag{5.1.9}$$

carrying the energy flux P. It was obtained by *Zakharov* [5.4] as the first example of a weakly turbulent Kolmogorov-like spectrum. Spectrally narrow pumping should generate at intermediate values of k a stationary distribution in the form of a chain of sharp peaks with the amplitudes dropping down like $n(k_j) \propto k_j^{-11/2}$ with $k_j = jk_0$ (see Fig. 3.14) [5.5]. With the growth of k such a pre-Kolmogorov solution goes over into a Kolmogorov solution.

According to the general formula (3.4.8), the flux P can be expressed in terms of the pumping characteristics. If an external environment generates waves with wave numbers of the order of k_0 with the growth-rate Γ_0, then the energy flux (i.e., the energy dissipation rate) can be evaluated by

$$P \simeq (\Gamma_0 c\lambda)^2 k_0^{-1} .$$

We see that in this case the index h equals $\alpha - m = 1 - 3/2 = -1/2$. So the long-wave part of the spectrum (5.1.9) is the one containing most of the energy

$$E = 4\pi \int \omega(k)n(k)k^2\,dk \propto k_0^{-1/2} .$$

Given a stationary pumping, the spectrum in the short-wave region is formed according to an "explosive" power law. It can be described in terms of the self-similar solution (4.3.10)

$$n(k,t) \propto (t_0 - t)^9 f\big(k(t_0-t)^2\big) .$$

For small $k(t_0-t)^2$ the distribution is quasi-stationary and almost a Kolmogorov one. The right tail of the Kolmogorov spectrum evolves according to the explosive power law $k_b \propto (t_0-t)^{-2}$ and reaches infinity within a finite time. In the case of a free decay of acoustic turbulence, the Kolmogorov short-wave asymptotics (right up to the damping region) also have an explosive evolution. Once the right boundary of the Kolmogorov asymptotics has reached infinity, the total energy starts to decrease (see Sect. 4.3 for details). It should be noted that the distribution also expands to the long-wave region (to $k < k_0$), but this is a self-decelerating process. The relaxation to the distribution (5.1.9) was first observed in the numerical simulations by *Zakharov* and *Musher* [5.6]. The nonstationary behavior was studied by *Falkovich* and *Shafarenko* [5.7] (see also Sect. 4.3.3).

For the Kolmogorov spectrum the applicability parameter (3.1.33) of weak turbulence has in this case the form

$$\xi^{-1} = a^2 k^3 t_{NL} \propto k^{5/2} .$$

Thus the applicability criterion $\xi^{-1} \gg 1$ will be satisfied increasingly better, as the distributions move towards large k.

Let us discuss the stability of the isotropic spectrum with regard to anisotropic perturbations. As we shall now see, the stability problem for such systems is non-trivial because of its closeness to the degenerate case. Indeed, in the limiting case of a linear dispersion law $\omega_k = ck$, the conditions of spatio-temporal synchronization $\omega(\boldsymbol{k}_1+\boldsymbol{k}_2) = \omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2)$ allow only interactions of waves propagating along a line. Waves traveling at different angles do not interact. Therefor, besides the energy integral

$$E = \int k n_{\boldsymbol{k}}\, d\boldsymbol{k}$$

and components of the momentum

$$\boldsymbol{\Pi}_i = \int \cos\theta_i\, k n_k\, d\boldsymbol{k}\ ,$$

the kinetic equation has (in the limit) $\omega_k \to ck$ an infinite set of integrals of motion

$$\int f(\zeta) k n_k\, d\boldsymbol{k}\ ,$$

where $f(\zeta)$ is an arbitrary function of the angular variables: $\zeta = (\theta, \phi)$. Expanding $f(\zeta)$ in angular harmonics, one can thus obtain an infinite series of integrals

$$I_l^0 = \int Y_l(\zeta) k n_k\, d\boldsymbol{k}\ . \tag{5.1.10}$$

It is interesting to clarify what happens to these integrals under the action of small dispersion of the wave velocity. To use the stability theory developed in Sect. 4.2, we shall approximate the dispersion law by the scale-invariant expression $\omega_k \propto k^{1+\varepsilon}$, $\varepsilon \ll 1$. Due to the dispersion, the interaction of waves with noncollinear wave vectors becomes possible. However, it is easy to see from the analysis of the synchronization conditions that waves with wave numbers of the same order essentially interact only within a narrow cone of angles $\theta_{\text{int}} \simeq \varepsilon^{1/2}$ [see (5.1.24) below]. One can prove that for angular harmonics changing only a little bit on the scale of the interaction angle θ_{int} (i.e., for $l \ll \varepsilon^{-1/2}$), there exist integrals of motion of the linearized kinetic equation that are similar to (5.1.10)

$$I_l = \int Y_l(\zeta) k^{1+p_l+\varepsilon} n_k\, d\boldsymbol{k}\ . \tag{5.1.11}$$

Accordingly, there are power corrections to the isotropic Kolmogorov spectrum that are the stationary solutions of the linearized equation and transfer the constant fluxes of the integrals (5.1.11). These fluxes have the same direction as the energy flux of the isotropic solution. The latter is thus structurally unstable with respect to the excitation of angular harmonics with $l \ll \varepsilon^{-1/2}$. In this section we shall give a rigorous proof of the instability of the isotropic spectrum of three-dimensional sound turbulence and we shall analytically derive the anisotropic Kolmogorov solution supporting two fluxes, namely, of energy and momentum. Such an analytical derivation is possible due to the presence of the small parameter ε.

To apply the criteria obtained in Sect. 4.2 to the analysis of sound turbulence, we thus model the near-sound dispersion law by the scale-invariant expression

$$\omega_k = c k^{1+\varepsilon}\ . \tag{5.1.12}$$

The use of this dispersion law instead of (5.1.3) is sensible only in the three-dimensional case when in the acoustic limit the use of different dispersion laws leads to the same Kolmogorov solutions.

Let us substitute ω_k and V_{k12} specified by (5.1.12) and (5.1.5) into the three-wave kinetic equation (2.1.12)

$$\begin{aligned}\frac{\partial n_k}{\partial t} &= \int \big[U_{k12}(n_1 n_2 - n_1 n_k - n_2 n_k) \\ &\quad - 2U_{1k2}(n_k n_2 - n_1 n_2 - n_1 n_k)\big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2, \\ U_{k12} &= \frac{B}{c}(q + \cos\theta_{k1} + \cos\theta_{12} + \cos\theta_{k2})^2 \\ &\quad \times \delta(k^{1+\varepsilon} - k_1^{1+\varepsilon} - k_2^{1+\varepsilon})\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\ .\end{aligned} \tag{5.1.13}$$

We shall restrict ourselves to the axially symmetric case when $n(\boldsymbol{k}) = n(k,\theta)$. Let us integrate (5.1.13) over $d\boldsymbol{k}_2$ using $\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)$ and over $d\cos\theta_1$ using $\delta(k^{1+\varepsilon} - k_1^{1+\varepsilon} - k_2^{1+\varepsilon})$, to obtain

$$\begin{aligned}\frac{1}{B}\frac{\partial n_k}{\partial t} &= \int_0^k (q + \cos\theta_{k1} + \cos\theta_{k2} + \cos\theta_{12})^2 \big(k^{1+\varepsilon} - k_1^{1+\varepsilon}\big)^{(2-\varepsilon)/(1+\varepsilon)} \\ &\quad \times k_1^2 (n_1 n_2 - n_k n_1 - n_k n_2)\, dk_1\, d\phi \\ &\quad - 2\int_k^\infty (q + \cos\theta_{k1} + \cos\theta_{k2} + \cos\theta_{12})^2 \\ &\quad \times \big(k_1^{1+\varepsilon} - k^{1+\varepsilon}\big)^{(2-\varepsilon)/(1+\varepsilon)} k_1^2 (n_2 n_k - n_1 n_k - n_1 n_2)\, dk_1\, d\phi\ .\end{aligned} \tag{5.1.14}$$

Here $\phi = \phi_1 - \phi_k$, and the arguments of $n_1 \equiv n(k_1, \theta_1, \phi)$ should be taken, in accordance with δ-functions from (5.1.13), on the surface given by

$$\begin{aligned}\frac{(\boldsymbol{k}\boldsymbol{k}_1)}{kk_1} \equiv \cos\theta_{k1} &= \frac{k^2 - k_1^2 - \big(k^{1+\varepsilon} - k_1^{1+\varepsilon}\big)^{2/(1+\varepsilon)}}{2kk_1} \\ &\approx 1 + \varepsilon\frac{k - k_1}{kk_1}\left[k_1 \ln\frac{k_1}{k} + (k - k_1)\ln\frac{k - k_1}{k}\right]\end{aligned} \tag{5.1.15a}$$

in the first integral, and

$$\begin{aligned}\cos\theta_{k1} &= \frac{\big(k_1^{1+\varepsilon} - k^{1+\varepsilon}\big)^{2/(1+\varepsilon)} - k^2 - k_1^2}{2kk_1} \\ &\approx 1 - \varepsilon\frac{k_1 - k}{kk_1}\left[k_1 \ln\frac{k_1}{k} - (k_1 - k)\ln\frac{k_1 - k}{k}\right]\end{aligned} \tag{5.1.15b}$$

in the second integral. For

$$n_2 = n\Big(\big|k^{1+\varepsilon} - k_1^{1+\varepsilon}\big|^{1/(1+\varepsilon)}, \theta_2, \phi\Big)\ ,$$

one should use

$$\sin\theta_{k2} = \sin\theta_{k1}\frac{k_1}{\big|k^{1+\varepsilon} - k_1^{1+\varepsilon}\big|^{1/(1+\varepsilon)}}\ .$$

Similar expressions with the substitution $k_1 \leftrightarrow k$ are valid for the second integral. Due to the smallness of the dispersion ($\varepsilon \ll 1$), the angles between interacting waves are small, except for narrow regions in the k-space which are close to the integration limits $k_1 = 0, k$ and $k_1 \to \infty$. However, these regions do not contribute to the collision integral because of the locality of the interaction, i.e., because of the convergence of the integrals with all the solutions obtained, see (5.1.18, 24). Thus, one can approximate all cosines by unity and use $\varepsilon = 0$ everywhere except for calculations involving the angular arguments of the occupation numbers n_1, n_2.

Let us obtain the stationary solutions of the kinetic equation (5.1.5) linearized with regard to a small deviation from the isotropic Kolmogorov solution. Let

$$n_k = k^{-9/2}[1 + A(k,\theta)] = k^{-9/2}[1 - k^{-p} f(\cos\theta)] .$$

Making use of the smallness of the interaction angle, we employ the differential approximation for the angular variables and use in (5.1.14) the expansion

$$f(\cos\theta_1) = f(\cos\theta) - \left(f' \cos\theta - f'' \sin^2\theta \cos^2\phi\right)\frac{\theta_{k1}^2}{2} . \tag{5.1.16}$$

Thus we obtain from (5.1.14) an equation for $f(\cos\theta) = f(z)$ and the parameter p. This equation is a factor multiplied by the converging integral (which we met in Sect. 3.2.1, after the substitution $x \to 1/x$ it coincides with $I_3 \approx 0.2$)

$$\left(\varepsilon z\frac{df}{dz} - \varepsilon\frac{1-z^2}{2}\frac{d^2f}{dz^2} + pf\right) \int\limits_0^1 x^2(1-x)^2[x\ln x + (1-x)\ln(1-x)]$$
$$\times \left[x^{-9/2} + (1-x)^{-9/2} - x^{-9/2}(1-x)^{-9/2}\right] dx = 0 .$$

In the parenthesis in front of the integral we have the Legendre equation possessing a regular solution of the form

$$f_l = P_l(\cos\theta), \qquad p_l = -\varepsilon\frac{l(l+1)}{2} , \tag{5.1.17}$$

where the P_l are Legendre polynomials. Thus, for not too high harmonics satisfying the differential approximation ($\varepsilon l^2 \ll 1$) we obtain a set of neutrally stable modes of the form

$$A(k,\theta) = k^{\varepsilon l(l+1)/2} P_l(\cos\theta) . \tag{5.1.18}$$

For $l = 1$ the solution (5.1.18) coincides with the Kats-Kontorovich drift mode. The formula (5.1.17) has been obtained by *L'vov* and *Falkovich* [5.8].

The existence of the set of stationary solutions (5.1.18) implies that in this approximation the Mellin functions $W_l(s)$ have (with an accuracy to terms of the order of $\varepsilon^2 l^4$) zeros in the points $s = p_l$. Does the exact expression (4.2.16a) for $W_l(s,\varepsilon)$ have a zero? At small ε it does due to the implicit function theorem [5.9]: if $W_l(s,0)$ has the zero $W_l(0,0) = 0$ and $[\partial W_l(s,\varepsilon)/\partial s]_{0,0} \neq 0$, then $W_l(s,\varepsilon)$ also has a zero for sufficiently small ε. The derivatives of the Mellin functions with respect to s may be calculated directly [5.11] to verify that for $l \ll \varepsilon^{-1/2}$ we have

$$\left(\frac{\partial W_l(s,\varepsilon)}{\partial s}\right)_{s=\varepsilon=0} > 0 ,$$

i.e., the implicit function theorem is applicable to answer our question affirmative. It is also easy to prove that in the zeros $s = p_l$ of the $W_l(s)$ functions their derivatives are also positive. As we already know, this implies that the fluxes of the integrals of motion (5.1.11) transferred by the modes (5.1.18) are also positive. Since the initial isotropic solution also transfers a positive energy flux, the modes (5.1.18) should be formed in the presence of an anisotropic source. Indeed, one can directly calculate that

$$W_l(0) = l(l+1)I_3 > 0 ,$$

i.e., criterion (4.2.55) is satisfied.

It is also easy to check if the function $W_l(s)$ has a zero rotation interval. It is convenient to start from the zeroth spherical harmonic for which the Mellin function can be calculated directly and is expressed via the gamma functions [5.10, 11]

$$W_0(s) = -\frac{8}{3} + \frac{1}{(s^2 - 1/4)(s + 3/2)} + \frac{\Gamma(-3/2)\Gamma(4+s)}{4\Gamma(s+5/2)}\,\frac{1 + \cos\pi s + \sin\pi s}{\cos\pi s} . \tag{5.1.19}$$

This function has the analyticity strip $\Pi(-1/2, 1/2)$ and the zero rotation interval $\sigma_-^0 = -1/2$, $\sigma_+^0 = 0$. At $\varepsilon l^2 \ll 1$ we have $W_l(s) \approx W_0(s)$ so that the functions $W_l(s)$ also have zero rotation intervals (σ_-^l, σ_+^l) close to (σ_-^0, σ_+^0). Thus, the conditions for structural instability are satisfied. This was first proved by *Balk* and *Zakharov* [5.10].

For $p_l < 0$, the zeros of the Mellin functions are located on the left of $s = 0$. Hence, the discussed instability is found in the region of large k, deep in the inertial interval. For three-dimensional sound, $h = \alpha - m = -1/2 < 0$, i.e., the instability is of the hard interval type. Thus, a weakly anisotropic source should generate in the inertial interval a distribution of the form

$$n(k,\theta) = \lambda P^{1/2} k^{-9/2}\left[1 + \sum_{l=1}^{L} c_l P_l(\cos\theta) k^{-\varepsilon l(l+1)/2}\right] . \tag{5.1.20}$$

As seen from (5.1.20), the higher the number of the angular harmonic, the quicker is the increase (with growing k) of its contribuiton to the stationary spectrum. Thus, a small anisotropy of a source located in the region of small k, leads to an essentially anisotropic spectrum in the short-wave region. This phenomenon was first predicted by *L'vov* and *Falkovich* [5.8].

It is interesting to have a closer look at the form of the stationary spectrum n the region of large k where the anisotropic part of the solution is no longer small and the linear approximation is inapplicable. At $l > 1$ it is only in the linear approximation that the quantities (5.1.11) are integrals of motion. One can come up with the hypothesis that in the inertial interval an essentially anisotropic spectrum should be determined by the fluxes of the first two integrals (of energy $l = 0$) and momentum ($l = 1$)) which are also conserved quantities within the nonlinear kinetic equation (5.1.13). Such a two-flux universal solution generalizing the Kolmogorov solution has been analytically constructed by *L'vov* and *Falkovich* [5.12]. According to the dimensional relation $Pk \propto R\omega_k$, the two-flux spectrum should have the form (4.1.5)

$$n_k = P^{1/2}k^{-9/2}F(y), \quad y = \frac{(\boldsymbol{R}\boldsymbol{k})\omega(k)}{Pk^2} . \tag{5.1.21}$$

Here $P, \boldsymbol{R}$ are the energy and momentum fluxes, respectively.

The form of the dimensionless function $F(y)$ may in this case be found using the smallness of the dispersion and employing the differential approximation in the variable y. Indeed, for k and k_1 we obtain from (5.1.14)

$$\begin{aligned} y_1 &= \frac{(\boldsymbol{R}\boldsymbol{k})\omega(k_1)}{Pk_1^2} \equiv \cos\theta_1 \left(\frac{k_1}{k_a}\right)^{\varepsilon} \\ &\approx \left[\cos\theta + \theta_{k1}\sin\theta\cos\phi + \left(\varepsilon\ln\frac{k_1}{k} - \frac{\theta_{k1}^2}{2}\right)\cos\theta\right]\left(\frac{k}{k_a}\right)^{\varepsilon}, \end{aligned} \tag{5.1.22}$$

i.e., $|y_1 - y| \ll y$. Now we expand in (5.1.14) the functions $F(y_1)$ and $F(y_2)$ up to terms of the order of ε. Then, we split the second integral up into two identical terms and make in one of them the substitution $k_1 \to k^2/k_1$ and in the other, $k_1 \to kk_1/(k-k_1)$ (Zakharov transformations, see Sects. 3.1, 2). Recalling then that $n_k \propto k^{-9/2}$ is an exact solution, we obtain to the first order in ε the equation

$$\left[\left(\frac{\partial F}{\partial y}\right)^2 + F\frac{\partial^2 F}{\partial y^2}\right]\left(\frac{k}{k_a}\right)^{\varepsilon}\sin^2\theta\, I_3 = 0 .$$

The equation for $F(y)$ is given by a factor multiplied by the converging integral I_3. The solution is

$$F(y) = \begin{cases} \sqrt{C_1 y + C_2} & \text{for } y > -C_2/C_1 \\ 0 & \text{for } y < -C_2/C_1 . \end{cases} \tag{5.1.23}$$

According to (5.1.21) and (5.1.23), the distribution n_k should vanish on a surface in the k-space. As a matter of fact, at $y \to -C_2/C_1$ the derivatives of $F(y)$ increase sharply and in the close vicinity of the surface (at $y + C_2/C_1 \lesssim \sqrt{\varepsilon}$) the applicability conditions of the differential approximation are violated. This tells us that it is not sufficient to consider only the first and second derivatives. The solution of the initial equation (5.1.14) should lead to a smooth, but (on the scale of the characteristic angle of interaction, i.e., $\sqrt{\varepsilon}$) rapidly decreasing function $F(y)$. At $y \to -\infty$ the function $F(y)$ should tend to zero. The integration constants C_1 and C_2 may be included into the definition of the fluxes R and P. If that is intended, then the constant C_1 must be considered to be positive, because the substitution $C_1 \to -C_1$ implies a simple rotation of the coordinate system $\theta \to \pi - \theta$. The two signs of C_2 specify two different families of solutions .

$$n_k = k^{-9/2}\left(\frac{R\omega_k\cos\theta}{k} + P\right)^{1/2}, \tag{5.1.24a}$$

$$n_k = k^{-9/2}\left(\frac{R\omega_k\cos\theta}{k} - P\right)^{1/2}. \tag{5.1.24b}$$

The first of these, (5.1.24a), corresponds to spectrum narrowing with the growth of k. In particular, it should describe a stationary distribution which is generated by a weakly anisotropic source located at $k = k_0$ and supports a small momentum flux $[R\omega(k_0) \ll Pk_0]$. Expansion of (5.1.24) in $R\omega(k)/(Pk)$ yields then at small k: in the zeroth order, the isotropic Kolmogorov solution; in the first order, the Kats-Kontorovich drift correction; in higher orders, the higher harmonics (5.1.17) whose contributions to the spectrum increase with k. At large k practically all waves are concentrated in the right hemisphere.

The solution (5.1.24) describes an expanding spectrum. Its width $\Delta\theta(k)$ increases with k according to the law

$$\cos\Delta\theta(k) = \frac{Pk}{R\omega_k} .$$

If on the boundary of the inertial interval (at $k = k_0$) $R\omega(k_0) \approx Pk_0$, then the initial width of the spectrum $\Delta\theta(k_0)$ may be very small. From below the quantity $\Delta\theta(k_0)$ is limited only by the interaction angle $\sqrt{\varepsilon}$, because at such a width the differential approximation used to obtain the solution (5.1.24) becomes invalid. Thus, the solution (5.1.24b) should presumably be generated by narrow sources with widths $\sqrt{\varepsilon} < \Delta\theta \ll \pi/2$. It is essential that in the limit of large k and at $-\pi/2 < \theta < \pi/2$ the solutions (5.1.24a) and (5.1.24b) coincide. The spectrum is entirely determined by the momentum flux and presents a wide jet whose angular form does not depend on the form of the boundary conditions

$$n_k \to k^{-(9+\varepsilon)/2}(R\cos\theta)^{1/2} .$$

As we see, the family of solutions (5.1.24) allows for a wide range of boundary conditions at small k (in the vicinity of a source): from isotropic to extremely narrow with a width of the order of the interaction angle. If such a solution is really generated by arbitrary pumping, then the universality hypothesis is still alive in spite of the structural instability of the isotropic Kolmogorov spectrum. But the hypothesis should be reformulated in a more sophisticated form: a steady spectrum in the inertial interval should be defined by the influxes of all motion integrals whose fluxes are directed from source to sink.

5.1.2 Two-Dimensional Acoustics with Positive Dispersion: Gravity-Capillary Waves on Shallow Water and Waves in Flaky Media

This section is devoted to the two-dimensional turbulence of sound with positive dispersion, see the dispersion law (5.1.1). We met it first in Sect. 1.2.5 when we studied short waves ($k \gg \sqrt{\varrho g/\sigma}$ with ϱ, σ, g are the density, surface tension coefficient and gravity acceleration, respectively) on shallow water ($kh \ll 1$), see (1.2.39). Such a dispersion also describes the evolution of sound in flaky media with a weak interaction between the layers. Since we again consider the long-wave case, the interaction coefficient is given by (5.1.5). We can obtain the kinetic equation in the same way as in the case of (5.1.7) and it has also a rather similar form

$$\begin{aligned}\frac{\partial n(\boldsymbol{k},t)}{b\partial t} = \int dk_1 d\theta_1 k k_1^2 (k-k_1)\Big\{&\delta\big(3ca^2 k k_1(k-k_1) - \theta_1^2 k k_1/2\big)\\ &\times \big[n(\boldsymbol{k}_1)n(\boldsymbol{k}-\boldsymbol{k}_1) - n(\boldsymbol{k})n(\boldsymbol{k}_1) - n(\boldsymbol{k})n(\boldsymbol{k}-\boldsymbol{k}_1)\big]\\ &-2\delta\big(3ca^2 k k_1(k_1-k) - \theta_1^2 k k_1/2\big)\\ &\times \big[n(\boldsymbol{k})n(\boldsymbol{k}_1-\boldsymbol{k}) - n(\boldsymbol{k}_1)n(\boldsymbol{k}) - n(\boldsymbol{k}_1)n(\boldsymbol{k}_1-\boldsymbol{k})\big]\Big\}\,.\end{aligned} \tag{5.1.25}$$

As usual the applicability criterion for this equation has the form (2.1.25)

$$bk^2N \ll c(ak)^2\,.$$

For isotropic distributions we can integrate (5.1.25) over angles to obtain (3.2.3)

$$\begin{aligned}\frac{\partial n(k,t)}{\partial t} &= \frac{2\pi b}{\sqrt{6}a}\\ &\times\Big\{\int_0^k k_1(k-k_1)\{n(k_1)n(k-k_1) - n(k)[n(k_1)+n(k-k_1)]\}\,dk_1\\ &-\int_k^\infty k_1(k_1-k)\{n(k)n(k_1-k) - n(k_1)[n(k)+n(k_1-k)]\}\,dk_1\Big\}.\end{aligned} \tag{5.1.26}$$

Its isotropic Kolmogorov solution (2.4.5)

$$n(k) = \lambda P^{1/2}k^{-3}\,, \qquad \lambda \approx (2b)^{-1/2}(2\pi)^{-1}\,, \tag{5.1.27}$$

carries the energy flux P. It was obtained by *Falkovich* [5.13] and *Musher* [5.14]. Spectrally narrow pumping at intermediate k should generate the stationary distribution (3.4.14–15) consisting of a chain of sharp peaks with the amplitudes obeying the law $n(k_j) \propto k_j^{-4}$, $k_j = jk_0$, see Fig. 3.13 [5.6]. With growing k such a pre-Kolmogorov solution goes over into a Kolmogorov solution.

According to the general formula (3.4.8), the flux P can be expressed in terms of the pumping characteristics (3.4.8). If an external environment generates waves with wave numbers of the order k_0 with the growth-rate Γ_0, then the energy flux (i.e., energy dissipation rate) can be evaluated according to

$$P \simeq (\Gamma_0 c\lambda)^2\,,$$

i.e., it is indepedendent of k_0. This is related to the fact that $h = 0$ in this case. Thus the energy is uniformly distributed over all scales of the inertial interval. The spectrum is formed in a "nonexplosive" process. The free decay takes place in two stages – see Fig. 4.10. The first corresponds to energy conservation and can be described in terms of the exponential self-similar solution (4.3.14). When the boundary of the Kolmogorov spectrum has reached the damping region, the energy starts to decrease like $E(t) \propto t^{-1}$. When going over towards large k, the applicability criterion for weak turbulence improves since in this case $\xi(k) \propto k^{\epsilon}$, see (3.1.33).

The stability of an isotropic spectrum of two-dimensional acoustic turbulence deserves special consideration for two reasons. First, in this case $h = m+d-s_0 = 1+2-3 = 0$, so that the stability criteria obtained in Sect. 4.2 are inapplicable. Second, in two dimensions the weak decay dispersion law cannot be modeled by the scale invariant formula $\omega_k \propto k^{1+\epsilon}$; the true expression (5.1.1) has to be used. This gives rise to large mathematical difficulties in the treatment of the stability problem, because the kinetic equation has a scale-invariant limit at $a \to 0$ only in the isotropic case, see (5.1.26). In the description of anisotropic perturbations, the dispersion length a enters the angular dependences of the occupation numbers and may not be separated out as a factor from the collision integral as in (5.1.26). Due to this, we shall treat the stability problem of Kolmogorov solutions with regard to isotropic perturbations analytically, while numerical simulations will have to be used for the anisotropic case.

Let us follow *Falkovich* [5.13] in the treatment of the isotropic case. At $h = 0$, the operator of the linearized collision integral has the scaling index zero. The eigenfunctions of such an operator are obviously the power functions

$$\delta n_s(k,t) = k^{-s}e^{W(s)t}\,,$$

where the Mellin function $W(s)$ is an eigenvalue.

Thus the stability problem reduces to the calculation of $W(s)$ and the study of the evolution of localized perturbations consisting of a superposition of eigenfunctions

$$\delta n(k,t) = \int_\gamma g(s)k^{-s}e^{W(s)t}\,ds\,, \tag{5.1.28}$$

where $g(s)$ and the contour γ are such that at $k \to 0, \infty$ we have for the perturbation $\delta n(k,t) \to 0$.

Let us linearize (5.1.26) with regard to small deviations from the Kolmogorov solution $n_k^0 = Nk^{-3}$. Substituting $n(k,t) = n_k^0 + \delta n(k,t)$ and using $D = \pi b\sqrt{2/3} = \pi\sqrt{2/3}B/c$ we have

$$\frac{a}{2DN}\frac{\partial \delta n(k,t)}{\partial t} = -\int_0^\infty \left\{ \frac{\delta n(k,t)k}{k_1^2} + \delta n(k_1,t) \right. \tag{5.1.29}$$
$$\left. \times \left[(k-k_1)^{-2} + (k+k_1)^{-2} - 2k^{-2}\right] k_1\, dk_1 \right\} .$$

Equation (5.1.29) has fundamental solutions of the form $k^{-s}e^{W(s)t}$. The eigenvalues are determined by the integral

$$W(s) = \frac{2DN}{a}\int_0^\infty \left[\frac{x^{1-s}}{(x-1)^2} + \frac{x^{1-s}}{(x+1)^2} - 2x^{1-s} - 2x^{-2}\right] dx$$
$$= \frac{2DN}{a}(1-s)\cot\frac{\pi s}{2} , \tag{5.1.30}$$

which converges in the strip $2 < \mathrm{Re}\, s < 4$. This implies that among the perturbations having the asymptotics $\delta n_k \propto k^{-s}$, only those with $\mathrm{Re}\, s \in (2,4)$ are local. For them the main wave interaction takes place between wave vectors of the same order. The evolution of nonlocal perturbations should not be discussed in terms of (5.1.29), but rather by implicitly introducing into the integration a cut-off on the scale of source and sink, taking into account the distortions of the stationary solution that occur because of finiteness of the inertial interval. We shall restrict ourselves to the study of the evolution of local perturbations. We assume the evolution of nonlocal perturbations to proceed in two steps. The fast first process will – within times determined by the scales of the source k_0 or sink k_d – lead to a rearrangement of the perturbation (e.g., an initially narrow packet will broaden and the slowly diminishing tails will damp and start to diminish rapidly). Further on it will become local and evolve in the manner discussed below.

As seen from (5.1.30), we have $W(3) = 0$ which corresponds to an indifferent stability of the Kolmogorov solution with regard to variations in the energy flux. It should be noted that $W(s)$ is complex in the strip $\mathrm{Re}\, s \in (2,4)$. This is due to the fact that the $\hat{L}$ operator of the kinetic equation linearized with regard to small deviations from the Kolmogorov spectrum is in general nonhermitian, in contrast to the weakly nonequilibrium case, see Sect. 4.1.

Among the fundamental solutions of the form $k^{-s}e^{W(s)t}$ there are solutions for which $\mathrm{Re}\, W(s) > 0$, see (5.1.30). Since the eigenfunctions k^{-s} do not satisfy the boundary conditions, this does not imply instability of the Kolmogorov solution. It is more convenient to impose on the perturbations physical conditions in terms of $F(x,t) = \delta n(k,t)/n_k^0$, where $x = \ln(k/k_0)$ [in the variable x the representation (5.1.28) goes over to the Fourier transformation of the perturbation]. Demanding finiteness of the perturbation energy

$$\delta E = \int_0^\infty ck^2 \delta n(k,t)\, dk \le M < \infty \tag{5.1.31}$$

leads to the condition $F(x) \to 0$ at $x \to \pm\infty$, with $F(x) \propto \exp(-\varepsilon_1 x)$ at $x \to +\infty$ and $F(x) \propto \exp(\varepsilon_2 x)$ at $x \to -\infty$ with $\varepsilon_1, \varepsilon_2 > 0$. It is also convenient to "translate" the analyticity strip by going over from s to $z = (s-3)/2$:

$$\frac{a}{2DN}W(z) = 2(1+z)\tan\pi z . \tag{5.1.32}$$

As we shall show now, the function $W(s)$ (5.1.30, 32) has such a structure that any perturbation localized in the k-space is swept over towards large k. Let us first consider hump-shaped perturbations given locally by $F(x,t) = F_0 \exp[-2zx + W(z)t]$ where z is a slow real function of x $(|dz/dx| \ll |z/x|)$, such that on the right of the maximum $0 \le z < 1/2$ and on the left $-1/2 < z \le 0$. As we are discussing local perturbations $(-1/2 < z < 1/2)$, the evolution of each section is entirely determined by wave vectors of the same order, i.e., by the quantity $W\{z(x)\}$. From (5.1.20) we see that on the real axis $\mathrm{sign}\, W(z) = \mathrm{sign}\, z$ holds at $z \in (-1/2, 1/2)$. This implies that a perturbation of the above type develops in the following way: the right slope increases, the left one decreases and the maximum value remains constant $[W(z=0) = 0]$. This is readily understood to correspond to a packet traveling to the right. Its top moves according to the law $k(t) = k_0 \exp[W'(0)t] = k_0 \exp(2DNt/a)$. The exponential character of the motion is readily appreciated if it is borne in mind that the condition $h = 0$ implies the independence of the characteristic nonlinear interaction time on k. Therefore, e.g., in its travel across the spectrum any wave doubles its wave vector during a period of time independent of k.

The arbitrary localized solution of (5.1.29) may be written in the form of (5.1.28) where the contour γ extends into the analyticity strip ranging from $-i\infty$ to $+i\infty$ and $g(z)$ is the Fourier transformation of the initial perturbation $F(x,0)$. Having aligned the contour γ with the axis $\mathrm{Re}\, s = 0$, $s = i\sigma$, we see that arbitrary localized perturbations are damped, since $a\mathrm{Re}\, W(z) = -2DN\sigma \tanh\sigma < 0$.

Now we discuss this in more detail. First we take the narrow Gaussian wave packet $g(z) = \exp\left[(z-z_0)^2/\Delta^2\right]$ with $(\Delta \ll |z_0| = |\kappa_0 + i\sigma_0|)$. We shall calculate the integral (5.1.28) by the saddle point method. The saddle point z_* is specified by

$$t\frac{\partial W(z_*)}{\partial z} + 2\frac{z_* - z_0}{\Delta^2} - 2x = 0 \tag{5.1.33}$$

i.e., $z_*(x,t)$ depends on x and t. Let us discuss the behavior of the maximum of the envelope. We shall designate the coordinate of the maximum coordinate by $x_0(t)$. As may be easily seen from (5.1.32–33), $x_0(0) = -\kappa_0/\Delta^2$ holds and the saddle point $z_*(x_0,t) = z_*^0(t)$ with $x = x_0$ is located at the imaginary axis

$$z_*^0(0) = i\sigma_0 , \quad aW(i\sigma_0) = 2DN(2i\tanh\sigma_0 - 2\sigma_0\tanh\sigma_0) .$$

At $t \ll \sigma_0\cosh^2\sigma_0$ (see below) we have

$$x_0(t) = -\frac{\kappa_0}{\Delta^2} + t\mathrm{Re}\frac{\partial W(z_*^0)}{\partial z} , \quad z_*^0(t) = -i\sigma_0 - it\Delta^2\left(\tanh\sigma_0 + \frac{\sigma_0}{\cosh^2\sigma_0}\right) .$$

The saddle point $z_*^0(t)$ moves along the imaginary axis towards zero. Since on the imaginary axis the velocity is $\mathrm{Re}\, W'(i\sigma) = 4DN/a\cosh^2\sigma > 0$, the narrow Gaussian packet travels towards large x, with a decreasing maximum of the envelope $[\mathrm{Re}\, W(i\sigma) < 0]$. As $\mathrm{Im}\, W(i\sigma) \neq 0$, the damping of the packet is accompanied by oscillations. At $\sigma_0 \ll 1$, the oscillation period is much smaller than the damping time.

But any perturbation is eventually transferred to large x leaving behind a trail damping out with time. Indeed, let us consider $t \to \infty$ and $x/t = v = $ const. For $x \to +\infty$, the function $g(z)$ is analytical at $\mathrm{Re}\, z \geq 0$. Shifting the integration contour to the right (to the saddle point which at $v > W'(0)$ is located on the real axis and is given by $W'(\kappa) = v$) we get

$$F(x,t) \propto \exp\left[t\left(W(\kappa) - \kappa W'(\kappa)\right)\right] .$$

It is easy to show that for the function (5.1.32) we have at $\kappa > 0$ the relation $W(\kappa) - \kappa W'(\kappa) < 0$ which corresponds to a system without a convective instability. With regard to the trail left behind it has to be noted that at $t \to \infty$ the saddle point is for finite x ($ax \ll 2DNt$) determined by the condition $W'(z_*) = 0$ or $\sin 2\pi z_* = -2\pi(1 + z_*)$. This transcendental equation has two roots. They are located in the left half-plane (approximately $z_* \approx -1/4 \pm 2i/5$). Since $\mathrm{Re}\, W(z_*) < 0$ we thus have

$$F(x,t) \propto \exp[x/2 + W(z_*)t] ,$$

showing that the trail is indeed decaying or damped as time progresses.

As we have seen above, the drift of the packet is determined by the quantity $W'(0)$. The same derivative is proportional to the energy flux of the Kolmogorov spectrum.

Summarizing, we conclude that small isotropic perturbations against the background of the Kolmogorov spectrum should be driven towards the damping region without growing in magnitude. If the perturbation have a structure oscillating in k-space [$\sigma_0 \neq 0$, $F \propto \exp(i\sigma_0 \ln k)$], the drift is accompanied by damping and oscillations in time.

Now we go over to discuss the stability of the isotropic spectrum of two-dimensional sound turbulence with regard to anisotropic perturbations. Again, like in the three-dimensional case, one can try to analytically derive the stationary anisotropic solutions of the linearized kinetic equation. The small parameter characterizing the approximation of the dispersion law to the linear one is in this case the quantity $(ak)^2$ rather than ε. As a consequence, the drift solution

$$A(k,\theta) = \frac{\omega_k(\boldsymbol{R}\boldsymbol{k})}{Pk^2} \propto (ak)^2 \cos\theta$$

has already for the first harmonic an index which is located outside the limits of the locality strip. The same holds for all subsequent harmonics; the linearized collision integral diverges for such solutions. The authors of Ref. 5.8 attempted an analytical derivation of the stationary anisotropic solutions by introducing a formal cut-off into the diverging integrals and requiring the most divergent terms to vanish. This lead to stationary solutions of the form

$$A_l(k,\theta) \propto (akl)^2 \cos(l\theta) . \tag{5.1.34}$$

However, these calculations fail to provide a rigorous proof. The structure of a stationary spectrum in the presence of an anisotropic source may be discussed with the help of computer simulations. Let us consider the two-dimensional kinetic equation (5.1.25) with an external source

$$\begin{aligned}\frac{\partial n(k,t)}{b\partial t} = \int \Big[&\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)(n_1 n_2 - n_k n_1 - n_k n_2) \\ &\times \delta\left(k - k_1 - k_2 + a^2(k^3 - k_1^3 - k_2^3)\right) - 2\delta(\boldsymbol{k}_1 - \boldsymbol{k} - \boldsymbol{k}_2) \\ &\times \delta\left(k_1 - k - k_2 + a^2(k_1^3 - k^3 - k_2^3)\right) \\ &\times (n_2 n_k - n_1 n_k - n_1 n_2)\Big] k k_1 k_2 \, d\boldsymbol{k}_1 d\boldsymbol{k}_2 + \Gamma_k n_k .\end{aligned} \tag{5.1.35}$$

As above, we have neglected the angular dependence in the interaction coefficient by using $\cos\theta_i \approx 1$. Let us integrate (5.1.35) over $d\boldsymbol{k}_2$ using the δ-function in the wave vectors and over $d\theta_1$ using the frequency δ-function. To first order in (ak) we obtain an equation formally coinciding with (5.1.26)

$$\begin{aligned}\frac{\partial n_k}{\partial t} = \frac{b2\pi}{\sqrt{6}a}\Big[&\int_0^k k_1(k - k_1)(n_1 n_2 - n_k n_1 - n_k n_2)\, dk_1 \\ &- 2\int_k^\infty (k_1 - k)k_1(n_k n_2 - n_1 n_2 - n_1 n_k)\, dk_1\Big] + \Gamma_k n_k .\end{aligned}$$

In this equation we took $n_k = n(k,\theta,t)$ and assumed

$$n_1 = n\left(k_1, \theta \pm \sqrt{6}a(k - k_1), t\right), \quad n_2 = \left(k - k_1, \theta \mp \sqrt{6}ak_1, t\right)$$

in the first integral and

$$n_1 = n\left(k_1, \theta \pm \sqrt{6}a(k_1 - k), t\right), \quad n_2 = n\left(k_1 - k, \theta \pm \sqrt{6}ak_1, t\right)$$

in the second one.

To remove the constant factor in front of the collision integral, we shall renormalize the occupation numbers via the substitution $n_k \to n_k\sqrt{6}a/b2\pi$. The resulting equation for the discrete case ($k = ik_0$, $\mu = \sqrt{6}ak_0$)

$$\begin{aligned}\frac{\partial n(i,\theta,t)}{\partial t} = &\sum_{j=1}^{i-1} i(i-j)\{n(j, \theta \pm \mu i \mp \mu j) n(i-j, \theta \mp \mu j) \\ &- n(i,\theta)[n(j,\theta \pm \mu i \mp \mu j) + n(i-j, \theta \mp \mu j)]\} \\ &- 2\sum_{j=i+1}^{L} j(j-i)\{n(i,\theta) n(j-i, \theta \pm \mu j) \\ &- n(j, \theta \pm \mu j \mp \mu i)[n(i,\theta) + n(j-i, \theta \pm \mu j)]\} \\ &+ \Gamma_i n(i,\theta) - 2n(i,\theta)\sum_{j=L}^{L+i} j(j-i) n(j-i, \theta \pm \mu j)\end{aligned} \tag{5.1.36}$$

was numerically solved by *Falkovich* and *Shafarenko* [5.14] in θ, i coordinates on a rectangular grid. At $i > L$ the occupation numbers were assumed to be zero. As mentioned in Sect. 3.4, [see (3.4.18)], this assumption gives rise to the last term in (5.1.36) which plays the role of nonlinear damping. Periodic boundary conditions in the coordinate θ were imposed. The dimensions of the coordinate grid were chosen to be equal to 100 points in k and 32 points in θ. The summations in (5.1.36) were performed separately for the upper and lower signs before μ. The integration path in the plane θ_1, j is a line with the slope μ; at sufficiently small μ it passes through between the nodes of the grid. Therefore, the values $n(j, \theta \pm \mu i \mp \mu j)$ and $n(i-j, \theta \mp \mu j$ were calculated using a linear approximation inbetween the two closest grid nodes taking into account the periodicity in θ.

The evolution of the two-dimensional spectrum was modeled starting from the isotropic solution. A weak angle modulation was imposed on the source ($\varepsilon = .01$)

$$\Gamma(i, \theta) = 100 \Delta_{i1}(1 + \varepsilon \cos m\theta) .$$

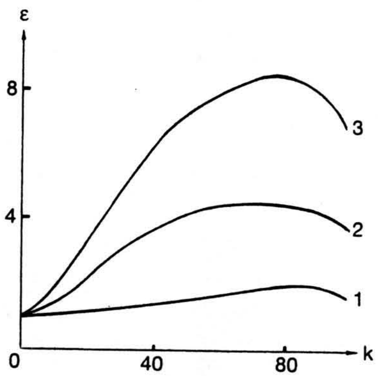

Fig. 5.1. Dependence of the relative angular modulation ε on k for different angular harmonics l and dispersion parameters μ: **1)** $l = 1$, $\mu = \pi/256$, **2)** $l = 2$, $\mu = \pi/256$, and **3)** $l = 1$, $\mu = \pi/128$

The results of the numerical simulations are represented in Fig. 5.1 showing the dependence of the relative depth of the modulation angle of the emerging spectrum [$n_{max} = \max_\theta n(k, \theta)$]

$$\varepsilon(k) = \frac{n_{max}(k) - n_{min}(k)}{n_{max}(k) + n_{min}(k)}$$

on the modules of the wave vector k. The quadratic dependence $\varepsilon = \varepsilon_0 + \nu k^2$ characterizes the initial section (for curve 3 up to $k \simeq 15$, $\varepsilon'''/\varepsilon'' \simeq 10^{-1}$), then the finiteness of $\mu i = \sqrt{6} ak$ starts to show itself. Comparison of curves 2 and 1 shows that under the same conditions the first harmonic increases by 1.02×10^{-2} and the second one, by 3.74×10^{-2}. This corresponds approximately to the relationship (5.1.34) $A_l \propto l^2$ [the difference by a factor of 3.7 rather than 4 seems to be due to the fact that for the second harmonic finiteness of μi begins to show itself at smaller $i = k/k_0$ and the growth of $\varepsilon(k)$ slows down]. The fact that the reduced growth of the $\varepsilon(k)$ is not only associated with influence of the right end of the inertial interval, but also with the finiteness of μi follows, e.g., from a comparison of curves 1 and 2. For curve 1, $\varepsilon(k)$ reaches its maximum at $k = 90$, and for 3 (corresponding to μ which is twice as high), at $k = 78$. The dependence of the occupation numbers on the angle is sinusoidal at all k. It reproduces the

form of the source to an accuracy not worse than 10^{-4}. This is due to the smallness of the angular modulation ($\varepsilon \ll 1$) and to the fact that the operator of the kinetic equation linearized against the background of the isotropic solution is diagonal with regard to the number of the angular harmonics. Increasing ε, we go over into the nonlinear region where different angular harmonics interact with each other more strongly. Figure 5.2 corresponding to $\varepsilon = 1/5$ shows the relative contributions of the first three angular harmonics of the emerging spectrum for different values of k.

$$n_l(k) = \int n(k, \theta) a^{il\theta} \, d\theta .$$

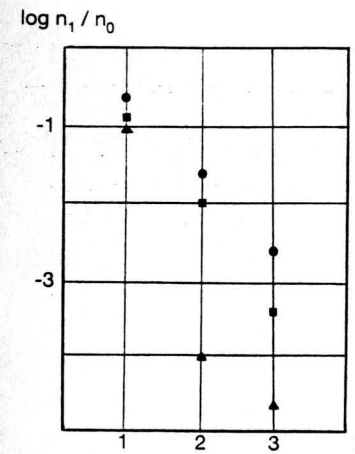

Fig. 5.2. Relative contributions of the first three harmonics: • $k = 2$, ■ $k = 40$, △ $k = 90$

The contributions from the higher harmonics are seen to increase with k. However, this does not imply that for large k the spectrum rapidly oscillates with the angle. Thus we have completed modeling the system at $\varepsilon = 1$, i.e., for a strongly anisotropic source. The dependence of the spectrum on θ is for all k smooth, see Fig. 5.3. The angular width of the spectrum $\Delta\theta$ decreases with the growth of the wave vector (Fig. 5.3).

It should be noted that up to $k = 50$, the quantity $d\Delta\theta/dk$ decreases with the growth of k. This possibly corresponds to $\Delta\theta$ approaching an asymptotically constant value. The increase of $d\Delta\theta/dk$ at $k > 50$ may be attributed to the influence of the right end of the inertial interval. Also up to $k \simeq 50$, the following effect is observed: in the direction of maximal values of the quantities $\Gamma(\theta)$ and $n_k(\theta)$, the spectrum decays slower than the isotropic one and the index $s(k) = \partial \ln[n(k, \theta)]/\partial \ln k$ decreases from $s(3) = 3.0$ to $s(50) = 2.4$ while in the direction $\theta = \pm\pi/2$ it increases: $s(3) = 3.0$ and $s(50) = 3.2$.

Such a behavior qualitatively corresponds to the properties of the two-flux solution (5.1.24a) obtained for three-dimensional sound.

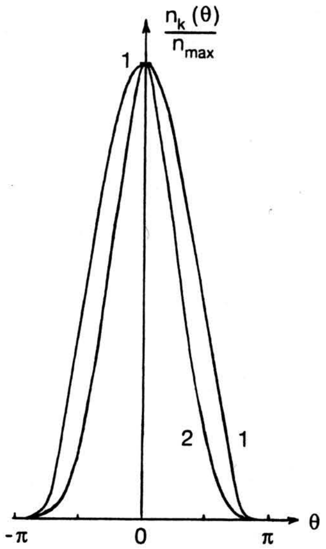

Fig. 5.3. Dependence of the stationary spectrum on the angle θ, 1) $k = 2$, 2) $k = 50$

It is of interest to note the nonmonotonic character of the relaxation of the anisotropic spectrum at large ε ($\varepsilon = 0.2, 1$). Thus, for $\varepsilon = 1$, the angular width $\Delta\theta$ for $k \le 5$ diminishes at the final relaxation stage, for $k > 5$, it grows with time. For $\varepsilon = 0.2$ the value $\Delta\theta$ at first grows and is then reduced down to stationary values. It should be borne in mind that the initial distribution was isotropic. Now we would like to make a general remark. The effect of an increased anisotropy of the spectrum for growing k may be more pronounced at the relaxation stage than in the steady state (in which such an effect may not be observed at all). This is the result of a nonmonotonic evolution of occupation numbers during the formation stage and of the inversely proportional dependence of the relaxation time on the amplitude of the source. Indeed, let us consider as a model the limiting case of nondispersive sound $\omega(k) = ck$ where only the waves with collinear wave vectors interact and relaxation occurs along the direction of each beam independently of the other beams. Clearly, towards the directions of increasing source amplitude, the spectrum is closer to the formed one, i.e., with growing k it decays from a certain moment of time onwards slower than in the directions into which the relative angular modulation of the spectrum $\varepsilon(k)$ is an increasing function of k. On the other hand, it is evident that in the stationary state of such a model the quantity ε is independent of k and is equal to the relative modulation of the source. In a system with the dispersion relation $d^2\omega/dk^2$ allowing for the interaction of waves moving at different angles, such an "intermediate" anisotropy (at intermediate times) may be less pronounced or nonexistent.

Taking into account the terms of the next order $\sim (ak)^2$ in the $n(k)$ and $V(k, k_1, k_2)$ does not change the conclusions arrived at [5.15].

Thus, the isotropic spectrum of turbulence is structurally unstable for two-and three-dimensional weak sound turbulence. For large k the spectrum should be essentially anisotropic.

5.1.3 Nondecay Acoustic Turbulence: Ion Sound, Gravity Waves on Shallow Water and Inertio-Gravity Waves

Long-Wave Acoustic Turbulence. This section is devoted to acoustic waves with a nondecay dispersion law

$$\omega(k) = ck(1 - a^2k^2) . \qquad (5.1.37)$$

The principal role in the interaction is played by four-wave scattering processes with the coefficient (1.1.29b)

$$\begin{aligned} T_p' = &- \frac{U_{-1-212}U_{-3-434}}{\omega_3 + \omega_4 + \omega_{3+4}} + \frac{V^*_{1+212}V_{3+434}}{\omega_1 + \omega_2 - \omega_{1+2}} \\ &- \frac{V^*_{131-3}V_{424-2}}{\omega_{4-2} + \omega_2 - \omega_4} - \frac{V^*_{242-4}V_{313-1}}{\omega_{3-1} + \omega_1 - \omega_3} \\ &- \frac{V^*_{232-3}V_{414-1}}{\omega_{4-1} + \omega_1 - \omega_4} - \frac{V^*_{141-4}V_{323-2}}{\omega_{3-2} + \omega_2 - \omega_3} . \end{aligned} \qquad (5.1.38)$$

Here we used the notation $(j \pm i) = k_j \pm k_i$.

The stationary solutions can be found with the help of Kats-Kontorovich transformations the definition of which has been given in Sect. 3.2.1. In the three-dimensional case the stationary solutions are determined by (3.2.6):

$$n_1(k, P) = \lambda_1 P^{1/3} k^{-11/3}, \qquad n_2(k, Q) = \lambda_2 Q^{1/3} k^{-10/3} .$$

Both of these solutions are local [5.16] and correspond to the general formulas (2.3.10) with the renormalized index $m = 1$ of the interaction coefficient. Defining the indices $h_1 = 1/3$ and $h_2 = -1/3$ in the usual manner [see (4.3.15)], we see that both spectra are formed by decelerating waves like (4.3.4) and that the fluxes may be expressed in terms of the pumping amplitude Γ_0 and typical wave number k_0:

$$P \propto \Gamma_0^{3/2} k_0^{1/2}, \qquad Q \propto \Gamma_0^{3/2} k_0^{-1/2} .$$

For weak nonlinearity in the region of large k the applicability parameter for the solution n_1 should be evaluated according to (3.1.33b):

$$\xi^{-1}(k) \propto k^{3+h_1} = k^{10/3} .$$

Then the criterion for weak nonlinearity $\xi \ll 1$ is obeyed at large k. In the same way, one can see that for small k weak turbulence goes over into strong turbulence.

The two-dimensional case corresponds to waves on shallow water. It should be recalled that to the zeroth order in the small parameter ak (where a is proportional to the fluid depth) the interaction coefficient $T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ vanishes on the resonance manifold $\omega(k) + \omega(k_1) = \omega(k_2) + \omega(k_3)$, $\mathbf{k} + \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$, see the end of Sect. 3.2.1. Such a remarkable property is connected with the integrability of the Kadomtsev-Petviashvili equation (1.5.4). So, to first order in the (small) water depth, there are neither nonlinear interaction nor an evolution of the spectrum to a stationary state. Considering also the subsequent expansion terms we can obtain the Kolmogorov solutions [5.16]

$$n_3(k,P) = \lambda_3 P^{1/3} k^{-10/3}, \qquad n_4(k,Q) = \lambda_4 Q^{1/3} k^{-3}$$

with $h_1 = 2/3$ and $h_2 = -1$ so that

$$P \propto \Gamma_0^{3/2} k_0, \quad Q \propto \Gamma_0^{3/2} k_0^{-3/2}$$

are seen to hold.

The stability problem has not yet been solved. It is natural to expect a long-wave structural instability of the spectra n_2 and n_4 with regard to an additional perturbation carrying a momentum flux, since for such a perturbation we have

$$\delta n(\boldsymbol{k}, \boldsymbol{R})/n(k,P) \propto 1/k \ .$$

Spectra carrying an energy flux should be stable, since

$$\delta n(\boldsymbol{k}, \boldsymbol{R})/n(k,P) \propto \omega(k)/k$$

holds and the phase velocity decreases with k.

Short-Wave Acoustic Turbulence. Concluding this section let us briefly consider short-wave acoustic turbulence. In this case, the linear term in the dispersion law is supplemented by a positive term inversely proportional to k

$$\omega(k) = ck + \omega_0^2/2ck \ . \tag{5.1.39}$$

This dispersion relation is of the nondecay type. Stationary Kolmogorov solutions can be obtained by the general formulas (3.2.5) with $\beta = -1$. For $d = 3$ (which corresponds to spin waves in antiferromagnets and to ultrarelativistic particles)

$$n_1(k,P) \propto k^{-5}, \qquad n_2(k,Q) \propto k^{-14/3}$$

is obtained. So far neither locality nor stability of this solution has been checked.

The two-dimensional case is of special interest. The dispersion relation (5.1.39) is encountered in different physical situations. The frequency gap ω_0 could be connected either with a magnetic field (for plasmas) or with a rotation (for fluids or gases). The latter case is apparently realized in the Earth's atmosphere. If the wavelength is larger than the height of the atmosphere (10 km), then the wave may be treated as being two-dimensional. Such motions (like inertio-gravity waves in the atmosphere) is described in the framework of the shallow water equations [5.17]. If, in addition to the above, the wavelength is smaller than the Rossby radius ($\simeq$ 3000 km for medium latitudes) then it is reasonable to neglect the dependence of the Coriolis parameter $f = 2\Omega \cos\alpha$ (see Sect. 1.3.2) on the latitude α. In this case we have $\omega_0 = f$. The contribution of rotations to (5.1.39) is so small that the waves could be referred to as being "gravitational" rather than "inertial", in spite of the fact that rotation gives rise to the dispersive term supplementing the linear acoustic term.

Proceeding from (3.2.5), we get

$$n_1(k,P) \propto k^{-14/3}, \qquad n_2(k,Q) \propto k^{-13/3} \ .$$

According to *Falkovich* and *Medvedev* [5.18], only the second solution is local. It carries the flux of the wave action towards the large-scale region. In contrast to the long-wave limit, the interaction coefficient (5.1.38) does not vanish on the resonant manifold.

5.2 Wave Turbulence on Water Surfaces

From Sect. 1.2.5 we see that there are several cases of wave turbulence for different interrelations both between wave length and water depth and between surface tension and gravity. Prominent examples of shallow-water waves were considered in Sect. 5.1 since they belong to acoustic-type waves. Other cases of capillary waves on shallow water will be considered in Sect. 5.2.3.

5.2.1 Capillary Waves on Deep Water

As mentioned in Sect. 1.1.4, for sufficiently short water waves (with wavelengthes not exceeding a centimeter) the restoring force should be entirely determined by surface tension. The dispersion relation and interaction coefficient are

$$\omega(k) = \sqrt{\frac{\sigma}{\varrho}}\, k^{3/2} \ , \tag{5.2.1}$$

$$\begin{aligned} V(\boldsymbol{k}, 12) = \frac{1}{8\pi}\left(\frac{\sigma}{4\varrho^3}\right)^{1/4} \Bigg[&(\boldsymbol{k}_1\boldsymbol{k}_2 + k_1k_2)\left(\frac{k_1k_2}{k}\right)^{1/4} \\ &+ (\boldsymbol{k}_1\boldsymbol{k} - k_1k)\left(\frac{k_1k}{k_2}\right)^{1/4} + (\boldsymbol{k}\boldsymbol{k}_2 - kk_2)\left(\frac{kk_2}{k_1}\right)^{1/4} \Bigg] \ , \end{aligned} \tag{5.2.2}$$

see also (1.2.41). This dispersion relation is of the decay type. There exists only a single isotropic stationary distribution (3.1.15b) as obtained by *Zakharov* and *Filonenko* [5.19]. It supports an energy flux

$$n(k) = \left(\frac{P}{a}\right)^{1/2} 8\pi \left(\frac{4\varrho^3}{\sigma}\right)^{1/4} k^{-17/4} \ . \tag{5.2.3}$$

Here a is the dimensionless constant from (3.1.13b) still to be evaluated. Determining the asymptotics of the interaction coefficient at $k_1 \ll k$ we obtain $m_1 = 7/2$ to see that the locality conditions (3.1.12) are satisfied. The index h is equal to $\alpha - m = 3/2 - 9/4 = -3/4$. Consequently, the energy flux is expressed in terms of the pumping characteristics [see (3.4.8)]:

$$P \propto \Gamma_0^2 \omega_0^{-1} .$$

Here ω_0 is the frequency of the waves generated and Γ_0 is the maximum growth-rate of linear instability. Knowledge of h also allows us to predict the character of nonstationary evolution. Since $h < 0$, short-wave Kolmogorov-like asymptotics (5.2.3) are reached in an "explosive" manner [5.20]

$$n(k,t) = (t_0 - t)^{17/3} f\left(k(t_0 - t)^{4/3}\right) .$$

Here f is a universal dimensionless function (see Sect. 4.3.1) and the dimensionless quantities k and t, respectively, are measured in units of an initial k_0 and of the typical nonlinear interaction time at $k = k_0$. At $k \to 0$ such a solution has quasi-stationary Kolmogorov-like asymptotics $f(x) \propto x^{-17/4}$. Before the front distribution falls exponentially: $f(x) \propto \exp(-x^\alpha)$ (see Sect. 4.3.1).

The dimensionless parameter of the nonlinearity level (3.1.33) has the k-dependence

$$\xi^{-1}(k) = \omega_k t_{NL} \propto k^{3/4} ,$$

hence, the weak-turbulence approximation is violated for small k.

The isotropic distribution is structurally unstable as shown in Sect. 4.2. The relative contribution of the anisotropic part grows with k like $\delta n(\boldsymbol{k})/n(k) \propto \omega_k/k \propto \sqrt{k}$. In the short-wave region the stationary spectrum is thus substantially anisotropic. For positive angles θ_k between the wave vector and momentum flux, the stationary distribution of short waves should be defined by the momentum flux:

$$n(\boldsymbol{k}) = k^{-4}\sqrt{R\cos\theta_k} ,$$

compare with (5.1.24).

Experiments with capillary waves generated in a wind-wave tunnel are discussed in *Lleonart* and *Blackman* [5.21].

5.2.2 Gravity Waves on Deep Water

This is the kind of wave which is excited by wind blowing over the surfaces of seas and oceans. The corresponding kinetic equation was first obtained by *Hasselmann* [5.22]. The dispersion relation of gravity waves $\omega(k) = \sqrt{gk}$ is of the nondecay type. The interaction coefficients (1.2.43) are

$$U_{k,12} = V_{-k12} = \frac{1}{8\pi}\left(\frac{g}{4\varrho^2}\right)^{1/4}\left[(\boldsymbol{k}_1\boldsymbol{k}_2 + k_1k_2)\left(\frac{k}{k_1k_2}\right)^{1/4} + (\boldsymbol{k}\boldsymbol{k}_1 + kk_1)\left(\frac{k_2}{kk_1}\right)^{1/4} + (\boldsymbol{k}\boldsymbol{k}_2 + kk_2)\left(\frac{k_1}{kk_2}\right)^{1/4}\right] , \qquad (5.2.4)$$

$$W(\boldsymbol{k}\boldsymbol{1},\boldsymbol{23}) = \frac{(kk_1k_2k_3)^{1/2}}{64\varrho\pi^2}[R(\boldsymbol{k}\boldsymbol{123}) + R(\boldsymbol{k}\boldsymbol{123}) - R(\boldsymbol{k}\boldsymbol{213}) - R(\boldsymbol{k}\boldsymbol{312}) - R(\boldsymbol{12k3}) - R(\boldsymbol{13k2})] , \qquad (5.2.5)$$

$$R(\boldsymbol{k}\boldsymbol{123}) = \left(\frac{kk_1}{k_2k_3}\right)^{1/4}[2(k + k_1) - |\boldsymbol{k} - \boldsymbol{k}_2| - |\boldsymbol{k} - \boldsymbol{k}_3| - |\boldsymbol{k}_1 - \boldsymbol{k}_2| - |\boldsymbol{k}_1 - \boldsymbol{k}_3|] . \qquad (5.2.6)$$

Eliminating three-wave processes, the resulting coefficient of the four-wave interaction has the form (1.1.30) and possesses the same homogeneity properties as $W(\boldsymbol{k}\boldsymbol{1},\boldsymbol{23})$, see also (1.1.43). Therefore we have in this case $\alpha = 1/2$, $d = 2$ and $m = 3$.

In the isotropic case there are two integrals of motion implying that there are also two isotropic Kolmogorov-like spectra (3.1.27–28). The first one was obtained by *Zakharov* and *Filonenko* [5.23]. It carries a constant energy flux to the short-wave region

$$n(k,P) = \varrho^{2/3}(P/a_1)^{1/3}k^{-4} . \qquad (5.2.7)$$

Here a_1 is a dimensionless constant defined by the integral (3.1.13b) not yet evaluated for this particular case. The index $h_1 = \alpha - 2m/3 = -3/2$ [see Sect. 3.4.1 and (4.3.15a)] is negative and the spectrum (5.2.7) is thus formed "explosively"

$$n(k,t) = (t_0 - t)^{8/3} f_1\left(k(t_0 - t)^{2/3}\right) .$$

Here f_1 is a universal dimensionless function (see Sect. 4.3.1 for details) of the dimensionless variables k (measured in the units of an initial k_0) and t (measured in the units of the typical nonlinear interaction time at $k = k_0$). The spectrum behind the front is (5.2.7) $f_1(x) \to x^{-4}$ at $x \to 0$. The right boundary of the Kolmogorov spectrum evolves according to the explosive law $k_b \propto (t_0 - t)^{-2/3}$. The value of t_0 is defined by the initial conditions.

Isotropic spectrum is stable with regard to angular modulations, since the contribution of the momentum-carrying additional term is proportional to the phase velocity $\omega(k)/k$ and thus decreases with k. With regard to the applicability condition of the weak turbulence approximation it is of note that the dimensionless nonlinearity parameter (3.1.33b) $\xi(k) \propto k^{-\alpha-h_1} = k$ grows with k. Thus the short-wave turbulence is a strong one which manifests itself in the observation of breakers and whitecaps.

The second stationary spectrum was obtained by *Zakharov* and *Zaslavskii* [5.24]. It carries a constant flux of wave action towards the long-wave region

$$n(k,Q) = g^{1/6}\varrho^{2/3}(Q/a_2)^{1/3}k^{-23/6} . \qquad (5.2.8)$$

The dimensionless constant a_2 equals the integral (3.1.22b) and has not yet been computed. The locality of both spectra (5.2.7–8) could be proved using the

$$T(k,k_1;k,k_1)=\begin{cases} k^2k_1 & \text{for } k<k_1\ ,\\ kk_1^2 & \text{for } k>k_1\ .\end{cases}$$

The index $h_2 = (\alpha - 2m)/3 = -11/6$ [see Sect. 3.4.1 and (4.3.15b)] is negative and spectrum (5.2.8) is thus formed by a decelarating relaxation wave

$$n(k,t) = t^{23/11} f_2(kt^{6/11})\ .$$

Here f_2 is a universal dimensionless function. Behind the front there exists the spectrum (5.2.8) with $f(x) \to x^{-23/6}$ at $x \to \infty$. The right boundary of the Kolmogorov spectrum moves according to $k_b \propto t^{-6/11}$. Considering both spectra (5.2.7–8), one can see that most of the wave energy is contained in the long-wave region. So the mean frequency of the distribution approximately corresponds to the left edge of the spectrum (5.2.8) and thus decreases like $\omega_b \propto t^{-3/11}$.

The isotropic spectrum (5.2.8) has been proved to be stable with regard to angular modulations, see Sect. 4.3.3. With regard to the applicability condition of the weak-turbulence approximation we would like to not that the dimensionless nonlinearity parameter (3.1.33c) $\xi(k) \propto k^{-\alpha-h_2} = k^{4/3}$ grows with k. So the waving becomes weaker as the structure moves towards large k.

In what way is the attenuation of waving effected when the wind abates? The short-wave part of the distribution has the Kolmogorov asymptotics (5.2.7) with constant energy flux implying that the energy decreases. The evolution of decaying turbulence should approach the self-similar regime $n(k,t) = t^{4/11} f_3(kt^{2/11})$ which describes the propagation into the long-wave region. Thus, in the isotropic case, the mean frequency of waving decreases like $\omega_E \propto t^{-1/11}$.

Several field and laboratory observations are summarized by *Phillips* [5.25], see also *Forristall* [5.26]. The findings presented therein agree with both versions (5.2.7–8), but do not allow to distinguish between them.

5.2.3 Capillary Waves on Shallow Fluids

Such waves exist on very shallow fluids with $h_0 \ll \sqrt{\sigma/\varrho g}$. For intermediate wave numbers with $\sqrt{\varrho g/\sigma} \ll k \ll h_0^{-1}$, the dispersion relation and interaction coefficient have the extremely simple form (1.2.40)

$$\omega(k) = \sqrt{\frac{\sigma h_0}{\varrho}}\,k^2,\quad V_{k12} = \frac{k^2}{8\pi}\left(\frac{\sigma}{4\varrho h_0}\right)^{1/4}.$$

The dispersion law is of the decay type. For our purposes it is sufficient to consider only three-wave processes which are seen to correspond to $\alpha = 2,\ d = 2,\ m = 2\ m_1 = 0$. A stationary Kolmogorov-like solution with constant energy flux was found by *Kats* and *Kontorovich* [5.27] to have the form

$$n(k) = P^{1/2} 8\sqrt{\pi}\left(\frac{\varrho h_0}{\sigma}\right)^{1/4} k^{-4}\ . \qquad (5.2.9)$$

According to criterion (3.1.12) it is local and it has been proved to be stable with regard to isotropic perturbations [5.28]. In this case, the index h is given by $h = \alpha - m = 0$ so that the energy is uniformly distributed over the scales in the inertial range. The spectrum is formed in a "nonexplosive" process. The free decay takes place in two steps, see Fig. 4.13. The first step conserves energy and can be described in terms of the exponential self-similar solution (4.3.14). When the boundary of the Kolmogorov spectrum has reached the damping region, the energy starts to decrease $E(t) \propto t^{-1}$. The applicability criterion for weak turbulence is violated when going over to large k, since we have in this case $\xi(k) \propto k^{-2}$, see (3.1.33).

With regard to anisotropic perturbations it is found that a locality strip exists only for even harmonics. Odd harmonics with the numbers $l = 2j+1$ evolve in a nonlocal manner, perturbations in the inertial range strongly interact with both left and right edges. The sign of the time derivative of the perturbation is seen to be proportional to $(-1)^j$. So the spectrum should be unstable with respect to harmonics with even j. The angular form of the spectrum should be substantially anisotropic in the inertial interval while index still be the same as in the isotropic case [5.28]. The transfer of energy is thus local while that of momentum is nonlocal in k-space,

5.3 Turbulence Spectra in Plasmas, Solids, and the Atmosphere

The media listed in the title are sometimes anisotropic due to electromagnetic fields (for plasmas and magnetics) or to rotation (for the atmosphere). However, in some situations it can be regarded as locally isotropic for wave turbulence. So, both isotropic and anisotropic Kolmogorov spectra may arise. Isotropic turbulence of ion sound waves was considered in Sect. 5.1. So we consider here an isotropic turbulence of plasmons.

5.3.1 Langmuir Turbulence in Isotropic Plasmas

For weak external fields, plasmas can be regarded as isotropic and we shall use (1.3.3–5)

$$\omega^2(k) = \omega_p^2(1 + 3k^2 r_D^2) \qquad (5.3.1)$$

for the Hamiltonian coefficients. Here ω_p and r_D are, respectively, plasma frequency and the Debye length

$$\omega_p^2 = \frac{4\pi\varrho_0}{m^2},\quad r_D^2 = \frac{T_e m}{4\pi\varrho_0 e^2}\ .$$

The coefficient of the three-wave interactions has the form

$$U_{k12} = V_{k12} = \frac{1}{8\sqrt{2\pi^3}\varrho_0}\left[\left(\frac{\omega_1\omega_2}{2\omega_k}\right)^{1/2} k\cos\theta_{12} + \left(\frac{\omega_k\omega_1}{2\omega_2}\right)^{1/2} k_2\cos\theta_1 + \left(\frac{\omega_k\omega_2}{\omega_1}\right)^{1/2} k_1\cos\theta_2\right] .$$

However, the dispersion relation (5.3.1) holds only in the long-wave range $kr_D \ll 1$ and is of the nondecay type. Using the transformation (1.1.28), one can obtain an efficient Hamiltonian (1.1.29). In the range of $kr_D \ll 1$ the interaction coefficients U and V become scale-invariant with the scaling index unity and the efficient four-wave interaction coefficient (1.1.29b) has the scaling index two, since $\omega(k) \approx \omega_p$

$$T_{k123} = \frac{1}{\omega_p}\left[V_{k+1,k1}V_{2+3,23} - V_{-k-1,k1}V_{-2-3,23} - V_{k2,k-2}V_{133-1} - V_{k3,k-3}V_{122-1} - V_{k22-k}V_{131-3} - V_{k33-k}V_{122-1}\right] . \quad (5.3.2)$$

Such a nonlinear interaction has an electronic origin. Let us briefly explain the conditions under which this interaction is the main one. Considering plasma turbulence, the following dimensionless parameters are usually introduced: kr_D, which corresponds to the plasmon dispersion and E/nT which equals the ratio of the wave energy to the thermal one and is a measure of the nonlinearity level. So the typical time for turbulence evolution due to wave-wave interactions can be estimated from the kinetic equation to be roughly

$$\frac{1}{t_{NL}} \sim \omega_p(kr_D)^2\left(\frac{E}{nT}\right)^2 .$$

It should be smaller than the time of induced scattering interactions with ions

$$\frac{1}{t_i} \sim \omega_p(kr_D)^3\frac{E}{nT} .$$

Thus we obtain

$$\frac{E}{nT} > kr_D .$$

It is necessary to demand the validity of the weak turbulence approximation. In the case of plasma turbulence that means that there should be no Langmuir collapse the typical time of which contains another small parameter, the mass ratio

$$\frac{1}{t_c} \sim \sqrt{\frac{mE}{MnT}} .$$

Requiring $t_{NL} \ll t_c$ we obtain as a criterion for the weak turbulence approximation

$$\frac{E}{nT} \gg (kr_D)^{-4/3}\left(\frac{m}{M}\right)^{1/3} .$$

Under such conditions, the Kolmogorov spectra of Langmuir turbulence can be obtained [5.29], see (3.1.29–30). The first one corresponds to a constant energy flux P towards the short-wave region

$$n(k) = (P/a_1)^{1/3}n^{2/3}k^{-13/3} . \quad (5.3.3)$$

According to (4.3.15a), an "energy-containing" region is a short-wave region, since we have in that case $h_1 = \alpha - 2m/3 = 2 - 4/3 = 2/3 > 0$. So the spectrum (5.3.3) is formed by the decelerating wave (4.3.4) and the right boundary of the spectrum k_{rb} evolves according to $k_{rb}(t) \propto t^{3/2}$. The dimensionless parameter of the nonlinearity is according to (3.1.33b) given by $\xi \simeq (\omega t_{NL})^{-1} \propto k^{-8/3}$ implying a weak short-wave turbulence.

The second spectrum carries the flux of the wave action Q towards small k

$$n(k) = (Q/b_1)\omega_p^{1/3}(r_D n)^{2/3}k^{-11/3} . \quad (5.3.4)$$

In this case $h_2 = (\alpha - 2m)/3 = -2/3$ and the action-containing region is the same as the pumping region (i.e., the right edge of the spectrum). The left boundary of the spectrum (5.3.4) moves with $k_{lb} \propto t^{-3/2}$. According to the estimate (3.1.33c), the dimensionless nonlinearity parameter behaves as $\xi \propto k^{-4/3}$. The long-wave turbulence should be strong. According to *Zakharov* [5.29] both spectra (5.3.3–4) are local.

If the external pumping generates waves with some k_0 and a growth rate equal to Γ, then the fluxes are estimated to be

$$P \propto \Gamma^{3/2}k_0, \qquad Q \propto \Gamma^{3/2}k_0^{-1} .$$

The stability problem for such spectra is not yet solved. Both spectra (5.3.3–4) have indices which are larger than the one of dispersion $\alpha = 2$. According to the criterion (3.1.22), that means that both fluxes have correct directions: the spectrum (5.3.3) transfers energy towards the short-wave region while the spectrum (5.3.4) transfers action towards the long-wave end. Therefore, it is natural to expect that both spectra are stable in the isotropic case. As far as structural stability is concerned, they can be supposed to be unstable, since the contributions of the drift corrections (4.1.9,14) with constant momentum flux $\boldsymbol{R}$

$$\frac{\delta n(\boldsymbol{k})}{n(k,P)} \propto \frac{(\boldsymbol{Rk})\omega_k}{Pk^2} \propto k$$

$$\frac{\delta n(\boldsymbol{k})}{n(k,Q)} \propto \frac{(\boldsymbol{Rk})}{Qk^2} \propto k^{-1}$$

increase when going over from pumping to the inertial intervals. In this case further investigations are necessary to clarify all details of interest.

Another case of Langmuir turbulence can be observed in a nonisothermal plasma ($T_e \gg T_i$) provided the conditions

$$\frac{T_i}{T_e}(kr_D)^2 < \frac{E}{nT} < (kr_D)^2 < \frac{m}{M}$$

are satisfied. Then the plasmons interact via virtual ion sound waves. The coefficient of the four-wave interaction is given by (1.3.14)

$$T_{k123} = -\frac{\omega_p(\cos\theta_{k1}\cos\theta_{23} + \cos\theta_{k3}\cos\theta_{12})}{8\pi^3 nT} .$$

Its scaling index is zero. All features of such a turbulence are the same as for optical turbulence (which corresponds to T_{k123} = const). So we shall treat them together.

5.3.2 Optical Turbulence in Nonlinear Dielectrics and Turbulence of Envelopes

As stated in Sect. 1.5, the nonlinear Schrödinger equation describes the behavior of envelopes of quite general high-freguency quasi-monochromatic waves. We referred to such turbulence of envelopes as optical turbulence. It is characterized by waves with a quadratic dispersion law like (5.3.1)

$$\omega(k) = \omega_0 + \beta k^2 .$$

The constant ω_0 may be eliminated by the canonical transformation (1.4.22). The only remaining memory of it is that three-wave processes are prohibited. The coefficient of the four-wave interaction is supposed to be a constant. Therefore, the correspondent kinetic equation has the form

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = T\int \delta(k^2 + k_1^2 - k_2^2 - k_3^2)\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3) \times n_k n_1 n_2 n_3 (n_k^{-1} + n_1^{-1} - n_2^{-1} - n_k^{-1})\, d\boldsymbol{k} d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 .$$

It is worthwhile to consider this equation for both two-dimensional media (which mainly corresponds to the envelopes of water waves) and three-dimensional ones. According to the general formula (3.1.9), its Kolmogorov solutions are equal to

$$n_1(k,P) = \lambda_1 P^{1/3} k^{-d-2m/3} \propto k^{-d} ,$$
$$n_2(k,Q) = \lambda_2 Q^{1/3} k^{-d-2m/3+\alpha/3} \propto k^{-d+2/3} . \qquad (5.3.5a)$$

Substituting these solutions into the kinetic equation, one can see that the only local solution is n_2 for both $d = 2$ and $d = 3$. Since the nonlinearity parameter behaves in this case like $\xi \propto k^{-8/3}$, the turbulence becomes strong at the long-wave region. It corresponds to the collapse of envelopes (or self-focusing) which provides a sink for Kolmogorov spectra with action flux [5.30].

In the two-dimensional case, the only local solution $n_2(k,Q) \propto k^{-4/3} \propto \omega_k^{-2/3}$ has an index 2/3 smaller than unity. The implications are that for this solution the action flux is directed towards large k. From the Frisch-Fournier criterion in follows that this spectrum is unstable, even in the isotropic situation.

Rigorous stability theory demonstrates the presence of instabilities with regard to both the zero and the first angular harmonics (Sect. 4.2.3). What are the characteristics of the stationary turbulence spectrum in this case? The answer is quite unexpected [5.32]. An external pump heats the wave system and creates a steady state which is close to thermodynamical equilibrium with a small nonequilibrium fraction carrying action flux towards small k. According to Sect. 4.1, such a solution should have the form (4.1.23). In numerical simulations [5.30] it was found that the stationary turbulent distribution is close to the more general solution

$$n(\omega_k) = \frac{T}{\omega + \mu + aQ\omega^2/T^3} \qquad (5.3.5b)$$

with temperature, chemical potential, and action flux. It should be noted that the collision integral logarithmically diverges in this case. Probably, it hints that the Kolmogorov-like addition should slightly deviate from a power form (e.g., with a logarithmic factor) to ensure convergence. For the small-scale part of the spectrum the formally defined Kolmogorov index $s = d = 2$ is just equal to the equilibrium index $s = \alpha = 2$. Hence, it is naturally to suppose the long-wave part of spectrum also to be close to the equilibrium distribution and to be slightly distorted either by a logarithmic factor (as in Sect. 3.2.2 where the coincidence index for plasmon-sound turbulence was considered) or by a small nonequlibrium additional contribution as for the long-scale part (5.3.5b).

The stability problem has not been solved for the three-dimensional case. The spectrum with an action flux carries a negative flux so it should be stable with regard to isotropic perturbations. In this case, $h_2 = (\alpha - 2m)/3 = 2/3$. In terms of the pumping characteristics the flux is given by

$$Q \simeq \Gamma^{3/2} k_0 .$$

Such a spectrum should be formed in an explosive way with the left boundary moving according to $k_{lb} \propto (t_0 - t)^{3/2}$. Its stability with regard to the first harmonics depends on the sign of the momentum flux carried by the drift Kolmogorov solution (4.1.14). If the momentum flux is negative, then the isotropic spectrum with an action flux is structurally unstable.

As far as a spectrum with energy flux is concerned, the collision integral diverges logarithmically for $d = 3$. It may be shown that the short-wave asymptotics of the stationary turbulent distribution slightly (e.g., logarithmically) differ from $k^{-3} \propto \omega_k^{-3/2}$, so the index might be close to 3/2. Numerical simulations [5.31] support such a hypothesis.

5.3.3 Spin Wave Turbulence in Magnetic Dielectrics

Spin waves in ferromagnetics and antiferromagnetics may have the dispersion law (1.4.9a, 21)

$$\omega(k) = \omega_0 + \beta k^2$$

hence, $\alpha = 2$. Spin waves in ferromagnets correspond to an interaction coefficient with $m = 2$. The Kolmogorov solutions (3.1.29c, 30c) coincide (up to a constant) with (5.3.3–4) for Langmuir waves with a nonlinearity induced by electrons. Thus everything said above about plasmons, holds also for magnons. For spin waves in antiferromagnets, such a case corresponds exactly to optical turbulence.

Now we shall discuss a somewhat detailed example of the Kolmogorov solution for spin waves whose dispersion is due to the exchange interaction (1.4.9a) and the interaction coefficient, to the magnetic dipole interaction (1.4.12). Such a "hybrid" is realized for an intermediate range of wave numbers k, which are (i) sufficiently large to allow it to neglect the gap ω_0 in the dispersion law (1.4.9a) and consider it to be of the decay type

$$\omega_k \propto k^2$$

and (ii) sufficiently small to neglect the four-wave exchange interaction (1.4.19b). The latter requirement, however, can always be satisfied assuming the turbulence level to be sufficiently small. The three-wave interaction coefficient (1.4.12) is in this case anisotropic

$$V_{k12} \propto \sin 2\theta_1 \exp(i\phi_1) + \sin 2\theta_2 \exp(i\phi_2). \tag{5.3.6}$$

Here θ_i, ϕ_i are, respectively, polar and azimuthal angle of $\boldsymbol{k}_i$ $(i = 1, 2)$ with the constant magnetization direction $\boldsymbol{M}$. The following fact found in [5.32] seemed a surprise: despite the angular dependence of the interaction coefficient, the three-wave kinetic equation (2.1.12) has an isotropic stationary solution. This is associated with the fact that on isotropic distributions $n(\boldsymbol{k}) \equiv n(\omega)$, the angular dependence is the same for all terms in the collision integral. Indeed, substituting (5.3.10) into (2.1.12), going over to the variable $\omega = k^2$ and integrating over angles, we get

$$\begin{aligned}\frac{\partial n(\omega, t)}{\partial t} \propto (1 + 2\cos^2\theta - 3\cos^4\theta)\Big\{\int_0^\omega [n(\omega')n(\omega - \omega') \\ - n(\omega)n(\omega - \omega') - n(\omega)n(\omega')]\, d\omega' - 2\int_0^\infty [n(\omega)n(\omega') \\ - n(\omega + \omega)n(\omega') - n(\omega + \omega')n(\omega)]\, d\omega'\Big\} \\ = I(\omega, \theta) \ .\end{aligned} \tag{5.3.7}$$

As we see, the angular dependence in the collision integral is separated out. Using Zakharov transformations (2.3.14), we obtain from (3.3.27) the stationary solution $n(\omega) \propto \omega^{-3/2}$, in accordance with the general formula $n(\omega) \propto \omega^{-(m+d)/\alpha}$ (here $m = 0$, $d = 3$, $\alpha = 2$). The solution is local as may be verified using (3.1.12) or (3.3.27). Solving $\operatorname{div} P = -\omega I(\omega, \theta)$, we find that the only nonzero component is the radial component of the flux, which depends only on the angle θ : $P \propto 1 + 2\cos^2\theta - 3\cos^4\theta$. The flux is maximal at $\cos\theta = \pm 1/\sqrt{3}$.

This example shows that Kolmogorov distributions in anisotropic media may also be obtained analytically (when the frequency and the interaction coefficient are not bihomogeneous). However, it is obvious that the efficiency of an analytical treatment depends in such a case strongly on the particular forms of dispersion law and interaction coefficient.

5.3.4 Anisotropic Spectra in Plasmas

Before considering specific examples of spectra in anisotropic media, a general remark should be made. The theory of such spectra is far from being complete and well developed. The matching problem is not yet solved. As seen in Sect. 3.3, there exist families of power-like stationary spectra a good understanding of the physical meanings of which will require further studies. Hence, we present here briefly some known theoretical results concerning different anisotropic spectra.

If a plasma is strongly magnetized, then the dispersion laws and interaction coefficients of ion sound and Langmuir waves are both bihomogeneous functions of the compoents of the wave vectors, i.e., they satisfy equations (3.3.1) in some angular intervals.

Ion Sound Turbulence in Magnetized Plasmas. Let us start from the ion sound which we considered in Sect. 5.1.3 for an isotropic plasma. Here we follow *Kuznetsov* [5.33] in considering the anisotropic case.

For sound waves $\omega(k_z, k_\perp) = \omega(p, q)$ and V_{k12} are according to (1.3.20) given by

$$\omega \propto pq^{-2}, \quad V \propto (pp_1p_2)^{1/2}\Theta(p)\Theta(p_1)\Theta(p_2) \ . \tag{5.3.8}$$

Thus, $a = 1$, $b = 1$, $u = 3/2$, $v = 0$. The presence of Θ-functions in the interaction coefficient implies that only waves with $p > 0$ are considered. The Kolmogorov solution supporting an energy flux is

$$n(p, q) = \lambda_1 P^{1/2} p^{-5/2} q^{-2} \ , \tag{5.3.9a}$$

and with momentum flux

$$n(p, q) = \lambda_2 R^{1/2} p^{-5/2} q^{-1} \ . \tag{5.3.9b}$$

Now we should verify locality of the resulting distributions. We shall substitute (5.3.8) into the kinetic equation (3.3.2). Consider first the convergence of the integral at zero, i.e., at $q_1 \ll q$ and $p_1 \ll p$. Using the δ-functions we integrate now over the variables p_2, q_2. The expression (3.3.3) for Δ_2 will simplify to

$$\Delta_2(q_1, p_1) \approx \frac{q}{2}\sqrt{4q_1^2 - q^2(p_1/p)^2} \ ,$$

and the integration will be only be performed for the region in which root in Δ_2 is positive. Let us gather all the terms that become infinite at $q_1 \to 0$, $p_1 \to 0$:

$$2\int dp_1 \int dq_1 \frac{qp^2 p_1}{\Delta_2(q_1, p_1)} n(p_1, q_1)$$
$$\times [n(p - p_1, q + qp_1/2p) + n(p + p_1, q - qp_1/2p) - 2n(p, q)] \ .$$

This expression takes into account the presence of two identical singularities in the first term (3.3.2a), namely at $(p_1, q_1) \to 0$ and $(p_1, q_1) \to (p, q)$. We see that, like for the isotropic solutions, see Sect. 3.1, the singularities are twice reduced. Let us expand the expression in square brackets up to the first nonvanishing terms (quadratic in the small parameters q_1/q and p_1/p). Introducing the dimensionless variables $\zeta = q_1/q$, $\eta = p_1/p$ and setting $n(p, q) \propto p^{-x} q^{-y}$, we reduce the integral to

$$q^{3-y} p^{3-x} \int_0 d\zeta\, \zeta^{4-y-x} \int_0 d\eta\, \eta^{3-x} \sqrt{1 - \eta^2} \ .$$

Whence, we obtain the convergence conditions $x < 4$ and $x + y < 5$. Both solutions (5.3.9), as well as the equilibrium solutions (3.3.9) (in the given case $n \propto p^{-1} q^{-2}, p^{-1}$) satisfy these conditions. In a similar way we obtain the convergence conditions for the integrals at $q_1 \to \infty$, $p_1 \to \infty$ with

$$\Delta_2(q_1, p_1) \approx \frac{q_1}{2} \sqrt{4q^2 - q_1^2 (p/p_1)^2} \ .$$

The most dangerous terms are arranged into the combination

$$p^2 n(p, q) \int dp_1 \int dq_1 \frac{q_1 p_1}{\Delta_2(q_1, p_1)} \left[\frac{\partial n(p_1, q_1)}{\partial p_1} - \frac{\partial n(p_1, q_1)}{\partial q_1} \frac{q_1}{2p_1} \right] \ .$$

Going over to the dimensionless variables $\zeta = q/q_1$, $\eta = \zeta^{-1} p/p_1$ we obtain

$$(x - y) p^{2-x} q^{2-y} \int_0 d\zeta\, \zeta^{x+y-3} \int_0^1 d\eta \frac{\eta^{x-2}}{\sqrt{1 - \eta^2}} \ .$$

The convergence of the integral is ensured for $x > 1$, $x + y > 2$. These conditions are satisfied for both distributions (5.3.9). By the way, in the given case we have for equilibrium distributions $x = -1$, $y = 0$; $x = -1$, and $y = -2$, i.e., the ultraviolet convergence conditions are not satisfied. This certainly does not mean divergence of the collision integral of which every term is identically zero in equilibrium. It implies simply that for any wave belonging to the Rayleigh-Jeans distribution, the principal role in the equilibrium state is played by the interaction with high-frequency waves. Since at $\omega \gtrsim T/\hbar$ the Rayleigh-Jeans decrease $n \propto \omega^{-1}$ goes over to the Planck decrease $n \propto \exp(-\hbar\omega/T$ [see (2.2.12)], the main interaction takes place with thermal waves (for which $\omega \simeq T/\hbar$).

Thus, for ion-sound turbulence in a magnetic field, both Kolmogorov distributions are local.

Langmuir Turbulence in Magnetized Plasmas. For the Langmuir waves propagating almost perpendicularly to a magnetic field, the dispersion relation has in the limiting cases of strong and weak fields, respectively, a similar dependence on p and q, see (1.3.22, 24)

$$\omega(p, q) \propto \frac{|p|}{q} \ .$$

We consider waves moving in the entire k-space, i.e., p varies from $-\infty$ to $+\infty$. The total momentum vanishes for the power solution $n \propto |p|^{-x} q^{-y}$ and (3.3.10d) should not exist. Indeed, the transformations (3.3.7) will cast the stationary kinetic equation into the form

$$\int\int_{-\infty}^{\infty} dp_1\, dp_2 \int\int_0^{\infty} dq_1\, dq_2\, U(p, p_1, p_2, q, q_1, q_2) \delta(p - p_1 - p_2)$$
$$\times \delta(p^a q^b - p_1^a q_1^b - p_2^a q_2^b) \Big[|p_1 p_2|^{-x} (q_1 q_2)^{-y} - |pp_1|^{-x} (qq_1)^{-y} \qquad (5.3.10)$$
$$- |pp_2|^{-x} (qq_2)^{-y} \Big] \left[1 - \left|\frac{p}{p_1}\right|^{\tilde{x}} \left(\frac{q}{q_1}\right)^{\tilde{y}} - \left|\frac{p}{p_2}\right|^{\tilde{x}} \left(\frac{q}{q_2}\right)^{\tilde{y}} \right] = 0 \ ,$$

see also (3.3.8). If we set $\tilde{x} = -1$, $\tilde{y} = 0$ like in (3.3.10b), the left-hand side of (5.3.10) will not vanish because of the presence of $|p/p_1|$ and $|p/p_2|$ in the second square bracket.

First we shall consider strong magnetic fields. In the angular range limited by the inequality $|\cos\theta_k| \ll \omega_p/\omega_H$, the interaction coefficient is given by (1.3.23a)

$$V \propto (\boldsymbol{q}_1[\boldsymbol{h}\boldsymbol{q}_2]) \sqrt{\frac{|p_1 p_2| q}{q_1 q_2 |p|}} \operatorname{sign} p \left[\frac{\operatorname{sign} p}{q} \left(\frac{q_1}{q_2} - \frac{q_2}{q_1} \right) + \frac{\operatorname{sign} p_1}{q_2} + \frac{\operatorname{sign} p_2}{q_1} \right] \ .$$

In this case $u = v = 1/2$ and

$$n(p, q) \propto P^{1/2} |p|^{-3/2} q^{-5/2} \qquad (5.3.11)$$

is the Kolmogorov solution with a constant energy flux. We can direct verify that it is local [5.34]. For a weak magnetic field, (1.3.24) holds for the interaction coefficient

$$V \propto \sqrt{\frac{p_1 p_2 q}{q_1 q_2 p}} \frac{(\boldsymbol{q}_1[\boldsymbol{h}\boldsymbol{q}_2])}{qq_1 q_2} \left[q^2 + qq_2 \operatorname{sign}(pp_2) + qq_1 \operatorname{sign}(pp_1) \right] \ .$$

Its indices $u = v = 1/2$ and the solution coincide with (5.3.11). Locality is also verified directly.

The common turbulence of Langmuir and ion-sound waves has been described in Sect. 3.2.2 as an example for the interaction of high- and low-frequency waves.

One also knows the case in which simultaneously the properties of low- and high-frequency wave interactions (see Sect. 3.2.2) and of bihomogeneity are observed. An example is the interaction of high-frequency Alfven waves ($\omega_k = k_z v_A$) with low-frequency magnetic sound ($\Omega_k = k_z c_s$) in a magnetized plasma at low pressure ($c_s \ll v_A$) [5.35]. The kinetic equations are obtained from magnetohydrodynamic equations and have the form

$$\frac{\partial n(k,t)}{\partial t} = -\int U(k_2,k,k_1)T(k_2,k,k_1)\,d\boldsymbol{k}_1 d\boldsymbol{k}_2\,,$$

$$\frac{\partial N(k,t)}{\partial t} = \int [U(k,k_1,k_2)T(k,k_1,k_2) - U(k_1,k,k_2)T(k_1,k,k_2)]\,d\boldsymbol{k}_1 d\boldsymbol{k}_2\,,$$

with

$$T(k,k_1,k_2) = N(k_1)n(k_2) - N(k)n(k_2) - N(k)N(k_1)\,,$$

$n(k)$, $N(k)$ are the densities of sound and Alfven waves, respectively, and

$$U = 2\pi|V_{k12}|^2\delta(k-k_1-k_2)$$

is a bihomogeneous function of the components of the wave vector

$$U(\lambda k_z, \mu k_\perp) = \lambda\mu^{-2}U(k_z,k_\perp)\,.$$

The resulting Kolmogorov solution is

$$n(k) = A_1 k_z^{-2}k_\perp^{-2}, \quad N(k) = B_1 k_z^{-2}k_\perp^{-2}\,,$$

and corresponds to a constant flux of high-frequency waves. The Kolmogorov spectrum with a constant energy flux has the form

$$n(k) = A_2 k_z^{-5/2}k_\perp^{-2}, \quad N(k) = B_2 k_z^{-5/2}k_\perp^{-2}\,.$$

For both spectra one can obtain the estimate $A_i c_s \sim B_i v_A$, $(i = 1,2)$. Therefore, for stationary turbulence, the energies of Alfven and sound waves are of the same order.

Checking the locality, we find that the integral over k_z converges, while the integral over $k_\perp$ is just about divergent, i.e., it diverges logarithmically. To assert that such a turbulence is local, it is necessary to consider also dispersion (i.e., the next terms in ω_k, Ω_k) [5.35].

5.3.5 Rossby Waves

Let us now discuss the turbulence of the barotropic Rossby waves introduced in Sect. 1.3.2. Their k-space is two-dimensional, which does not allow to take over the results of the cases discussed above. Therefore we shall consider this example in more detail. We shall restrict ourselves to the region of small-scale ($k \gg k_0$) motions close to *zonal* ones (for which $k_x \equiv p \ll k_y \equiv q$). In this limit the dispersion relation (1.3.31) and the interaction coefficient (1.3.54) are bihomogeneous functions of the wave vector components [5.36–37]

$$\omega \propto pq^{-2}\,, \tag{5.3.12a}$$

$$V \propto (pp_1p_2)^{1/2}(q_1^{-1}+q_2^{-1}-q^{-1}) \tag{5.3.12b}$$

so that we have $a = 1$, $b = -2$, $u = 3/2$, and $v = -1$.

The kinetic equation for Rossby waves takes the form

$$\begin{aligned}\frac{\partial n}{\partial t} = \int\!\!\int_0^\infty dp_1 dp_2 \int\!\!\int_{-\infty}^\infty dq_1 dq_2 \\ \times \left[U(k,k_1,k_2)(n_1n_2 - nn_1 - nn_2)\right. \\ \left. - 2U(k_1,k,k_2)(nn_2 - nn_1 - n_1n_2)\right]\,,\end{aligned} \tag{5.3.13}$$

where

$$U(k,k_1,k_2) = |V|^2\delta(p-p_1-p_2)\delta(q-q_1-q_2)\delta(pq^{-2} - p_1q_1^{-2} - p_2q_2^{-2}),$$
$$n = n(p,q,t), \quad n_i = n(p_i,q_i,t), \quad \boldsymbol{k} = (p,q)\,.$$

Equation (5.3.13) has three general integrals of motion: the energy

$$E = \int \omega n\, dpdq$$

and the *zonal* and *meridional* momentum components

$$\text{(zonal)} \int pn\, dpdq \quad \text{and} \quad \text{(meridional)} \int qn\, dpdq\,,$$

respectively. However, it would be wrong to suppose that this equation must have three stationary Kolmogorov solutions of the type $n \propto p^{-x}|q|^{-y}$ corresponding to the fluxes of the three conserved quantities. Because of its parity with regard to q, the meridional component of the total momentum is identically zero on power distributions so that there should be no corresponding Kolmogorov distribution. Indeed, let us use the transformations (3.3.7) to reduce the stationary kinetic equation to

$$\begin{aligned}I(p,q) = \int\!\!\int_0^\infty dp_1 dp_2 \int\!\!\int_{-\infty}^\infty dq_1 dq_2\, (q_1^{-1}+q_2^{-1}-q^{-1})^2 \\ \times \delta(p-p_1-p_2)\delta(q-q_1-q_2)\delta\left(\frac{p}{q^2}-\frac{p_1}{q_1^2}-\frac{p_2}{q_2^2}\right) \\ \times \left[(p_1p_2)^{-x}(q_1q_2)^{-y} - (pp_1)^{-x}(qq_1)^{-y} - (pp_2)^{-x}(qq_2)^{-y}\right] \\ \times \left\{1 - \left|\frac{q_1}{q}\right|^{2y-2}\left(\frac{p_1}{p}\right)^{2x-4} - \left|\frac{q_2}{q}\right|^{2y-2}\left(\frac{p_2}{p}\right)^{2x-4}\right\}pp_1p_2 = 0\end{aligned} \tag{5.3.14}$$

This equation may have two solutions if the expressions in braces coincide with the arguments of the first or third δ-function. The first solution corresponds to the choice $2x - 4 = 1$, $2y - 2 = 0$ and should transfer the constant flux R_x of the zonal component of momentum:

$$n(p,q) \propto R_x^{1/2}p^{-5/2}|q|^{-1}\,. \tag{5.3.15a}$$

The second solution corresponds to the constant energy flux P

$$n(p,q) \propto P^{1/2} p^{-5/2} \,. \tag{5.3.15b}$$

The second δ-function may not be utilized for obtaining an extra stationary solution, as the choice $2y-2=1$, $x=2$ reduces only the integral over the region $q_2/q>0$ to zero, cf. (3.3.20).

It is interesting to mention the existence of an additional integral of motion recently obtained for Rossby waves [5.38]. It has the simple form $\int(\omega^3/q^2)n\,dpdq$. The corresponding Kolmogorov distribution $n(p,q)\propto p^{-7/2}q^3$ is nonlocal.

Substituting (5.3.15) into (5.3.13), it is easy to verify that the local distribution is only (5.3.15b). The energy flux in the k-space has two components proportional to the derivatives of the collision term (5.3.14) with respect to the solution indices. Calculating $\partial I/\partial x$ and $\partial I/\partial y$ at $x=5/2$, $y=0$ we see that $P_x>0$, $P_y<0$, i.e., the flux is directed towards larger p and smaller $|q|$. Considering the form of the dispersion law (5.3.12a), we conclude that the Kolmogorov distribution transfers the energy flux into the high-frequency region [5.39].

The treatment of baroclinic Rossby waves is complicated by the presence of various vertical harmonics, see (1.3.61). If only the waves corresponding to $l=1$ are excited, then (1.3.61) coincides with (3.3.22) and the problem reduces to the solved one.

It is worthwhile to point out that anisotropic Kolmogorov spectra are more often found to be nonlocal than isotropic ones. In [5.40], a model of nonlocal turbulence of Rossby waves has been proposed. It suggests that the waves from the inertial interval interact mainly with low-frequency waves. Such an interaction is almost elastic and redistributes the high-frequency waves over the surface of the constant frequency $p-\omega(p,q)=$ const. However, for the $\omega(p,q)$ of (5.3.12a), the corresponding surface is not closed. Thus, the local interaction may pump waves from the source to the region of sufficiently large q of the damping area. As a result, the evolution of the drift turbulence probably leads to the separation of the spectrum into two well-distinguished components; a small-scale component concentrated on a certain line in $\boldsymbol{k}$-space and a large-scale zonal flow with its level determined by the interaction with the short-wave turbulence. Further investigations (possibly involving computer simulations) are necessary to find out whether drift turbulence is local or nonlocal.

6. Conclusion

There may be a Moral, though some say not;
I think there's a moral, though I don't know what.

A. Milne "Now We Are Six"

Methodological Guide

The reader who has worked his way through the book up to this place should have found out that the Kolmogorov spectra of wave turbulence are fairly easy to handle. Indeed, detailed treatment of any new case will not require much further efforts. The answers to most questions, including rather subtle ones, may be obtained from dimensional analysis and from simple asymptotic estimates. Now, what is the procedure to be followed in such an analysis? Let us formulate the program to be carried out. This outline will simultaneously serve as a methodological guide for this volume:

1. If we don't even know the dispersion relation of the waves, we have to determine from a dimensional analysis the form of the dependence of the frequency ω on the wave vector k; in particular, the index of the dispersion relation α, see Sect. 1.1.4 (1.1.31). With the parameters of our problem we can now construct several different expressions having the dimension of a frequency. If these expressions depend differently on the wave vector, then scale-invariance and the universal Kolmogorov spectra of turbulence may only be observed in the region of k-space where one of the contributions to the frequency is by far larger than all others. The only exception occurs when the frequency consists of two terms, a constant $\omega(0)$ and the dispersion $\delta\omega(k)$. Then, in the nondecay case, the constant term $\omega(0)$ does not enter the resonance conditions for four-wave processes

$$\omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2)=\omega(\boldsymbol{k}_3)+\omega(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3)$$

and Kolmogorov solutions exist when $\omega(0)$ and $\delta\omega$ are of the same order. However, such dispersion relations are frequently of the decay type in one region of k-space and of the nondecay type in another. Consequently they have to be treated differently. An example is the dispersion law $\omega(k)=\omega(0)+\beta k^2$ which goes for $\beta k^2>2\omega(0)$ over to the decay type.

2. From the form of the dispersion relation it is seen whether three- or four-wave interactions, see (1.1.24, 29), are dominant. For different interaction types there are different numbers of the integrals of motion (see Sect. 2.2.1) and one can

also expect a different number of Kolmogorov spectra (Sect. 3.1) for them. From a dimensional analysis we can now extract the scaling index m of the respective interaction coefficient, Sect. 1.1.4 (1.1.32). Knowing α, m and the space dimension d, we may utilize one of the formulas (3.1.4–6) to obtain the Kolmogorov spectrum.

3. In the next step we have to verify the locality of the interaction of the Kolmogorov spectrum (i.e., stationary locality). For example, for three-wave interactions the verification of locality by the aid of (3.1.12) requires the knowledge of the index m_1 determining the asymptotics of the interaction coefficient

$$|V(\boldsymbol{k}, \boldsymbol{k}-\boldsymbol{\kappa}, \kappa)|^2 \propto \kappa^{m_1} k^{2m-m_1} \quad \text{at} \quad \kappa \ll k\ . \tag{6.1}$$

From this step onwards it is therefore not sufficient to rely on dimensional analysis. It is necessary to carry out a real calculation. However, this calculation is a rather simple one since we do not need to obtain the exact expression for the interaction coefficient but only the index of its asymptotics.

In most cases the asymptotic index m_1 may also be found from a dimensional analysis. For example, if the frequency parameter ω_k depends on the medium parameter that oscillates for the wave $a\kappa$ with $\kappa \to 0$, then the interaction Hamiltonian

$$\int V(k, k-\kappa, \kappa) a(\kappa) a(k) a^*(k-\kappa) \approx \int V(k, k, \kappa) a(\kappa) a(k) a^*(k)$$

may be represented as

$$\int \delta\omega_k a_k a_k^*\ .$$

As a consequence, the k-dependence of $V(k, k, \kappa)$ should be similar to the one of the frequency, namely

$$V(k, k, \kappa) \propto k^{\alpha} \kappa^{m-\alpha}\ . \tag{6.2}$$

Comparing (6.2) and (6.1) we obtain

$$m_1 = 2(m-\alpha) = -2h\ . \tag{6.3}$$

For example, a sound wave with a small wave vector corresponds to a small density variation which is almost constant in space. Since, the frequency depends on the density we have $m_1 = 2(m-\alpha) = 2(3/2-1) = 1$, compare with (5.1.6). It should be noted that for two-dimensional sound in (6.3), one should substitute the index of the initial interaction coefficient $3/2$, rather than the angle-averaged one 1. For shallow-water capillary waves, the waves alter the fluid height which in turn determines the frequency so that (6.3) is correct, see Sect. 3.1.2. But for deep-water capillary waves, the frequency does not depend on the depth so that (6.3) cannot be used. Indeed, according to (1.2.41b) we have in this case $m_1 = 7/2$ while (6.3) would lead to $m_1 = 3/2$.

4. Thus we have made sure that the turbulence is local. Consequently the corresponding Kolmogorov solution exists. Now, (4.2.17) yields the index h. Knowing h, we can immediately answer two important questions simultaneously. Namely, with regard to

(i) the type of the dependence of the flux absorbed by the turbulence on the pumping frequency (3.4.8);

(ii) the character of the nonstationary behavior of the wave system. In particular, it is in the isotropic case possible to describe the formation process of a Kolmogorov distribution, see Sect. 4.3.

5. Certainly, we can only speak about the formation of Kolmogorov spectra after having checked their stability (at least with regard to isotropic perturbations). For this purpose it is sufficient to compare the index of the Kolmogorov solution with the index of the respective equilibrium distribution. This allows to determine the sign of the flux, see (3.1.13b, 22). In line with the Fournier-Frisch criterion (see Sect. 3.1.3 and the end of Sect. 4.2), positive-flux spectra are stable in the short-wave region and negative-flux spectra in the long-wave region.

6. It is only when assessing the problem of structural stability of an isotropic spectrum that we will need to know the exact expression of the coefficient $V(k, k_1, k_2)$ or $T(k, k_1, k_2, k_3)$ of the interaction Hamiltonian. This expression must be substituted into (4.2.16) for the Mellin functions $W_l(s)$. It is sufficient to analyse the first few harmonics.

In general, the dispersion relation is nondegenerate (see the end of Sect. 2.2.1) and the interaction coefficient does not vanish on the resonance surface. So we cannot expect any anisotropic integrals of motion except the one for momentum. Therefore, it is in this case sufficient to consider only the first angular harmonic ($l = 1$) which is proportional to the cosine. Now we have to check the sign of the zero of the Mellin function. Positiveness of $W_l(0)$ implies according to (4.2.55) structural instability of the isotropic solution against perturbations of the form of the l-th harmonic. In this case determination of the growth rate of the anisotropic perturbation contribution with k makes it necessary to find that zero of the $W_l(s)$ function which is closest to the point $s = 0$. For the first harmonic it can be found from a dimensional analysis, in line with (4.1.9, 14). If $W_l(p) = 0$, the relative perturbation term behaves like $A_l(k) = \delta n(k)/n(k) \propto k^{-p}$.

Let us take $W_l(0) < 0$ to hold, a condition which does not ensure stability. We have to calculate the rotation $\kappa_l(0)$ of the Mellin function $W_l(i\sigma)$ around the imaginary axis (as σ moves from $-\infty$ to $+\infty$). According to the Balk-Zakharov criterion (see Sect. 4.2.2) the isotropic spectrum is stable with regard to the given harmonic, provided $\kappa_l(0) = 0$ holds.

In the general case, when we are concerned with the first harmonic only, we can proceed in a much simpler way. The zero of the $W_1(s)$ function (i.e., the index of the indifferently stable mode) may be found from a dimensional analysis following (4.1.9, 14). Whether this mode may be excited (leading to structural instability of the isotropic solution) is according to the Falkovich criterion (see Sect. 4.2) determined by the sign of the momentum flux $W_1'(p)$ it carries.

7. Finally, we have to determine the applicability conditions for the obtained results. For this purpose, we have to determine scale and amplitude ranges to

which the weak turbulence approximation is applicable. This may be done following a dimensional analysis according to (1.1.34, 35) and (2.1.14, 15, 21, 25, 27, 30).

With the recipe laid out in items 1. to 7. and the material provided in the preceding chapters of this volume the reader should have all tools at his disposition to investigate in details turbulence and Kolmogorov spectra.

A. Appendix

A.1 Variational Derivatives

Without giving a strict justification, we shall explain several simple rules for calculating variational derivatives. They follow from the fact that $\delta/\delta f(r)$ generalizes the notion of the partial derivative $\partial/\partial f(r_n)$ with discrete r_n to continuous variables.

1. The variational derivatives of a linear functional of the form $I = \int \phi(r')f(r')\,dr'$ are calculated by

$$\frac{\delta I}{\delta f(r)} = \int \phi(r')\frac{\delta f(r')}{\delta f(r)}\,dr' = \int \phi(r')\delta(r-r')\,dr' = \phi(r)\ . \qquad (A1.1)$$

To obtain this formula, one can mentally substitute $\delta/\delta f(r)$ by $\partial/\partial f(r_n)$, simultaneously replacing the integration by a summation to return after differentiation to the continuous version

$$\frac{\delta}{\delta f(r)}\int \phi(r')f(r')\,dr' \to \frac{\partial}{\partial f(r_n)}\sum_m \phi(r_m)f(r_m) = \phi(r_n) \to \phi(r)\ .$$

Symbollically this result may be represented by

$$\frac{\delta f(r')}{\delta f(r)} = \delta(r-r')\ . \qquad (A1.2)$$

2. If the function f in the functional is affected by differential operators, then application of the rule (A1.2) requires first to bring them to the left-hand-side and to integrate by parts. For example,

$$\frac{\delta}{\delta f(r)}\int \phi\nabla f\,dr' = -\frac{\delta}{\delta f(r)}\int f\nabla\phi\,dr' = -\nabla\phi\ . \qquad (A1.3)$$

Here we assumed that the product $f(r')\phi(r')$ vanishes on the boundary of the integration region.

The variational derivative of nonlinear functionals is calculated following a procedure similar to the one for the partial differentiation of a complex function:

$$\frac{\delta}{\delta f(r)}\int F[f(r')]\,dr' = \int \frac{\delta F}{\delta f(r')}\frac{\delta f(r')}{\delta f(r)}\,dr' = \frac{\delta F}{\delta f(r)}\ .$$