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Wave Turbulence Under Parametric Excitation

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10 Nonlinear Kinetics of Parametrically Excited Waves

10.1 General Equations

In the previous chapters of this book the nonlinear theory of the parametric excitation of waves was given in the mean–field approximation. It was called the S-theory, after the amplitudes of the interaction of wave pairs, $S(\boldsymbol{k}, \boldsymbol{k}')$, which plays a decisive role in it. As has been shown in Chap. 9, the S-theory is in good qualitative and quantitative agreement with the whole set of experiments on parametric excitation of magnons in ferromagnets and antiferromagnets. At the same time there are some other experiments and it is possible to realize special experiments which show a necessity of overcoming the framework of the S-theory. Indeed, the S-theory describes only the total characteristics of the parametrically excited waves and does not allow for the width of its distribution in $\omega, \boldsymbol{k}$-space. In the approximation of the S-theory (the first order of perturbation theory with respect to $\mathcal{H}_{\rm int}$) the stationary state of parametric waves is singular:

$$n(\boldsymbol{k},\omega), \quad \sigma(\boldsymbol{k},\omega) \propto \delta(\omega-\omega_{\rm p}/2)\delta[\omega_{\rm NL}(\boldsymbol{k})-\omega_{\rm p}/2]\,, \tag{10.1.1}$$

where $n(\boldsymbol{k},\omega)$ and $\sigma(\boldsymbol{k},\omega)$ are the Fourier transform of the non-simultaneous normal and anomalous correlators (see Sect. 6.4.1, eqs. (6.4.1–4)). Actually, singularities of the parametric wave distributions (10.1.1) never occur. To confirm this point it is sufficient to attempt to estimate the effect of the next order in $\mathcal{H}_{\rm int}$ with the help of the kinetic equation [10.1]. Its collision term becomes divergent for the solution (10.1.1). This means that the mutual scattering of the parametric waves should broaden their distribution function (10.1.1).

There have been many attempts to improve the approximation of the S-theory, allowing for the second-order terms in $\mathcal{H}_{\rm int}$ like derivation of the kinetic equation. But this approach is not correct because the kinetic equation can be applied only if the wave distribution has sufficiently large width in frequencies to guarantee the stochastization of wave phases; at the same time the distribution (10.1.1) has only one frequency $\omega=\omega_{\rm p}/2$. As a rule, in attempts to overcome the framework of second order approximation in $\mathcal{H}_{\rm int}$ it is necessary to use a diagrammatic technique, which is the regular method to formulate a perturbation theory with respect to $\mathcal{H}_{\rm int}$.

There are several types of diagrammatic techniques applied to describe non-equilibrium systems (see, e.g., [10.2, 3]). For classical problems it is more natural to use the diagrammatic technique (DT) suggested by *Wyld* in 1961 [10.2] which has become a regular procedure for investigation of developed hydrodynamic turbulence. The Wyld DT is very similar to the well known Feynman DT for quantum electrodynamics and other field theories: the rules of diagram reading are the same in both DTs, the Dyson equation for the Green's function is also the same. The principal feature of the Wyld DT (as well as that of any technique for strongly non-equilibrium systems [10.3]) consists in constructing two diagram series for the Green's functions (6.4.1) and for the pair correlators (6.4.4). In the thermodynamic equilibrium the Green's function $G(\boldsymbol{k},\omega)$ and double correlator $n(\boldsymbol{k},\omega)$ related by the universal relationship (by the fluctuation-dissipation theorem [10.3]) and two types of function reduce to one type. Under parametric excitation of waves there is no such relation.

Our goal in this chapter is to describe the consistent nonlinear theory of parametric excitation of waves which takes into account not only the mean–field S-interaction of pairs, but also the T^2-scattering of parametric waves from each other and their interaction with the thermal bath of the thermal waves. The latter interaction leads to the damping of the parametric waves. This theory was named S,T^2-theory [10.4–6]. The formalism of the S,T^2-theory is essentially more complicated than that of the S-theory. To keep the review of the S,T^2-theory within reasonable limits, the discussion in this chapter presupposes a higher standard of knowledge than in the previous part of this book. In particular, the reader is assumed to be familiar with the ideas of the Feynman diagrammatic technique [10.3]. A systematic derivation of the main equation of the S,T^2-theory, making use of the Wyld DT for non-equilibrium processes was carried out by *L'vov* [10.6]. To describe this procedure let us consider the motion equation (6.4.8) for $b_j \equiv b(\boldsymbol{k}_j,\omega_j)$ (Fourier transform of the canonical variables $b(\boldsymbol{k}_j,t)$):

$$\begin{aligned}b(\boldsymbol{k},\omega) =&G_0(\boldsymbol{k},\omega)[-hV(\boldsymbol{k})b^*(-\boldsymbol{k},\omega_\mathrm{p}-\omega)\\&-\frac{1}{2}\sum_{\boldsymbol{k}+\boldsymbol{k}_1=\boldsymbol{k}_2+\boldsymbol{k}_3}\int T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_3)b^*(\boldsymbol{k}_1)b(\boldsymbol{k}_2)b(\boldsymbol{k}_3)\\&\times\delta(\omega+\omega_1-\omega_2-\omega_3)\,d\omega_1 d\omega_2 d\omega_3+f(\boldsymbol{k},\omega)]\,.\end{aligned}\tag{10.1.2}$$

Here $G_0(\boldsymbol{k},\omega) = [\omega - \omega(\boldsymbol{k}) + i\gamma(\boldsymbol{k})]^{-1}$ is the zeroth Green's function of (6.4.8), which describes the response of the field $b(\boldsymbol{k},\omega)$ to the external force $f(\boldsymbol{k},\omega)$ at $T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_3)$=0 and $hV(\boldsymbol{k})$=0. There is a Langevin random force $f(\boldsymbol{k},\omega)$, which simulates the interaction of a wave system with the thermal bass, in the right-hand part of this equation. To develop the diagrammatic technique one can obtain the formal solution of these equations in the form of a series in degrees of $f(\boldsymbol{k},\omega)$:

$$\begin{aligned}b(\boldsymbol{k},\omega) =&b_1(\boldsymbol{k},\omega)+b_3(\boldsymbol{k},\omega)+b_5(\boldsymbol{k},\omega)+\dots,\\b_1(\boldsymbol{k},\omega) =&G_0(\boldsymbol{k},\omega)f(\boldsymbol{k},\omega)+hV(\boldsymbol{k})G_0^*(-\boldsymbol{k},\omega_\mathrm{p}-\omega)f^*(-\boldsymbol{k},\omega_\mathrm{p}-\omega),\\b_3(\boldsymbol{k},\omega) =&G_0(\boldsymbol{k},\omega)\int T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_3)G^*(\boldsymbol{k}_1,\omega_1)\\&\times G(\boldsymbol{k}_2,\omega_2)G(\boldsymbol{k}_3,\omega_3)f^*(\boldsymbol{k}_1,\omega_1)f(\boldsymbol{k}_2,\omega_2)f(\boldsymbol{k}_3,\omega_3)\\&\times d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\omega_1 d\omega_2 d\omega_3/(2\pi)^3+\dots\\\Rightarrow&G_0[TG^*G^2f^*f^2]+hVG_0^*[T^*G(G^*)^2f(f^*)^2],\\b_5(\boldsymbol{k},\omega)\Rightarrow&G_0\{TG^*Gf^*f[G_0(TG^*G^2f^*f^2)]\}+\dots\end{aligned}\tag{10.1.3}$$

Then one can build a series for $b(\boldsymbol{k},\omega)f^*(\boldsymbol{k}_1,\omega_1)$, $b(\boldsymbol{k},\omega)f(\boldsymbol{k}_1,\omega_1)$, $b(\boldsymbol{k},\omega)b^*(\boldsymbol{k}_1,\omega_1)$, and $b(\boldsymbol{k},\omega)b(\boldsymbol{k}_1,\omega_1)$ then can average over the Gaussian ensemble of the random force f. Using definitions (6.4.1) for the normal and anomalous Green's functions $G(\boldsymbol{k},\omega)$ and $L(\boldsymbol{k},\omega)$, definitions (6.4.4) for normal and anomalous double correlators $n(\boldsymbol{k},\omega)$ and $\sigma(\boldsymbol{k},\omega)$, one can derive a series for these functions. The next step in the derivation of the Wyld DT is the Dyson summation of a weakly linked (reducible) diagram which results in the following system of the Dyson equations for the Green's functions:

$$\begin{aligned}G(\boldsymbol{k},\omega) =&G_0(\boldsymbol{k},\omega)[1+\Sigma_\mathrm{c}(\boldsymbol{k},\omega)G(\boldsymbol{k},\omega)+\Pi_\mathrm{c}(\boldsymbol{k},\omega)L^*(\boldsymbol{k},\omega)]\,,\\L(\boldsymbol{k},\omega) =&G_0^*(-\boldsymbol{k},\omega_\mathrm{p}-\omega)[\Pi_\mathrm{c}^*(-\boldsymbol{k},\omega_\mathrm{p}-\omega)G(\boldsymbol{k},\omega)\\&+\Sigma_\mathrm{c}^*(-\boldsymbol{k},\omega_\mathrm{p}-\omega)L^*(\boldsymbol{k},\omega)]\,,\end{aligned}\tag{10.1.4}$$

and the Wyld equations for double correlators:

$$\begin{aligned}n(\boldsymbol{k},\omega) =&[|G(\boldsymbol{k},\omega)|^2+|L(\boldsymbol{k},\omega)|^2][\Sigma_\mathrm{d}(\boldsymbol{k},\omega)+f^2(\boldsymbol{k},\omega)]\\&+G(\boldsymbol{k},\omega)\Pi_\mathrm{d}(\boldsymbol{k},\omega)L^*(-\boldsymbol{k},\omega_\mathrm{p}-\omega)\\&+L(\boldsymbol{k},\omega)\Pi_\mathrm{d}^*(\boldsymbol{k},\omega)G^*(\boldsymbol{k},\omega),\\\sigma(\boldsymbol{k},\omega) =&G(\boldsymbol{k},\omega)L(\boldsymbol{k},\omega)[\Sigma_\mathrm{d}(\boldsymbol{k},\omega)+f^2(\boldsymbol{k},\omega)]\\&+L(-\boldsymbol{k},\omega_\mathrm{p}-\omega)G(-\boldsymbol{k},\omega_\mathrm{p}-\omega)\\&\times[\Sigma_\mathrm{d}(-\boldsymbol{k},\omega_\mathrm{p}-\omega)+f^2(-\boldsymbol{k},\omega_\mathrm{p}-\omega)]\\&+G(\boldsymbol{k},\omega)\Sigma_\mathrm{d}(\boldsymbol{k},\omega)G(-\boldsymbol{k},\omega_\mathrm{p}-\omega)+L^2(\boldsymbol{k},\omega)\Sigma_\mathrm{d}^*(\boldsymbol{k},\omega)\,.\end{aligned}\tag{10.1.5}$$

Here $f^2(\boldsymbol{k},\omega)$ is the random force correlator (6.4.12); the following notations are introduced for mass operators (MO) of the compact diagram sums:

$\Sigma_\mathrm{c}(\boldsymbol{k},\omega)$: normal causal mass operator,
$\Pi_\mathrm{c}(\boldsymbol{k},\omega)$: anomalous causal mass operator,
$\Sigma_\mathrm{d}(\boldsymbol{k},\omega)$: normal distributive mass operator,
$\Pi_\mathrm{d}(\boldsymbol{k},\omega)$: anomalous distributive mass operator.

Zakharov and *L'vov* [10.7] described the derivation procedure of these equations in detail. To close (10.1.4, 5), the mass operators must be expressed in terms of the Green's functions and correlators. These expressions are partially summarized but, in fact, they are infinite series of the perturbation theory. However, it is sufficient to retain first-order diagrams for small amplitudes. Diagrams of second order in the interaction of waves are retained in the S,T^2-theory. In this approximation the simplest distributive mass operators are those which determine the distribution function of parametric waves in accordance with (10.1.5):

$$\begin{aligned}\Sigma_{\rm d}(\boldsymbol{k},\omega) =&2\int[|T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_3)|^2 n_1 n_2 n_3 + 2T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_3)\\ &\times T^*(\boldsymbol{k},-\boldsymbol{k}_2;-\boldsymbol{k}_1,\boldsymbol{k}_3)\sigma_1^*\sigma_2 n_3]\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\\ &\times\delta(\omega+\omega_1-\omega_2-\omega_3)d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\omega_1 d\omega_2 d\omega_3/(2\pi)^3\,,\\ \Pi_{\rm d}(\boldsymbol{k},\omega) =&2\int[T^2(\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_3\boldsymbol{k}_4)\sigma_1^*\sigma_2\sigma_3 + 2T(\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_3,\boldsymbol{k}_3)\\ &\times T(-\boldsymbol{k},\boldsymbol{k}_2,\boldsymbol{k}_1,-\boldsymbol{k}_3)n_1 n_2\sigma_3]\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\\ &\times\delta(\omega+\omega_1-\omega_2-\omega_3)d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\omega_1 d\omega_2 d\omega_3/(2\pi)^3\,.\end{aligned} \tag{10.1.6}$$

Here $n_j = n(\boldsymbol{k}_j,\omega_j), \sigma_j = \sigma(\boldsymbol{k}_j,\omega_j)$. The expressions for the causal mass operator, which defines the causal Green's function, are, in accordance with (10.1.4), somewhat more complicated:

$$\begin{aligned}\Sigma_{\rm c}(\boldsymbol{k},\omega) =& 2\int T(\boldsymbol{k},\boldsymbol{k}')n(\boldsymbol{k}')d\boldsymbol{k}'d\omega'/2\pi\\ &+2\int\{|T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_3)|^2]G_1^* n_2 n_3\\ &+n_1(G_2 n_3+n_2 G_3)] + 2T(\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_3,\boldsymbol{k}_4)T^*(\boldsymbol{k},\boldsymbol{k}_2;\boldsymbol{k}_1,\boldsymbol{k}_3)\\ &\times[\sigma_1^*\sigma_2 G_3 + (L_1^*\sigma_2+\sigma_1^* L_{\bar 2} n_3]\}\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\\ &\times\delta(\omega+\omega_1-\omega_2-\omega_3)d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\omega_1 d\omega_2 d\omega_3/(2\pi)^3\,,\end{aligned} \tag{10.1.7a}$$

$$\begin{aligned}\Pi_{\rm c}(\boldsymbol{k},\omega) =& hV(\boldsymbol{k}) + \int S(\boldsymbol{k},\boldsymbol{k}')\sigma(\boldsymbol{k}')d\boldsymbol{k}'d\omega'/2\pi\\ &+2\int\{T^2(\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_3,\boldsymbol{k}_4)\}[L_1^*\sigma_2\sigma_3+\sigma_1^*(L_2\sigma_3+\sigma_2 L_3)]\\ &+2T(\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_3,\boldsymbol{k}_4)T(\boldsymbol{k},-\boldsymbol{k}_2;\boldsymbol{k}_1,\boldsymbol{k}_3)[n_1 n_2 L_{\bar 3}\\ &+(G_1^* n_2+n_2 G_2)\sigma_3]\}\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\\ &\times\delta(\omega+\omega_1-\omega_2-\omega_3)d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\omega_1 d\omega_2 d\omega_3/(2\pi)^3\,,\end{aligned} \tag{10.1.7b}$$

Here $\bar{\boldsymbol{j}} \equiv -\boldsymbol{k}_j, \omega_{\rm p}-\omega_j$. If there were random inhomogeneities, i.e. defects, pores, deformation fields, in a medium, then the elastic scattering from them leads to an additive contribution to the Hamiltonian,

$$\mathcal{H}_{\rm el} = \int g(\boldsymbol{k},\boldsymbol{k}')\eta(\boldsymbol{k}-\boldsymbol{k}')b^*(\boldsymbol{k},t)b(\boldsymbol{k}',t)d\boldsymbol{k}d\boldsymbol{k}'\,, \tag{10.1.8}$$

where $g(\boldsymbol{k},\boldsymbol{k}')$ is the interacting amplitude which characterizes the scattering of waves from the inhomogeneities, and $\eta(\boldsymbol{k})$ is their amplitude. For point defects we have the following expression: $(2\pi)^3\eta(\boldsymbol{k}) = v_0\sum_{r_{\rm n}}\exp(i\boldsymbol{k}\cdot\boldsymbol{r}_{\rm n})$, where $\boldsymbol{r}_{\rm n}$ are the coordinates of defects, and v_0 is the volume of the elementary cell. The detailed analysis [10.6, 7] shows that in interesting cases it is enough, as a rule, to retain the lowest diagrams in $g(\boldsymbol{k},\boldsymbol{k}')$ which give an additive contribution into mass operators (c is the concentration of defects):

$$\begin{aligned}\Sigma_{\rm c,el}(\boldsymbol{k},\omega) &= c\int|g(\boldsymbol{k},\boldsymbol{k}')|^2 G(\boldsymbol{k}',\omega)d\boldsymbol{k}'\,,\\ \Pi_{\rm c,el}(\boldsymbol{k},\omega) &= c\int[g(\boldsymbol{k},\boldsymbol{k}')]^2 L(\boldsymbol{k}',\omega)d\boldsymbol{k}\,,\\ \Sigma_{\rm d,el}(\boldsymbol{k},\omega) &= c\int|g(\boldsymbol{k},\boldsymbol{k}')|^2 n(\boldsymbol{k}',\omega)d\boldsymbol{k}'\,,\\ \Pi_{\rm d,el}(\boldsymbol{k},\omega) &= c\int[g(\boldsymbol{k},\boldsymbol{k}')]^2\sigma(\boldsymbol{k}',\omega)d\boldsymbol{k}'\,.\end{aligned} \tag{10.1.9}$$

10.2 Limit of the S-Theory

10.2.1 Form of the Green's Function

One can represent the solution of the Dyson equations (10.1.3,4) in the form, similar to (6.4.5)

$$\begin{aligned}G(\boldsymbol{k},\omega) &= [\omega_{\rm p}-\omega-\omega_{\rm NL}(\boldsymbol{k})-i\Gamma(\boldsymbol{k})]/\Delta(\boldsymbol{k},\omega),\\ L^*(\boldsymbol{k},\omega) &= \Pi_{\rm c}^*(-\boldsymbol{k},\omega_{\rm p}-\omega)/\Delta(\boldsymbol{k},\omega),\end{aligned} \tag{10.2.1a}$$

$$\begin{aligned}\Delta(\boldsymbol{k},\omega) &= [\omega-\omega_{\rm NL}(\boldsymbol{k})+i\Gamma(\boldsymbol{k})][\omega_{\rm p}-\omega-\omega_{\rm NL}(\boldsymbol{k})-i\Gamma(\boldsymbol{k})] - |\Pi_{\rm c}(\boldsymbol{k},\omega)|^2,\\ \omega_{\rm NL}(\boldsymbol{k}) &= \omega(\boldsymbol{k}) + {\rm Re}\{\Sigma_{\rm c}[\boldsymbol{k},\omega(\boldsymbol{k})]\}\,,\\ \Gamma(\boldsymbol{k}) &= \gamma(\boldsymbol{k}) - {\rm Im}\{\Sigma_{\rm c}[\boldsymbol{k},\omega(\boldsymbol{k})]\}\,.\end{aligned} \tag{10.2.1b}$$

It is obvious that ${\rm Im}G(\boldsymbol{k},\omega)$ and $L(\boldsymbol{k},\omega)$ in the middle of the frequency packet, i.e., when $\omega{=}\omega_{\rm p}/2$ have the Lorentzian form with the width $\Delta k = \nu/v$ (v is the group velocity and $\nu^2{=}\Gamma^2-|\Pi|^2$). This width is essentially less than Γ/v because of the compensation of damping by the pump according to (10.10a). The packet width of the frequency $\omega_{\rm NL}(\boldsymbol{k})$ in the center of the packet, i.e. on the resonance surface, is even smaller: $\Delta\omega \approx \nu^2/\Gamma$. The normal Green's function $G(\boldsymbol{k},\omega)$ looks like the free Green's function far from the resonance surface, but the function must have the renormalized frequency and damping:

$$G(\boldsymbol{k},\omega)) \to 1/[\omega-\omega_{\rm NL}(\boldsymbol{k})+i\Gamma(\boldsymbol{k})] \tag{10.2.2}$$

and the anomalous Green's function $L(\boldsymbol{k},\omega)$ is small: $|L(\boldsymbol{k},\omega)|^2 \ll |G(\boldsymbol{k},\omega)|$.

Concluding this section it is necessary to note that the Green's functions (10.2.1a) coincide with (6.4.15), if one puts

$$\begin{aligned} \omega_{\rm NL} =& \omega(\boldsymbol{k}) + 2\int T(\boldsymbol{k},\boldsymbol{k}')n(\boldsymbol{k}')d\boldsymbol{k}' , \quad \Gamma(\boldsymbol{k}) = \gamma(\boldsymbol{k}) , \\ \Pi_{\rm c}(\boldsymbol{k},\omega) =& P(\boldsymbol{k}) = hV(\boldsymbol{k}) + \int S(\boldsymbol{k},\boldsymbol{k}')\sigma(\boldsymbol{k}')d\boldsymbol{k}' , \end{aligned} \tag{10.2.3}$$

i.e. neglects terms proportional to T^2 in (10.1.7). This is an approximation of the S-theory for the Green's functions!

10.2.2 Separation of the Waves into Parametric and Thermal

This problem has been discussed in Sect. 6.4.3 in the mean–field approximation of the basic S-theory. By analogy with (6.4.19) let us put $n_{\rm p}(\boldsymbol{k},\omega) = n(\boldsymbol{k},\omega) - n_{\rm T}(\boldsymbol{k},\omega)$, where

$$n_{\rm T}(\boldsymbol{k},\omega) = \frac{2\Gamma(\boldsymbol{k})n_0(\boldsymbol{k})}{[\omega_{\rm NL}(\boldsymbol{k})-\omega]^2 + \Gamma^2(\boldsymbol{k})} . \tag{10.2.4}$$

Here $n_{\rm p}$ and $n_{\rm T}$ are the distribution functions of parametric and thermal waves. It is necessary to note that functions $\Gamma(\boldsymbol{k})$ and $\omega_{\rm NL}(\boldsymbol{k})$ in (10.2.4) are not to be calculated in the thermodynamic equilibrium but using the real spectrum $n_{\rm T}(\boldsymbol{k},\omega)$, $n_{\rm p}(\boldsymbol{k},\omega)$ and $\sigma(\boldsymbol{k},\omega)$ which was calculated in the presence of pumping. In this definition $n_{\rm T}$ is everywhere a smooth function of $\boldsymbol{k}$ but it moves to the equilibrium spectrum asymptotically beyond the resonance surface for $[\omega_{\rm NL}(\boldsymbol{k}) - \omega_{\rm p}/2] \gg \Gamma$. As for the quantities

$$n_{\rm p}(\boldsymbol{k}) = \int n_{\rm p}(\boldsymbol{k},\omega)d\omega/2d\pi , \quad \sigma(\boldsymbol{k}) = \int \sigma(\boldsymbol{k},\omega)d\omega/2\pi , \tag{10.2.5}$$

they rapidly decrease beyond the resonance surface. It may be shown that $n_{\rm p}(\boldsymbol{k})$ and $\sigma(\boldsymbol{k}) \propto 1/[\omega_{\rm NL}(\boldsymbol{k}) - \omega_{\rm p}/2]^2$. It must be noted that in the above arguments we have nowhere used the actual form of the interaction of waves. The definitions of parametric and thermal waves in (10.2.4) are, therefore, valid for every interaction, particularly in cases where three-wave processes and interactions with phonons, etc. are essential. Now let us use the method developed for studying the packet of parametric waves at relatively small amplitudes of pumping, where the scattering of the thermal waves by each other forms the main contribution to the mass operator. It is necessary to substitute $n(\boldsymbol{k},\omega) = n_{\rm T}(\boldsymbol{k},\omega) + n_{\rm p}(\boldsymbol{k},\omega), \sigma(\boldsymbol{k},\omega)$ into the expressions for $\Sigma_{\rm c}$, $\Pi_{\rm c}$, $\Sigma_{\rm d}$ and $\Pi_{\rm d}$ and to study the obtained expressions. The result obtained in the zero approximation in $n_{\rm p}, \sigma$ is known from (10.2.4a). Formula

$$\omega_{\rm T}(\boldsymbol{k}) = \omega(\boldsymbol{k}) + 2\int T(\boldsymbol{k},\boldsymbol{k}')n_{\rm T}(\boldsymbol{k}',\omega')d\boldsymbol{k}'d\omega'/2\pi$$

describes the frequency dependence of the waves on the medium temperature in the first order in $\mathcal{H}_{\rm int}$. Let us assume that this dependence has already been included in the definition of $\omega(\boldsymbol{k})$, so that

$$\omega_{\rm NL}(\boldsymbol{k}) = \omega(\boldsymbol{k}) + 2\int T(\boldsymbol{k},\boldsymbol{k}')n_{\rm p}(\boldsymbol{k}')d\boldsymbol{k}' . \tag{10.2.6}$$

Expressions (10.1.6) for $\Sigma_{\rm d}$ and $\Pi_{\rm d}$ are quadratic in the amplitudes of the interaction Hamiltonian $T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_3)$. At the first stage of the investigation this allows one to calculate them in the zero approximation in the amplitude of the parametric turbulence. The applicability framework of this approximation and the effects appearing for large amplitudes will be considered below. In this approximation $\Pi_{\rm d}$=0 and

$$\begin{aligned} \Sigma_{\rm d} =& 2\int]T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_3)]^2 n_{\rm T}(\boldsymbol{k}_1)n_{\rm T}(\boldsymbol{k}_2)n_{\rm T}(\boldsymbol{k}_3)\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_3-\boldsymbol{k}_4) \\ &\times \delta[\omega(\boldsymbol{k})+\omega(\boldsymbol{k}_1)-\omega(\boldsymbol{k}_2)-\omega(\boldsymbol{k}_3)]d\boldsymbol{k}_1d\boldsymbol{k}_2d\boldsymbol{k}_3 . \end{aligned} \tag{10.2.7}$$

This expression is the contribution to the correlator of random force $f^2(\boldsymbol{k})$, arising due to four-wave scattering. Let us assume that the contribution (10.2.7) has already been included in $f^2(\boldsymbol{k})$. Then in equations (10.1.5) it is necessary to put not only $\Pi_{\rm d}$=0 but also $\Sigma_{\rm d}$=0. Substituting the expressions (10.2.1a, 3) for the Green's functions into (10.1.5) for double correlators (at $\Pi_{\rm d}=\Sigma_{\rm d}=0$) it is easy to see that the resulting equations coincide with (6.4.16), which has been obtained in the temperature S-theory in Sect. 6.4. That means the approximation of the basic S-theory may be obtained from the S, T^2-theory if we neglect the influence of parametric waves in the expressions for mass operators of second order in the vertex $T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_3)$. The parameter of this approximation will be given later.

10.3 Nonlinear Theory for Parametric Excitation of Waves in Random Media

Except for some rare cases, real media possess various inhomogeneities which destroy their ideal translation symmetry. The nature of the inhomogeneities in ferromagnetic crystals and their influence on the ferromagnetic resonance and spin waves were studied in a large number of works: see, e.g., *Spark's* monograph [10.8] and *Schlöman's* paper [10.9]. A more detailed information can be found in *Gurevich* [10.10]. Many experimental works (see [10.10, 12 – 14]) are devoted to the question of the influence of inhomogeneities on parallel pumping of magnons. This section will discuss the theory of the phenomenon and make a comparison with experiment.

10.3.1 General Equations in the S, g^2-Approximation

Let us formulate the so-called S, g^2*-approximation* in the nonlinear theory for the parametric excitation of waves in random media [10.15]. We start with the diagrammatic equations (10.1.5–9) and (10.2.1) in which we will take into account:

(1) interaction of the parametric and thermal waves which leads to damping of parametric waves and to a dependence of the spectrum on the temperature,

(2) mean–field S-interaction between the parametric wave pairs, resulting in a renormalization of the pumping,

(3) elastic scattering of waves from inhomogeneities in Born's approximation proportional to $|g(\boldsymbol{k},\boldsymbol{k}')|^2$.

In such a S, g^2-theory we obtain the following equations:

$$\begin{aligned} \Gamma(\boldsymbol{k},\omega) &= \gamma(\boldsymbol{k}) - \operatorname{Im}\Sigma_{\mathrm{c,el}}(\boldsymbol{k},\omega)\,, \quad \Pi_{\mathrm{c}}(\boldsymbol{k},\omega) = P(\boldsymbol{k}) + \Pi_{\mathrm{c,el}}(\boldsymbol{k},\omega), \\ \Sigma_{\mathrm{d}}(\boldsymbol{k},\omega) &= \Sigma_{\mathrm{d,el}}(\boldsymbol{k},\omega)\,, \quad \Pi_{\mathrm{d}}(\boldsymbol{k},\omega) = \Pi_{\mathrm{d,el}}(\boldsymbol{k},\omega), \end{aligned} \tag{10.3.1}$$

where $\gamma(\boldsymbol{k})$ and $P(\boldsymbol{k})$ are damping and pumping of parametric waves in the basic S-theory and where the MOs Σ, Π are given by (10.1.9). It is easy to see that if $n(\boldsymbol{k},\omega)$ and $\sigma(\boldsymbol{k},\omega) \propto \delta(\omega-\omega_{\mathrm{p}}/2)$, Σ_{d}, and Π_{d} are also $\propto \delta(\omega-\omega_{\mathrm{p}}/2)$. This means that elastic scattering of parametric waves on inhomogeneities does not disturb the uniform character of the parametric turbulence of waves inherent in the S-theory. With the one-frequency approximation the Wyld equations (10.1.5) may be integrated over the modulus k. As a result we have:

$$\begin{aligned} n(\Omega) &= \frac{\pi k^2(\Omega)\Gamma(\Omega)}{v(\Omega)\nu^3(\Omega)}\left\{\Gamma(\Omega)\Sigma_{\mathrm{d}}(\Omega) + \operatorname{Im}\left[\Pi_{\mathrm{c}}^*(\Omega)\Pi_{\mathrm{d}}(\Omega)\right]\right\}, \\ \Gamma(\Omega)\sigma(\Omega) &+ i\Pi_{\mathrm{c}}(\Omega)n)\Omega) = 0\,, \quad \nu^2(\Omega) = \Gamma^2(\Omega) - |\Pi_{\mathrm{c}}(\Omega)|^2\,; \end{aligned} \tag{10.3.2a}$$

$$\begin{aligned} \Gamma(\Omega) &= \gamma(\Omega) + \pi c\int\left|g(\Omega,\Omega_1)\right|^2\frac{\Gamma(\Omega_1)k^2(\Omega_1)}{\nu(\Omega_1)v(\Omega_1)}d\Omega_1\,, \\ \Pi_{\mathrm{c}}(\Omega) &= P(\Omega) + \pi c\int g(\Omega,\Omega_1)g(\bar{\Omega},\bar{\Omega}_1)\frac{\Pi_{\mathrm{c}}(\Omega_1)k^2(\Omega_1)}{\nu(\Omega_1)v(\Omega_1)}d\Omega_1\,, \end{aligned} \tag{10.3.2b}$$

$$\begin{aligned} \Sigma_{\mathrm{d}}(\Omega) &= \Sigma_{\mathrm{d,el}}(\Omega) = c\int\left|g(\Omega,\Omega_1)\right|^2 n(\Omega_1)d\Omega_1\,, \\ \Pi_{\mathrm{d}}(\Omega) &= \Pi_{\mathrm{d,el}}(\Omega) = c\int g(\Omega,\Omega_1)g(\bar{\Omega},\bar{\Omega}_1)\sigma(\Omega_1)d\Omega_1\,. \end{aligned} \tag{10.3.2c}$$

Here $n(\Omega)$ and $\sigma(\Omega)$ are the distribution functions $n_{\mathrm{p}}(\boldsymbol{k},\omega)$ and $\sigma(\boldsymbol{k},\omega)$ (6.4.21) integrated in ω and module k and depending only on the angular coordinates $\Omega = \Theta, \varphi$, on the resonant surface; $\bar{\Omega} = \pi - \Theta, \varphi + \pi$; $k(\Omega)$ and $v(\Omega)$ are the wave vector and group velocity of the waves at the point on the resonant surface with the angular coordinate Ω.

Equations (10.3.2) represent the S, g^2-theory, a closed system of integral equations which enables one to describe the system of interacting parametric waves in a medium with random inhomogeneities. They were obtained in [10.15] for the first time.

Elastic scattering of waves may be characterized by the decrement of damping. From the conventional perturbation theory it follows:

$$\gamma_{\mathrm{el}}(\boldsymbol{k}) = \pi c\int\left|g(\boldsymbol{k},\boldsymbol{k}_1)\right|^2\delta\left[\omega(\boldsymbol{k}) - \omega(\boldsymbol{k}_1)\right]d\boldsymbol{k}_1 \simeq \pi c\Delta^2 k^2 g^2/v\,, \tag{10.3.3}$$

where $g \approx g(\boldsymbol{k},\boldsymbol{k}_1)$, and $\Delta^2 \le 4\pi$ is a characteristic scattering space angle (Δ is the scattering angle). Thus, the S, g^2-theory includes three dimensionless parameters: the degree of homogeneity of the medium $\gamma_{\mathrm{el}}/\gamma$, the scattering angle Δ, and the supercriticality p.

10.3.2 Distribution Function of Parametric Waves

From (10.3.2) there follows the integral relation

$$\int\gamma(\Omega)n(\Omega)d(\Omega) + \operatorname{Im}\left\{\int hV^*(\Omega)\sigma(\Omega)d\Omega\right\} = 0\,, \tag{10.3.4}$$

which describes the energy balance in the system of parametric waves: the total dissipation of energy due to external relaxation mechanisms is equal to the total energy flow into all the pairs. Elastic scattering does not enter into this relation because it occurs with frequency conservation and, hence, does not expel energy out of a system of parametric waves. Such a scattering results in two effects: isotropization of the distribution functions and destruction of the phase correlations in the pairs, which leads to a decrease of the ratio $\sigma(\Omega)/n(\Omega)$.

Let us briefly discuss now the simplest and interesting limiting case where $\gamma_{\mathrm{el}} \gg \gamma$. Then it follows from (10.3.2)

$$\sigma(\Omega) = -in(\Omega)\left[\Pi_{\mathrm{c}}(\Omega)/\Gamma(\Omega)\right], \quad n(\Omega) = N/4\pi\,; \tag{10.3.5}$$

$$\Gamma(\Omega) = 4\pi^2 c\left\langle\left|g(\Omega,\Omega_1)\right|^2\frac{k^2(\Omega_1)}{v(\Omega_1)}\right\rangle_{\Omega_1}, \quad \langle f(\Omega)\rangle_\Omega \equiv \int f(\Omega)\frac{d\Omega}{4\pi}; \tag{10.3.6}$$

$$\begin{aligned} \Pi_{\mathrm{c}}(\Omega) &= P(\Omega) + \langle K(\Omega,\Omega_1)\Pi_{\mathrm{c}}(\Omega_1)\rangle_{\Omega_1}\,, \\ K(\Omega,\Omega_1) &\equiv \frac{g(\Omega,\Omega_1)g(\bar{\Omega},\bar{\Omega}_1)k^2(\Omega_1)}{\langle|g(\Omega_1,\Omega_2)|^2 k^2(\Omega_2)/v(\Omega_2)\rangle_{\Omega_2}v(\Omega_1)}\,. \end{aligned} \tag{10.3.7}$$

With the exception of the so-called *degenerate cases*, in which the operator $K(\Omega,\Omega_1)$ has the eigenvalue 1 (for instance, at $K = 1$), it follows from (10.3.7) that $\Pi_{\mathrm{c}} \approx \Pi \approx P$. Allowing for (10.2.5, 6) this yields

$$|P|^2 \simeq \gamma_{\mathrm{el}}\gamma\,, \qquad |\sigma(\Omega)| \simeq n(\Omega)\sqrt{\gamma/\gamma_{\mathrm{el}}}\,, \tag{10.3.8}$$

i.e. the destruction of phase correlations and, accordingly, the increase of the excitation threshold of waves. Instead of the estimate $\langle h_{\rm th} V\rangle = \langle\gamma\rangle$, which would have held for $|\sigma|=n$, it follows from (10.3.4) that

$$h_{\rm th}^2 \langle |V(\Omega)|^2\rangle \simeq \langle \gamma(\Omega)\rangle \gamma_{\rm el}\,. \qquad (10.3.9)$$

This formula has been qualitatively confirmed in a direct experiment by *Smirnov* and *Petrov* [10.4] who independently measured the values $h_{\rm th}$, $\gamma_{\rm el}$ and $\langle\gamma(\Omega)\rangle$ in the antiferromagnet $CsMnF_3$. A quantitative analysis of the behavior of parametric waves requires knowledge of the function $g(\boldsymbol{k}, \boldsymbol{k}_1)$ which is determined by the destructive character of the inhomogeneity of the medium. After that there are no fundamental difficulties to carry out this analysis.

10.3.3 Behavior of Parametrically Excited Waves Beyond the Threshold

In the study of the behavior of parametrically excited waves the most interesting case is that of large scattering intensities, where a stronger influence of two-magnon scattering may be expected. In the simple model, with $S(\Theta,\Theta') \simeq V(\Theta)V(\Theta')$, $g(\boldsymbol{k})=g$, and $\gamma(\boldsymbol{k})=\gamma$, it follows from (10.3.2)

$$SN = (15\gamma_{\rm el}/8)\sqrt{p-1}\,, \qquad h_{\rm th} V = \sqrt{15\gamma\gamma_{\rm el}/8}\,. \qquad (10.3.10)$$

In the case of scattering from point defects, when $g(\boldsymbol{k}, \boldsymbol{k}_1)$ is proportional to $(\boldsymbol{k}\boldsymbol{k}_1)$, one can obtain for the fluctuation of the exchange constant an expression close to (10.3.10)

$$SN = (9\gamma_{\rm el}/8)\sqrt{p-1}\,. \qquad (10.3.11)$$

For intense small-angle scattering, when $\gamma\Delta^2 > \gamma$, one can obtain [10.3] :

$$SN = \frac{45}{16}\gamma_{\rm el}\Delta^2\sqrt{p-1}\,. \qquad (10.3.12)$$

In all the described cases the excitation level of parametric waves for strong scattering from inhomogeneities proves to be a factor of $\gamma_{\rm el}\Delta^2/\gamma$ greater (at the same supercriticality) than in uniform medium. This is caused by partial destruction of the phase correlations, which leads to a weakening of the phase mechanism of the amplitude limiting. The specific form of the dependence N on supercriticality in (10.3.10–13) is not universal, but originates from the specific form of the function $S(\Theta,\Theta_1)$. In other cases (see, e.g., [10.16]) the dependence of N on h is more complicated and it reproduces these dependences only qualitatively.

In all the described cases one can obtain from (10.3.2) for the nonlinear susceptibility the following expression

$$\chi = \left[2V^2(h^2-h_{\rm th}^2) + i h_{\rm th}\sqrt{h^2-h_{\rm th}^2}\right]/h^2\,. \qquad (10.3.13)$$

This result coincides with the known expressions (5.5.35) for χ, which were obtained within the basic S-theory for an uniform medium. It means that the dependences of the nonlinear susceptibilities χ' and χ'' on the value $(h/h_{\rm th})$ are not significantly changed by the scattering of waves from inhomogeneities. Of course, the value of the threshold field in a nonuniform medium itself is greater than that in an uniform medium. The literal coincidence of the formulae for χ should not be overestimated. In more complicated situations (see, e.g., [10.16]) the dependence $\chi(h/h_{\rm th})$ resembles (10.3.13) only qualitatively.

10.4 Consistent Nonlinear Theory for Parametric Excitation of Waves

We know that the S-theory takes into account only the mean–field interaction of $\pm\boldsymbol{k}$–pairs of parametrically excited waves. In order to describe phenomena beyond the framework of such approximation it is necessary to include into the formalism of the theory the scattering of individual parametric waves and their interaction with the thermal bath. This was possible with the help of the Wyld diagrammatic technique [10.2]. An account of such a consistent theory (S,T^2*–theory*) based on the Wyld DT is given in my book [10.3] in Russian. Here I represent only a short review of its results.

10.4.1 Spectral Density of Parametrically Excited Waves

As has already been pointed out, the elastic scattering does not change the number and frequency of parametric waves but only destroys their coupling to the pump, the process leading to the dephasing of wave pairs, and to the isotropization of the parametric wave distribution. If the frequency of the elastic scattering distributed is greater than all other relaxation frequencies, the distribution of the parametric waves is isotropic, and the influence of the parametric pumping and of the parametric wave scattering from each other can be taken into account as small perturbations. A simple equation appears in this approximation for the distribution function of parametric waves in frequencies. One can solve it analytically and isolate the only stable solution from the stationary solutions. We discuss here the results of the S,T^2-theory for this case, which looks at first sight complicated. In the mass operator we will take into consideration the contributions of elastic scattering $\Sigma_{\rm el}$, $\Pi_{\rm el}$, the contribution of the interaction of parametric waves with thermal waves $\gamma(\boldsymbol{k})$ and $f^2(\boldsymbol{k},\omega)$ and the contribution of the interaction of parametric waves among themselves, (10.1.6). In the limit $\gamma_{\rm el} \gg \gamma$ the equations (10.1.4, 5) have the isotropic solution $n(\boldsymbol{k},\omega) - n(k,\omega)$. It allows one to integrate them in the general form not only in k but also in the solid angle Ω. Ultimately, after some modifications, we obtain the equation for the spectral density of parametric waves $n(\omega) = \int n(\boldsymbol{k},\omega)d\boldsymbol{k}$,

$$\begin{aligned} n(\epsilon) \equiv& n(\omega - \omega_{\mathrm{p}}/2) = \frac{\Gamma_1}{\epsilon^2+\eta^2}\left\{\frac{4\pi^2k^2n_0}{v}\right. \\ &\left. + \frac{T^2}{kv}\int n(\epsilon_1)n(\epsilon_2)n(\epsilon_3)\delta(\epsilon+\epsilon_1-\epsilon_2-\epsilon_3)d\epsilon_1 d\epsilon_2 d\epsilon_3\right\}. \end{aligned} \tag{10.4.1}$$

Here T^2 is the mean value of the square $T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_3)$. Γ_1 has the order of magnitude $(\gamma_{\mathrm{el}})^2/\gamma$. The second term in (10.4.1) may be neglected at small supercriticality. In this case the distribution $n(\epsilon)$ is a Lorentzian with width $\eta = \eta_{\mathrm{T}}$, which may be determined by integrating (10.4.1) in ϵ. Taking (10.3.12) into account we have

$$\eta_{\mathrm{T}} = \frac{\Gamma_1\gamma}{kv}\frac{4\pi^2k^3n_0}{N} \simeq \frac{r}{\sqrt{p-1}}. \tag{10.4.2}$$

Here r is the small parameter (6.4.28c), characterizing the influence of the thermal bass: $r = SN_{\mathrm{T}}/kv$, $N_{\mathrm{T}} = 4\pi^2 n_0 k^3$. In the opposite case of large supercriticality, the thermal term in (10.4.1) can be neglected. *Cherepanov* and *L'vov* [10.17] showed that this equation has a one-parametric set of solutions. However, only one of them is stable. It is a *spectral soliton*:

$$Tn(\epsilon) = \frac{kv}{2\Gamma_1\cosh(\pi\epsilon/2\eta)}, \qquad \eta \simeq \frac{(\gamma_{\mathrm{el}})^2}{\gamma}\sqrt{\frac{\gamma}{kv}(p-1)}. \tag{10.4.3}$$

In almost uniform crystals, when $\gamma_{\mathrm{el}} \ll \gamma$, the study of spectral solitons becomes very complicated because of the anisotropy of $n(\boldsymbol{k},\omega)$. Therefore *Krutensko* et al. [10.18] limited themselves to an axially symmetric situation which is realized in isotropic and cubic ferromagnets. In the region of supercriticalities $p_1 < p < p_2$ (here $p_1 - 1 \simeq 2p_{\mathrm{s}}(\gamma_{\mathrm{el}}/\gamma)^{3/4}$, $p_2 - 1 \simeq p_{\mathrm{s}}(\gamma_{\mathrm{el}}/\gamma)^{3/4}$, $p_{\mathrm{s}} \simeq kv/\gamma$) the broadening in ω is determined by the T^2–scattering of parametric waves and the line shape of $n(\omega)$ is close to (10.4.3) with the effective width, η_{int}:

$$\eta_{\mathrm{int}} = \gamma\left(\frac{\gamma_{\mathrm{el}}}{\gamma}\right)^{1/4}\sqrt{\frac{p-1}{p_{\mathrm{s}}}}. \tag{10.4.4}$$

Broadening in angles is determined by elastic two-wave scattering. Both thermal fluctuations and two-wave scattering may be neglected at greater supercriticalities $p > p_2$. Then

$$\Delta\Theta(\boldsymbol{k}) \approx \sqrt{\frac{\eta_{\mathrm{int}}}{\gamma}}, \quad \Delta\omega \approx \eta_{\mathrm{int}} \approx \gamma\left[\frac{p-1}{p_{\mathrm{s}}}\right]^{2/3}. \tag{10.4.5}$$

The line shape of $n(\omega)$ remains similar to that in (10.4.3). In the applicability framework of the theory ($p \simeq p_{\mathrm{s}}$) $\Delta\Theta(\boldsymbol{k}) \simeq \pi$. However, such supercriticalities are of interest only from an academic point of view since auto-oscillations arise first, leading to sharp broadening of the spectrum in $\Delta\Theta(\boldsymbol{k})$ and $\Delta\omega(\boldsymbol{k})$.

In conclusion we point out that the spectral solitons (10.4.3) at the parametric excitation of the magnons in YIG have been experimentally investigated in detail by *Krutensko* et al. [10.18]. Good qualitative and quantitative agreement with the above described theory has been observed (see Sect. 9.4.2 and Figs. 9.12, 13). One can see that the data concerning $n(\omega)$, which are given in Fig. 9.13 in "straightening" coordinates (chosen in such a way that the Gaussian in coordinates 1, the Lorentzian in coordinates 2, and function (10.4.3) in coordinates 3 will be straight lines) lie on a straight line only in coordinates 3. This and other facts give reason to believe that the theory elaborated on describes the reality well.

10.4.2 Structure of the Distribution Function in *k*-Space

It has already been pointed out that the T^2-scattering of parametric waves leads to the finite width of the distribution function $n(\boldsymbol{k},\omega)$ of parametric waves, not only in ω but also in k. There $\Delta\omega \simeq \nu^2/2\gamma$, which is much less than $\Delta\omega(\boldsymbol{k}) \simeq \nu$, when the supercriticality is not large. It gives reason to think that the study of the distribution function structure $n(\boldsymbol{k}) = \int n(\boldsymbol{k},\omega)d\boldsymbol{k}$ may be restricted to the so called *one-frequency turbulence* approximation. This is the assumption that

$$n(\boldsymbol{k},\omega) = n(\boldsymbol{k})\delta(\omega-\omega_{\mathrm{p}}/2), \sigma(\boldsymbol{k},\omega) = \sigma(\boldsymbol{k})\delta(\omega-\omega_{\mathrm{p}}/2). \tag{10.4.6}$$

Such an approximation allows one to analyze equations of the S,T^2-theory effectively in practically all interesting cases, as will be shown in this section. In the one-frequency approximation the Wyld equations (10.1.5) have the simple form:

$$\begin{aligned} n_{\mathrm{p}}(\boldsymbol{k}) =& \frac{2}{\Delta^2(\boldsymbol{k})}\Big\{\Gamma^2(\boldsymbol{k})\Sigma_{\mathrm{d}}(\boldsymbol{k}) \\ &+ \Big[\omega_{\mathrm{NL}}(\boldsymbol{k}) - \frac{\omega_{\mathrm{p}}}{2}\Big]\mathrm{Re}\Big\{\Pi_{\mathrm{c}}^*(\boldsymbol{k})\Pi_{\mathrm{d}}(\boldsymbol{k})\Big\} + \Gamma(\boldsymbol{k})\mathrm{Im}\Big\{\Pi_{\mathrm{c}}^*(\boldsymbol{k})\Pi_{\mathrm{d}}(\boldsymbol{k})\Big\}\Big\}, \\ \sigma(\boldsymbol{k}) =& \frac{1}{\Delta^2(\boldsymbol{k})}\Big\{2\Pi_{\mathrm{c}}(\boldsymbol{k})\Sigma_{\mathrm{d}}(\boldsymbol{k})\big[\omega_{\mathrm{NL}}(\boldsymbol{k}) - \frac{\omega_{\mathrm{p}}}{2} - i\Gamma(\boldsymbol{k})\big] \\ &+ \Pi_{\mathrm{c}}^2(\boldsymbol{k})\Pi_{\mathrm{d}}^*(\boldsymbol{k}) + \big[\omega_{\mathrm{NL}}(\boldsymbol{k}) - \frac{\omega_{\mathrm{p}}}{2} - i\Gamma(\boldsymbol{k})\big]^2\Pi_{\mathrm{d}}(\boldsymbol{k})\Big\}, \\ \Delta(\boldsymbol{k}) =& \big[\omega_{\mathrm{NL}}(\boldsymbol{k}) - \frac{\omega_{\mathrm{p}}}{2}\big]^2 + \nu^2(\boldsymbol{k}), \quad \nu^2(\boldsymbol{k}) = \Gamma^2(\boldsymbol{k}) - |\Pi_{\mathrm{c}}(\boldsymbol{k})|^2. \end{aligned} \tag{10.4.7}$$

Here the mass operators $\Sigma_{\mathrm{d}}(\boldsymbol{k})$ and $\Pi_{\mathrm{d}}(\boldsymbol{k})$ are determined by the equations:

$$\Sigma_{\rm d}(\boldsymbol{k},\omega) = \Sigma_{\rm d}(\boldsymbol{k})\delta(\omega-\omega_{\rm p}/2)\,, \quad \Pi_{\rm d}(\boldsymbol{k},\omega) = \Pi_{\rm d}(\boldsymbol{k})\delta(\omega-\omega_{\rm p}/2)\,. \quad (10.4.8)$$

The values $\Sigma_{\rm d}(\boldsymbol{k})$ and $\Pi_{\rm d}(\boldsymbol{k})$ can be taken on the resonant surface: $\Sigma_{\rm d}(\boldsymbol{k}) \Rightarrow \Sigma_{\rm d}(\Omega)$, $\Pi_{\rm d}(\boldsymbol{k}) \Rightarrow \Pi_{\rm d}(\Omega)$. Contributions to the mass operators $\Sigma_{\rm d}$ and $\Pi_{\rm d}$ arise due to elastic two-wave scattering (see expressions (10.3.2) for $\Sigma_{\rm d,el}$ and $\Pi_{\rm d,el}$) and due to four-wave scattering of parametric waves:

$$\Sigma_{\rm d}(\Omega) = \Sigma_{\rm d,el}(\Omega) + \Sigma_{\rm d,int}(\Omega)\,, \Pi_{\rm d}(\Omega) = \Pi_{\rm d,el}(\Omega) + \Sigma_{\rm d,el}(\Omega)\,. \quad (10.4.9)$$

Expressions for $\Sigma_{\rm d,int}(\Omega)$ and $\Pi_{\rm d,int}(\Omega)$ follow from (10.1.6):

$$\begin{aligned}\Sigma_{\rm d,int}(\Omega) =2 \int & \big[|T(\Omega,\Omega_1;\Omega_2,\Omega_3)|^2 n(\Omega_1)n(\Omega_2)n(\Omega_3) \\ & + 2\Gamma(\Omega,\Omega_1;\Omega_2,\Omega_3)\Gamma^*(\Omega,\bar{\Omega}_2;\bar{\Omega}_1,\Omega_3)\sigma^*(\Omega_1) \\ & \times \sigma(\Omega_2)\sigma(\Omega_3)\delta(\boldsymbol{n}+\boldsymbol{n}_1-\boldsymbol{n}_2-\boldsymbol{n}_3)d\Omega_1 d\Omega_2 d\Omega_3\,, \\ \Pi_{\rm d,int}(\Omega) =2 \int & T(\Omega,\Omega_1;\Omega_2,\Omega_3)\big[T(\Omega,\Omega_1;\Omega_2,\Omega_3)\sigma^*(\Omega_1) \\ & \times \sigma(\Omega_2)\sigma(\Omega_3) + 2T(\Omega,\bar{\Omega}_2;\bar{\Omega}_1,\Omega_3)n(\Omega_1)n(\Omega_2) \\ & \times \sigma(\Omega_3)\big]\delta(\boldsymbol{n}+\boldsymbol{n}_1-\boldsymbol{n}_2-\boldsymbol{n}_3)\,d\Omega_1 d\Omega_2 d\Omega_3\,.\end{aligned} \quad (10.4.10)$$

Here $\boldsymbol{n} = \boldsymbol{k}/k$, $\boldsymbol{n}_j = \boldsymbol{k}_j/k$. The values $\Sigma_{\rm c,int}$ and $\Pi_{\rm c,int}$, which are the renormalization of the pumping and damping (on the parametric wave scattering) are small (in comparison with the $\gamma(\boldsymbol{k})$ and $P(\boldsymbol{k})$) and will not be taken into consideration. It is very simple to analyze solutions of one-frequency equations of the S,T^2-theory in a rough form. First of all, assuming that $\Sigma_{\rm d,int}$=0, $\Pi_{\rm d,int}$=0, we can verify that $\nu(\Omega)$=0 for those directions where $n(\Omega) \neq 0$. This means that the distribution of parametric waves in the $\boldsymbol{k}$-space is singular: $n(\boldsymbol{k}) \neq 0$ only on the resonance surface which satisfies the condition of external stability of the basic S-theory. The distribution $n(\Omega)$ on this surface and the integral quantity N are defined by (10.4.7 - 10), which reduces to equations of the basic S-theory in the considered approximation. Next, integrating the first of Eqs. (10.4.7), one gets an estimate for the quantity ν/γ which characterizes the relative part of damping not compensated by the pumping:

$$(\nu/\gamma)^3 \simeq (TN)^2/(\gamma k v) \simeq (\gamma/kv)(p-1)\,. \quad (10.4.11a)$$

It is necessary to remember that the spectral width in ω–space $\Delta\omega \simeq \nu^2/2\gamma$; so

$$\Delta\omega \simeq \gamma[(\gamma/kv)(p-1)]^{2/3}\,. \quad (10.4.11b)$$

This result is in a good agreement with the experimental data for parametric magnons in YIG [10.18] shown in Fig. 9.13.

From the one-frequency equations (10.4.7 - 10) it follows that the distribution $n(\boldsymbol{k})$ in the modulus k close to the resonant surface is the squared

Lorentzian with width $\Delta\omega(\boldsymbol{k}) \approx \nu$ in $\omega(\boldsymbol{k})$ and a width of the order of ν/γ in $\Theta(\boldsymbol{k})$ (under conditions of axial symmetry). It may be seen from (10.4.7–10) that the relative difference of their coefficients from those of the basic S-theory for the parameter $(\nu/\gamma)^2$ is small and, hence, for total values (like the total number of parametric waves N, etc.) the difference in the approximate results of the basic S-theory and the accurate results of the S,T^2-theory is also small for the same parameter. In particular, from (10.4.7) follows that $1-|\sigma|/n \approx \nu^2/2\gamma$, i.e. at $\nu \ll \gamma$ the phase correlations in pairs are retained almost completely. An analysis of the diagrams which were neglected when solving (10.4.6) and which are proportional to T^3, T^4, etc. shows [10.1] that they are arranged in a series with the parameter $\lambda = (\Gamma/\nu)(TN/kv)$ and, consequently, for $\nu \le \gamma\cdot\lambda \le \sqrt{\gamma/kv} \ll 1$. This means that the equations of the S,T^2-theory are correct and that the integral quantities N, χ' and χ'' are well described by the corresponding formulae of the S-theory right up to the amplitudes $h \approx h_{\rm s}$ which is determined by the condition

$$h_{\rm s} = h_{\rm th}\sqrt{kv/\gamma}\,. \quad (10.4.12)$$

As a specific example of a solution of the one-frequency equations of the S,T^2-theory, parallel pumping of the magnons in a cubic ferromagnet for $\boldsymbol{M}\|[100]$ and [111] was considered by *Cherepanov* and *L'vov* [10.19]. Since the cubic anisotropy is very small, the magnon distribution on the resonant surface comprises a set of long stripes with $\Delta\varphi \gg \Delta\Theta$ which are stretched along the equator. Analyzing the distribution in Θ one can therefore consider the distribution in φ to be isotropic. From this assumption it follows that

$$\Delta\Theta \approx \frac{\nu_0}{\gamma} \approx \left[\frac{\gamma}{kv}(p-1)\right]^{1/3} \quad (10.4.13)$$

and the distribution in φ for $\boldsymbol{M}\|[100]$ has the form of a smeared cross with

$$\Delta\varphi = \sqrt{2}\nu_0/\nu_1\,, \quad \nu^2(\varphi) = \nu_0^2 + \nu_1^2\big[\sum_j \sin^2(\varphi-\varphi_j) - 2\big]\,. \quad (10.4.14)$$

However, for $\boldsymbol{M}\|[111]$, when the distribution $n(\varphi)$ has a shape of a smeared star of six vertices, the estimate for ν_0 and consequently for all the quantities connected with it, is quite different [10.19]:

$$\nu_0 \approx \gamma[\mu^2(p-1)/\gamma kv]^{1/5}\,, \ \mu = \partial^2\gamma(\varphi)/\partial\varphi^2\,. \quad (10.4.15)$$

In some cases (e.g. for real $T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_3)$ and for spherical symmetry of the problem) the contribution to the parametric magnon scattering proportional to T^2 is almost completely canceled. Then the T^3 scattering should be taken into account and it is possible to prove that $\nu \approx \gamma\sqrt{\gamma/kv}(p-1)^{3/8}$. The latter two examples show that the general estimates (10.4.11) for $\Delta\omega(\boldsymbol{k})$ $=\nu$ and for $\Delta\omega$ (10.4.13) for $\Delta\Theta$ may prove to be incorrect in some specific situations because of unexpected cancellations. Therefore, in spite of the basic understanding of the main statements of the S, T^2-theory, the investigation of the parametric excitation of waves in other media may still lead to the discovery of new effects.

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