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Wave Turbulence Under Parametric Excitation

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4 Nonlinear Dynamics of Narrow Packets of Spin Waves

Chapter 1 contained the introduction to the general theory of nonlinear wave dynamics within the classical Hamiltonian formalism. "General" here implies that this theory describes nonlinear processes irrespective of the nature of waves and the type of the nonlinear media where their propagation takes place. Clearly, this approach, alongside with the evident advantages has certain limitations. To avoid them one must use the concrete laws of waves dispersion as well as the $\boldsymbol{k}$-dependences of the interaction Hamiltonian amplitudes providing specific information about a practical problem. In this chapter the attempt is made to overcome such limitations on the description of the nonlinear dynamics of spin waves in magnetically ordered dielectrics. In this attempt we shall proceed from the general theory developed in Chap. 1 and use the magnon Hamiltonians calculated in Chap. 3.

Section 4.1 describes elementary interaction processes involving three and four magnons in ferromagnets. This section illustrates the general theory of the three- and four-wave processes developed in Sects. 1.4.1, 2 and 1.5.1. Section 4.2 is much more independent. Here, the theory of wave self-focusing (an elementary introduction to this theory is given in Sects. 1.5.3, 4) is presented in connection with magnetoelastic waves in antiferromagnets. This problem is very interesting from the experimental point of view and rather peculiar from the theoretical angle. In Sect. 4.3 various methods of parametric magnon excitation in ferro- and antiferromagnets are described. They have all been given thorough experimental study.

4.1 Elementary Processes of Spin Wave Interaction

4.1.1 Three-Magnon Processes

Two simple examples of the behavior of magnetodielectrics will be considered here to illustrate the general theory of three-wave processes.

The first example is the confluence of two magnons in an isotropic ferromagnetic. Let two spin waves with wave vectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and canonical amplitudes b_1 and b_2 be excited in the ferromagnet. Then according to the first formula of (1.4.4) they bring about the excitation of the third magnon with the canonical amplitude b_3:

$$b_3 = \frac{V^*_{3,12} b_1 b_2}{2[\omega_1 + \omega_2 - \omega_3 - i\gamma_3]} . \tag{4.1.3}$$

By substituting the explicit expression (3.1.18, 20) for the amplitude $V(\mathbf{3},\mathbf{12})$ into (4.1.3) and expressing the canonical amplitudes $b(\boldsymbol{k})$ in terms of the amplitudes of magnetization oscillations $m(\boldsymbol{k})$ using (3.8.8) we obtain

$$\frac{m_3}{M} = \frac{m_1 m_2}{4M^2}\left[\frac{k_{1z}k_{1+}}{k_1^2} + \frac{k_{2z}k_{2+}}{k_2^2}\right]\frac{M}{\omega_1 + \omega_2 - \omega_3 - i\gamma_3} . \tag{4.1.4}$$

For simplicity we take in this formula $\omega_{\rm ex}(ak_j)^2 \gg \omega_{\rm M}$. This enables us to simplify the expressions for $V(\mathbf{3},\mathbf{12})$ assuming in (3.1.19, 20) $v_j = 0$ and $u_j = 1$. It should be recalled that (4.1.4) is in agreement with the general (whatever the nature of the waves) estimation (1.4.9, 11) of the wave conversion under the resonance condition: $x_3 \simeq x_1 x_2(\omega/\gamma)$, if x_j denotes the dimensionless amplitude of a spin wave $x_j = m_j/M$, and if ω is replaced by $\omega_{\rm M}$, the frequency of the dipole–dipole interaction is responsible for the conversion process. The exact formula (4.1.4) as compared with the estimation for x_3 gives the value of the numerical factor and determines how the efficiency of the transformation depends on the orientation of the wave vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ of the initial waves. In particular, if $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are directed along the magnetization ($\boldsymbol{k}_j || \boldsymbol{M}$, $k_{j+} = k_{jx} + k_{jy} = 0$) or transverse to the magnetization direction ($\boldsymbol{k} \perp \boldsymbol{m}$, $k_{jz} = 0$) the conversion efficiency becomes zero.

It must be noted that to observe the nonlinear process described here experimentally is not easy, first because it is rather difficult to generate and record monochromatic spin waves and second because even at low temperatures their path length is fairly short, of the order of a few millimeters. It is much easier to observe the confluence of two Walker modes into one. Their amplitudes in resonance will be connected by a relation similar to (4.1.4) to an accuracy of the numerical factor of the order unity.

The second example is the decay instability of the monochromatic spin wave which splits into two waves. The conservation laws in this process have the form:

$$\omega(\boldsymbol{k}_1) = \omega(\boldsymbol{k}_2) + \omega(\boldsymbol{k}_3) , \quad \boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3 . \tag{4.1.5a, b}$$

The resonance conditions (4.1.5), as it will be recalled, are not always fulfilled in the decay processes. Accordingly, the laws of wave dispersion are subdivided into decay and non-decay. If the dispersion law $\omega(\boldsymbol{k})$ is not of the form k^x, then one part of the spectrum may turn out to be a decay one and the other a non-decay one. This is the case, for instance, of Langmuir waves

in non-isothermal plasma, of waves on a liquid surface with the dispersion law (1.2.18) and for spin waves in cubic ferromagnets with the dispersion law (3.1.5, 6). For the interpretation of the experimental results it will later be important to know the values of the wave vector $\boldsymbol{k}_{\rm S}$ determining the boundary of the decay processes in cubic ferromagnets.

Let us calculate $k_{\rm S}$, confining ourselves to the cases when the magnetization $\boldsymbol{M}||[100]$ or $\boldsymbol{M}||[111]$. Then in the law of dispersion (3.1.5) the factor $\beta = 0$ (see Table 3.1). First consider processes where (4.1.5) $\boldsymbol{k}_2 = \boldsymbol{k}_3 = 2\boldsymbol{k}_1$. Then from (4.1.5a) and (3.1.5) we obtain the expression for $\boldsymbol{k}_1$

$$\omega_{\rm ex}(ak_1^2) = 2\sqrt{A^2 - B^2} , \tag{4.1.6}$$

where $A = \omega_{\rm H} - \omega_{\rm M} N_z - \alpha\omega_{\rm a} + B$, and $B = [\omega_{\rm M}\sin^2\Theta]/2$. A more detailed analysis of the dispersion law (3.1.5) reveals that the value of the wave vector $\boldsymbol{k}_1$ (4.1.6) is boundary in the fact that under $k < k_1 = k_{\rm S}$ decay processes for spin waves with $\omega(\boldsymbol{k}_3)$ are forbidden, while they are allowed under $k > k_{\rm S}$. Sect. 4.3, which is devoted to the methods of parametric excitation of magnons in magnets, will give a detailed treatment of the decay instability of a magnon with $k = 0$, i.e. homogeneous precession of the magnetization into two other magnons, into the nuclear magnons in antiferromagnets, etc. By way of example, the decay instability of a monochromatic spin wave into two spin waves will be briefly outlined. Under resonance (4.1.4) the threshold amplitude $b_{1\rm th}$ according to (1.4.19) is $|V_{1,23}b_{1,th}|^2 = \gamma_2\gamma_3$; γ_2 and γ_3 here are the damping decrement in the final state. Substituting here the interaction amplitudes $V(\mathbf{1},\mathbf{2},\mathbf{3})$ (3.1.18, 20) and taking for simplicity $\boldsymbol{k}_1$ (the wave vector of the initial wave with the finite amplitude b_1) to be perpendicular to $\boldsymbol{M}(\sin\Theta = 1)$, we obtain the critical value of b_1:

$$|b_{1,\rm th}| = \frac{2\gamma}{\omega_{\rm M}}\sqrt{\frac{2M}{g}} , \quad \gamma = \sqrt{\gamma_2\gamma_3} \tag{4.1.7}$$

corresponding to the threshold of decay into two magnons propagating through the optimum angles $\Theta_2 = \pi - \Theta_3 = \pi/4$, $\varphi_2 = \varphi_3$. To the amplitude (4.1.7) corresponds the critical value of the precession angle

$$\Psi_{1\rm th} = m_{1\rm th}/M = b_{1\rm th}\sqrt{2g/M} = 4\gamma/\omega_{\rm M} . \tag{4.1.8}$$

Thus a spin wave with amplitude b and the precession angle Ψ exceeding the critical values of (4.1.7, 8) is unstable with respect to its splitting into two others.

4.1.2 Modulation Instability of Spin Waves

The linear theory of the modulation instability of a plane wave of arbitrary nature has been considered in Sect. 1.5.1. This is a decay instability of the second order of the wave (with the wave vector $\boldsymbol{k}_0$) splitting into two other waves (with the wave vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$). This instability is due to the four-wave processes

$$\begin{aligned} \omega(\boldsymbol{k}_0)+\omega(\boldsymbol{k}_0) &= \omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2)\ , \\ \boldsymbol{k}_0+\boldsymbol{k}_0 &= \boldsymbol{k}_1+\boldsymbol{k}_2\ , \quad \boldsymbol{k}_{1,2}=\boldsymbol{k}_0 \pm \boldsymbol{\kappa}\ . \end{aligned} \tag{4.1.9}$$

The threshold amplitude of the initial wave $b_{0\mathrm{th}}$ according to (1.5.9) is determined by the damping of the waves in the final state γ_1 and γ_2:

$$|T_{\mathbf{00},\mathbf{12}} b^2_{\mathbf{0},\mathrm{th}}| = \sqrt{\gamma_1\gamma_2}\ . \tag{4.1.10}$$

The modulation instability is not inevitable. In addition to the usual threshold condition $|b_0| > |b_{0\mathrm{th}}|$ the criterion (1.5.13) must be fulfilled:

$$T_{\mathbf{00},\mathbf{00}} \hat{L}\kappa^2 < 0\ , \tag{4.1.11}$$

where $\hat{L}\kappa^2$ is the quadratic form (1.5.11) determined by the dispersion law $\omega(\boldsymbol{k})$. On the basis of the specific dispersion laws of magnons and the four-magnon amplitudes (presented in Chap. 3) the modulation instability of magnons in ferro- and antiferromagnets can be studied [4.1].

It can be done most easily for the case of easy-plane antiferromagnets. The dispersion law of their magnons is nearly isotropic (and, accordingly, $\hat{L}\kappa^2 > 0$) and the four-magnon amplitudes T are negative (see (3.2.12)) and are continuous functions of the wave vectors. The above developed theory can therefore be directly applied, and according to the criterion (4.1.11) spin waves of finite amplitude in the easy-plane antiferromagnets are unstable.

Also unstable are spin waves in ferromagnets whose Hamiltonian includes only the exchange interaction. It can be seen from (3.1.10, 21) that L in this case is given by

$$\hat{L}\kappa^2 = \omega_{\mathrm{ex}}(a\kappa)^2 > 0\ , \quad T_{\mathbf{00}} = -(g/M)\omega_{\mathrm{ex}}(ak_0)^2 \tag{4.1.12}$$

and the instability criterion (4.1.11) is fulfilled. It follows from (1.5.11) and (4.1.12) that the instability is maximum when

$$(\kappa/k_0)^2 = -2|b|^2 T_{\mathbf{00}}/L \simeq 2g|b|^2/M \simeq 2\Delta M_z/M\ , \tag{4.1.13}$$

where ΔM_z is the variation of the static magnetization caused by the existence of the initial wave b_0. The threshold amplitude of the spin waves is given by (4.1.10) and, consequently,

$$(\Delta M_z/M)_{\mathrm{th}} = \gamma g/(2\sqrt{2}MT_{\mathbf{00}}) = \gamma/[2\sqrt{2}\omega_{\mathrm{ex}}(ak_0)^2]\ . \tag{4.1.14}$$

Thus, if its amplitude is big enough the "exchange" spin wave in ferromagnets is always unstable with respect to the process (1.5.11) with small $(\kappa/k_0)^2$ in the vicinity of $\gamma/\omega_{\mathrm{ex}}(k_0a)^2$. This instability brings about long period longitudinal ($\boldsymbol{\kappa}||\boldsymbol{k}_0$) and transverse ($\kappa \perp \boldsymbol{k}$) amplitude modulations of the travelling spin wave. The case of long spin waves in ferromagnets is more complicated because of the long-range magnetic dipole interaction leading to the non-analytical dependence of $T(\mathbf{12},\mathbf{34})$ on its arguments. Then (1.5.11) cannot be applied to spin waves in ferromagnets. Instead, it follows from (1.5.7, 8) (for more detail, see *Zakharov* et al. [4.1]):

$$\nu = -\gamma \pm \sqrt{[\hat{L}\kappa^2 + 2T(\boldsymbol{\kappa})|b|^2]^2 - 4|F(\boldsymbol{\kappa})b^2|^2}\ . \tag{4.1.15}$$

Here it is taken that $\kappa \ll k$ and the designations $T(\boldsymbol{\kappa})$ and $F(\boldsymbol{\kappa})$ are introduced for the following functions depending on the direction of $\boldsymbol{\kappa}$ only:

$$2T(\boldsymbol{\kappa}) = T(\boldsymbol{k}_0+\boldsymbol{\kappa},\boldsymbol{k}_0) + T(\boldsymbol{k}_0-\boldsymbol{\kappa},\boldsymbol{k}_0) - T(\boldsymbol{k}_0,\boldsymbol{k}_0)\ , \tag{4.1.16}$$

$$2F(\boldsymbol{\kappa}) = T(\boldsymbol{k}_0,\boldsymbol{k}_0,\boldsymbol{k}_0+\boldsymbol{\kappa},\boldsymbol{k}_0-\boldsymbol{\kappa})\ . \tag{4.1.17}$$

Note that if $T(\mathbf{12},\mathbf{34})$ analytically depends on the wave vectors, then $T(\boldsymbol{\kappa}) = F(\boldsymbol{\kappa}) = T(\mathbf{00})$ under $\kappa \to 0$ and the expression (4.1.15) is safely transformed into (1.5.11). Using (3.1.5,6), we can obtain

$$F(\boldsymbol{\kappa}) = T(\boldsymbol{\kappa}) = [g/(2\pi)^3 M][\varphi(\boldsymbol{\kappa})\omega_{\mathrm{M}}\cos^2\Theta - \omega_{\mathrm{ex}}(ak_0^2)]\ , \tag{4.1.18}$$

$$\begin{aligned} \varphi(\boldsymbol{\kappa}) &= (\Omega - \omega_{\mathrm{M}}\sin^2\Theta)/(\Omega + \omega_{\mathrm{M}}\sin^2\Theta)\ , \\ \hat{L}\kappa^2 &= \omega_{\mathrm{ex}}(a\kappa)^2 + \kappa^2\omega_{\mathrm{M}}\sin^2\Theta/2k_0^2\ , \\ \Omega &= gH - \omega_{\mathrm{M}}N_z\ , \quad \cos\Theta = \kappa_z/|\kappa|\ . \end{aligned} \tag{4.1.19}$$

At $F = T$ the necessary condition of instability, as follows from (4.1.15, 19) is $TL^{-1} < 0$. Note that $\hat{L}\kappa^2$ is always positive, and $T(\boldsymbol{\kappa}) > 0$ only if $\omega_{\mathrm{ex}}(ak_0)^2 < \omega_{\mathrm{M}}$, therefore an external magnetic field larger than $4\pi M(N_z+1)$ suppresses the modulation instability of the travelling wave with $\boldsymbol{k}||\boldsymbol{M}$. If $H < 4\pi M(N_z+1)$, then $\varphi(\boldsymbol{\kappa})$ will be negative for some directions which leads to an instability with small κ, first of all with those satisfying the equation $\hat{L}\kappa^2 = \gamma\kappa/k_0$. The threshold of this instability is given by the expression (4.1.14), where T is satisfied by (4.1.16).

When the spin wave propagates transversely ($\boldsymbol{k} \perp \boldsymbol{M}$) there can arise instabilities with large κ unlike in the case of the longitudinal propagation ($\boldsymbol{k}||\boldsymbol{M}$). This happens because the decay of the wave with $\boldsymbol{k} \perp \boldsymbol{M}$ is accompanied by a decrease of its dipole–dipole energy which can compensate the decreased exchange energy under some κ. However, the instability threshold with large κ, has the same order of magnitude as the threshold for $\kappa \ll k$. For simplicity we shall consider only the latter case. The instability analysis under arbitrary ω_{M}, Ω and k_0 would be too cumbersome. Here only one interesting case will be presented: $\omega_{\mathrm{M}} \gg \Omega$, $\omega_{\mathrm{ex}}(ak_0)^2$. Other specific cases were considered by *Zakharov* et al. [4.1]. In our example we obtain from (3.1.5,6) and (3.1.22)

$$\begin{aligned}
\hat{L}\kappa^2 &= \frac{\omega(k_0)}{k_0^2}\frac{\kappa^2\Omega}{64\pi^3 M[\Omega+\omega_{\mathrm{ex}}(ak_0^2)]}\left[\frac{\omega_{\mathrm{ex}}\sin^2\Theta}{\Omega+\omega_{\mathrm{ex}}(ak_0)^2}+\cos\Theta\right],\\
T(\boldsymbol{\kappa}) &= \frac{g\omega_{\mathrm{M}}^2(\cos^2\Theta-2N_z)}{64\pi^3 M[\Omega+\omega_{\mathrm{ex}}(ak_0^2)]},\\
F(\boldsymbol{\kappa}) &= \frac{g\omega_{\mathrm{M}}^2(N_z-\cos^2\Theta)}{64\pi^3 M[\Omega+\omega_{\mathrm{ex}}(ak_0^2)]}.
\end{aligned} \tag{4.1.20}$$

Since the sign of $\hat{L}\kappa^2$ can be changed, spin waves are unstable whatever the sign of F and T. The threshold amplitude is

$$(\Delta M_z/M)_{\mathrm{th}} = 8\gamma[\Omega+\omega_{\mathrm{ex}}(ak_0^2)]/(\omega_{\mathrm{M}}^2|N_z-\cos^2\Theta|)\ . \tag{4.1.21}$$

It is clear that for long spin waves ($\omega_{\mathrm{ex}}(ak_0)\ll\omega_{\mathrm{M}}$) propagating in a sample where the internal field is small ($\Omega\ll\omega_{\mathrm{M}}$) the instability threshold decreases significantly.

In conclusion, let us estimate the spin wave intensity leading to modulation instability. The estimate for the energy flux in spin waves follows from (4.1.14)

$$P_{\mathrm{th}} = \omega b_{\mathrm{th}}\partial\omega/\partial k \simeq \gamma\omega_{\mathrm{ex}}a^2\ kM/\mathrm{g}\ . \tag{4.1.22}$$

For a monocrystal YIG $\gamma\simeq 10^6\mathrm{s}^{-1}$, $M\simeq 100$ Oe, $\omega_{\mathrm{ex}}a^2\simeq 0.1\ \mathrm{cm}^2\mathrm{s}^{-1}$. If we put $k=10$ cm, then $P_{\mathrm{th}}\simeq 10^{-3}$ Wt · cm. It must be noted that the expression (4.1.23) sets the upper limit to the energy flux which can be transferred by a spin wave inside the crystal. This flux is usually much smaller that the maximum energy flux (0.1 Wt · cm) for a sound wave in ferromagnets (*Gurevitch*, [4.2]), therefore the critical values of the flux (4.1.23) can be reached as the sound is transformed to a spin wave in a nonuniform magnetic field (*Schlömann* et al. [4.3]).

4.2 Self-Focusing of Magnetoelastic Waves in Antiferromagnets

4.2.1 Structure of Basic Equations

The magnetic subsystem of a crystal is considerably nonlinear, which results in the nonlinearity of the system of other quasi-particles interacting with magnons. For example, in most solids under practically attainable deformations the nonlinear acoustic effects are small. This is due to the imperfections in the crystals leading to the failure of the sample under relative deformations u much less than unity ($u\simeq 10^{-4}-10^{-6}$). In antiferromagnets because

of the exchange amplification of the magnetostriction (Sect. 3.2) the nonlinearity and the dispersion of magnetoelastic waves may be considerable. The law of acoustic dispersion in this case has the following form:

$$\omega^2 = (V_{\mathrm{S}}+Q_1u+Q_2^2u^2)^2k_x + 2V_{\mathrm{S}}^4Dk_x + 2V_{\mathrm{S}}ak^2\ . \tag{4.2.1}$$

Here, the nonlinearity (which can be considered as the renormalization of the sound velocity) is taken into account, as well as the dispersion of the wave velocity and the diffraction (we assume the elastic wave to be weakly non-unidimensional $k^2=k_x^2+k_\perp^2$, $k_\perp\ll k_x$). In isotropic media $a=V_{\mathrm{S}}/2$; in crystals because of the anisotropy of the elasticity and the magnetostriction the value of a can be either positive or negative (for more detail see *Turitsyn* and *Falkovich* [4.4]). At not too high excitation levels the evolution of magnetoelastic waves can be subdivided into quick and slow evolution. Quick evolution is the transfer of the initial excitation with the sound velocity and slow evolution is due to the small effects of nonlinearity, dispersion and diffraction. To study these effects the reference system moving with the sound velocity is convenient. To change over to this system we substitute in (4.2.1) $\omega\Rightarrow\Omega-V_{\mathrm{S}}k_x$, $\Omega\ll V_{\mathrm{S}}k_x$, and perform the inverse Fourier transform

$$u_t(\boldsymbol{r},t) = \int u(\boldsymbol{k},\Omega)\exp(i\boldsymbol{kr}-i\Omega t)\,d\boldsymbol{k}d\Omega\ .$$

We obtain the following equation:

$$\begin{aligned}
&u_t+Q_1uu_x+Q_2u^2u_x+Du_{xxx}+a\int_{-\infty}^{x}\Delta_\perp u\,dx' = 0\ ,\\
&u_t=\partial u/\partial t\ ,\quad u_x=\partial u/\partial x\ ,\quad u_{xxx}=\partial^3u/\partial^3x\ .
\end{aligned} \tag{4.2.2}$$

The derivation of this equation is surely not precise and only serves as an illustration. The rigorous derivation of (4.2.2) from the Landau-Lifshitz equations and the theory of elasticity were obtained by *Turitsyn* and *Falkovich* in [4.4], where the expressions for the constants Q_j, D and a were calculated for antiferromagnets with rhombohedral structure. Note also that in some cases $Q=0$, i.e. the first nonvanishing nonlinearity is the cubic one. The dispersion factor $D\simeq(V_{\mathrm{S}}^2-V_{\mathrm{M}}^2)$ is proportional to the difference of the sound velocity and the limiting velocity of magnons. Therefore in antiferromagnets where $V_{\mathrm{S}}>V_{\mathrm{M}}$ (or, in other words, the Debye temperature is greater than the Néel temperature), $D>0$, i.e. the dispersion is negative, and the velocity of magnetoelastic waves drops as the wave vector increases (*Ozhogin* et al. [4.5]).

Under unidimensionality (when the initial perturbation is uniform over the plane perpendicular to the line of propagation) (4.2.2) is reduced depending on the type of nonlinearity either to the Korteveg-de Vries (KdV) equation

$$u_t+Quu_x+Du_{xxx} = 0\ , \tag{4.2.3}$$

or to the modified KdV (MKdV) equation

$$u_t + Qu^2 u_x + Du_{xxx} = 0 \ . \tag{4.2.4}$$

4.2.2 Properties of Unidimensional Equations

The one-dimensional equations (4.2.3-4) have the following remarkable property: If any solution of such an equation for $u(x,t)$ (soliton) is treated as the potential of the linear Schrödinger equation

$$\left[\frac{\partial^2}{\partial t^2} + u(x,t)\right]\Psi(x,t) = E\Psi(x,t) \ , \tag{4.2.5}$$

the energy spectrum E does not depend on time. In other words, the evolution $u(x,t)$ by virtue of the equations KdV (4.2.3) or MKdV (4.2.4) leads to isospectrum conversion of the Schrödinger equation potential. This fact, discovered by *Gardner* at al. [4.6], forms the basis of the method of the inverse problem of scattering and its further generalizations [4.7, 8]. The connection with (4.2.5) enables us to describe for (4.2.3–4) the evolution of the arbitrarily localized initial perturbation $u(x,0)$. To this end, the spectrum (4.2.5) must be found with the potential $u(x,0)$. The number of discrete levels in the spectrum is equal to the number of solitons resulting from the decay of the initial perturbation in the limit $t \to \infty$. The solitons, as in (1.5.24), under unidimensionality are asymptotic states. It is the study of (1.5.24) and (4.2.3) that revealed the basic role of solitons in nonlinear wave dynamics (see [4.7]). Such localized stationary pulses may appear beginning with the amplitudes under which the effects of the nonlinearity uu_x and the dispersion u_{xxx} are comparable. At such amplitudes plane waves are no longer well-defined objects since their interaction time becomes of the order of the scattering time of the wave packets due to the dispersion of the group velocities. Unfortunately, the soliton as a fundamental subject of description instead of the plane wave did not meet our expectations since solitons are very often unstable. As this instability increases, it often results in collapse (or, in other words, self-focusing) which therefore proves to be of no less importance in the nonlinear dynamics of waves than solitons.

4.2.3 Stability of Solitons and Self-Focusing Theorem

Let us consider the weakly non-unidimensional generalization of the MKdV equation (4.2.4):

$$\frac{\partial}{\partial x}(u_t + Du_{xxx} + Qu^2 u_x) = au_{yy} + cu_{zz} \ . \tag{4.2.6}$$

In the easy-plane antiferromagnets the case $Q > 0$ occurs most often (see [4.5]), so here we shall discuss only this case. The plane solitons

$$u_0(x - Vt) = \frac{\sqrt{2V/Q}}{\cosh[(x-Vt)\sqrt{V/D}]}$$

travel with supersonic velocity ($V > 0$; it should be recalled that (4.2.4, 6) have been written in the reference system moving at the sound velocity). As can be seen from the expression for u_0, the solitons exist only at $D > 0$, i.e. for antiferromagnets where $V_S > V_M$ (when $D < 0$), the stationary solution has the form of a running "domain wall"

$$u_t = \frac{\sqrt{2V/Q}}{\coth[(x-Vt)\sqrt{V/D}]} \ .$$

The stability of unidimensional solitons with respect to transverse perturbation was first studied by *Kadomtsev* and *Petviashvili* [4.9]. They showed solitons to be stable only under $a < 0$, $c < 0$; if the sign of either one (or both) of these values is positive the plane solitons are unstable with respect to crimping. Qualitatively, this instability is due to the fact that a local increase of the soliton amplitude leads to its decreased velocity in the same place. This part of the soliton falls behind; the wave front is bent. This in turn leads to a concentration of energy and the further increase of the amplitude in the place of initial fluctuation etc. At $a > 0$, $c > 0$ the self-focusing of the elastic wave is possible. To illustrate this, let us represent (4.2.6) as

$$u_t = \frac{\partial}{\partial x}\frac{\delta\mathcal{H}}{\delta u} \ , \quad \mathcal{H} = \frac{1}{2}\int (Du_x^2 + aw_y^2 + cw_z^2 - 2u^4)\, d\boldsymbol{r} \ , \tag{4.2.7}$$

where $\mathcal{H}w_x = u$. As (1.5.32), (1.4.29), equations (4.2.7) are Hamiltonian and retain the Hamiltonian $\mathcal{H}$.

As in (1.5.3), we obtain the second derivative of the positive value proportional to the square of the characteristic transverse size of the beam with respect to time:

$$\frac{\partial^2}{\partial t^2}\int r^2 u^2 d\boldsymbol{r} = 16a\mathcal{H} - 8aD\int u_x^2 d\boldsymbol{r} \ . \tag{4.2.8}$$

Clearly, at $D > 0$ and $a = c > 0$ any distribution with the negative Hamiltonian $\mathcal{H}$ is self-focused. When the sign of even one of the D values is negative, (4.2.8) reveals the defocusing of the beam. Indeed, if $D > 0$, $a > 0$, a sufficient condition for the diffusion of the transverse imperfections is $\mathcal{H} < 0$. At $a < 0$

$$\frac{\partial}{\partial t^2}\int r^2 u^2 d\boldsymbol{r} = 16a\left[\frac{1}{2}\int a(\nabla_\perp w)^2 dr - \int u^4 d\boldsymbol{r}\right] > 0 \tag{4.2.9}$$

and, consequently, any initial distribution is diffuse.

4.2.4 Evolution of Magnetoelastic Waves in the Absence of a Linear Bond Between Magnons and Phonons

The system of equations describing the joint evolution of the stress tensor u and the angle of rotation of the antiferromagnetism vector φ ($\alpha = V_M/V_S$) in dimensionless variables has the form:

$$u_{tt} - u_{xx} = (\varphi^2)_{xx} , \quad \varphi_{tt} - \alpha^2\varphi_{xx} + \varphi = 2u\varphi . \tag{4.2.10}$$

The further reductions (4.2.10) depend on the way the magnetic or elastic subsystem of the crystal are excited. Under parametric excitation of spin waves by a high-frequency electromagnetic field we may pass from $\varphi(x,t)$ to the smooth envelope $\Psi(x,t)$:

$$\varphi = \Psi \exp[ikx - it\sqrt{1+\alpha^2k^2}] + \text{c.c.}$$

after which we get the system first obtained by *Zakharov* [4.10] for the interaction of the Langmuir and ion-sound waves in a plasma:

$$u_{tt} - u_{xx} = |\Psi|^2_{xx} ,$$

$$i(\Psi_t + \alpha^2 ik\Psi_x) + \frac{1}{2}\alpha^2\Psi_{xx} = u\Psi . \tag{4.2.11}$$

Within this system there exist stable unidimensional solitons, and the non-unidimensional generalizations (4.2.11) describe the instability of the solitons with respect to the transverse crimping and (in the case of three-dimensions) collapse of waves.

The physical situation may be different, e.g. the slow sound wave motions of $u(x,t)$ are excited (say, by a piezotransducer). The system (4.2.10) has soliton solutions. In the initial variables

$$u_0 = -2\cosh^{-2}\left[\frac{x-Vt}{\sqrt{\alpha^2-V^2}}\right], \quad \varphi_0 = \sqrt{2(1-V^2)}\cosh^{-1}\left[\frac{x-Vt}{\sqrt{\alpha^2-V^2}}\right]$$

(unlike the *solitons of the envelopes* in (4.2.11)). These solitons, however, are unstable even in the one-dimensional geometry. In this case, too, we can prove the possibility of the collapse which by virtue of unidimensionality does not mean self-focusing, but wave overturn over finite time.

Obviously, the unlimited increase of the amplitude (or its derivatives) cannot be physically meaningful, since in this case we will fall outside the scope of applicability of the employed equations (1.5.23), (4.2.6) or (4.2.10). Here, the very possibility of the wave amplitude increasing to the point when nonlinearity is no longer small is important.

4.3 Methods of Parametric Excitation of Spin Waves

Theoretically, the parametric excitation of waves is simply a result of the developing decay instability of the first or second order as shown in Sects. 1.4.2 and 1.5.1. Therefore to obtain the threshold of the parametric excitation we have to calculate only the amplitude of the corresponding elementary process and the damping decrement of the waves in their final state.

4.3.1 Transverse Pumping of Spin Waves in FM

Transverse pumping of spin waves in ferromagnets, or *Suhl's instability of the first order* [4.11] are the usual notions for the decay of the homogeneous precession of the magnetization into a pair of spin waves (magnons) with the wave vectors $\boldsymbol{k}$ and $-\boldsymbol{k}$ and the frequencies $\omega(\boldsymbol{k}) = \omega(-\boldsymbol{k}) = \omega_p/2$. This process is described by the following terms of the Hamiltonian $\mathcal{H}_3$:

$$\mathcal{H}_3 = \frac{1}{2}\sum_{\boldsymbol{k}}[V(\boldsymbol{0},\boldsymbol{k},-\boldsymbol{k})b^*(0)b(\boldsymbol{k})b(-\boldsymbol{k}) + \text{c.c.}] \tag{4.3.1}$$

with the interaction amplitudes (3.1.19, 20):

$$\begin{aligned} V(\boldsymbol{0},\boldsymbol{k},-\boldsymbol{k}) &= 2\tilde{V}(\boldsymbol{k})u^2(\boldsymbol{k}) + 2u(\boldsymbol{k})\tilde{V}(\boldsymbol{k})V^*(\boldsymbol{k}) \\ &= \pi g\sqrt{2gM}c(\boldsymbol{k})\sin 2\Theta(\boldsymbol{k})\exp[i\varphi(\boldsymbol{k})] , \\ c(\boldsymbol{k}) &= 1 + \omega(\boldsymbol{k})/[A(\boldsymbol{k}) + |B(\boldsymbol{k})|] . \end{aligned} \tag{4.3.2}$$

It has been assumed here that the magnetization $\boldsymbol{M}$ is directed along the rotation axis of an ellipsoidal sample so that the polarization of the homogeneous precession is circular and accordingly $v_0 = 0$. In accordance with (1.4.21) the threshold amplitude of the homogeneous precession $b_{\text{th}}(0)$ with respect to its decay into a pair of magnons $\pm\boldsymbol{k}$ is given by

$$|V(\boldsymbol{0},\boldsymbol{k},-\boldsymbol{k})|b_{\text{th}}(0) = \gamma(\boldsymbol{k}) . \tag{4.3.3}$$

Consequently the pair of magnons for which the ratio $\gamma(\boldsymbol{k})/|V(\boldsymbol{0},\boldsymbol{k},-\boldsymbol{k})|$ is a minimum are the first to be excited (as b increases). If $\gamma(\boldsymbol{k})$ does not depend on the angles $\Theta(\boldsymbol{k})$ and $\varphi(\boldsymbol{k})$ over the resonant surface $\omega(\boldsymbol{k}) = \omega_p/2$ (for the time being we shall not go into the details of this dependence), the first to be excited are the pairs for which $|V(\boldsymbol{0},\boldsymbol{k},-\boldsymbol{k})|$ is maximum. Their wave vectors lie on two circles (meridians) of the resonant surface and have $0 < \varphi(\boldsymbol{k}) < 2\pi$, $\Theta(\boldsymbol{k}) = \Theta_M$, $\pi - \Theta_M$, where Θ_M is a little less than $\pi/4$. This can be seen if we identically rewrite $|V(\boldsymbol{0},\boldsymbol{k},-\boldsymbol{k})|$ from (4.3.2) in the form

$$\begin{aligned} |V(\boldsymbol{0},\boldsymbol{k},-\boldsymbol{k})| &= V_1(\Theta) = (\pi g/\omega_p)\sqrt{2gM}\sin 2\Theta \\ &\times \left(\omega_p + \omega_M\sin^2\Theta + \sqrt{\omega_p^2 + \omega_M^2\sin\Theta}\right) . \end{aligned} \tag{4.3.4}$$

This (when $\omega_p > \omega_M$) gives

$$b_{th} = \min_{\Theta(\boldsymbol{k})} b_{th}(\boldsymbol{k}) \simeq (\gamma/\omega_M)\sqrt{2M/g} . \tag{4.3.5}$$

To the canonical amplitude of the homogeneous precession b_{th} corresponds the critical precession angle Ψ_{th}:

$$\Psi_{th} = m_{th,1}/M = \sqrt{2g/M} b_{th} \simeq 2\gamma/\omega_M . \tag{4.3.6}$$

For YIG (at $\omega_M = 1700$ Oe, $\gamma = 0.5$ Oe) $\Psi_{th} \simeq 2.5 \cdot 10^{-3}$.

Now the term "transverse pumping" must be explained. Usually the homogeneous precession is excited by a SHF magnetic field with a polarization $\boldsymbol{h}_\perp$ directed transverse to the constant magnetic field $\boldsymbol{H}$ (and accordingly to the stationary magnetization $\boldsymbol{M}$). The Hamiltonian of this interaction is due to the Zeeman energy $-\boldsymbol{h} \cdot \boldsymbol{M}(\boldsymbol{r},t) = -\boldsymbol{h}_\perp \cdot \boldsymbol{m}_\perp$ and has the form

$$\mathcal{H}_p^{(1)} = U[h_+(t)b_0^* + \text{c.c.}] , \tag{4.3.7}$$

where $U = -\sqrt{gM_0/2}$, $h_+ = h_x + ih_y$. This Hamiltonian describes the well-known phenomenon of ferromagnetic resonance (FMR)

$$\frac{\partial b_0}{\partial t} + i\omega_0 b_0 + \gamma_0 b_0 = -i\frac{\delta \mathcal{H}_p^{(1)}}{\delta b_0^*} = -iuh_+(t) . \tag{4.3.8}$$

If the SHF field has right-hand polarization, then

$$\begin{aligned} h_+(t) &= h_+ \exp(-i\omega t) , \\ b_0(t) &= \frac{\sqrt{gM_0/2}h_+ \exp(-i\omega t)}{\omega_0 - \omega - i\gamma_0} . \end{aligned} \tag{4.3.9}$$

Under the linear polarization $h_y = 0$, $h_+ = h_x = 2h\cos\omega t$, and

$$\begin{aligned} b_0(t) &= b_0(\omega)\exp(-i\omega t) + b_0(-\omega)\exp(i\omega t) , \\ b_0(\omega) &= h\sqrt{gM_0/2}(\omega - \omega_0 + i\gamma_0) . \end{aligned} \tag{4.3.10}$$

In the vicinity of the resonance when $|\omega - \omega_0| \ll \omega_0$ the term $b_0(\omega)$ describing the clockwise polarized part of b_0 is the principal one and the expression for the critical amplitude h_{th} corresponding to the instability threshold of the homogeneous precession (4.3.3) is the same for the clockwise circular and linear polarization of h:

$$h_{th} = b_{th}\sqrt{(\omega - \omega_0)^2 + \gamma_0^2}/U \simeq \gamma(\boldsymbol{k})\sqrt{(\omega - \omega_0)^2 + \gamma_0^2}/g\omega_M . \tag{4.3.11}$$

For exact resonance ($\omega = \omega_0$)h_{th} is very small. Taking for estimation $\gamma_0/g \simeq$ 1 Oe, we have $h_{th} \simeq 0.002$ Oe. In experiments at the frequency 10^{10}s^{-1} the amplitude of the SHF field h reached values up of to 10 Oe. Therefore the amplitude of the precession b_{th} attains the critical values b_{th} not only under

FMR but also far from it. In such a case this instability is more naturally treated as a parametric instability of the external field with respect to decay into two waves with the wave vectors $\boldsymbol{k}$ and $-\boldsymbol{k}$. The equation (1.4.13) can therefore easily be rewritten in a form where the amplitude b_0 is expressed in terms of the amplitude of the pumping field h:

$$\begin{aligned} &\partial b(\boldsymbol{k},t)/\partial t + [\gamma(\boldsymbol{k}) + i\omega(\boldsymbol{k})]b(\boldsymbol{k},t) \\ &\qquad + ih\exp(i\omega_p t)\tilde{V}(\boldsymbol{k})b^*(-\boldsymbol{k},t) = 0 , \\ &\partial b^*(-\boldsymbol{k},t)/\partial t + [\gamma(\boldsymbol{k}) - i\omega(\boldsymbol{k})]b^*(-\boldsymbol{k},t) \\ &\qquad - ih\exp(-i\omega_p t)\tilde{V}(\boldsymbol{k})b(\boldsymbol{k},t) = 0 . \end{aligned} \tag{4.3.12}$$

Here the value $\tilde{V}(\boldsymbol{k})$ has been introduced which is the *effective amplitude of the magnon interaction with pumping.* In terms of this value the threshold amplitude h_{th} can be easily expressed:

$$h_{th}|\tilde{V}(\boldsymbol{k})| = \gamma(\boldsymbol{k}) . \tag{4.3.13}$$

For the above treated case of circular polarization

$$\tilde{V}(\boldsymbol{k}) = \sqrt{\frac{gM}{2}}\frac{\tilde{V}(\boldsymbol{k})}{\omega_p - \omega_0}\left[1 + \frac{A(\boldsymbol{k}) - |B(\boldsymbol{k})|}{\omega(\boldsymbol{k})}\right] . \tag{4.3.14}$$

From (4.3.13, 14) the expression (4.3.11) for h_{th} can be obtained again. In the case of linear polarization far from the resonance the initial equations (1.4.13) must allow besides the term $V(\boldsymbol{0},\boldsymbol{k},-\boldsymbol{k})b_0(\omega)$ (which has now become non-resonant) also another non-resonant term $U(\boldsymbol{0},\boldsymbol{k},-\boldsymbol{k})b_0^*(-\omega)$. As a result (4.3.12) holds true, but the expression for $\tilde{V}(\boldsymbol{k})$ will change:

$$\begin{aligned} \tilde{V}(\boldsymbol{k}) &= \sqrt{\frac{gM_0}{2}}\left(\frac{V(\boldsymbol{0},\boldsymbol{k},-\boldsymbol{k})}{\omega_p - \omega_0} - \frac{U(\boldsymbol{0},\boldsymbol{k},-\boldsymbol{k})}{\omega_p + \omega_0}\right) \\ &= \sqrt{\frac{gM_0}{2}}\tilde{V}(\boldsymbol{k})\left(1 + \frac{2[A(\boldsymbol{k}) - |B(\boldsymbol{k})|]}{\omega_p}\right) \\ &\quad + \left(\frac{1}{\omega_p - \omega_0} + \frac{[\omega_p + 2A(\boldsymbol{k})]\exp[2i\varphi(\boldsymbol{k})]}{\omega_p[\omega_p + \omega_0]}\right) . \end{aligned} \tag{4.3.15}$$

It is significant that the first and second terms in this expression describing the contributions of the right-hand and left-hand polarization respectively to the linear pumping depend differently on the azimuthal angle of the spin waves $\varphi(\boldsymbol{k})$. Therefore the maximum of the modulo of the whole expression is achieved not over the whole meridian with the fixed $\Theta(\boldsymbol{k})$ and any $\varphi(\boldsymbol{k})$ as in the case of the circular polarization, but in two points $\varphi(\boldsymbol{k}) = 0, \pi$ corresponding to the excitation of the two pairs of the magnons $\boldsymbol{k}$ and $-\boldsymbol{k}$ on the plane of the pumping $\boldsymbol{h}$ and $\boldsymbol{M}$. This circumstance, i.e. the maximum interaction with the pumping not of a whole group of pairs with different $\varphi(\boldsymbol{k})$, but only of a finite number (in our case it is two) makes lateral pumping rather promising for future experimental studies of magnon behavior above threshold.

Concluding this section we shall give the expression for the minimum threshold of magnon excitation by transverse pumping with $\Theta(\boldsymbol{k}) = \pi/4$, $\varphi(\boldsymbol{k}) = 0, \pi$ in the case of $\gamma(\boldsymbol{k}) = \gamma(k)$ and $\omega_{\rm M} \ll \omega_{\rm p}$:

$$h_{\rm th} = 2H\gamma(\boldsymbol{k})(\omega_{\rm p} - \omega_0)/(g\omega_{\rm M}) . \qquad (4.3.16)$$

For monocrystals YIG with $\gamma(\boldsymbol{k}) = g \cdot 0.1$ Oe at the frequency $\omega_{\rm p} = 2\pi \cdot 10^{10}{\rm s}^{-1}$ we obtain $h_{\rm th} \simeq 0.3$ Oe. This value can be easily achieved experimentally.

4.3.2 Parallel Pumping of Spin Waves in FM

This type of parametric instability appearing at $\boldsymbol{h} \parallel \boldsymbol{M}$ was predicted and observed by *Morgenthaler* [4.12] and *Schlomann* et al. [4.13]. In order to describe this instability let us discuss the longitudinal part of the Zeeman energy $-h_z m_z$ which results in the Hamiltonian

$$\mathcal{H}_{p2} = \mathcal{H}_{p1} + \mathcal{H}_{\rm p} , \qquad (4.3.17)$$

$$\mathcal{H}_{\rm p} = \frac{1}{2}\sum_{\boldsymbol{k}}[h(t)V(\boldsymbol{k})b^*(-\boldsymbol{k})b^*(\boldsymbol{k}) + {\rm c.c.}] . \qquad (4.3.18)$$

$$\mathcal{H}_{p1} = \sum_{\boldsymbol{k}} U(\boldsymbol{k})b^*(\boldsymbol{k})b(\boldsymbol{k})[h^*(t) + h(t)] , \qquad (4.3.19)$$

$$\begin{aligned} V(\boldsymbol{k}) &= 2gu(\boldsymbol{k})v(\boldsymbol{k}) = gB(\boldsymbol{k})/\omega(\boldsymbol{k}) \\ &= [g\omega_{\rm M}/2\omega(\boldsymbol{k})]\sin^2\Theta(\boldsymbol{k})\exp[2i\varphi(\boldsymbol{k})] , \\ U(\boldsymbol{k}) &= gu^2(\boldsymbol{k}) = g[A(\boldsymbol{k}) + \omega(\boldsymbol{k})]/2\omega(\boldsymbol{k}) \qquad (4.3.20) \\ &= \frac{1}{2}g\left(1 + \sqrt{1 + \frac{\omega_{\rm M}^2\sin^2\Theta(\boldsymbol{k})}{4\omega^2(\boldsymbol{k})}}\right) , \end{aligned}$$

Here $h(t) = h\exp(-i\omega_{\rm p}t)$, $h_z = h(t) + h^*(t)$. The Hamiltonian $\mathcal{H}_{p1}$ describes processes which do not change the total number of magnons. Due to the condition of time synchronism this term may prove to be important only at low frequencies Ω of $h(t)$. The frequency Ω approximates the spin-wave relaxation frequencies giving the accuracy of the magnetization oscillation frequency $\omega(\boldsymbol{k})$. The origin of the Hamiltonian $\mathcal{H}_{\rm p}$ describing the parametric excitation of the pair of the spin waves by the longitudinal field can easily be illustrated by the following simple geometric consideration. Due to the magnetic dipole interaction and the crystallographic anisotropy the magnetization, at any point, precesses along the elliptic cone (formally it is revealed in the circular variables $a(\boldsymbol{k}) \sim m_x + im_y$ not being normal so that in order to diagonalize the quadratic term of the Hamiltonian the

u-v-transformation is necessary). Since the length of the vector $\boldsymbol{M}$ remains constant the base of the cone is not flat, which results in the appearance of a variable longitudinal component (z-component) of the vector $\boldsymbol{M}$ changing with the doubled precession frequency $2\omega(\boldsymbol{k})$. Clearly, those waves can be excited by a magnetic field with frequency $2\omega(\boldsymbol{k})$ polarized along z.

From the expressions (4.3.12) and (4.3.18) the threshold of the parametric pumping of magnons is readily obtained:

$$h_{\rm th}|V(\boldsymbol{k})| = \gamma(\boldsymbol{k}) . \qquad (4.3.21)$$

Here $V(\boldsymbol{k})$ is the amplitude of the $\pm k$-magnon-pumping interaction (4.3.20). Evidently, $|V(\boldsymbol{k})|$ is maximum for magnon pairs with wave vectors on the equator of the resonance surface: $0 \le \varphi(\boldsymbol{k}) < 2\pi$, $\Theta(\boldsymbol{k}) = \pi/2$. If $\gamma(\boldsymbol{k})$ depends slowly on $\Theta(\boldsymbol{k})$, these very pairs have the minimum excitation threshold

$$gh_{\rm th} = \gamma\omega_{\rm p}/\omega_{\rm M} . \qquad (4.3.22)$$

The order of magnitude of the parallel pumping field (4.3.22) is the same as that of the threshold field of the lateral pumping (4.3.13)-(4.3.15) far from FMR.

4.3.3 "Oblique" Pumping of Spin Waves in FM

"*Oblique pumping*" is a method of parametric excitation of spin waves in ferromagnets intermediate between the lateral and parallel pumping when the linearly polarized SHF field $\boldsymbol{h}(t)$ is at some angle Θ_0 to the magnetization. Therefore both lateral and longitudinal components of pumping must be simultaneously allowed for. The resulting effective amplitudes of the magnon interaction with the oblique pumping $V'(\boldsymbol{k}, \Theta_0)$ will have the form

$$V'(\boldsymbol{k}, \Theta_0) = \tilde{V}(\boldsymbol{k})\sin\Theta_0 + V(\boldsymbol{k})\cos\Theta_0 , \qquad (4.3.23)$$

where $\tilde{V}(\boldsymbol{k})$ and $V(\boldsymbol{k})$ are given by (4.3.15) and (4.3.20) respectively. As it can be seen from these expressions, the threshold of parametric excitation of the spin waves is minimum only for the pair of waves with $\varphi(\boldsymbol{k}) = 0$ and some $\Theta(\boldsymbol{k})$ depending on Θ_0.

4.3.4 Suhl Instability of the Second Order in FM

If we decrease the pumping frequency $\omega_{\rm p}$ or increase the constant magnetic field $\boldsymbol{H}$ and, accordingly, the gap in the magnon spectrum, it is easy to make the pumping frequency less than the double minimum (with respect to $\varphi(\boldsymbol{k})$ and $\Theta(\boldsymbol{k})$) gap in the magnon spectrum. Then the condition of the parametric resonance

$$\omega_{\mathrm{p}} = \omega(\boldsymbol{k}) + \omega(-\boldsymbol{k}) \tag{4.3.24}$$

will be fulfilled for no $\boldsymbol{k}$ and, consequently, three-wave decay processes (4.3.24) of the external field into two waves will be forbidden. In this case we can observe the *second-order Suhl instability of the homogeneous precession of magnetization* [4.11]. The conditions for parametric resonance for this instability have the following form:

$$2\omega_{\mathrm{p}} = \omega(\boldsymbol{k}) + \omega(-\boldsymbol{k}) \ . \tag{4.3.25}$$

Since the interaction Hamiltonian of the magnons with the external field (4.3.18) is proportional to the first power of h, the direct decay process of two quanta of the pumping into two magnons in ferromagnets will not take place. Only the indirect process involving the uniform precession is possible: the external field h spins it up in resonance $\omega_{\mathrm{p}} \simeq \omega_0$ and then the following four-magnon parametric process takes place:

$$2\omega_0 = \omega(\boldsymbol{k}) + \omega(-\boldsymbol{k}) \ . \tag{4.3.26}$$

Clearly, the threshold amplitude of the precession $b_{0,\mathrm{th}}$ is given by the condition

$$|S(\boldsymbol{0},\boldsymbol{k})(b_{0,\mathrm{th}})^2| = \gamma(\boldsymbol{k}) \ , \tag{4.3.27}$$

where b_0 is connected with h by (4.3.9) or (4.3.10). Substituting $S(\boldsymbol{0},\boldsymbol{k})$ from (3.1.24) into (4.3.27) we have for the cubic ferromagnets:

$$(gh_{\mathrm{th}}/\omega_{\mathrm{M}})^2 = \gamma(\boldsymbol{k})/\omega(\boldsymbol{k}) \ . \tag{4.3.28}$$

If for the first-order parametric processes $gh_{\mathrm{th}}/\omega_{\mathrm{M}} \simeq \gamma(\boldsymbol{k})/\omega_{\mathrm{M}}$, for the second-order ones the threshold $gh_{\mathrm{th}}/\omega_{\mathrm{M}} \simeq \sqrt{\gamma(\boldsymbol{k})/\omega_{\mathrm{M}}}$, and the threshold amplitude h_{th} far away from ferromagnetic resonance appears significantly greater. When the resonance condition $\omega_{\mathrm{p}} = \omega_0$ is fulfilled, however, it follows from (4.3.28) that $gh_{\mathrm{th}} \simeq \sqrt{\gamma_0\gamma(\boldsymbol{k})}$, i.e. the amplitude $h_{\perp\mathrm{th}}$ of the process under consideration has the same order of magnitude as the amplitude $h_{\|\mathrm{th}}$ of the parallel pumping. It is important for the parametric process (4.3.26) that the instability threshold (4.3.28) is minimum for the magnons at the poles of the resonance surface $[\Theta(\boldsymbol{k}) = 0, \pi]$, i.e. at a single pair of points.

4.3.5 Parallel Pumping in "Easy-Plane" Antiferromagnets

As has already been mentioned in Chap. 2 when studying nonlinear properties of magnons, "easy-plane" antiferromagnets are most interesting since they have magnon branches with frequencies lying in the convenient range below 50 gHz. The mechanism of the parametric excitation of magnons by parallel pumping is similar here to the mechanism of transverse pumping by a linearly polarized field or of the oblique pumping of magnons in ferromagnets: both the direct magnon excitation by the field h in the lower branch $h \to a(\boldsymbol{k}) + a(-\boldsymbol{k})$ and their indirect excitation, the decay of magnons with $\boldsymbol{k} = 0$ (homogeneous precession) in the upper branch $b_0 \to a(\boldsymbol{k}) + a(-\boldsymbol{k})$, this precession b_0 being linearly excited by the nonresonant field $h^* \to b$, $h \to b^*$. Accordingly, the total Hamiltonian of the system including the interaction with the pumping has the form:

$$\begin{aligned}
\mathcal{H} &= \sum_{\boldsymbol{k}}[\omega(\boldsymbol{k})a^*(\boldsymbol{k})a(\boldsymbol{k}) + \Omega(\boldsymbol{k})b^*(\boldsymbol{k})b(\boldsymbol{k})] + 2hU(b_0 + b_0^*)\cos\omega_{\mathrm{p}}t \\
&\quad + \frac{1}{2}\sum_{\boldsymbol{k}}[hV(\boldsymbol{k})a^*(\boldsymbol{k})a^*(-\boldsymbol{k})\exp(-i\omega_{\mathrm{p}}t) + \mathrm{c.c.}] + \mathcal{H}_3 \ , \\
\mathcal{H}_3 &= \sum_{2+3=1}[V^{(1)}_{1,23}b_1^*a_2a_3 + V^{(2)}_{1,23}a_1^*b_2a_3 + \mathrm{c.c.}] \\
&\quad + \frac{1}{2}\sum_{1+2+3=0}[U_{123}b_1^*a_2^*a_3^* + \mathrm{c.c.}] \ .
\end{aligned} \tag{4.3.29}$$

In this equation $\omega(\boldsymbol{k})$ and $\Omega(\boldsymbol{k})$ are the frequencies of the quasi-ferromagnetic and quasi-antiferromagnetic branches of the spectrum (see (3.2)). The amplitudes U and $\tilde{V}(\boldsymbol{k})$ describe the linear and parametric interactions of the field h with antiferro- and ferromagnons, and, finally, the amplitudes $V^{(1)}(\boldsymbol{1};\boldsymbol{2},\boldsymbol{3})$, $V^{(2)}(\boldsymbol{1};\boldsymbol{2},\boldsymbol{3})$ and $U(\boldsymbol{1},\boldsymbol{2},\boldsymbol{3})$ describe three-magnon processes of special interest for us (for them see (3.2.10)). Calculating b_0 from the equations of motion (4.3.8), we obtain the expressions for the effective amplitude of the interaction between the magnons of the lower branch and the pumping:

$$\begin{aligned}
\tilde{V}(\boldsymbol{k}) &= V(\boldsymbol{k}) - U\left[\frac{V^{(1)}(0,\boldsymbol{k},-\boldsymbol{k})}{\Omega_0 - \omega_{\mathrm{p}}} - \frac{U(0,\boldsymbol{k},-\boldsymbol{k})}{\Omega_0 + \omega_{\mathrm{p}}}\right] \\
&= \frac{g^2}{2\omega(\boldsymbol{k})}\left(H_D + \frac{2H\Omega_0^2}{\Omega_0^2 - \omega_{\mathrm{p}}^2}\right) \ ,
\end{aligned} \tag{4.3.30}$$

where H is the external constant field, H_D is the Dzyalochinsky field. The result (4.3.30) was obtained by *Ozhogin* [4.14]. It differs essentially from ferromagnets in its spherical symmetry: $\tilde{V}(\boldsymbol{k})$ does not depend on the direction of k and therefore under the spherical symmetry of magnon damping in the above-threshold state pairs with all directions of $\boldsymbol{k}$ must be excited, i.e. "the whole resonance surface is excited". Taking into account that in easy-plane antiferromagnets $\omega_0^2 = g^2H(H+H_D)$ (see (3.2.5)) and usually H and H_D are of the same order, clearly $\tilde{V}(\boldsymbol{k}) \simeq g$, i.e. has the same order of magnitude as $V(\boldsymbol{k})$ in the method of the parallel pumping in ferromagnets (4.3.21). Spin wave damping in good samples of antiferromagnetic $MnCO_3$ under helium temperatures is about $10^6\ \mathrm{s}^{-1}$, so the threshold field h_{th} is of the order of 0.1 Oe. This value is readily obtained experimentally.

4.3.6 Parametric Pumping of Nuclear Magnons

It should be recalled that the concept of *nuclear magnons* in antiferromagnets has been briefly described in Sect. 3.2.3. Their dispersion law $\omega_n(\boldsymbol{k})$ is given by (3.2.14). In [4.5] the thresholds $h_{\text{th}}^{(nn)}$ and $h_{\text{th}}^{(en)}$ of the processes

$$\omega_{\text{p}} = \omega_n(\boldsymbol{k}) + \omega_n(-\boldsymbol{k}) \ , \quad \omega_{\text{p}} = \omega_n(\boldsymbol{k}) + \omega_e(-\boldsymbol{k}) \ . \qquad (4.3.31a,b)$$

were calculated. The first one is the parametric excitation of two nuclear magnons and the second one is the parametric excitation of one electron and one nuclear magnon in "easy–plane" antiferromagnets. In our notation

$$h_{\text{th}}^{(nn)} = \gamma_n(\boldsymbol{k})/|V^{(nn)}(\boldsymbol{k})| \ , \quad h_{\text{th}}^{(en)} = \sqrt{\gamma_e(\boldsymbol{k})\gamma_n(\boldsymbol{k})}/|V^{(en)}(\boldsymbol{k})| \ . \ (4.3.32)$$

Here $\gamma_n(\boldsymbol{k})$ and $\gamma_e(\boldsymbol{k})$ are the damping decrements of the electron and nuclear magnons respectively: $V^{(nn)}(\boldsymbol{k})$ and $V^{(en)}(\boldsymbol{k})$ are the effective amplitudes of the magnon interaction with the pumping in the processes (4.3.31a) and (4.3.31b) respectively. At $\omega_{\text{p}} \ll \Omega_e(0)$ [4.5]:

$$\begin{aligned} V^{(nn)}(\boldsymbol{k}) &= \frac{g^2(2H+H_D)[\omega_n^2(0)-\omega_n^2(\boldsymbol{k})]}{2\omega_e^2(\boldsymbol{k})\omega_n(\boldsymbol{k})} \ , \\ V^{(en)}(\boldsymbol{k}) &= \frac{g^2(2H+H_D)\sqrt{\omega_n^2(0)-\omega_n^2(\boldsymbol{k})}}{2\omega_e(\boldsymbol{k})\sqrt{\omega_e(\boldsymbol{k})\omega_n(\boldsymbol{k})}} \ . \end{aligned} \qquad (4.3.33)$$

In the designation of the present section the interaction amplitude of the external field with two electron magnons (4.3.30) at $\omega_{\text{p}} \ll \Omega_e(\boldsymbol{k})$ has the form:

$$V(\boldsymbol{k}) = V^{(ee)}(\boldsymbol{k}) = g^2(2H+H_D)/[2\omega_e(\boldsymbol{k})] \ . \qquad (4.3.34)$$

Obviously, the relation

$$V_{\text{th}}^{(ee)}(\boldsymbol{k})V^{(nn)}(\boldsymbol{k}) = [\tilde{V}^{(en)}(\boldsymbol{k})]^2 \ . \qquad (4.3.35)$$

is satisfied. A direct consequence of (4.3.35) is the relation for the threshold fields $h_{\text{th}}^{(ee)}h_{\text{th}}^{(nn)} = [h_{\text{th}}^{(en)}]^2$ corresponding to the excitation of the nuclear and electron magnons with the same values of the wave vector. This relation probably has no important physical meaning, but it can be useful for numerical estimations.

In conclusion, note that all the above processes have been observed and investigated experimentally. Thus, in $MnCO_3$ at $T \simeq 2$ K and frequency $\omega(\boldsymbol{k}) = 0.9$ gGz, $h_{\text{th}}^{(nn)} \simeq (0.08 - 0.15)$ Oe and $h_{\text{th}}^{en} \simeq (0.4 - 1.0)$ Oe depending on the value of the external field H.

5 Stationary Nonlinear Behavior of Parametrically Excited Waves. Basic *S*-Theory

5.1 History of the Problem

The development of the parametric instability in a continuous medium (if the system's size is big enough in comparison with the wavelength of the excited waves) results in a great number of waves simultaneously excited and intensively interacting. The state of the waves is determined mostly by some particular factors: the dispersion law of the waves and the nonlinear and dissipative properties of the medium. These factors can vary considerably in different cases.

It became clear in the early seventies that there is one simple and at the same time very important specific case when one can formulate a general nonlinear theory of the parametric excitation of waves. This is the case of pumping by a spatially homogeneous field with $\boldsymbol{k}_{\text{p}} = 0$ or by a very long wave $k_{\text{p}} \ll k', k''$. Then the characteristics of the state above threshold may be assumed to be statistically homogeneous. This is the case in most experiments on the parametric excitation of spin waves in magnetically ordered dielectrics. Note that these are the most "pure" experiments in the physics of nonlinear waves because it is relatively simple to carry them out (in comparison, say, with experiments on plasma or the experimental study of nonlinear optics) and the high quality of the dielectrics. Experiments on the ferrimagnetic YIG - Yttrium Iron Garnet - are the most successful because of its many unique properties. Most of the experiments that are described in this book have been performed on YIG. Experiments on antiferromagnets $MnCO_3$, $CsMnF_3$ and $FeBO_3$ have also produced valuable results and are described here.

The first experimental data on the behavior of the spin waves above the threshold as well as the first models of their behavior above the threshold were obtained in the early 60s. The researchers who tried to suggest such models primarily aimed at obtaining the mechanism limiting the increasing amplitude of unstable spin waves. The first step in this direction was made by *Suhl* in [5.1]. He showed that under the pumping of spin waves by the homogeneous precession of the magnetization the principal limiting