Wave Turbulence Under Parametric Excitation

6 Advanced S-Theory: Supplementary Sections

The previous chapter treated the main concepts of the S-theory of the parametric excitation of waves in terms of spatially homogeneous pumping. Yet, a variety of interesting situations has not been treated.

In particular, nonlinear behavior of the system of parametric waves (with the exception of their general characteristics) was found to be very sensitive to the fine details of the functions $V(\boldsymbol{k})$ and $S(\boldsymbol{k},\boldsymbol{k}')$, specifying the interaction Hamiltonian. Only the simplest assumptions concerning these functions have been made, but these are not always valid. New details of the nonlinear behavior of the system of parametric waves caused by a complex form of the interaction Hamiltonian will be described in Sect. 6.1.

In the basic S-theory the damping of parametric waves was assumed to be independent of their amplitudes. If this assumption does not hold the damping of waves is nonlinear. Some mechanisms of the nonlinear damping will be considered in Sect. 10.2. The nonlinear damping of parametric waves can easily be allowed for within the S-theory. This is done in Sect. 6.2.

We also believe that the amplitude of the pumping field is the external parameter of the theory given by the experimentator. This is really so, if the role of pumping is played by an external field (electric or magnetic) whose energy is substantially higher than the energy of parametrically excited waves. If pumping is one of the oscillatory modes of the sample itself (e.g. the uniform precession of the magnetization under the parametric excitation of spin waves), its energy can as a rule be compared with the energy of the system of parametrically excited waves. In this case the feedback effect on the pumping must be taken into account. To this end, the equation for the pumping amplitude must be added to the basic equations of the basic S-theory and their simultaneous solutions must be obtained. This will be done in Sect. 6.3 for the processes of the parametric instability of the first and second order.

In Sect. 6.4 the fine effects will be studied caused by the interaction of parametrically excited waves with a thermal bath, which is a medium in thermal equilibrium with the temperature T. The basic S-theory described in Chap. 5 may be called a limiting case of the S-theory treating a thermal bath temperature tending to zero.

In Chap. 5 the pumping was assumed to be spatially homogeneous and the sample in which waves are excited parametrically was taken to be unbounded. Clearly this is not always correct and it would be interesting to study how the spatial inhomogeneity influences the nonlinear behavior of the system of parametrically excited waves. Some steps in this direction have been made in Sect. 6.5. The corrections to the basic S-theory obtained in this section are small if the characteristic size of the inhomogeneity L is great compared with the mean free path of parametric waves $|\partial\omega(\boldsymbol{k})/\partial\boldsymbol{k}|/\gamma(\boldsymbol{k})$.

The basic S-theory can easily be generalized to the "asymmetrical case" when the waves from different spectrum branches enter into the parametric pairs, e.g. electronic and nuclear spin waves in magnets, Langmuir and ionic-sonic waves in nonisothermic plasma, etc. An introduction into the asymmetrical S-theory describing such situations is given in Sect. 6.6. I consider the predicted (but not yet experimentally observed) phenomenon of the *correlation instability of waves* to be of special interest.

One more interesting problem is to study how the incoherence of the pumping influences the nonlinear behavior of the system of parametric waves. This problem is discussed in Sect. 6.7. The basic S-theory in the case of coherent pumping developed in Chap. 5 holds true in the limit where the linewidth of the pumping generator $\Delta\omega_{\mathrm{p}}$ is small compared with the damping decrement of parametric waves γ. Special attention is devoted to the opposite case $\Delta\omega_{\mathrm{p}} > \gamma$. It has been shown in particular that the coupling of waves and the phase mechanism of amplitude limitation is retained also in this case, although it becomes $\gamma/\Delta\omega_{\mathrm{p}}$ times less effective (under the same supercriticality).

The physical situations treated and generalized in the above mentioned seven subsections by no means exhaust the diversity of the general picture. The S-theory can be developed further and further, as I hope it will be. In particular, the nonlinear theory of the parametric excitation of waves by the plane wave of pumping determined on the boundary of the sample must be developed. This is possible if the ideas of Sects. 6.3, 6.5 and, perhaps, 6.6 are combined. The theory must be developed further also for the cases which will be studied in Sects. 6.5–7.

And, finally, the results outlined in the sections of this chapter are to a great extent independent of each other and have only one very important point in common: they are treated on the basis of the basic S-theory presented in Chap. 5. Therefore it is not necessary to read and study Chap. 6 section by section, the reader can choose the problems interesting to him and skip the others or leave them till later.

6.1 Ground State Evolution of System with Increasing Pumping Amplitude

Studying the basic S-theory we tacitly made some simplifying assumptions about the form of the functions $T(\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_3,\boldsymbol{k}_4)$, describing the four-wave interaction. It was done to avoid digression from the presentation of the main concept of the S-theory. Now we shall return to this problem.

The first assumption. The formula (5.4.12) gives the definition of the functions $T(\boldsymbol{k},\boldsymbol{k}_1)$ and $S(\boldsymbol{k},\boldsymbol{k}_1)$, which is given below

$$T(\boldsymbol{k},\boldsymbol{k}_1)=T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k},\boldsymbol{k}_1)/2\ ,\quad S(\boldsymbol{k},\boldsymbol{k}_1)=T(\boldsymbol{k},-\boldsymbol{k};\boldsymbol{k}_1,-\boldsymbol{k}_1)/2\ . \tag{6.1.1}$$

The interaction amplitudes $T(\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_3,\boldsymbol{k}_4)$, are the functions of four vector arguments $\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3$ and $\boldsymbol{k}_4$. However, in a spatially homogeneous medium (we shall confine ourselves to this case throughout the book) they are defined over the hypersurface $\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}_3+\boldsymbol{k}_4$. Therefore these amplitudes are the functions of three vector arguments. The functions of two vector arguments $T(\boldsymbol{k},\boldsymbol{k}_1)$ and $S(\boldsymbol{k},\boldsymbol{k}_2)$ are defined by the additional reduction (6.1.1) of these functions. Trying to use the future theory (the untrusting reader is free to think that we dreamt it up) one can suggest the following definitions of the functions T and S:

$$\begin{aligned}T(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{\kappa})&=T(\boldsymbol{k}+\boldsymbol{\kappa},\boldsymbol{k}_1-\boldsymbol{\kappa},\boldsymbol{k}-\boldsymbol{\kappa},\boldsymbol{k}+\boldsymbol{\kappa})/2\ ,\\ S(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{\kappa})&=T(\boldsymbol{k}+\boldsymbol{\kappa},-k+\boldsymbol{\kappa},k_1+\boldsymbol{\kappa},-k+\boldsymbol{\kappa})/2\ ,\end{aligned} \tag{6.1.2}$$

such that

$$T(\boldsymbol{k},\boldsymbol{k}_1)=T(\boldsymbol{k},\boldsymbol{k}_1,0)\ ,\quad S(\boldsymbol{k},\boldsymbol{k}_1)=S(\boldsymbol{k},\boldsymbol{k}_1,0)\ . \tag{6.1.3}$$

It must be noted here that the functions $T(\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_3,\boldsymbol{k}_4)$ are not necessarily continuous functions of their arguments. For example, the calculations of the Hamiltonian of the spin wave interaction in ferromagnets performed in Chap. 3 (see (3.1.22)) for $c(\boldsymbol{k})$ and $c(0)$) show that the limits

$$\lim T(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{\kappa}),\quad \lim S(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{\kappa})\quad \text{at}\quad \kappa\to 0 \tag{6.1.4}$$

may fail to exist at all (may depend on the direction of the vector $\boldsymbol{\kappa}$) and, second, may fail to coincide with the values of these functions at $\kappa=0$. Therefore the definition of the functions $S(\boldsymbol{k},\boldsymbol{k}_1)$ and $T(\boldsymbol{k},\boldsymbol{k}_1)$ is ambiguous. Either the definition (6.1.3) is correct, or $S(\boldsymbol{k},\boldsymbol{k}_1)$ and $T(\boldsymbol{k},\boldsymbol{k}_1)$ denote the limits (6.1.4), but then how is the vector $\boldsymbol{\kappa}$ directed? We have no answers to these questions (in Chap. 5 and in general) and we can affirm only that the basic S-theory holds true if the limits of (6.1.4) exist and equal to the values of (6.1.3).

For other cases the S-theory must be correspondingly generalized. This future theory must be spatially non-homogeneous and allow for the shape of the sample. One of its basic elements will take into account the long-range dipole-dipole interaction of the parametrically excited waves with the eigenmodes (electrostatic or magnetostatic) of the sample. Since at present there is no such theory, the four-wave amplitude of interaction $T(\boldsymbol{k}_1, \boldsymbol{k}_2; \boldsymbol{k}_3, \boldsymbol{k}_4)$ throughout this book will be assumed to be a continuous function of its arguments. Then the limits (6.1.4) exist and it is easy to define the functions $T(\boldsymbol{k}, \boldsymbol{k}_1)$ and $S(\boldsymbol{k}, \boldsymbol{k}_1)$.

The second assumption. In the analysis of the ground state of the system of parametric waves the function $S(\Omega, \Omega_1)$ was assumed in Sect. 5.5 to be real. Usually this is so, though not always. In Sect. 6.1.1 the behavior of the system of parametric waves will be studied in the case when the function $S(\Omega, \Omega_1)$ is complex.

The third assumption. Studying the step-by-step excitation of parametric waves in Sect. 5.5.4 we basically assumed that $S(\Omega, \Omega_1)$ is not only continuous but everywhere a differentiable function of its arguments. However, the calculation of this function performed in Sect. 3.1.4 showed (the details will be given later) that in cubic ferromagnets this is not the case. The non-analytical contribution to the function $S(\Omega, \Omega_1)$ can qualitatively change the nonlinear behavior of parametric waves. This will be considered in detail in Sects. 6.1.2, 3.

6.1.1 Ground State of Parametric Waves for Complex Pair Interaction Amplitudes

It must be recalled that by using the invariant phase (5.5.8) and the substitution (5.5.10) the equation of the basic S-theory can be reduced to the form of (5.5.11) where the amplitude of the pumping wave interaction $V(\Omega)$=const and real. This can be done with unrestricted generalization. It can naturally be expected that under small supercriticality only one group of pairs will be excited in the ground state. Under the axial symmetry of the problem these pairs are located in parallels on the resonant surface Θ_1 and $\pi - \Theta_1$, at arbitrary φ. Their polar angle Θ_1 can be defined by the geometrical interpretation of the condition of the external stability discussed at the end of Sect. 5.5.1. In the terms of (5.5.11) this is formulated in the following way: the whole line $\Gamma(x)$ ($x = \cos\Theta$) is below the line $|\tilde{P}(x)|$ and is tangent to it at the points where $N(x)$=0. This means that the location of the first pair (i.e. the value $x_1 = \pm\cos\Theta_1$) is determined from the following condition:

$$d[\Gamma(x) - |\tilde{P}(x)|]/dx = 0 \quad \text{at} \quad x = x_1 \,. \tag{6.1.4}$$

The imaginary part of the function $\tilde{S}(x, x_1)$ ($\mathrm{Im}\{\tilde{S}(x, x_1)\}$) results in the changed expression for $\tilde{P}(x)$. Instead of (5.5.14) one can obtain:

$$|\tilde{P}(x)|^2 = N_1[S(x_1, x_1) - Re\{\tilde{S}(x_1, x)\}]^2 + [\Gamma_1 - N_1 \mathrm{Im}\{\tilde{S}(x_1, x)\}]^2 \,. \tag{6.1.5}$$

From the condition (6.1.4) we have:

$$N_1 d[\mathrm{Im}\{\tilde{S}(x_1, x)\}]/\,dx = d\Gamma(x)/\,dx \,. \tag{6.1.6}$$

Thus, unlike before, $d\Gamma(x)/\,dx$=0 only at the threshold point. Afterwards the pair "leaves" the point x_1 (x_1 is the coordinate of the pair at p=0) and the *departure* $\Delta x = x - x_1$, being proportional to N_1. This results in a further weakening of the wave interaction with the pumping and is, in addition to the phase mismatch, one more cause of the limitation of the pair amplitude. Simultaneously solving (5.5.11) and (6.1.6) we obtain at small N_1 (for more detail, see [6.1]):

$$N_1^2 = \left[h^2V_1^2 - \Gamma^2\right]\left\{S_{11}^2 + \frac{[d\mathrm{Im}\{\tilde{S}(x_1, x)\}/dx]^2 h^2 V_1^2}{[d\Gamma(x)/dx]^2}\right\}^{-1} \,. \tag{6.1.7}$$

Comparing this result with (5.5.12) we can see that this departure leads to the appearance of an additional term in the denominator of (6.1.7), generally speaking, of the same order of magnitude. It must be noted that under the excitation of the magnons by parallel pumping in ferromagnets, the first group of pairs appears at symmetrical position of the resonant surface, i.e. at its equator. As the supercriticality increases, the pairs do not leave the equator because they "do not know" where to go: northwards or southwards. Formally this is reflected in the fact that in (6.1.6) $d\,\mathrm{Im}\{S(x, x_1)\}/\,dx = 0$ at x=x_1=0 ($\cos\Theta_1$=0). On the contrary, under the transverse pumping of magnons the first group of the pairs is excited at $\Theta = \Theta_1$ and $\pi - \Theta_1$, $\Theta_1 \simeq \pi/4$. Because of absence of symmetry in this case it is quite possible for magnons "to go" to the equator or to poles. The dependence of the location of the pairs on the supercriticality has not yet been studied experimentally or theoretically for particular cases. Here we only point out the possibility of such a situation.

6.1.2 The Second and Intermediate Thresholds

In order to compare later the theoretical results with the experimental data we shall study a concrete example of the parametric excitation of magnons in ferromagnets by parallel pumping. First of all, the particular form of the function $\tilde{S}(x, y)$ for this case must be obtained. From the definition of the function $S(\boldsymbol{k}, \boldsymbol{k}')$ (5.4.12) and (5.5.9, 10) it follows that the function $\tilde{S}(x, y)$ is the even function of each of the arguments

$$\tilde{S}(x, y) = \tilde{S}(-x, y) = \tilde{S}(x, -y) = \tilde{S}(-x, -y) \tag{6.1.8a}$$

and its real part $\mathrm{Re}\{\tilde{S}(x, y)\}$ is symmetrical about the permutation x, y. The calculations for ferromagnets show that $\mathrm{Im}\{\tilde{S}(x, y)\}$=0. Thus

$$\mathrm{Re}\{\tilde{S}(x,y)\} = \tilde{S}(x,y) = \tilde{S}(y,x) \ . \qquad (6.1.8b)$$

Since the problem is axially symmetrical, $S(\Theta,\Theta',\varphi-\varphi')$ is independent of $(\varphi-\varphi')$ if Θ=0 or Θ'= 0. Therefore the integral (5.5.9) must become zero if $x \to 1$ or $y \to 1$ and function $\tilde{S}(x,y)$ accordingly remains finite at $x,y \to 1$, in spite of the singular factor $f(x) = 1-x$ in the denominator of (5.5.10). Thus, $\tilde{S}(x,y) = \tilde{S}(y,x)$ are finite.

The behavior of $\tilde{S}(x,y)$ under small x,y will be studied on the basis of the explicit form of (3.1.24, 26) for $S(\boldsymbol{k},\boldsymbol{k}')$. Some terms of these formulae are proportional to $\cos^2[\Theta(\boldsymbol{k}+\boldsymbol{k}')]$ and $\cos^2[\Theta(\boldsymbol{k}-\boldsymbol{k}')]$. Each of these terms as a result of the integration over φ according to (5.5.9) gives the non-analytical contribution into $\tilde{S}(x,y)$, proportional to the powers of $|x \pm y|$. Taking this fact and the above discussed properties of symmetry (6.1.8) into account the expansion of $\tilde{S}(x,y)$ under small x and y to the accuracy of the terms $\propto x^2, y^2$ will take the form

$$\tilde{S}(x,y) = S[1+(1/2)a(|x+y|+|x-y|)+b(x^2+y^2)] \ . \qquad (6.1.9)$$

Computer calculations for YIG at room temperature according to (3.1.24, 26) show that within the range of magnetic fields $H - H_{\mathrm{th}}$ from 100 Oe to 500 Oe the coefficient a changes within the limits 2.6 – 2.1 and the coefficient b changes from −1.8 to 1.3 becoming zero at $H = H_{\mathrm{th}} - 180$ Oe (note that under $H = H_{\mathrm{th}}$, $\omega(\boldsymbol{k},H) = \omega_{\mathrm{p}}/2$ at $\boldsymbol{k} \perp \boldsymbol{M}$ and $k \to 0$).

Now we can further analyze the problem of the generation of the second group of pairs we started to consider in Sect. 5.5.3 on the basic S-theory. Substituting (6.9) into the expression (5.5.16) for $p(x)$ and expanding it in terms of x, we obtain

$$\begin{aligned} p(x) &= p(0) + |x|p' + x^2p''/2 \ , \quad p(0) = 1 + \frac{\Gamma''}{a^2\Gamma} \ , \\ p' &= \frac{-b\Gamma''}{a^3\Gamma} \ , \quad p'' = \frac{3\Gamma''^2 + \Gamma\Gamma^{(4)}}{6a^2\Gamma^2} - \frac{2\Gamma'' b^2}{\Gamma a^4} \ . \end{aligned} \qquad (6.1.10)$$

Here $\Gamma'' = \Gamma^{(2)}$ and $\Gamma^{(4)}$ denote the second and fourth derivatives of $\Gamma(x)$ at $x = x_1 = 0$. It is clear that the fact is that $\tilde{S}(x,y)$ is non-analytical (a=0) leads to the finiteness of $p(0)$. The case when at x=0 the minimum of the function $p(x)$ is realized will subsequently be considered. For the time being we shall assume $p < 0$. This happens in YIG in the fields $H > H_{\mathrm{th}} + 180$Oe when the coefficient a is positive. It can be seen from (6.1.10) that

$$|x_2| = -p'/p'' \ , \qquad p_2 = p(0) - 2p'^2/p'' \ . \qquad (6.1.11)$$

This expression holds true under small b when $|x_2| \ll 1$ and we can use the result of the expansion of (5.5.16) in the form of (6.1.10).

Now let us turn to the range of magnetic fields with negative coefficient a in the expansion (6.1.9) for $\tilde{S}(x,y)$. We shall subsequently see that in this case with increasing supercriticality the distribution function of parametric

waves over the resonant surface could be quite different and not result in the generation of the second group of pairs. Under some $p = p_{\mathrm{th}}$ (this value of the supercriticality is called the *intermediate threshold*) the regular part appears in $N(x)$ in the vicinity of the equator: $N(x) \neq 0$ in the range $|x| < \delta$ the width of the distribution $N(x)$ being zero at $p = p_{\mathrm{th}}$ and further increasing as the supercriticality increases.

To find p_{th} let us substitute the expression (6.1.9) for $\tilde{S}(x,y)$ in the expression (5.5.14) for the self-consistent pumping. As a result, we obtain:

$$|\tilde{P}(x)|^2 = \Gamma^2\{1 + x^2[a^2 + 2ab|x| + b^2x^2](p-1)\} \ . \qquad (6.1.12a)$$

Expanding $\Gamma(x)$ into a series we obtain:

$$\Gamma^2(x) = \Gamma^2 + \Gamma\Gamma''x^2 + (3\Gamma''^2 + \Gamma\Gamma^{(4)})x^4/12 \ . \qquad (6.1.12b)$$

The comparison of the (6.1.12a) and (6.1.12b) shows that at $p_2 = p(0)$ given by (6.1.10) the coefficients of x^2 of these dependences become equal: at the point x=0 second order tangency of the lines $|\tilde{P}(x)|$ and $\Gamma(x)$ occurs – the values of these functions coincide as well as the values of their first and second derivatives. Therefore $p_{\mathrm{th}} = p(0)$. At $p > p(0)$ this results in the appearance of the instability region at $|x| \le \delta$ (where $|\tilde{P}(x)| > \Gamma(x)$).

The boundary $\delta_1(p)$ of the instability region is obtained from the comparison of the two formulae (6.1.12):

$$p - p(0) = \frac{2b(p-1)|\delta_1|}{a\Gamma^2} + \left[\frac{b^2(p-1)}{a^2} - \frac{2\Gamma''^2 + \Gamma\Gamma^{(4)}}{12\Gamma^2}\right]\delta_1^2 . \qquad (6.1.13a)$$

As the smallness of δ_1 is $p - p(0)$ we can substitute p in the right-hand side of (6.1.13a) for $p(0)$. Then (6.1.13a) can be represented in the form similar to (6.1.10):

$$p - p(0) = p'\delta_1 + p''\delta_1^2/2 \ . \qquad (6.1.13b)$$

As is clear from this equation under $p' > 0$ the instability region is limited and $\delta_1 \propto [p - p(0)]$. If p'=0, then for δ_1 being limited it is necessary that $p'' > 0$, then $\delta_1 \propto \sqrt{p - p(0)}$.

The subsequent section, which is based on the study of *Zautkin, L'vov* and *Podivilov* [6.2], shows that the development of the instability results in a region term in the distribution function of parametric waves with the width over x equal to $\delta(p)$, and under the small supercriticality above the intermediate threshold $p(0)$ the dependence $\delta(p)$ can be obtained from the equation similar to (6.1.13):

$$p - p(0) = 6[p'|\delta| + p''\delta^2/2] \ . \qquad (6.1.14)$$

It will also be shown that an considerable change in the distribution function above the intermediate threshold (i.e. when $p > p(0)$) does not seriously affect the total characteristics of the system of parametric waves such as $\chi'\,\chi'', N$, etc.

6.1.3 Nonlinear Behavior of Non-Analytic Pair Interaction Amplitudes

First of all we obtain the nonlinear equations for the regular term of the distribution function of parametric waves $N(x)$ and their phases $\Psi(x)$. To this end, we calculate $\tilde{P}(x)$ by using (5.5.11) and (6.1.9). Under $0 \le x \le \delta$ (δ is the distribution width):

$$\tilde{P}(x) = A + Bx^2 + 2aS[x\int_0^x \sigma(y)\,dy + \int_x^\delta y\sigma(y)\,dy]\,, \tag{6.1.15a}$$

$$A = hV + S(\Sigma_0 + b\Sigma_2)\,, \quad B = bS\Sigma_0\,, \quad \Sigma_0 = \int_{-\delta}^{\delta} \sigma(y)\,dy\,, \tag{6.1.15b}$$

$$\Sigma_2 = \int_{-\delta}^{\delta} y^2\sigma(y)\,dy\,, \quad \sigma(y) = N(y)\exp[-i\Psi(y)]\,. \tag{6.1.15c}$$

At $x > \delta$ the expression for $\tilde{P}(x)$ has another form, i.e.:

$$\tilde{P}(x) = A + Bx + aS\Sigma_0 x\,. \tag{6.1.15d}$$

In combination with (5.5.11) this results in a closed system of integral equations for $N(x)$ and $\Psi(x)$. Proceeding from the above we can obtain the integrodifferential equations for these values which are easier to analyze. It follows from (6.1.15) that the functions $\tilde{P}(x)$ and $d\tilde{P}(x)/\,dx$ are continuous at $x = \delta$, whereas:

$$\tilde{P}(\delta) = hV + S\Sigma_0(1 + a\delta + b\delta^2) + bS\Sigma_2]\,, \tag{6.1.16a}$$

$$d\tilde{P}(x)/\,dx = S\Sigma_0(a + 2b\delta) \quad \text{at} \quad x = \delta\,. \tag{6.1.16b}$$

Subsequent differentiation yields

$$d^2\tilde{P}(x)/\,dx = 2bS\Sigma_0\,, \quad \text{at } x > \delta\,; \tag{6.1.16c}$$

$$d\tilde{P}(x)/\,dx = 2bS\Sigma_0 + 2aS\sigma(x)\,, \quad \text{at } 0 < x < \delta\,. \tag{6.1.16d}$$

At $x < \delta$ the equations of the S-theory (5.5.11) in combination with (6.1.15, 16) yield the required system of equations:

$$2aSN(x) = -\,2bS\int_{-\delta}^{\delta} N(y)\exp\{-i[\Psi(y) - \Psi(x)]\}dy + \Gamma(x)\frac{d^2\Psi}{dx^2} + \frac{d\Gamma}{dx}\frac{d\Psi}{dx} + i\left[\frac{d^2\Gamma}{dx^2} - \Gamma(x)\left(\frac{d\Psi}{dx}\right)^2\right]\,. \tag{6.1.17}$$

These equations can most easiest be solved for the case b=0 when the integrated term is absent. Then the x-dependence of the phase Ψ has the form:

$$\frac{d\Psi(x)}{dx} = \Psi(0) + \int_0^x \sqrt{\frac{d^2\Gamma(y)}{\Gamma(y)dy^2}}\,dy\,. \tag{6.1.18a}$$

The distribution $N(x)$ contains the singular term due to the salient point of the first derivative of $\Psi(x)$ at x=0:

$$\begin{aligned} N(x) &= N_1\delta(x) + N_2(x)\,, \\ N_1 &= N_{1\mathrm{cr}} = \sqrt{\Gamma\Gamma''}/aS\,, \quad \Gamma'' = d^2\Gamma(x)/dx^2 \text{ at } x = 0\,, \\ N_2(x) &= \left[\frac{d^3\Gamma(x)}{dx^3}\Gamma(x) + \frac{d\Gamma(x)}{dx}\frac{d^2\Gamma(x)}{dx^2}\right]/2aS\sqrt{\frac{\Gamma(x)d^2\Gamma(x)}{dx^2}}\,. \end{aligned} \tag{6.1.18b}$$

This solution is realized under the supercriticalities p above the intermediate threshold $p(0)$. Under $p < p(0)$ the ordinary singular solution of the basic S-theory (5.5.13a) is realized, which in the case under consideration is increased as p increases according to the law

$$N_1(p) = \Gamma(0)\sqrt{p-1}/|S|\,. \tag{6.6.19}$$

Using (6.1.10b) for $p(0)$ one can see that the value of the function $N_1(p)$ under p=$p(0)$ coincides with the value $N_{1,\mathrm{cr}}$ in (6.1.18). Therefore in our solution the number of parametric waves $N_1(p)$ under $p < p(0)$ increases according to (6.1.19) and at $p > p(0)$ stops at the threshold level $N_{1,\mathrm{cr}}$ (6.1.18c). According to (6.1.18) $N_2(x)$=0 at x=0 and linearly increases with $|x|$ at small x

$$N_2(x) = |x|\left[\Gamma\Gamma^{(4)} + (\Gamma^{(2)})^2\right]/4aS\sqrt{\Gamma\Gamma^{(2)}}\,, \tag{6.1.20}$$

where $\Gamma^{(n)}$ is the derivative of $\Gamma(x)$ with respect to x of the order n at x=0.

When $b \neq 0$ the integrodifferential equation (6.1.17) can be solved under small supercriticalities above the intermediate threshold, i.e. under $p - p(0) \ll 1$. To this end, the theory of perturbations with respect to δ (the width of the continuous part of the distribution function) should be employed assuming that the value δ must be small. We shall present the result of the corresponding calculations

$$\begin{aligned} \frac{d\Psi}{dx} &= \mathrm{sign}x\sqrt{\frac{d^2\Gamma(x)}{\Gamma(x)dx^2}}\left[1 + (2\delta|x| - \delta^2)\frac{b^2}{a^2}\right]\,, \\ N_1 &= N_{1\mathrm{cr}}[1 - \delta^2b^2/a^2]\,, \qquad N_2(x) = N_{21} + N_{22}x\,, \\ N_{21} &= N_{1\mathrm{cr}}[-b/a + 3\delta^2b^2/a^2] - N_{22}\delta^2 b\,, \\ N_{22} &= N_{1\mathrm{cr}}[(\Gamma\Gamma^{(4)} + (\Gamma^{(2)})^2/4\Gamma\Gamma^{(2)} + \delta^2b^2/a^2] \end{aligned} \tag{6.1.21}$$

This solutions differ from (6.1.18) obtained under b=0 in two significant characteristics. First, the singular part of the N-distribution in (6.1.21) decreases as the supercriticality increases and apparently must become zero under some high p; second, the regular part of the $N(x)$-distribution at x=0 is nonzero and increases with the increasing p.

Now we shall find how the width of the distribution δ depend on the supercriticality $p - p(0)$. To this end, we shall make use of the fact that $|\tilde{P}(x)| = \Gamma(x)$ at $x \leq \delta$. Therefore

$$|\tilde{P}(\delta)|^2 = \Gamma^2(\delta), \tag{6.1.22a}$$

$$d|\tilde{P}(\delta)|/dx = d\Gamma^2(x)/dx \qquad \text{at} \quad x = \delta \tag{6.1.22b}$$

$$d^2|\tilde{P}(\delta)|^2/d^2x = d^2\Gamma^2(x)/dx^2 \qquad \text{at} \quad x = \delta - 0 \tag{6.1.22c}$$

At $x > \delta$ the inequality $|\tilde{P}(x)|^2 \leq \Gamma^2(x)$ holds. Taking into account the relations (6.1.22a, b) we have

$$d^2|\tilde{P}|^2/dx^2 \leq d^2\Gamma^2/dx \qquad \text{at} \quad x = \delta + 0 .$$

A much more detailed analysis based on the condition of the stability of the obtained solution in the wave package narrowing (decrease of δ) shows that the equality sign must appear in the last formula

$$d^2|\tilde{P}|^2/dx^2 = d^2\Gamma^2/dx^2 \qquad \text{at} \quad x = \delta + 0 . \tag{6.1.22d}$$

Substituting into (6.1.22a, b and c) the expressions for $\tilde{P}(\delta)$, $\tilde{P}'(\delta)$ and $\tilde{P}''(\delta + 0)$ from (6.1.16a, b and c) after simple transformations we obtain

$$Xf^2 + 2Yf = \Gamma^2 - h^2V^2 , \quad Xff_1 + Yf_1 = \Gamma\Gamma' , \tag{6.1.23a, b}$$

$$X[f_1^2 + 2ff_2] + 2Yf_2 = \Gamma'^2 + \Gamma\Gamma'' , \tag{6.1.23c}$$

$$X = |S\Sigma_0|^2 , \quad Y = hVS\mathrm{Re}\{\Sigma_0\} , \tag{6.1.23d}$$

$$f(x) = S(0,x)/S(0,0) = 1 + ax + bx^2 , \tag{6.1.23e}$$

$$f_1(x) = f'(x) = a + 2bx \quad f_2(x) = f''/2 = b . \tag{6.1.23f}$$

The values of all functions f, f_1, f_2 and Γ,Γ', Γ'' are taken at the point $x = \delta$. In these equations the term $b\Sigma_2$ in comparison with Σ_0 was neglected. Using the solution (6.1.21) we can estimate $\Sigma_2 \simeq -b\delta^3\Sigma_0/3a$. Therefore our approximation holds true if $b\delta^3 \ll 3a$ and can be employed either in the case of small b and arbitrary δ or in the case of arbitrary b and small δ. This approximation considerably simplifies the situation: the equations (6.1.23) become closed and specify the dependences of δ and Σ_0 on hV. This enables one to obtain not only the width of the package of parametric waves δ, but also the total characteristic of the system, such as the nonlinear susceptibilities χ' and χ'' without the explicit solution of the original integrodifferential equations (6.1.17) and subsequent integration according to the formulae

$$\chi' = -\frac{2}{h}\int V(x)\mathrm{Re}\sigma(x)dx , \quad \chi'' = -\frac{2}{h}\int V(x)\mathrm{Im}\sigma(x)dx , \tag{6.1.24a}$$

Indeed, in our notation (6.1.24a) are reduced to the form

$$\chi' = -2V\mathrm{Re}\Sigma_0/h , \quad \chi'' = 2V\mathrm{Im}\Sigma_0/h \tag{6.1.24b}$$

i.e. are expressed in terms of the value Σ_0, whose dependence on the hV is given by (6.1.23). In order to find the dependence of the package width δ on the supercriticality let us eliminate X and Y from (6.1.23). This results in:

$$f_1^3(h^2V^2 - \Gamma^2) = f[ff_1(\Gamma'^2 + \Gamma\Gamma'') - 2\Gamma\Gamma'(f_1^2 + ff_2)] . \tag{6.1.25a}$$

Substituting in the above the explicit form of the functions f,f_1 and f_2 from (6.1.23 e and f) we can obtain accurately to the terms not higher than δ^3:

$$p - p(0) = 6[p'(1 + a\delta^3/3 - 5b\delta/a) + p''^2/2] . \tag{6.1.26a}$$

The coefficients $p(0)$, p' and p'' are given by (6.1.10b); a and b are the expansion coefficients of the function $\tilde{S}(x,y)$ in (6.1.9). Under b=0, when p'=0 in (6.1.26a) we can limit ourselves to the terms linear in δ. Then

$$\delta = [p - p(0)]/6p' . \tag{6.1.26b}$$

If b=0, then p'=0 and

$$\delta^2 = [p - p(0)]/3p'' . \tag{6.1.26c}$$

Both these cases are described by the interpolation formula (6.1.14), whose form is simpler than (6.1.26a). Comparison of (6.1.26) with (6.1.13b) enables us to draw a qualitative conclusion about the width of the nonlinear package of the parametric waves δ being smaller by a factor from 6 to $\sqrt{6}$ than the width of the instability region δ_1. The S-theory generally shows the trend towards the narrowing of the parametric wave package at the nonlinear stage of its evolution and this is one of its manifestations.

Seeking the solution of (6.1.23) with respect to X and Y we can easily obtain

$$|S\Sigma_0|^2 = X = [h^2V^2 - \Gamma^2]/f^2 + 2\Gamma\Gamma'/ff , \tag{6.1.27a}$$

$$-hVS\mathrm{Re}\Sigma_0 = -Y = [h^2V^2 - \Gamma^2]/f - \Gamma\Gamma'/f_1 . \tag{6.1.27b}$$

The right-hand parts of these equations are the functions of δ which depends on the supercriticality according to (6.1.25a) or (6.1.26a). At $p - p(0) \ll 1$ (6.1.27) can be reduced to the accuracy of the second order of δ to the following form:

$$\begin{aligned} |S\Sigma_0|^2 &= \Gamma^2(0)[p - 1 - ap'\delta^2] , \\ -hVS\ \mathrm{Re}\Sigma_0 &= \Gamma^2(0)[p - 1 - a\delta^2/3] \end{aligned} \tag{6.1.28}$$

It should be recalled that at $p < p(0)$ when $N(x) \propto \delta(x)$, these values are described by the following simple formulae of the basic S-theory:

$$|S\Sigma_0|^2 = -hVS \,\mathrm{Re}\Sigma_0 = \Gamma^2(0)[p-1] \tag{6.1.29}$$

It can easily be seen from the comparison of the above formulae with (6.1.28) that the significant qualitative rearrangement of the wave distribution function $N(x)$ above the intermediate threshold (i.e. at $p - p(0) \ll 1$) produces a small effect on the total characteristic of the wave system (Σ_0, χ' and χ'') if $p - p(0)$ is small. Indeed, at $b \neq 0$ the difference of (6.1.28) and (6.1.29) is proportional to $[p - p(0)]^2$. If b=0, then p'=0 and the difference between (6.1.28) and (6.1.29) is proportional to $[p - p(0)]^{3/2}$. Therefore this rearrangement of the distribution function above the threshold $p(0)$ is not easily detected in experiments aimed at obtaining the total characteristics of the system of the parametric waves (Σ_0, χ' and χ'') . Nevertheless, the above-described rearrangements of the parametric wave distribution function under increasing supercriticality have been detected and studied by *Zautkin* et al. in [6.2] under parallel pumping of magnons in YIG (see also Sect. 9.6.2. and Fig. 9.26).

6.2 Influence of Nonlinear Damping on Parametric Excitation

It is assumed in the basic S-theory that the damping of the parametric waves $\gamma(\boldsymbol{k})$ is independent of their number. In Sect. 1.3.3 it has already been mentioned that this assumption is an ideal case, generally speaking $\gamma(\boldsymbol{k})$ is the functional of $n(\boldsymbol{k}')$. According to (1.3.10) under small $n(\boldsymbol{k}')$ it can be assumed that

$$\gamma(\boldsymbol{k}) = \gamma_0(\boldsymbol{k}) + \int \eta(\boldsymbol{k}, \boldsymbol{k}')n(\boldsymbol{k}')\,d\boldsymbol{k}' \,. \tag{6.2.1}$$

Since we are mainly interested in the qualitative influence of the nonlinear damping on the behavior of parametric waves, in this section we shall confine ourselves mostly to the linear dependence (6.2.1). The subsequent results can readily be generalized to the more complex models of the nonlinear damping.

6.2.1 Simple Theory

For simplicity, the function $\eta(\boldsymbol{k}, \boldsymbol{k}')$ will be assumed in this subsection to be a continuous function of its arguments [6.3]. Its values over the resonant surface will enter the fundamental equations of the basic S-theory (5.5.6):

$$\gamma(\Omega) = \gamma_0(\Omega) + \int \eta(\Omega, \Omega')n(\Omega')d\Omega' \,. \tag{6.2.2}$$

As is known, under small supercriticality in the ground state there is only one group of pairs (located in one pair of points on the line or over the entire resonant surface depending on the symmetry of the problem). In this case

$$\eta(N) = \gamma_0 + \eta N \,, \qquad N = \int n(\Omega)d\Omega \,, \quad \eta = \int \eta(\Omega, \Omega')n(\Omega')d\Omega'/N \,. \tag{6.2.3}$$

The total number of pairs N and their phase Ψ is obtained from the following equations generalizing (5.5.7):

$$hV \sin\Psi = \gamma(N) \,, \quad -hV\cos\Psi = SN \,. \tag{6.2.4}$$

From this equation and from (6.2.3) the dependence of N on the supercriticality p (p is the ratio of the power of the pumping to its threshold value $p = h^2V^2/\gamma^2$) can readily be obtained:

$$N = \frac{\gamma_0}{|S|}\frac{-c \pm \sqrt{p(c^2+1)-1}}{c^2+1} \,, \quad c = \frac{\eta}{|S|} \,. \tag{6.2.5}$$

In the absence of the nonlinear damping (at c=0) this formula goes over to (5.5.7). Under large positive c (6.2.5) changes into

$$N = \gamma_0[\sqrt{p} - 1]/\eta = (hV - \gamma_0)/\eta \,. \tag{6.2.6}$$

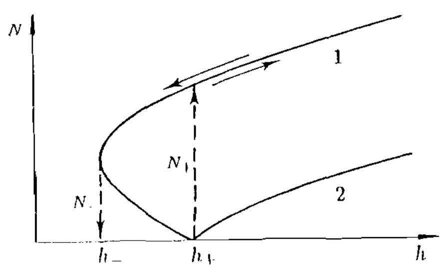
This expression trivially follows from (6.2.4) at $S = 0$ ($\cos\Psi = 0$, $\sin\Psi = 1$, $hV = \gamma(N) = \gamma_0 + \eta N$) and is a condition of the energy balance in the absence of the phase mechanism of the amplitude limitation. In this case the stationary value of N is entirely due to the increase of damping as the N increases.

Under negative nonlinear damping (i.e. under $\eta < 0$) the dependence $N(p)$ becomes ambiguous (see Fig. 6.1). *Hard excitation* of waves arises: at the instability threshold (at p=1) the wave number increases abruptly from zero to N, where

$$N_+ = N(p=1) = 2c\gamma_0/|S|(c^2+1) \,. \tag{6.2.7}$$

The further increase of p results in the increasing number of waves according to (6.2.5). As p decreases from the values of $p > 1$ to $p < 1$ (but $p > p_-$, where $p_- = 1/(c^2+1)$) the number of waves decreases according to (6.2.5) to N_-=$N_-/2$ and then abruptly falls to zero. Therefore the hard excitation of the waves inevitably is accompanied by the hysteresis of the dependence $N(p)$.

With $N(p)$ known, the susceptibility χ of the parametric wave system can readily be calculated. Its imaginary part χ'' is proportional to the power



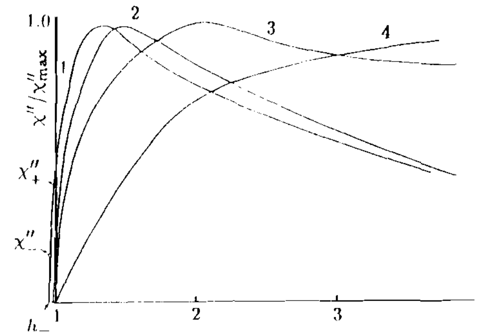


Fig. 6.1. (left) Hard excitation of parametric waves under negative nonlinear damping: theoretical dependences (6.2.5) of the total number of parametric waves N on the pumping amplitude h. The line (2) is unstable

Fig. 6.2. (right) Theoretical dependences (6.2.8) of imaginary part of nonlinear susceptibility χ'' on dimensionless pumping amplitudes h/h_{th} at different values of nonlinear damping: (1) negative nonlinear damping ($\eta = -0.25|S|$); (2) linear damping ($\eta = 0$); (3 and 4) positive nonlinear damping at $\eta/|S|$ equals 0.25 and 1.0, respectively

absorbed by the waves $W_+ = \omega_p \chi'' h^2/2$ and is connected with their total number N by the following formula:

$$\begin{aligned}\chi'' &= (2/h^2)\int \gamma(x)N(x)\,dx \\ &= \frac{2V^2}{|S|}\frac{\sqrt{p(c^2+1)-1}}{p(c^2+1)}\sqrt{p(c^2+1)^2 - [\sqrt{p(c^2+1)-1} - c]^2}\ .\end{aligned} \quad (6.2.8)$$

The dependences $\chi''(p)$ for different $c = \nu/|S|$ are plotted in Fig. 6.2. At $c > 0$ the finite slope $\chi''(p)$ appears under p=1, and the finite value of $\chi'(\infty) = V^2\eta/(\eta^2 + S^2)$. It is interesting to note that at $c < 1$ ($\eta < S$) the maximum value $\chi''_{\max} = V^2/|S|$, it is independent of η and coincides with the maximum value of χ'' calculated by the basic S-theory without the nonlinear damping (at c=0). As η increases the position of the maximum p is shifted to the greater p: $p_m = 2S^2/(|S|-\eta)^2$. At $\eta > |S|$ the susceptibility χ'' increases monotonically with the increase of p.

For $\eta < 0$, as already said, the hard excitation of the parametric waves takes place. Naturally, it is accompanied by the hysteresis of the dependence $\chi''(p)$. When p assumes the threshold value of $p = 1$ from the side of the smaller p the susceptibility abruptly changes at p=1 from zero to the following value

$$\chi_+ = -\frac{4V^2}{S}\frac{c\sqrt{(c^2+1)^2 - 4c^2}}{(c^2+1)^2}\ . \quad (6.2.9)$$

As the amplitude of the pumping decreases to some values $p_- \simeq 1/(c^2+1)$ (which is below the threshold p_+) than the reverse abrupt change of the

susceptibility takes place. In the case of $|c| < 1$ this reverse change is half as great as the direct change

$$\chi_- = \chi_+/2 = 2V^2\eta/S^2\ . \quad (6.2.10)$$

This hard excitation phenomenon and the hysteresis of χ'' was discovered by *Le Gall, Lemaire* and *Sere* in the YIG under parametric excitation of the magnons by parallel pumping at the frequency of 9.8 hHz [6.4].

In conclusion, we shall briefly discuss the region of applicability of the obtained results. Obviously, (6.2.3) for damping holds true if $\eta N \ll \gamma_0$. Under $0 < c \ll 1$ this condition is satisfied within the wide range of the supercriticality values $p \ll 1/\sqrt{c}$, and at $c \gg 1$ is satisfied only when the supercriticality values are only insignificantly above the threshold, i.e. at $p - 1 \ll 1$. In the case of negative nonlinear damping the applicability criterion of the results obtained is still more stringent. The condition $\eta N \ll \gamma_0$ brings about the requirement $|c| \ll 1$. In this case the above formulae hold true at $p \ll 1/|c|$.

The applicability of the simple S-theory with the negative nonlinear damping can be significantly extended if instead of (6.2.3a) with $\eta < 0$ the more realistic model dependence $\eta(N)$ is used:

$$\gamma(N) = \gamma_1 + \gamma_2^2/(\gamma_2 + \eta N)\ . \quad (6.2.11)$$

It holds true qualitatively for the common mechanisms of the negative nonlinear damping of magnons even when $\eta N > \gamma_2$ [6.5].

6.2.2 Influence of Non-Analyticity on Nonlinear Damping

The subsequent study of different mechanisms of nonlinear damping (Sect. 11.2) will show that in some cases the function $\eta(\boldsymbol{k}, \boldsymbol{k}')$ is non-analytical under $\boldsymbol{k} \to \boldsymbol{k}'$. In the region $|\boldsymbol{k} - \boldsymbol{k}'| = \kappa \ll k, k'$ in some approximation it resembles $1/\kappa$, i.e. it has an integrated singularity.

By way of example, let us consider the problem of the parametric excitation of the spin waves in ferromagnets (for more detail, see [6.6]). As shown in Sect. 5.5.2 in this case under η=0 (or when the dependence $\eta(\boldsymbol{k}, \boldsymbol{k}')$ is analytical) and when the supercriticalities are not too high, the waves are excited only on the equator of the resonant surface (i.e. $N(x) \propto \delta(x)$). Therefore it should be expected that at $\eta N \ll \gamma_0$ the distribution function $n(\boldsymbol{k}) = n(k, x, \kappa)$ will be non-zero only under small x and k, approaching the radius of the equator k_0. The calculations of the nonlinear damping performed in Sect. 11.2 showed that in this range, i.e. at $x_1, x_2 \ll 1$ and $|\boldsymbol{k}_{1,2} - \boldsymbol{k}_0| \ll k_0$ the function $\eta(\boldsymbol{k}_1, \boldsymbol{k}_2)$ can be represented in the following form

$$\eta(\boldsymbol{k}_1, \boldsymbol{k}_2) = \eta\{\sin^2(\frac{\psi_1 - \psi_2}{2}) + (\frac{\alpha_1 - \alpha_2}{4}) + (\frac{k_1 - k_2}{2k_1})^2\}^{-1/2}\ . \quad (6.2.12)$$

Here $\alpha = \pi/2 - \Theta$ and η is the analytical function of the magnetic field, k and other parameters of the problem. The entire non-analytical part of the function $\eta(\boldsymbol{k}_1, \boldsymbol{k}_2)$ is enclosed in braces.

In order to simplify the problem further note that the expected width of the packet $n(\boldsymbol{k})$ is significantly less in module k than in polar angle Θ. It can be assumed that $(k-k_0)[\partial\omega(\boldsymbol{k})/\gamma(\boldsymbol{k})\partial \boldsymbol{k}] \simeq \alpha$. Then the last term in braces in (6.2.12) can be neglected, since it is by a factor of $[k_0\gamma(\boldsymbol{k})]/[\partial\omega(\boldsymbol{k})/\partial\boldsymbol{k}]$ smaller than the last but one. Let us also take into account the axial symmetry of the problem and average (6.2.12) over the difference of the azimuthal $\varphi_1 - \varphi_2$. This yields

$$\gamma(\alpha) = \gamma_0 + \eta \int \ln|\alpha - \beta| N(\beta)\, d\beta\,. \tag{6.2.13}$$

In this formula the numerical factor of the integral of the order of unity has been dropped as unimportant. The integral equations of the basic S-theory (5.5.6) we shall represent as

$$[P(\alpha)\exp[i\Psi(\alpha)] - i\gamma(\alpha)]N = 0\,, \tag{6.2.14a}$$

$$P(\alpha) = hV(\alpha) + \int S(\alpha,\beta)N(\beta)\exp[-i\Psi(\beta)]\, d\beta\,. \tag{6.2.14b}$$

Our task therefore consists in the simultaneous solution of the integral equations (6.2.13, 14). We shall be interested mainly in the qualitatively new results due to the non-analyticity of the function of the nonlinear damping in (6.2.13), and therefore the problems associated with (6.2.14b) will be significantly simplified. To this end, let us choose a function $S(\alpha,\beta)$ in the factorized form

$$S(\alpha,\beta) = Sf(\alpha)f(\beta)\,, \quad f(\alpha) = V(\alpha)/V\,, \quad V = V(0)\,. \tag{6.2.15}$$

Then the dependence $P(\alpha)$ will easily be found

$$\begin{aligned} P(\alpha) &= Pf(\alpha)\,, \quad P = hV + S\Sigma\,, \\ \Sigma &= \int f(\alpha)N(\alpha)\exp[-i\Psi(\alpha)]\, d\alpha\,. \end{aligned} \tag{6.2.16}$$

This enables us to obtain from (6.2.13, 14a) a closed equation for $N(\alpha)$:

$$\gamma_0 + \eta \int \ln|\alpha-\beta| N(\beta)\, d\beta = |P| f(\alpha)\,. \tag{6.2.17}$$

From (6.2.14a) it also follows that

$$\Psi(\alpha) = \Psi\,, \qquad \mathrm{Re}\{P\exp\Psi\} = 0\,. \tag{6.2.18}$$

Let us first consider the case of **positive nonlinear damping**: $\eta > 0$. The general solution of this equation localized in some range $-a < \alpha < a$ and equal to zero when $|\alpha| > a$ has the following form

$$N(\alpha) = \frac{1}{\pi\eta}\int_{-a}^{a} \sqrt{\frac{a^2-t^2}{a^2-\alpha^2}}\frac{P'(t)dt}{t-\alpha} + \frac{A}{\sqrt{a^2-\alpha^2}} \tag{6.2.19}$$

where $P' = Pf'$ is the derivative of the pumping with respect to the angle, A denotes an arbitrary constant and the integral has the meaning of the Cauchy's principal value. As can easily be shown, the solution limited for the both end points of the interval exists only if the equality

$$\int_{-a}^{a} \frac{P'(t)\, dt}{\sqrt{a^2-t^2}} = 0$$

is satisfied, which is ensured by the evenness of the function $P(t)$. This solution has the following form:

$$N(\alpha) = \frac{1}{\pi\eta}\int_{-a}^{a} \sqrt{\frac{a^2-\alpha^2}{a^2-t^2}}\frac{P'(t)dt}{t-\alpha}\,. \tag{6.2.20}$$

It must be recalled that our consideration holds true when the excited packet is narrow ($a \ll 1$). Therefore the integral in (6.2.20) must be calculated expanding $P(t)$ under small t:

$$P(t) = P[1-(1/2)f''t^2]\,, \quad f'' = d^2f/d\alpha^2 \text{ at } \alpha = 0\,, \quad P'(t) = -Pf''t\,.$$

The form of the packet in this case is as follows:

$$N(\alpha) = (|P|f''/\pi\eta)\sqrt{a^2-\alpha^2}\,. \tag{6.2.21a}$$

This expression can be conveniently represented as

$$N(\alpha) = (2N/\pi a)\sqrt{1-(\alpha/a)^2}\,, \tag{6.2.21b}$$

$$N = \int N(\alpha)d\alpha = |P|f''a^2/2\eta \tag{6.2.21c}$$

where N is the total number of waves in the packet. Substituting (6.2.21b) in the initial equation (6.2.17) we obtain the integrated relation

$$|P| = \gamma_0 + \eta N\ln(2/a)\,. \tag{6.2.22}$$

From (6.2.16, 18) we can readily obtain in the approximation of narrow packets the usual relations of the basic S-theory

$$\Sigma = N\exp[-i\Psi]\,, \quad SN = -hV\cos\Phi\,, \quad |P|^2 = (hV)^2 - (SN)^2\,. \tag{6.2.23}$$

The relations (6.2.21c, 22, 23) close the problem of the self-consistency and make it possible to determine the dependences of $|P|$, N, Ψ and a on the supercriticality. This can be done as follows. At the first stage the weak (logarithmic) dependence of $|P|$ on a in (6.2.22) can be neglected if we assume

$$|P| = \gamma_0 + \tilde{\eta} N \ , \quad \tilde{\eta} = \eta \ln(2/a) = \text{const} \ . \qquad (6.2.24)$$

Substituting this expression into (6.2.23), we obtain the formulae of the simple theory developed in the previous section. For the supercriticality dependences N and χ'' this yields (6.2.5, 8), η being replaced by $\tilde{\eta}$. By substituting the dependence $N(p)$ into (6.2.21c), we can obtain the dependence of the packet width a on the supercriticality p.

Now let us consider the case of **negative nonlinear damping**. Under $\eta < 0$ equation (6.2.17) (corresponding to the mean-field approximation) admits only the solutions with the integrated singularity, i.e.

$$N(\alpha) = A/\sqrt{a^2 - \alpha^2} + Pf''\sqrt{a^2 - \alpha^2}/\pi\eta \ .$$

By substituting this solution into the equation of the energy balance we find the following connection between P, a and A:

$$-\frac{P^2 a^2}{\eta} \ln\left(\frac{2}{a}\right) - A \ln\left(\frac{2}{a}\right) = -\left(\frac{Pf''}{\eta}\right)\frac{P-\gamma}{P} \ . \qquad (6.2.25)$$

From the condition $\int N(\alpha)\, d\alpha = N$ we can obtain the second connection between these three parameters. Therefore we have a one-parameter set of solutions. Which of them is actually realized? Note that for each of these solutions the renormalized damping and pumping coincide for all α and not only for $-a \le \alpha \le a$. This implies that any of the solutions is indifferently stable with respect to the appearance of the waves outside the integral $[-a, a]$. Consequently, the solution with the maximum (under the given supercriticality) width of a will be realized. Allowing for the condition $N(\alpha) \ge 0$ we find that the maximum width corresponds to the choice $A = -2f''aP/\pi\eta$ and is equal to

$$a^2 = -\eta f'' N/2P \ , \qquad (6.2.26)$$

i.e. it coincides with the expression (2.2.21c) for the case $\eta > 0$ (it must only be assumed that $c < 0$ and the general sign must be changed); and the dependence of the total number of waves N on the dimensionless pumping power p is still given by (6.2.5) but now with $c < 0$.

In conclusion, compare the influences of the non-analyticity of the main functions of the S-theory $S(\boldsymbol{k}_1, \boldsymbol{k}_2)$ and $\eta(\boldsymbol{k}_1, \boldsymbol{k}_2)$ that determine the nonlinear renormalization of the self-consistent pumping $P[n(\boldsymbol{k}), \Psi(\boldsymbol{k})]$ and the self-consistent damping $\eta[n(\boldsymbol{k})]$. In both cases the non-analyticity results in the broadening of the parametric wave packet over the polar angle. However, the non-analyticity of the function S proved to be much weaker than the non-analyticity of the function η (under the axial symmetry S is characterized by a discontinuity of the derivatives while η has a logarithmic singularity). Therefore the broadening of the packet when S is non-analytic is substantially weaker: it occurs only when the supercriticality p is above the intermediate threshold $p(0)$. At the same time the width of the packet δ increases rather slowly, i.e. proportionally to $[p - p(0)]^n$, n=2 or 3/2. When η is non-analytic the packet broadening starts immediately above the threshold of the parametric excitation, i.e. under $p > 1$ and is faster, proportionally to $(p-1)$ under the positive nonlinear damping and is discontinuous when the nonlinear damping is negative. On the other hand, the non-analyticity of the two functions S and η is so weak that they practically do not influence the general characteristics of the system of parametric waves such as N, χ'', χ', etc. It must be added that the weak dependence of the general characteristics of nonlinear systems on the fine properties of the functions describing the interaction is not characteristic of the problem of parametric wave excitation, but is typical of the systems that can be described within approximations like the mean-field.

6.3 Parametric Excitation Under the Feedback Effect on Pumping

6.3.1 Hamiltonian of the Problem

For definiteness, we shall consider this problem for the case of the nonlinear theory of the ferromagnetic resonance [6.7]. This theory must describe the amplitude dependence of the uniform precession of the magnetization (UP) c_0 on the frequency ω_p and the amplitude $h_\perp$ of the external magnetic field $\boldsymbol{h}_\perp$ (oriented transverse to the magnetization $\boldsymbol{h}_\perp \perp \boldsymbol{M}$) when the amplitude $h_\perp$ is so large that the nonlinear effects must be taken into consideration. In addition to the nonlinear frequency shift of the UP proportional to $|c_0|^2$, the most important of these effects are the processes of parametric excitation of magnons by the uniform precession (see the description in Sect. 4.3.1, 4) and the feedback effect of the parametric magnons on the uniform precession. All the above processes are described by the Hamiltonian

$$\mathcal{H} = \omega_0 c_0 c_0^* + \mathcal{H}_{00} + \mathcal{H}_\perp + \sum_{\boldsymbol{k}} \mathcal{H}_{0\boldsymbol{k}} + \sum_{\boldsymbol{k}} \omega(\boldsymbol{k}) c(\boldsymbol{k}) c^*(\boldsymbol{k}) + \mathcal{H}_\text{S} \ , \qquad (6.3.1a)$$

$$\mathcal{H}_{00} = T_{00} c_0^2 c_0^* \ , \quad \mathcal{H}_\perp = [h_\perp \exp(-i\omega_\text{p} t) U c_0^* + \text{c.c.}] \ , \qquad (6.3.1b)$$

$$\begin{aligned} \mathcal{H}_{0\boldsymbol{k}} = {} & (1/2)[V^*(0,\boldsymbol{k},-\boldsymbol{k}) c^* c(\boldsymbol{k}) c^*(-\boldsymbol{k}) + \text{c.c.}] \ , \\ & + [S(0,\boldsymbol{k}) c_0^* c_0^* c(\boldsymbol{k}) c(-\boldsymbol{k}) + \text{c.c.}] + T(0,\boldsymbol{k}) c_0^* c_0 c^*(\boldsymbol{k}) c(\boldsymbol{k}) \ , \end{aligned} \qquad (6.3.1c)$$

$$\begin{aligned} \mathcal{H}_\text{S} = {} & \sum_{1,2} T(\boldsymbol{k}_1, \boldsymbol{k}_2) c^*(\boldsymbol{k}_1) c^*(\boldsymbol{k}_2) c(\boldsymbol{k}_2) \\ & + \frac{1}{2} \sum_{1,2} S(\boldsymbol{k}_1, \boldsymbol{k}_2) c^*(\boldsymbol{k}_1) c^*(-\boldsymbol{k}_1) c(\boldsymbol{k}_2) c(-\boldsymbol{k}_2) \ . \end{aligned} \qquad (6.3.1d)$$

Here $\mathcal{H}_{00}$ describes the nonlinear eigen shift of the UP frequency, $\mathcal{H}_{\perp}$ gives the UP interaction with the uniform magnetic UHF field with the frequency ω_p (for simplicity, we assume this field to be circularly polarized in the plane perpendicular to $\boldsymbol{M}$) and, finally, $\mathcal{H}_{0k}$ describes the interaction of the uniform precession with the magnons. The first two terms describe the parametric interaction of the first and second order respectively and the last term describes the nonlinear frequency shift. $\mathcal{H}_S$ is the diagonal in pairs Hamiltonian of the spin wave (magnon) interaction in the basic S-theory (5.4.17). The nonlinear theory of the ferromagnetic resonance based on the Hamiltonian (6.3.1) was first developed in 1971 by *Starobinets* and *L'vov* [6.7]. This theory was based on the earlier theory of the ferromagnetic resonance suggested by *Suhl* [6.8]. The Suhl theory, however, allowed only for the magnon-UP interaction and neglected the interaction between the magnons (magnon-magnon interaction), though the latter is considerable under large wave amplitudes and cannot be neglected. *Schlömann* [6.9] wrote: "This approximation (the exclusion of the magnon-magnon interaction) was intended for the mathematical simplification of the problem and, generally speaking, cannot be justified. Assuming this approximation we, probably, lose the major part of the important physical information".

From the theoretical viewpoint it is clear that as the number of the parametric magnons increases above the threshold the energy of their interaction $\mathcal{H}_S$ at first becomes equal and then exceeds the energy of the magnon interaction with UP. The power of the pumping under which the Hamiltonian $\mathcal{H}_S$ becomes important depends on the properties of the spin system and the type of the nonlinear processes and, as a rule, may be exceeded in an experiment. At the same time it was commonly accepted that the amplitude limitation at the parametric excitation is mostly due to the reverse effect of UP on magnons (described by the Hamiltonian $\mathcal{H}_{0k}$). This opinion is based on the "freezing" UP above the threshold which has been predicted by the "magnon-UP-theory" and was applied to the ferrite power limiters.

As shown by *Schlömann* et al. [6.10] the increase of the UP amplitude almost stops under the ferromagnetic resonance for the first-order processes. On the other hand, some facts cannot be explained only by the magnon-UP interaction. In particular, the actual behavior of the nonlinear susceptibilities χ', χ'' far from the resonance differs from the theoretically predicted one when the pumping power considerably exceeds the threshold value (*Damon* [6.11]). It must be specially emphasized that the real part of the susceptibility χ' is practically unchanged above the threshold whereas according to the theory of the magnon-UP interaction it must decrease as $1/h_{\perp}^2$ ($h_{\perp}$ is the amplitude of the microwave magnetic field). We must also mention the experiments performed by *Gurevich* and *Starobinets* [6.12] on "saturation" of the ferromagnetic resonance (in the case of second order processes) testifying to the "increased UP amplitude above the threshold".

The above-mentioned and some other facts are readily explained in the subsequently presented theory of *L'vov* and *Starobinets* [6.7] simultaneously taking into account the magnon-UP interaction H_{0k} as well as the magnon-magnon interaction in the approximation of the diagonal Hamiltonian $\mathcal{H}_S$ in the basic S-theory.

It must also be noted that the Hamiltonian (6.3.1) refers not only to the nonlinear theory of the ferromagnetic resonance. It can equally describe also other cases of parametric excitation of waves where the pumping field (or some other eigenmode of the medium oscillations acting as pumping) is linearly excited by an external inducing force. Thus, under parallel pumping of magnons in ferromagnets the variable c_0 is proportional to the amplitude of magnetic field $h_{\perp}$ in the resonator; ω_0 is the eigenfrequency of the resonator; T_{00}=0, because the eigen nonlinearity of the resonator usually is vanishingly small ($T_{00} \ll T(\boldsymbol{k}_1, \boldsymbol{k}_2)$); S_{0k}=0, if $\boldsymbol{h} \parallel \boldsymbol{M}$; $V(0, \boldsymbol{k}, -\boldsymbol{k}) \propto Q$, where Q is the filling factor of the resonator by the sample, the amplitude of magnetic field in the waveguide serves as h , the U-factor describes the coupling between the fields in the waveguide and the resonator.

6.3.2 General Analysis of the Equations of Motion

The equations of motion with the Hamiltonian (6.3.1)

$$\left[\frac{\partial}{\partial t} + \gamma_0\right] c_0 = -i\frac{\delta\mathcal{H}}{\delta c_0^*}, \quad \left[\frac{\partial}{\partial t} + \gamma(\boldsymbol{k})\right] c(\boldsymbol{k}) = -\frac{\delta\mathcal{H}}{\delta c^*(\boldsymbol{k})} \qquad (6.3.2a, b)$$

can be represented in the following form usual for the basic S-theory (5.4.16)

$$\left\{\frac{\partial}{\partial t} + \gamma_0 + i\left[\omega_{\rm NL}(0) - \omega_{\rm p}\right]\right\} c_0(t) + iP_0 c_0^*(t) = 0\ , \qquad (6.3.3a)$$

$$\left\{\frac{\partial}{\partial t} + \gamma(\boldsymbol{k}) + i\left[\omega_{\rm NL}(\boldsymbol{k}) - \frac{\ell\omega_{\rm p}}{2}\right]\right\} c(\boldsymbol{k}), t) + iP(\boldsymbol{k})c^*(-\boldsymbol{k}, t) = 0\ , \qquad (6.3.3b)$$

$$\omega_{\rm NL}(0) = \omega_0 + 2T_{00}|c_0|^2 + 2\sum_{\boldsymbol{k}} T(0, \boldsymbol{k})|c(\boldsymbol{k})|^2\ , \qquad (6.3.4a)$$

$$\omega_{\rm NL}(\boldsymbol{k}) = \omega(\boldsymbol{k}) + 2\sum_{1} T(\boldsymbol{k}, \boldsymbol{k}_1)|c(\boldsymbol{k}_1)|^2 + 2T(0, \boldsymbol{k})|c_0|^2\ . \qquad (6.3.4b)$$

Here c_0 is the amplitude of the spatially homogeneous pumping whose role in the problem of the nonlinear ferromagnetic resonance is played by the uniform precession of the magnetization, $\omega_{\rm NL}(0)$ and $\omega_{\rm NL}(\boldsymbol{k})$ describe the nonlinear frequency of the uniform precession and the parametric magnons, ℓ=1,2 respectively for the processes of the first and second order. The expressions for the values P_0, $P(\boldsymbol{k})$ denoting the "complex energy fluxes" into the pumping and the $\boldsymbol{k}$-pair of magnons depend on the type of the process under consideration. For the first-order processes

$$P_0 = h_\perp U + \frac{1}{2}\sum_{\boldsymbol{k}} V(0,\boldsymbol{k},-\boldsymbol{k})c(\boldsymbol{k})c(-\boldsymbol{k}) \,, \tag{6.3.5a}$$

$$P(\boldsymbol{k}) = V(0,\boldsymbol{k},-\boldsymbol{k})c_0 + \sum_{\boldsymbol{k}'} S(\boldsymbol{k},\boldsymbol{k}')c(\boldsymbol{k}')c(-\boldsymbol{k}') \,. \tag{6.3.5b}$$

The second-order instability threshold is much higher than the first-order one. However, if the frequency $(\omega_p/2)$ is outside the magnon spectrum, first-order processes are forbidden and second-order instability can be observed with $\omega(k) = \omega_p$. For it

$$\begin{aligned} P_0 &= \frac{hU}{c_0^*} + \sum_{\boldsymbol{k}} S(0,\boldsymbol{k})c(\boldsymbol{k})c(-\boldsymbol{k}) \,, \\ P(\boldsymbol{k}) &= S(0,\boldsymbol{k})c_0^2 + \sum_{\boldsymbol{k}'} S(\boldsymbol{k},\boldsymbol{k})c(\boldsymbol{k}')c(-\boldsymbol{k}') \,. \end{aligned} \tag{6.3.6}$$

Our task is now reduced to the study of solutions of the coupled equations (6.3.3-6) depending on the amplitude of the external field $h_\perp$. It is clear that the equations (6.3.3b) for parametric magnons assuming c_0=const. fully coincide with the equations describing the parametric instability described in detail in Chap. 5. By substituting the solution of these equations into (6.3.3a) we obtain one complex equation specifying the dependence $c_0(h_\perp)$. This is a rather complicated task requiring computer processing and actually to carry it out is worthwhile only for some important cases (e.g. for YIG monocrystals). This, however, has not yet been done. Here we shall only qualitatively analyze the behavior of the parametric magnons and the pumping above the instability threshold of the nonlinear pumping resonance. Taking into account that the total characteristics of the parametric wave system are not very sensitive to the particular forms of the functions $V(0,k,-k)$, $S(0,k)$ and $S(k_1,k_2)$ we shall assume that $V(0,k,-k)$ and $S(0,k)$=const. Then for the first-order processes from (5.5.7) we obtain

$$N_1 = \sqrt{V_{01}^2 n_0 - \gamma_1^2}/|S_{11}| \,, \quad \sin(\Psi_1 - \Psi_0/2) = \gamma_1/\sqrt{V_{01}^2 n_0} \,. \tag{6.3.7}$$

Similarly, for second-order processes we have

$$N_1 = \sqrt{S_{01}^2 n_{01}^2 - \gamma_1^2}/S_{11} \,, \quad \sin(\Psi_1 - \Psi_0) = \gamma_1/|S_{01}|n_0 \,. \tag{6.3.8}$$

Here N_1 is the total number of parametric waves, $V_{01} = V(0,k_1,-k_1)$, $S_{01} = S(0,k)$, S_{11} is the mean value of $S(k_1,k_2)$ over part of the resonant surface occupied by parametric waves.

6.3.3 First-Order Processes

The stationary solution of (6.3.2a, 7) for the uniform precession has the following form:

$$c\big[\omega_{NL}(0) - \omega_p + i\gamma_{NL}(0)\big] = hU \,, \tag{6.3.9}$$

where $\omega_{NL}(0)$ and $\gamma_{NL}(0)$ are the frequency and damping of the uniform precession renormalized due to the interaction with the parametric waves:

$$\begin{aligned} \omega_{NL}(0) &= \omega_0 + 2T_{00}|n_0|^2 + 2T_{01}N_1 - S_{11}N_1^2/2|n_0| \,, \\ \gamma_{NL}(0) &= \gamma_0 + \gamma_1 N_1/2n_0 \,, \quad n_0 = |c_0|^2 \,. \end{aligned} \tag{6.3.10}$$

Here T_{01} is the mean value of $T(0,\boldsymbol{k})$. Now we have the complete system of equations (5.5.7, 9) required for calculating the stationary state. Subsequently, these equations should be rendered dimensionless which will facilitate the estimation of the relative value of the different terms. Introducing to this end the following dimensionless values

$$\begin{aligned} &x_0 = \frac{n_0}{n_{th}} \,, \quad x_1 = \frac{N_1}{n_{th}} \,, \quad n_{th} = \frac{\Gamma_1}{V_{01}^2} \,, \quad p = \Big(\frac{h}{h_{th}}\Big)^2 \,, \\ &h_{th} = \frac{\gamma_0\gamma_1}{UV_{01}} \,, \quad \delta = \frac{(\omega_0 - \omega_p)}{\gamma_0} \,, \quad d = \frac{2\gamma_0 S_{11}}{V_{01}^2} \,, \\ &a = \frac{T_{00}\gamma_1^2}{S_{11}\gamma_0^2} \,, \quad b = \frac{2T_{01}\gamma_1}{S_{11}\gamma_0} \,, \end{aligned} \tag{6.3.11}$$

we reduce the initial equations to the form

$$\gamma_1|d|x_1 = 2\gamma_0\sqrt{x_0 - 1} \,, \quad (\delta_{NL}^2 + \gamma_{NL}^2)x_0 = p\gamma_0^2 \,, \tag{6.3.12}$$

$$\begin{aligned} \gamma_{NL}^2 &= \gamma_0^2[1 + \sqrt{x_0 - 1}/|d|x_0] \,, \\ \delta_{NL} &= \delta - (x_0 - 1)/dx_0 + b\sqrt{x_0 - 1} + dax_0 \,. \end{aligned} \tag{6.3.13}$$

These equations contain the small parameter $|d| \simeq \gamma/\omega_M \simeq 10^{-3} - 10^{-4}$, which in the theory not allowing for the interaction of parametric waves was assumed to be zero.

1 Behavior of Uniform Precession in Resonance. As follows from (6.3.12, 13), the amplitude x_0 is reaches its maximum under $\delta_{NL} = 0$ and is given by the following equation

$$\sqrt{x_0 - 1} + |d|x_0 = d\sqrt{px} \,. \tag{6.3.14}$$

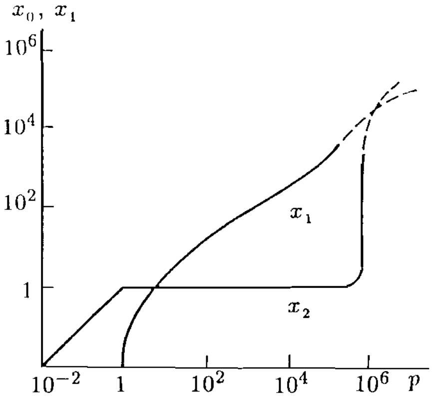
The smallness of d leads to the appearance of two characteristic regions of the solution of this equation: the region of small supercriticality $|d|\sqrt{p} \ll 1$, where

$$x = 1/(1 + d^2p) \,, \qquad x_1 = 2(\sqrt{p} - 1)\gamma_0/\gamma_1 \,, \tag{6.3.15}$$

and the region of large supercriticality $|d|\sqrt{p} \simeq 1$, where

$$x_0 = 1/(1 - d^2 p) , \qquad x_1 = 2\gamma_0(\sqrt{p} - 1)/\gamma_1 - d^2 p) . \tag{6.3.16}$$

These relations hold true as long as $x < |d|^{-2/3}$. Under other supercriticalities, as follows from (6.3.14), the amplitude of the uniform pumping must linearly increase $x_0 = p$ (Fig. 6.3), but in this case we enter the region $x_0 \simeq |d|^{-1}$ where the initial equations (6.3.2) are no longer valid.



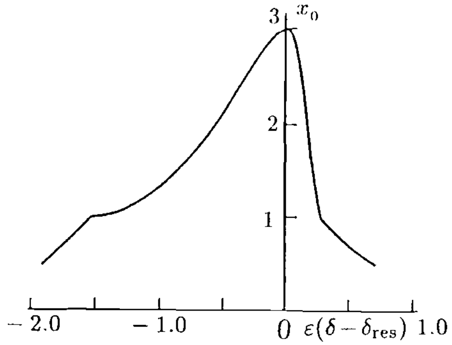


Fig. 6.3. (left) Theoretical dependences (6.3.16) of the square of the homogeneous precession amplitude $x_0 = |c_0|^2/n_{th}$ and number of magnons N_1/n_{th} on the dimensionless power of the pumping $p = (h/h_{th})^2$ for first-order processes. Dashed lines denote the region where theory is not valid

Fig. 6.4. (right) Theoretical form (6.3.18) of the nonlinear resonant curve (x_0 versus p) at large pumping power for first-order processes

2 Form of Resonant Curve. When the exact resonance is impossible or is absent, the threshold condition has the following form

$$p = p_{th} = (1 + \delta^2)$$

At $p \to p_{th} > |d|^2$ let us obtain from (6.3.12, 13) the following relation

$$x_0 = 1 + d^2(p - \delta^2)/(1 - \delta d)^2 , \tag{6.3.17}$$

which holds as long as $x_0 < 1/|d|^{2/3}$. The relation (6.3.17) shows that within the limit of large p the amplitude of the homogeneous pumping increases linearly

$$x_0 = \frac{|d|^2(1 + \delta^2)}{(1 - d\delta)^2} \frac{p}{p_{th}} .$$

Far from resonance, when $(1 - d\delta) \simeq 1$, the proportionality factor approaches unity, i.e. its order of magnitude is the same as in the linear mode. The form of the resonant curve of the nonlinear resonance is appreciated after performing an identical transformation of (6.3.17) yielding:

$$x_0 = \{1 - d^2[(dp - \delta)/(1 - d\delta)]^2\}/(1 - d^2\delta) . \tag{6.3.18}$$

It is readily seen that the resonant value x_0 is given by (6.3.16) and is attained at $\delta_{res} = dp$ which corresponds to the condition of the nonlinear resonance above the threshold. The resonant curve (6.3.16) shown in Fig. 6.4 is significantly different from the usual Lorentzian curve of the linear resonance. The half-width of the curves at a half-height (on the left and on the right of the resonance) equals

$$\delta_\pm = \delta_{res} \pm \delta_{1/2} = \pm[1 - d^2 p]/[d(\sqrt{2} \mp 1)] .$$

so that the ratio of $\delta_-/\delta_+ = 6$.

3 Nonlinear susceptibility. From (6.3.5b, 9) one can obtain

$$\chi = 2U c_0/h = 2U^2/[(\omega_{NL} - \omega_p) - i\gamma_{NL}] . \tag{6.3.19}$$

In the dimensionless form

$$\chi' = \chi_0 \frac{(\omega_{NL} - \omega_p)x_0}{\gamma p} , \quad \chi'' = \chi_0 \frac{\gamma_{NL} x_0}{\gamma p} , \tag{6.3.20}$$

where $\chi_0 = 2U^2/\gamma_0$ denotes the resonant susceptibility of the homogeneous pumping to the transverse SHF field in the linear mode. Now let us consider the behavior of the nonlinear susceptibilities far from the resonance ($\delta \gg 1$), i.e. in the region of the *additional absorption*. It follows from (6.3.17, 20) that:

$$\chi'' = \chi_0 \sqrt{p - \delta^2}/(p|1 - d\delta|) , \quad \text{at} \quad p - p_{th} > |d| . \tag{6.3.21}$$

Hence, in particular, it follows that above the threshold χ'' linearly increases and attains the maximum

$$\chi''_{max} = \chi_0/(2\delta|1 - d\delta|) \tag{6.3.22}$$

under $p = 2\delta \simeq 2p_{th}$. The decrease χ'' under large supercriticality according to (6.3.22) obeys the law $1/\sqrt{p}$. The real part of the nonlinear susceptibility changes within a much narrower range. From (6.3.16, 20) we obtain

$$\chi' = (\chi_0/\delta)[1 + (1 - \delta^2/p)/(d\delta - 1)] , \quad \text{at} \quad p - p_{th} \gg 1 . \tag{6.3.23}$$

In the theory not allowing for parametric wave interaction, $d = 0$ and χ' according to (6.3.23) falls off to zero as $1/p$. If this interaction is allowed for, this result changes drastically, i.e. under large supercriticality $\chi' \to$

$\chi_0 d/|1-d\delta|$, that is the order of magnitude of the susceptibility is the same as below the threshold. This accounts for the fact that the phase constants of the ferrite SHF equipment are practically independent of the power level [6.13].

6.3.4 Second-Order Processes

By using (6.3.2) we can obtain (6.3.9) for the amplitude of the homogeneous pumping. However, the expressions for $\omega_{\rm NL}$ and $\gamma_{\rm NL}$ for second-order processes will differ from the corresponding expressions (6.3.10) obtained for the first-order processes, i.e.

$$\omega_{\rm NL}=\omega_0+2T_{00}n_0+2T_{01}N_1-S_{11}N_1^2/n_0\,,\quad \gamma_{\rm NL}=\gamma_0+\gamma_1N_1/n_0\,. \qquad (6.3.24)$$

These relations together with (6.3.8, 9) set the complete system of equations for the definition of the stationary mode of second-order processes. It is convenient to represent it in the dimensionless form:

$$\begin{aligned} x_0^2 &= 1+\left(\frac{r^2\gamma_1^2x_1^2}{\gamma_0^2}\right)\left(\Omega_{\rm NL}^2+\gamma_{\rm NL}^2\right)x_0=\gamma_0^2p\,,\\ \gamma_{\rm NL} &= \gamma_0\left\{1+\sqrt{x_0^2-1}/rx_0\right\}\,,\\ \Omega_{\rm NL} &= \delta\gamma_0+\gamma_1\left[\frac{2x_0T_{00}}{S_{01}}+2\sqrt{x_0^2-1}\frac{T_{01}}{S_{11}}-\frac{(x_0^2-1)S_{11}}{S_{01}x_0}\right]\,. \end{aligned} \qquad (6.3.25)$$

Here x_0 and x_1 as before denote the dimensionless amplitudes $n/n_{\rm th}$ and $N/n_{\rm th}$, but now the threshold amplitude $n_{\rm th}=\gamma_1/|S_{01}|$ and the smallest parameter d is replaced by the coefficient $r=\gamma_0S_{11}/\gamma_1S_{01}\simeq 1$. The latter determines the drastic difference between first-order and second-order processes.

The dependence of the resonance amplitude x_0 on p is given by the following equation

$$rx_0+\sqrt{x_0^2-1}=r\sqrt{px_0}\,. \qquad (6.3.26)$$

If the interaction of parametric waves is not taken into account, then $r=0$ and $x_0^2=1$. Actually, no "freezing" of the amplitude of the homogeneous pumping takes place and even at small $(p-1)$ the amplitude x_0 significantly differs from unity:

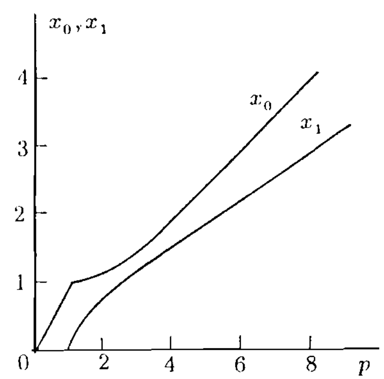
$$x_0^2=1+r^2(\sqrt{p}-1)/2 \quad \text{at} \quad \sqrt{p}-1<1\,. \qquad (6.3.27)$$

Hence it is clear that we can avoid taking into account the interaction of parametric waves only near the threshold itself when $(\sqrt{p}-1)\ll 1/r\simeq 1$. For the first-order process the corresponding condition is satisfied in a

much wider pumping range $(\sqrt{p}-1)\ll 1/d\simeq 10^3-10^4$. Under large supercriticalities $(p\gg 1)$

$$x_0=[r\sqrt{p}/(r+1)]^2+(r+1)/(r^2p)\,. \qquad (6.3.28)$$

The dependences $x_0(p)$, $x_1(p)$ are shown in Fig. 6.5. Figure 6.6 shows nonlinear resonant curves $x_0(\delta)$ computer processed according to (6.3.25) for various powers p of the external pumping field. Subsequently, in Sect. 9.2.4 we shall compare this theory with the experimental data of *Gurevich* and *Starobinets* [6.12] on the nonlinear ferromagnetic resonance.



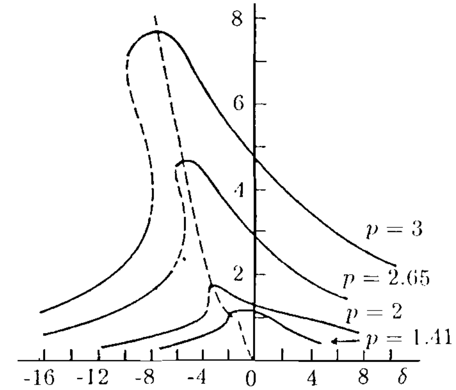


Fig. 6.5. (left) Solution of (6.3.25, 26) for the square of the homogeneous precession amplitude $x_0=|c_0|^2/n_{\rm th}$ and number of magnons $N_1/n_{\rm th}$ on the dimensionless power of the pumping $p=(h/h_{\rm th})^2$ for second-order processes

Fig. 6.6. (right) Nonlinear resonant curve at various pumping power $p=(h/h_{\rm th})^2$ for second-order processes which is the solution of (6.3.25)

6.4 Nonlinear Theory of Parametric Wave Excitation at Finite Temperatures

6.4.1 Different Time Correlators and Frequency Spectrum

It will be shown later that the interaction of the parametric wave system with the thermal bath leads to a non-trivial behavior of its non-simultaneous correlators $n(\boldsymbol{k},t,\tau)$ and $\sigma(\boldsymbol{k},t,\tau)$. The definition of these functions generalizes (5.4.2, 7) for the non-simultaneous (different time) correlators:

$$n(\boldsymbol{k},t)=n(\boldsymbol{k},t,0)\,,\qquad \sigma(\boldsymbol{k},t)=\sigma(\boldsymbol{k},t,0)\,. \qquad (6.4.1)$$

That is

$$n(\boldsymbol{k},t,\tau)\delta(\boldsymbol{k}-\boldsymbol{k}_1) = \langle b(\boldsymbol{k},t+\tau/2)b^*(\boldsymbol{k}_1,t-\tau/2)\rangle\,,$$
$$\sigma(\boldsymbol{k},t,\tau)\delta(\boldsymbol{k}+\boldsymbol{k}_1) = \langle b(\boldsymbol{k},t+\tau/2)b(\boldsymbol{k}_1,t-\tau/2)\rangle \exp(i\omega_{\rm p}t)\,. \tag{6.4.2}$$

In the present section we shall be interested only in the stationary state of the parametric wave system. Therefore the argument t of the correlators will be dropped, and we shall assume

$$n(\boldsymbol{k},t,\tau) = n(\boldsymbol{k},\tau)\,, \qquad \sigma(\boldsymbol{k},t,\tau) = \sigma(\boldsymbol{k},\tau)\,. \tag{6.4.3}$$

In order to avoid ambiguity the time argument difference in the non-simultaneous non-stationary correlators $n(\boldsymbol{k},t)$ and $\sigma(\boldsymbol{k},t)$ (6.4.1) is denoted by the Latin t and the time difference in the non-simultaneous stationary correlators $n(\boldsymbol{k},\tau)$ and $\sigma(\boldsymbol{k},\tau)$ (6.4.3) is designated by the Greek letter τ .

In theory, it is more convenient to deal with Fourier transforms of the non-simultaneous correlators

$$n(\boldsymbol{k},\omega) = \int n(\boldsymbol{k},\tau)\exp(i\omega\tau)\,d\tau\,,$$
$$\sigma(\boldsymbol{k},\omega) = \int (\boldsymbol{k},\tau)\exp(i\omega\tau)\,d\tau\,. \tag{6.4.4}$$

They are connected with the Fourier transforms of the canonical variables

$$b(\boldsymbol{k},\omega) = \int b(\boldsymbol{k},t)\exp(i\omega t)\,dt,\;\; b(\boldsymbol{k},\omega) = \int b(\boldsymbol{k},t)\exp(-i\omega t)\,dt\,. \tag{6.4.5}$$

in the following way

$$n(\boldsymbol{k},\omega)\delta(\boldsymbol{k}-\boldsymbol{k}_1)\delta(\omega-\omega_1) = \langle b(\boldsymbol{k},\omega)b^*(k_1,\omega_1)\rangle\,, \tag{6.4.6a}$$

$$\sigma(\boldsymbol{k},\omega)\delta(\boldsymbol{k}+\boldsymbol{k}_1)\delta(\omega+\omega_1-\omega_{\rm p}) = \langle b(\boldsymbol{k},\omega)b(k_1,\omega_1)\rangle\,. \tag{6.4.6b}$$

These relations can be verified by the direct substitution of the definitions (6.4.5) in (6.4.2). The relations (6.4.6) illustrate the physical meaning of the functions $n(\boldsymbol{k},\omega)$ and $\sigma(\boldsymbol{k},\omega)$ - these are the power spectra of the normal and abnormal correlators describing the spectral density of the wave energy at various frequencies.

6.4.2 Basic Equations of Temperature S-Theory

According to the results obtained in Sect.1.3.3 let us take into account the interaction of the system of parametric waves with the thermal bath, adding the Langevin random force $f(\boldsymbol{k},t)$ (1.3.6) into (5.2.2). This yields

$$[\partial/\partial t + \gamma(\boldsymbol{k}) + i\omega(\boldsymbol{k})]b(\boldsymbol{k},t) + ihV(\boldsymbol{k})b^*(-\boldsymbol{k},t)\exp(-i\omega_{\rm p}t)$$
$$= \frac{-i}{2}\sum_{k+1=2+3} T(\boldsymbol{k},\boldsymbol{1};\boldsymbol{2},\boldsymbol{3})b_1^*b_2b_3 + f(\boldsymbol{k},t)\,. \tag{6.4.7}$$

Here, as usual, $\boldsymbol{j} = \boldsymbol{k}_j$,$b_j = b(\boldsymbol{k}_j,t)$. In Fourier representation the equations assume the following form:

$$\{i[\omega(\boldsymbol{k})-\omega] + \gamma(\boldsymbol{k})\}b(\boldsymbol{k},\omega) + ihV(\boldsymbol{k})b^*(-\boldsymbol{k},\omega_{\rm p}-\omega)$$
$$= -\frac{i}{2}\sum_{k+1=2+3}\int [T(\boldsymbol{k},\boldsymbol{1};\boldsymbol{2},\boldsymbol{3})b_1^*b_2b_3$$
$$\times\,\delta(\omega+\omega_1-\omega_2-\omega_3)]\,d\omega_1\,d\omega_2\,d\omega_3 + f(\boldsymbol{k},\omega)\,. \tag{6.4.8}$$

Here and in the following $b_j = b(\boldsymbol{k}_j,\omega)$ which differs from the above assumed denotation $b_j = b(\boldsymbol{k}_j,t)$. This ambiguity will not lead to confusion since the context always shows the function of which additional argument (t or ω) is b_j.

Deriving the equations of the S-theory for the correlations $n(\boldsymbol{k},\omega)$ and $\sigma(\boldsymbol{k},\omega)$ we shall use almost the same procedure as in Sect. 5.4.2 when we derived the equations for the simultaneous correlators $n(\boldsymbol{k},t)$ and $\sigma(\boldsymbol{k},t)$. We shall first multiply (6.4.8) by $b^*(\boldsymbol{k}',\omega')$, then by $b(\boldsymbol{k}',\omega)$ and each time we shall average the obtained equations splitting the fourth-order correlators through the product of pair correlators. This yields the following equations:

$$\gamma(\boldsymbol{k})n(\boldsymbol{k},\omega) + {\rm Im}\{P(\boldsymbol{k}),\sigma^+(\boldsymbol{k},\omega)\} = {\rm Im}\{G(\boldsymbol{k},\omega)\}f^2(\boldsymbol{k},\omega),$$
$$\{-i[\omega_{\rm p} - \omega_{\rm NL}(\boldsymbol{k}) - \omega_{\rm NL}(-\boldsymbol{k})] + [\gamma(\boldsymbol{k})+\gamma(-\boldsymbol{k})]\}\sigma^+(\boldsymbol{k},\omega)$$
$$-\,iP^+(\boldsymbol{k})[n(\boldsymbol{k},\omega) + n(-\boldsymbol{k},\omega_{\rm p}-\omega)]$$
$$= [L^*(\boldsymbol{k},\omega) + L^*(-\boldsymbol{k},\omega_{\rm p}-\omega)]f^2(\boldsymbol{k},\omega)\,. \tag{6.4.9}$$

Here, as in the basic S-theory, $\omega_{\rm NL}(\boldsymbol{k})$ and $P(\boldsymbol{k})$ are the frequency and pumping renormalized by the interaction (in the first-order theory of perturbation with respect to the interaction Hamiltonian $\mathcal{H}_4$):

$$\omega_{\rm NL}(\boldsymbol{k}) = \omega(\boldsymbol{k}) + 2\int T(\boldsymbol{k},\boldsymbol{k}_1)n(\boldsymbol{k}_1,\omega_1)\,d\boldsymbol{k}_1 d\omega_1/2\pi\,,$$
$$P(\boldsymbol{k}) = hV(\boldsymbol{k}) + \int S(\boldsymbol{k},\boldsymbol{k}_1)\sigma(\boldsymbol{k}_1,\omega_1)\,d\boldsymbol{k}_1 d\omega_1/2\pi\,. \tag{6.4.10}$$

Two new very important functions $G(\boldsymbol{k},\omega)$ and $L(\boldsymbol{k},\omega)$ emerged in (6.4.9). They are called the *normal* and *abnormal Green's functions* and they are defined by the following equations:

$$G(\boldsymbol{k},\omega)\delta(\boldsymbol{k}-\boldsymbol{k}_1)\delta(\omega-\omega_1) = \langle b(\boldsymbol{k},\omega)f^*(\boldsymbol{k}_1,\omega_1)\rangle/f^2(\boldsymbol{k},\omega)\,,$$
$$L(\boldsymbol{k},\omega)\delta(\boldsymbol{k}+\boldsymbol{k}_1)\delta(\omega+\omega_1-\omega_{\rm p}) = \langle b(\boldsymbol{k},\omega)f(\boldsymbol{k}_1,\omega_1)\rangle/f^2(\boldsymbol{k},\omega)\,. \tag{6.4.11}$$

Here $f^2(\boldsymbol{k},\omega)$ is the Fourier transform of the non-simultaneous correlator of the Langevin random force. According to (1.3.6) it is independent of frequency and equals to

$$f^2(\boldsymbol{k},\omega) = f^2(\boldsymbol{k}) = 2\gamma(\boldsymbol{k})n_0(\boldsymbol{k})\,. \tag{6.4.12}$$

In (6.4.9) and henceforth the following designations are used:

$$G^+(\boldsymbol{k},\omega) = G^*(-\boldsymbol{k},\omega_p - \omega)\,, \quad L^+(\boldsymbol{k},\omega) = L(-\boldsymbol{k}\omega_p - \omega)\,, \ldots \tag{6.4.13}$$

Here the symbol + denotes the complex conjugation and the substitution $\boldsymbol{k} \to -\boldsymbol{k}\,, \omega \to \omega_p - \omega$.

The equations for the Green's functions G and L can be derived from (6.4.8) in the same way as (6.4.9) for the correlators n and σ. Namely, (6.4.8) must be multiplied by $f^*(\boldsymbol{k}',\omega')$ and $f(\boldsymbol{k}',\omega')$ and the averaging must be performed by splitting the fourth-order correlators into various products of the pair ones. This yields:

$$\begin{aligned}[\omega - \omega_{NL}(\boldsymbol{k}) + i\gamma(\boldsymbol{k})]G(\boldsymbol{k},\omega) - P(\boldsymbol{k})L^+(\boldsymbol{k},\omega) = 1\,,\\ -P^+(\boldsymbol{k})G(\boldsymbol{k},\omega) + [\omega_p - \omega - \omega_{NL}(\boldsymbol{k}) - i\gamma(\boldsymbol{k})]L^+(\boldsymbol{k},\omega) = 0\,.\end{aligned} \tag{6.4.14}$$

These equations can be solved for the Green's functions

$$G(\boldsymbol{k},\omega) = \frac{\omega_p - \omega - \omega_{NL}(\boldsymbol{k}) - i\gamma(\boldsymbol{k})}{\Delta(\boldsymbol{k},\omega)}\,, \quad L(\boldsymbol{k},\omega) = \frac{P(\boldsymbol{k})}{\Delta(\boldsymbol{k},\omega)}\,, \tag{6.4.15}$$

$$\Delta(\boldsymbol{k},\omega) = [\omega - \omega_{NL}(\boldsymbol{k}) + i\gamma(\boldsymbol{k})][\omega_p - \omega - \omega_{NL}(\boldsymbol{k}) - i\gamma(\boldsymbol{k})] - |P(\boldsymbol{k})|^2.$$

The solution of (6.4.9) for the pair correlators (allowing for (6.4.12, 15)) can be represented as

$$\begin{aligned}n(\boldsymbol{k},\omega) &= n_0(\boldsymbol{k})\frac{2\gamma(\boldsymbol{k})\{[\omega_p - \omega - \omega_{NL}(\boldsymbol{k})]^2 + \gamma^2(\boldsymbol{k}) + |P(\boldsymbol{k})|^2\}}{|\Delta(\boldsymbol{k},\omega)|^2}\,,\\ \sigma(\boldsymbol{k},\omega) &= n_0(\boldsymbol{k})\frac{2P(\boldsymbol{k})\gamma(\boldsymbol{k})[\omega_p - 2\omega_{NL}(\boldsymbol{k}) + 2i\gamma(\boldsymbol{k})]}{|\Delta(\boldsymbol{k},\omega)|^2}\,.\end{aligned} \tag{6.4.16}$$

These equations together with (6.4.10) expressing $P(\boldsymbol{k})$ and $\omega_{NL}(\boldsymbol{k})$ in terms of $n(\boldsymbol{k},\omega)$ and $\sigma(\boldsymbol{k},\omega)$ are the basic equations for the so called *temperature S-theory* which takes into account the finite temperature of thermal bath.

6.4.3 Separation of Waves into Parametric and Thermal

Equations (6.4.16) describe the waves over the entire $\boldsymbol{k},\omega$-space. In the absence of the pumping they describe the state of the thermodynamic equilibrium at the temperature T. Indeed, under $hV = 0$ it follows from (6.4.16) that $\sigma(\boldsymbol{k},\omega) = 0$ and $n(\boldsymbol{k},\omega) = n_0(\boldsymbol{k},\omega)$, where

$$n_0(\boldsymbol{k},\omega) = \frac{2\gamma(\boldsymbol{k})n_0(\boldsymbol{k})}{[\omega(\boldsymbol{k}) - \omega]^2 + \gamma^2(\boldsymbol{k})}\,. \tag{6.4.17}$$

Here $n_0(\boldsymbol{k}) = T/\omega(\boldsymbol{k})$ are the Raleigh-Jeans distributions. This formula describes the occupation numbers of the *thermal waves*, i.e. waves excited by the thermal bath with the temperature T. Obviously,

$$\int n(\boldsymbol{k},\omega)\,d\omega/2\pi = n_0(\boldsymbol{k})\,. \tag{6.4.18}$$

The pumping changes the occupation numbers $n(\boldsymbol{k},\omega)$ not only in the region of parametric resonance but also over the entire $\boldsymbol{k}$-space. Therefore the following question arises: What waves must be considered parametrically excited and which waves must be as before referred to as thermal waves? The qualitative answer is: Parametric waves are the pairs whose phases are correlated with the pumping phase, other waves could be called thermal if the $n(\boldsymbol{k})$ for them does not differ greatly from the equilibrium level. The formula for the numbers of parametric waves n_p can be written as follows

$$n_p(\boldsymbol{k},\omega) = n(\boldsymbol{k},\omega) - n_T(\boldsymbol{k},\omega)\,, \; n_T(\boldsymbol{k},\omega) = \frac{2\gamma(\boldsymbol{k})n_0(\boldsymbol{k})}{[\omega(\boldsymbol{k}) - \omega]^2 + \gamma^2(\boldsymbol{k})}. \tag{6.4.19}$$

Here $n_T(\boldsymbol{k},\omega)$ is defined by analogy with (6.4.17). But unlike in (6.4.19) $n_T(\boldsymbol{k},\omega)$ it is defined not by means of the thermodynamic equilibrium spectrum $\omega(\boldsymbol{k})$, but using the real wave frequency $\omega_{NL}(\boldsymbol{k})$, calculated in the presence of pumping. In such a case $n(\boldsymbol{k},\omega)$ asymptotically tends to $n_T(\boldsymbol{k},\omega)$ as it recedes from the resonant surface.

6.4.4 Two-Dimensional Reduction of Basic Equations

The obtained equations (6.4.16) may seem rather complicated. This is a system of essentially nonlinear integral equations in the four-dimensional $\boldsymbol{k},\omega$-space. However, because the packets $n_p(\boldsymbol{k},\omega)$ and $\sigma(\boldsymbol{k},\omega)$ are narrow with respect to $\omega(\boldsymbol{k})$ and ω, (6.4.16) can be reduced to the two-dimensional integral equations over the resonant surface and effectively analyzed afterwards. Indeed, the main dependence of $n_p(\boldsymbol{k},\omega)$ and $\sigma(\boldsymbol{k},\omega)$ on ω and $\omega(\boldsymbol{k})$ in (6.4.16) is explicitly specified, therefore the dependences $\gamma(\boldsymbol{k})$, $P(\boldsymbol{k})$ and $n_0(\boldsymbol{k})$ on the module of k may be neglected replacing those functions by their values on the resonant surface $\gamma(\Omega)$, $P(\Omega)$ and $n_0(\Omega)$. This enables one explicitly to integrate (6.4.16) with respect to ω and $(\omega\boldsymbol{k})$ and to obtain the closed two-dimensional equations

$$\begin{aligned}n_p(\Omega) &= \pi k^2(\Omega)n_0(\Omega)|P(\Omega)|^2/v(\Omega)\nu(\Omega)\,,\\ \sigma(\Omega) &= i\pi k^2(\Omega)n_0(\Omega)\gamma(\Omega)P(\Omega)/v(\Omega)\nu(\Omega)\,,\\ \nu^2(\Omega) &= \gamma^2(\Omega) - |P(\Omega)|^2\end{aligned} \tag{6.4.20}$$

for the values $n_p(\Omega)$ and $\sigma(\Omega)$, integrated in ω and k and depending only on the angular coordinates $\Omega = \Theta, \varphi$ on the resonant surface:

$$\begin{aligned}n_p(\Omega) &= [k^2(\Omega)/2\pi v(\Omega)]\int n_p(\boldsymbol{k},\omega)\,d\omega(\boldsymbol{k})\,d\omega/2\pi\,,\\ \sigma(\Omega) &= [k^2(\Omega)/2\pi v(\Omega)]\int \sigma(\boldsymbol{k},\omega)\,d\omega(\boldsymbol{k})\,d\omega/2\pi\,.\end{aligned} \tag{6.4.21}$$

Here $k(\Omega)$ and $v(\Omega)$ are the wave vector and the group velocity of the waves at the point of resonant surface with the angular coordinate Ω . Substituting $\sigma(\Omega)$ from (6.4.21) into the usual expression of the basic S-theory for the self-consistent pumping $P(\Omega)$ we shall obtain the nonlinear integral equation for $P(\Omega)$ on the resonant surface:

$$P(\Omega) = hV(\Omega) + i\pi n_0(\Omega) \int \frac{S(\Omega,\Omega')\gamma(\Omega')P(\Omega')k^2(\Omega')\,d\Omega'}{\sqrt{[\gamma^2(\Omega') - |P(\Omega')|^2]}v(\Omega')} . \quad (6.4.22)$$

On solving this equation we can determine from (6.4.20) the integrated characteristics of the parametric waves $n_p(\Omega)$ and $\sigma(\Omega)$ on the resonant surface, and then using (6.4.16) we can study the distribution structure of $n_p(\boldsymbol{k},\omega)$ and $\sigma(\boldsymbol{k},\omega)$ with respect to ω and $\omega_{NL}(\boldsymbol{k})$ near it.

6.4.5 Distribution of Parametric Waves in k

On integrating (6.4.16) only with respect to ω and taking into account (6.4.19) we obtain

$$\begin{aligned} n(\boldsymbol{k}) &= n_0(\boldsymbol{k})\frac{[\omega_{NL}(\boldsymbol{k}) - \omega_p/2]^2 + \gamma^2(\Omega)}{\Delta(\boldsymbol{k})}, \quad n_p(\boldsymbol{k}) = n_0(\boldsymbol{k})\frac{|P(\Omega)|^2}{\Delta(\boldsymbol{k})}, \\ \sigma(\boldsymbol{k}) &= n_0(\boldsymbol{k})\frac{P(\Omega)[\omega_{NL}(\boldsymbol{k}) - \frac{\omega_p}{2} + i\gamma(\Omega)]}{\Delta(\boldsymbol{k})}, \\ \Delta(\boldsymbol{k}) &= [\omega_{NL}(\boldsymbol{k}) - \frac{\omega_p}{2}]^2 + \nu^2(\Omega) . \end{aligned} \quad (6.4.23)$$

It can be seen from (6.4.23) that the distribution of the $n_p(\boldsymbol{k})$ and $\sigma(\boldsymbol{k})$ in their eigen frequencies $\omega_{NL}(\boldsymbol{k})$ (or in module k) has the form of the Lorentzian function with the maximum on the resonant surface and the halfwidth $\nu(\Omega)$. This value will be calculated a bit later (see (6.4.30, 31, 33)). The equations (6.4.23) can be identically transformed to the following form:

$$\begin{aligned} \gamma(\Omega)n(\boldsymbol{k}) + \mathrm{Im}\{P^*(\Omega)\sigma(\boldsymbol{k})\} &= \gamma(\Omega)n_0(\boldsymbol{k}), \\ \{\gamma(\Omega) + i[\omega_{NL}(\boldsymbol{k}) - \omega_p/2]\}\sigma(\boldsymbol{k}) + iP(\omega)n(\boldsymbol{k}) &= 0 . \end{aligned} \quad (6.4.24)$$

These equations were first intuitively written and treated in detail by *Zakharov* and myself in 1971 [6.14]. The equations (6.4.24) differ from the basic equations of the S-theory (5.4.11) only in the inhomogeneous term $\gamma(\Omega)n_0(\boldsymbol{k})$ describing the influence of the thermal bath with the non-zero temperature on the system of parametric waves.

6.4.6 Spectrum of Parametric Waves

The function of parametric wave distribution in actual frequencies, i.e. the quantity

$$n_p(\Omega,\omega) = [k(\Omega)/v(\Omega)] \int n(\boldsymbol{k},\omega)d\omega(\boldsymbol{k}) \quad (6.4.25)$$

can also be readily obtained. To this end, (6.4.16a) must be integrated with respect to $\omega_{NL}(\boldsymbol{k})$ neglecting the dependence of the function $\gamma(\boldsymbol{k})$,$P(\boldsymbol{k})$ and $n_0(\boldsymbol{k})$ on the modulus k and substituting in (6.4.16a) their values on the resonant surface. This yields:

$$\begin{aligned} n(\Omega,\omega) &= \sqrt{2}\gamma^3(\Omega)n_0(\Omega)k^2(\Omega)/\delta^3(\Omega,\omega), \\ \delta^6(\Omega,\omega) &= \left\{\nu^2(\Omega) + \sqrt{\nu^4(\Omega) + 2\gamma^2(\Omega)(\omega - \omega_p/2)^2}\right\} \\ &\quad \times \left\{\nu^4(\Omega) + 4\gamma^2(\Omega)(\omega - \omega_p/2)^2\right\} . \end{aligned} \quad (6.4.26)$$

The width of this function in ω is $\nu^2(\Omega)/2\gamma(\Omega)$. The line shape differs from the Lorentzian shape which has wider wings. The dependence of the function $\nu(\Omega)$ on Ω and the supercriticality p will be obtained in Sect. 6.4.8 (see (6.4.30, 31 , 33)).

6.4.7 Heating Below Threshold

Below the threshold of the parametric instability (i.e. under $hV(\Omega) < \gamma(\Omega)$) the wave interaction can be neglected and in all formulae we can assume $P(\Omega) = hV(\Omega)$. Then formulae (6.4.20a, c) will describe the heating below the threshold and the phase correlation of waves by the pumping

$$n_p(\Omega) = \frac{\pi k^2(\Omega)n_0(\Omega)|hV(\Omega)|^2}{v(\Omega)\sqrt{\gamma^2(\Omega) - |hV(\Omega)|^2}}, \quad \sigma(\Omega) = in_p(\Omega)\frac{hV(\Omega)}{\gamma(\Omega)}. \quad (6.4.27)$$

Clearly, below the threshold the phase correlation is not complete (i.e. $|\sigma| < n$), the number of parametric waves n at $hV \ll \sigma$ increases proportionally to the pumping power. Then (approaching the threshold) the correlation becomes complete ($|\sigma| \to n_p$), n_p tends to infinity and the wave interaction resulting in the limitation of their amplitude must be taken into account.

6.4.8 Influence of Thermal Bath on Total Characteristics

This problem can be best studied for the case of the spherical symmetry when (6.4.22) becomes algebraic:

$$P(1 - ir\gamma/\nu) = hV, \quad \nu^2 = \gamma^2 - |P|^2 . \quad (6.4.28a,b)$$

Here r is the small parameter characterizing the influence of the thermal bath:

$$r = SN_{\mathrm{T}}/kv\,, \quad N_{\mathrm{T}} = 4\pi^2 n_0(\boldsymbol{k})k^3\,. \tag{6.4.28c}$$

In this formula N_{T} is approximately equal to the number of thermal waves inside the resonant surface. In typical experiments on the ferromagnetic YIG $r = 10^{-2} - 10^{-3}$. The parameter r for the Heisenberg ferromagnet can be theoretically estimated as

$$r \simeq (ak)^2 T/T_{\mathrm{c}}\,, \tag{6.4.29}$$

where T denotes the thermal bath temperature, T_{c} is the Curie temperature, a designates the lattice constant and k is the radius of the resonant surface.

The solution of (6.4.27) $P(p)$ is substituted in (6.4.26a) and for $p = (hV/\gamma) > 1$ we obtain

$$SN = \sqrt{p-1}\left[1 + \frac{r^2}{2(p-1)}\right], \quad \nu = \frac{r\gamma}{\sqrt{p-1}} \quad \text{at } \frac{r^2}{(p-1)} \ll 1\,. \tag{6.4.30}$$

It is clear that within the applicability of these formulae the dependence of $N(p)$ does not differ much from the dependence predicted by the basic S-theory. But the basic difference consists in the appearance of the finite width of the packet $n(\boldsymbol{k},\omega)$ in the eigenfrequency $\omega_{\mathrm{NL}}(\boldsymbol{k})$ ($\delta\omega_{\mathrm{NL}}(\boldsymbol{k}) = \nu$) and in the actual frequency ($\delta\omega = \nu^2/2\gamma$). The width ν below the threshold increases as $\gamma\sqrt{p-1}$ and then decreases according to (6.4.30) as $1/\sqrt{p-1}$. The maximum value $\nu_{\max} = \sqrt{r}$ is attained on the threshold (at $p = 1$).

The situation is quite different when the number of parametric waves is limited by the nonlinear damping. In this case $P = hV = \gamma + \eta N$. On substituting this relation in (6.4.20a) we obtain

$$\nu = r\gamma p\eta/S[\sqrt{p} - 1]\,. \tag{6.4.31}$$

Hence

$$\delta\omega(\boldsymbol{k}) = \frac{r\gamma p\eta}{S[\sqrt{p}-1]} \qquad \delta\omega = \frac{r^2\gamma p^{3/2}\eta^2}{S^2[\sqrt{p}-1]^2}\,. \tag{6.4.32}$$

We must, however, draw attention to the limited applicability of these expressions because in their derivation the nonlinear behavior of the heated waves far from the resonant surface was not allowed for.

In conclusion, the results will be presented for the case of axial symmetry when, according to the basic S-theory, only one group of equivalent pairs (on the equator of the resonant surface) is excited. In this case above the threshold [6.14]:

$$SN = \gamma\sqrt{p-1}\left[1 + p^2 \exp\left(-\frac{\sqrt{p-1}}{r}\right)\right], \quad \nu^2(x) = \nu_0^2 + c\gamma^2 x^2\,,$$

$$\nu_0 = \gamma \exp\left(-\frac{\sqrt{p-1}}{r}\right) \qquad x = \cos\Theta\,, \quad c \simeq 1 \tag{6.4.33}$$

at $p < \exp(\sqrt{p-1}/2r)$. Clearly, in this case the influence of the thermal bath is exponentially small.

Finally, a general remark. In the basic S-theory the solution of the stationary equations was highly ambiguous. This ambiguity was eliminated by the condition of the external stability, from which it followed, in particular, that parametric waves are excited on the equator of the resonant surface $\omega_{\mathrm{NL}}(\boldsymbol{k}) = \omega_{\mathrm{p}}/2$. This ambiguity can be removed by taking into account the wave interaction with the thermal bath: the solution of (6.4.16) is unique (and, naturally, is concentrated over the resonant surface). As a result of the thermal bath influence all the possible instabilities obtain the "initial impact" and develop removing the ambiguity of parametric waves' state.

6.5 Introduction to Spatially Inhomogeneous S-Theory

In describing the nonlinear behavior of parametric waves the statistical properties of the wave field have been assumed to be spatially homogeneous. In terms of the correlation functions this implies that

$$\langle c(\boldsymbol{k})c^*(\boldsymbol{k}')\rangle \propto \delta(\boldsymbol{k}-\boldsymbol{k}')\,, \quad \langle c(\boldsymbol{k})c(-\boldsymbol{k}')\rangle \propto \delta(\boldsymbol{k}-\boldsymbol{k}')\,.$$

If we abandon this assumption, and take the space inhomogeneity to be smooth compared with the wavelength, then the values

$$n(\boldsymbol{k},\boldsymbol{k}') = \langle c(\boldsymbol{k})c^*(\boldsymbol{k}')\rangle\,, \quad \sigma(\boldsymbol{k},\boldsymbol{k}') = \langle c(\boldsymbol{k})c(-\boldsymbol{k}')\rangle\,. \tag{6.5.1}$$

will no longer be proportional to $\delta(\boldsymbol{k}-\boldsymbol{k}')$: with respect to $(\boldsymbol{k}-\boldsymbol{k}')$ they will be concentrated in a narrow layer with the width of $1/L$, where L is the characteristic size of the inhomogeneity. In the present section the S-theory equations for $n(\boldsymbol{k},\boldsymbol{k}',t)$ and $\sigma(\boldsymbol{k},\boldsymbol{k}',t)$ will be obtained and analyzed for the case of the smooth inhomogeneity.

6.5.1 Basic Equations

In order to obtain the equations of the S-theory for the correlators (6.5.1), let us obtain the derivatives

$$\frac{\partial n(\boldsymbol{k},\boldsymbol{k}',t)}{\partial t} = \left\langle c(\boldsymbol{k},t)\frac{\partial c^*(\boldsymbol{k}',t)}{\partial t}\right\rangle + \left\langle \frac{\partial c(\boldsymbol{k},t)}{\partial t}c^*(\boldsymbol{k}')\right\rangle\,. \tag{6.5.2}$$

As in the derivation of the basic equations of the spatially homogeneous S-theory (5.4.11) the dynamical equations of motion (5.2) will be used and we shall "split" the fourth order correlators into the paired correlators according to the rule generalizing (5.4.10):

$$
\begin{aligned}
&\langle c_1^* c_2^* c_3 c_4 \rangle = n_{31} n_{42} + n_{41} n_{32} + \sigma_{12}^* \sigma_{34}\,, \\
&\langle c_1^* c_2 c_3 c_4 \rangle = n_{21} \sigma_{3,-4} + n_{31} \sigma_{2,-4} + n_{41} \sigma_{2,-3}\,, \\
&\langle c_1^* c_2^* c_3^* c_4 \rangle = n_{41} \sigma_{2,-3} + n_{42} \sigma_{1,-3}^* + n_{43} \sigma_{1,-2}^*\,, \\
&n_{12} = n(\boldsymbol{k}_1, \boldsymbol{k}_2)\,, \quad \sigma_{12} = \sigma(\boldsymbol{k}_1, \boldsymbol{k}_2)\,.
\end{aligned} \tag{6.5.3}
$$

As a result, it follows from (5.2.2, 3) and (6.5.2)

$$
\begin{aligned}
&\left\{\frac{\partial}{\partial t} + \gamma(\boldsymbol{k}) + \gamma(\boldsymbol{k}') + i[\omega(\boldsymbol{k}) - \omega(\boldsymbol{k}')]\right\} n(\boldsymbol{k}, \boldsymbol{k}', t) \\
&= -i \sum_{1,2,3} \{T_{\boldsymbol{k}1,23}(n_{2\boldsymbol{k}'} n_{31} + n_{3\boldsymbol{k}'} n_{21} + \sigma_{1,-\boldsymbol{k}'}^* \sigma_{2,-3}) \delta(\boldsymbol{k} + \mathbf{1} - \mathbf{2} - \mathbf{3}) \\
&\quad - T_{23,\boldsymbol{k}'1}(n_{\boldsymbol{k}2} n_{13} + n_{\boldsymbol{k}3} n_{12} + \sigma_{\boldsymbol{k},-1}^* \sigma_{2,-3}) \delta(\boldsymbol{k}' + \mathbf{1} - \mathbf{2} - \mathbf{3})\}\,, \\
&\left\{\frac{\partial}{\partial t} + \gamma(\boldsymbol{k}) + \gamma(\boldsymbol{k}') + i[\omega(\boldsymbol{k}) + \omega(\boldsymbol{k}')] - i\omega_{\mathrm{p}}\right\} \sigma(\boldsymbol{k}, \boldsymbol{k}', t) \\
&= -i \sum_{1,2,3} \{T_{\boldsymbol{k}1,23}(n_{\boldsymbol{k}'1} \sigma_{23} + n_{21} \sigma_{3\boldsymbol{k}'} + n_{2,3} \sigma_{2,\boldsymbol{k}'}) \delta(\boldsymbol{k} + \mathbf{1} - \mathbf{2} - \mathbf{3}) \\
&\quad - T_{\boldsymbol{k}'1,23}(n_{\boldsymbol{k}1} \sigma_{23} + n_{\boldsymbol{k}1} \sigma_{\boldsymbol{k}3} + n_{3,1} \sigma_{\boldsymbol{k},2}) \delta(-\boldsymbol{k}' + \mathbf{1} - \mathbf{2} - \mathbf{3})\}\,.
\end{aligned} \tag{6.5.4}
$$

Subsequently, for simplicity we shall assume the spatial homogeneity to be unidimensional (along the z-axis). Then

$$
\begin{aligned}
n(\boldsymbol{k}, \boldsymbol{k}', t) &= n(\boldsymbol{k}_\perp, k_z, k_z', t) \delta(\boldsymbol{k}_\perp - \boldsymbol{k}_\perp'), \\
\sigma(\boldsymbol{k}, \boldsymbol{k}', t) &= \sigma(\boldsymbol{k}_\perp, k_z, k_z', t) \delta(\boldsymbol{k}_\perp - k_\perp').
\end{aligned} \tag{6.5.5}
$$

Taking into account that under the fixed $\boldsymbol{k}_\perp$ the packets $n(\boldsymbol{k}_\perp, \boldsymbol{k}_z, \boldsymbol{k}_z', t)$ and $\sigma(\boldsymbol{k}_\perp, \boldsymbol{k}_z, \boldsymbol{k}_z', t)$ are concentrated in the narrow layer $\delta k \ll k$, it is possible to expand $\omega(\boldsymbol{k}_\perp, \boldsymbol{k}_z)$ in (6.5.4) into the series of $k_z - k_z^0$ (k_z^0 is the center of the packet: $k_z^0 = f(\boldsymbol{k}_\perp)$), and the dependence T on $k_z - k_z^0$ can be neglected. The obtained equations can be reduced after the transition to the $\boldsymbol{r}$-representation with respect to the z-coordinate.

$$
\begin{aligned}
&\left\{\frac{\partial}{\partial t} + 2\gamma(\boldsymbol{k}) + v(\boldsymbol{k}) \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z'}\right) + i\left[\omega_{\mathrm{NL}}(\boldsymbol{k}, z) - \omega_{\mathrm{NL}}(\boldsymbol{k}, z'\right]\right\} n(\boldsymbol{k}, z, z', t) \\
&\qquad + i[P(\boldsymbol{k}, z', z') \sigma(-\boldsymbol{k}, z, z', t) - P^*(\boldsymbol{k}, z, z) \sigma(\boldsymbol{k}, z', z, t)] = 0\,, \\
&\left\{\frac{\partial}{\partial t} + 2\gamma(\boldsymbol{k}) + v(\boldsymbol{k}) \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z'}\right)\right. \\
&\qquad \left. + i\left[\omega_{\mathrm{NL}}(\boldsymbol{k}, z) + \omega_{\mathrm{NL}}(\boldsymbol{k}, z') - \omega_{\mathrm{p}}\right]\right\} \sigma(\boldsymbol{k}, z', z, t) \\
&\qquad + i[P(\boldsymbol{k}, z', z') n(-\boldsymbol{k}, z', z, t) + P(-\boldsymbol{k}, z, z, t) n(\boldsymbol{k}, z, z', t)] = 0
\end{aligned} \tag{6.5.6}
$$

Here

$$
\begin{aligned}
&(2\pi)^3 n(\boldsymbol{k}, z, z') = \\
&= \int n(\boldsymbol{k}_\perp, k_z, k_z') \exp\{i[k_z z - k_z' z' - k_z^0 (z - z')]\}\, dk_z dk_z'\,, \\
&(2\pi)^3 \sigma(\boldsymbol{k}, z', z) = (2\pi)^3 \sigma(-\boldsymbol{k}, z, z') \\
&= \int \sigma(\boldsymbol{k}_\perp, k_z, k_z') \exp\{i[k_z z - k_z' z' - k_z^0 (z - z')]\}\, dk_z dk_z'\,, \\
&\omega_{\mathrm{NL}}(\boldsymbol{k}, z) = \omega(\boldsymbol{k}) + 2 \int T(\boldsymbol{k}, \boldsymbol{k}') n(\boldsymbol{k}', z, z)\, d\boldsymbol{k}', \\
&P(\boldsymbol{k}, z, z') = hV(\boldsymbol{k}) + \int S(\boldsymbol{k}, \boldsymbol{k}') \sigma(\boldsymbol{k}', z', z) d\boldsymbol{k}'\,.
\end{aligned} \tag{6.5.7}
$$

For simplicity we dropped the terms with the anti-Hermitian parts S and T and the diffraction proportional to ω'' since these terms are usually insignificant.

In the case of the space homogeneity (6.5.6) go over into the basic equations of the S-theory. If the pumping amplitude h changes slowly (in comparison with the wavelength $1/k$) in space then in (6.5.6) h depends on z. It is important that the equations considered above admit the factorized solution of the form

$$
n(\boldsymbol{k}, z, z') = A^*(\boldsymbol{k}, z) A(\boldsymbol{k}, z')\,, \quad \sigma(\boldsymbol{k}, z, z') = A(\boldsymbol{k}, z') A(-\boldsymbol{k}, z), \tag{6.5.8}
$$

where $A(\boldsymbol{k}, z)$ satisfy the equation generalizing (5.4.15):

$$
\begin{aligned}
&\left\{\frac{\partial}{\partial t} + \gamma(\boldsymbol{k}) + v(\boldsymbol{k}) \frac{\partial}{\partial z}\right. \\
&\qquad \left. + i\left[\omega_{\mathrm{NL}}(\boldsymbol{k}, z, t) - \frac{\omega_{\mathrm{p}}}{2}\right]\right\} A(\boldsymbol{k}, z, t) + iP(\boldsymbol{k}, z, t) A^*(-\boldsymbol{k}, z, t) = 0, \\
&P(\boldsymbol{k}, z, t) = h(z) V(\boldsymbol{k}) + \int S(\boldsymbol{k}, \boldsymbol{k}') A(\boldsymbol{k}', z, t) A(-k', z, t) d\boldsymbol{k}'\,, \\
&\omega_{\mathrm{NL}}(\boldsymbol{k}, z, t) = \omega(\boldsymbol{k}) + 2 \int T(\boldsymbol{k}, \boldsymbol{k}') |A(\boldsymbol{k}', z, t)| d\boldsymbol{k}'\,.
\end{aligned} \tag{6.5.9}
$$

6.5.2 Parametric Threshold in Inhomogeneous Media

In studying the problem, one should take into account the energy flux from the range of the positive increment. Therefore unlike for the case of space homogeneity, even the problem of obtaining the threshold of the parametric instability proves to be sufficiently meaningful.

Evidently, in order to calculate the threshold we can confine ourselves in (6.5.9) to the terms linear in the wave amplitude, i.e.

$$
\begin{aligned}
&\left\{\frac{\partial}{\partial t} + \gamma(\boldsymbol{k}) + v(\boldsymbol{k}) \frac{\partial}{\partial z} + i\left[\omega_{\mathrm{NL}}(\boldsymbol{k}, z, t) - \frac{\omega_{\mathrm{p}}}{2}\right]\right\} A(\boldsymbol{k}, z, t) \\
&\qquad + ihV(\boldsymbol{k}, z, t) A^*(-\boldsymbol{k}, z, t) = 0\,.
\end{aligned} \tag{6.5.10}
$$

Obviously, there is no point in studying these equations under the arbitrary dependence $h(z)$. The problem of the decay instability of the homogeneous wave ($h(z) = h$) in the non-homogeneous medium has been thoroughly investigated in connection with the laser problems [6.15-16]. We are interested in a different case of inhomogeneous pumping in the homogeneous medium. This situation is often observed under the parametric excitation of spin waves and waves on the water surface [6.17]. It would be natural to consider two opposite cases, i.e. fast and slowly decreasing field of the pumping. In the first case the dependence $h(z)$ can be approximated by the rectangle

$$h(z) = h \quad \text{under } 0 < z < L \quad h(z) = 0 \quad \text{at } z < 0 \text{ or } \ z > L \,. \tag{6.5.11}$$

In the second case

$$h(z) = h/[1 + z^2/2L^2] \,. \tag{6.5.12}$$

1 Threshold of Parametric Instability in a Plate. A more general presentation of the problem (6.5.11) about the parametric instability in the plane layer has been given in some studies (e.g., [6.18]). For the subsequent study of the nonlinear stage of the parametric wave excitation in the plane layer we shall briefly outline some results of the linear theory. Obviously, the boundary conditions have the following form:

$$A(\boldsymbol{k}, 0) = A(-\boldsymbol{k}, L) = 0 \,. \tag{6.5.13}$$

It can be assumed without loss of generality that in (6.5.10) $\omega(\boldsymbol{k}) = \omega_{\text{p}}/2$. Then

$$A(\boldsymbol{k}, z) = A \sin(kz) \quad A(-\boldsymbol{k}, z) = A \sin[\kappa(L - z)], \tag{6.5.14}$$

κ and the threshold value of $h = h_{\text{th}}$ being given by the following equations

$$h_{\text{th}} V(\boldsymbol{k}) \cos(\kappa L) = -\gamma(\boldsymbol{k}), h_{\text{th}} V(\boldsymbol{k}) \sin(\kappa) L = \kappa v(\boldsymbol{k}) \,. \tag{6.5.15}$$

Hence

$$[h_{\text{th}} V(\boldsymbol{k})]^2 = \gamma(\boldsymbol{k})^2 + [\pi b v(\boldsymbol{k})/L]^2 \,. \tag{6.5.16}$$

The numerical coefficient b depends on the ratio of the mean free path v/γ to the size of the layer L and varies in the range from unity (at $v \ll \gamma L$) to the half of the quantity in the opposite limiting case. Note that in the threshold formula (6.5.16) the squares of the eigen damping of waves $\gamma(\boldsymbol{k})$ and the effective damping ($\pi b v/L$) caused by the energy flux from the pumping region (or by the absorption on the boundary of the sample) are added.

2 Threshold Under Smoothly Non-Homogeneous Pumping (6.5.12). To calculate this threshold let us take in (6.5.10) $\partial/\partial t = 0$ and drop $A(-\boldsymbol{k}, z)$. Then

$$\left[\gamma^2 - h^2(z)V^2 - v^2\frac{\partial^2}{\partial z^2} + \frac{\partial h(z)}{h(z)\partial z^2}\left(\gamma + v\frac{\partial}{\partial z}\right)\right] A(\boldsymbol{k}, z) = 0 \,. \tag{6.5.17}$$

Here and in the following for brevity the argument $\boldsymbol{k}$ of the functions γ ,V and v is dropped. The boundary conditions for this equation have the form $A(\boldsymbol{k}, \infty)$=$A(\boldsymbol{k}, -\infty)$ =0. Taking into account, first, that for the smoothly decreasing pumping $L \gg v/\gamma$ and, second, that the solution is concentrated in the region of the size l satisfying the following inequality

$$L \gg l \gg v/\gamma \,, \tag{6.5.18}$$

(6.5.17) can be reduced to the form

$$\left[\gamma^2 + h^2V^2 + \frac{(hV)^2}{L^2}\left(z - \frac{\gamma v}{2n^2V^2}\right) - v^2\frac{\partial^2}{\partial z^2}\right] A(\boldsymbol{k}, z) = 0 \,. \tag{6.5.19}$$

The solution of the equation we are interested in has the form

$$A(\boldsymbol{k}, z) = A \exp[-(\gamma/2vL)(z - v/2\gamma)^2] \tag{6.5.20}$$

under the threshold value of the pumping amplitude

$$h_{\text{th}} V = \gamma + v/2L \,. \tag{6.5.21}$$

The characteristic size of the solution is

$$l = \sqrt{Lv/\gamma} \tag{6.5.22}$$

and the inequality (6.5.18) necessarily holds true.

Note that the maximum of the packet $A(\boldsymbol{k}, z)$ (6.5.20) is shifted to the right by the distance equal to the mean free path. The maximum of the packet $A(-\boldsymbol{k}, z)$ is shifted to the left:

$$A(-\boldsymbol{k}, z) = A \exp[-(/2vL)(z + v/2)] \,. \tag{6.5.23}$$

The maximum of their product obviously remains at the point $z = 0$ where the pumping amplitude is at maximum (6.5.12). That is

$$A(\boldsymbol{k}, z)A(-\boldsymbol{k}, z) = A^2 \exp[z^2 + v^2/4\gamma^2] \,.$$

Interestingly, the obtained expression for the threshold (6.5.21) under the continuous inhomogeneity differs greatly for the corresponding expression (6.5.16) for the rectangular profile $h(z)$. To explain this difference, let us obtain the expressions for the profiles from the following simple considerations. As seen from (6.5.10), the characteristic size of the region, l, where the parametrically excited waves are concentrated, is given by the expression

$$(v/l)^2 = (hV)^2 - \gamma^2 \; . \tag{6.5.24}$$

In its turn, l is of the order of magnitude of a size of the region, where $h(z) > \gamma$. For (6.5.11) this is, naturally, the pumping scale L given by (6.5.11) and then from (6.5.24) follows the estimate of h_{th} coinciding with (6.5.16). If the pumping amplitude continuously decreases, $h(z)V = hV/(1+z/L)^n$, then $\gamma^2(l/L)^n = (hV)^2 - \gamma^2$. Combining this relation with (6.5.24), we obtain

$$l \simeq L(v/\gamma L)^{2/(n+2)} , \quad (h_{\text{th}}V)^2 - \gamma^2 \simeq (v/L)^2(\gamma L/v)^{4/(n+2)} , \tag{6.5.25}$$

hence at $n = 2$ we get the estimate coinciding with (6.5.21) and at $n \gg 1$ we get the result (6.5.16) for the "rectangular" pumping (6.5.11).

3 Excitation Threshold of Oblique Waves in a Plate. In order to calculate the excitation threshold of the waves propagating at an angle Θ with the direction of the non-homogeneity, it is sufficient to allow for the dependences $V(\Theta)$, $v(\Theta)$ $=v\cos\Theta$ and $\gamma(\Theta)$ in (6.5.16). By minimizing the expression (6.5.16) for the threshold pumping amplitude we obtain the threshold of generation and location Θ_1 of the wave pair first to be excited. Here are some simple examples.

A. For the threshold to be minimum for the waves propagating along the non-homogeneity, the maximum $V(\Theta)$ at $\Theta = 0$ must be sufficiency sharp:

$$\left[\frac{\partial^2 V}{V\partial\Theta^2}\right]_{\Theta=0} > 2\left[\frac{h_{\text{th}}^2 V^2 - \gamma^2}{h_{\text{th}}^2 V^2}\right]_{\Theta=0} = 2\frac{(\pi b v/L)^2}{\gamma^2 + (\pi b v/L)^2} \; . \tag{6.5.26}$$

B. If the wave-pumping interaction amplitude is isotropic, i.e $V(\Theta) = V$, then the waves propagating across the non-homogeneity are the first to be excited. Their distribution is homogeneous in space and it is therefore described by the usual formulae of the basic S-theory. It must be noted that under $\Theta = \pi/2$ the term with the group velocity in (6.5.10) becomes zero. Thus, we must allow for the diffraction effects described by the term $\omega''\partial^2/\partial z^2$ and which we had failed to take into account in (6.5.10). All this results in the following for a plate of thickness L at $V(\Theta) = V$:

$$h_{\text{th}}^2 V^2 = \gamma^2 + (\pi^2\omega''/2L) \; . \tag{6.5.27}$$

6.5.3 Stationary State in Non-Homogeneous Media

The instability treated in the previous section is absolute; thus, under $h > h_{\text{th}}$ the wave amplitudes are limited by their nonlinear interaction. The resulting distribution is described by (6.5.9) with the corresponding boundary conditions. Let us take some interesting examples confining ourselves to the consideration of the parametric excitation of waves in the plane (6.5.11).

1 Stationary conditions under low supercriticality. Let the pumping amplitude be not too great and let only one set of pairs be excited of the parallels Θ and $\pi - \Theta$. Then the stationary equations (6.5.9) are reduced to

$$\left\{v\frac{d}{dz} + \gamma + i\left[\omega_{\text{NL}}(\boldsymbol{k}) - \frac{\omega_{\text{p}}}{2}\right]\right\} A(\boldsymbol{k},z) = -iP(\boldsymbol{k})A^*(-\boldsymbol{k},z) \; . \tag{6.5.28}$$

By substituting here the expressions for $\omega_{\text{NL}}(\boldsymbol{k})$ and $P(\boldsymbol{k})$ and selecting (without loss of generality) $\omega(\boldsymbol{k}) = \omega_{\text{p}}/2$ we obtain

$$\begin{aligned} &\left\{v\frac{d}{dz} + \gamma + 2i[T_1|A(\boldsymbol{k},z)|^2 + T_2|A(-\boldsymbol{k},z)|^2]\right\}A(\boldsymbol{k},z) \\ &\quad = -\,i[hV + 2S_1A(\boldsymbol{k},z)A(-\boldsymbol{k},z)]A^*(-\boldsymbol{k},z) \; . \end{aligned} \tag{6.5.29}$$

Let us pass to the amplitude phase variables:

$$A(\boldsymbol{k},z) = a(z)\exp[-i\varphi(z)] , \quad A(-\boldsymbol{k},z) = b(z)\exp[-i\psi(z)] \; . \tag{6.5.30}$$

The equation for the phase difference $(\varphi - \psi)$ in this case is split out and we come to the following closed system of equations:

$$\begin{aligned} &v\frac{da}{dz} + \gamma a = hVb\sin\Phi , \quad -v\frac{db}{dz} + \gamma b = hVa\sin\Phi , \\ &v\frac{d\Phi}{dz} + 2S(a^2 - b^2) + \frac{hV(a^2-b^2)}{ab}\cos\Phi = 0 , \\ &\Phi = \varphi + \psi , \quad S = T_2 - T_1 + S_1 \; . \end{aligned} \tag{6.5.31}$$

Equations (6.5.31) can be immediately verified to have the following integral of motion

$$I = ab[Sab + hV\cos\Phi] = \text{const} \; . \tag{6.5.32}$$

On the boundaries of the sample $ab = 0$, therefore $I = 0$ and in the volume of the plane

$$Sab + hV\cos\Phi = 0 \; . \tag{6.5.33}$$

2 Amplitude and profile of distributions $a(z)$ and $b(z)$ under low supercriticality. These values can be obtained by making use of the perturbation theory with respect to the parameter $\delta h/h_{\text{th}} \ll 1$ ($\delta h = h - h_{\text{th}}$). According to (6.5.14) at $h = h_{\text{th}}$, $\Phi = \Phi_0 = \pi/2$ and:

$$a = a_0 = \sqrt{N/2}\sin[\kappa z] , \quad b = b_0 = \sqrt{N/2}\sin[\kappa(L-z)] . \tag{6.5.34}$$

Here N is the number of parametric waves which in the linear theory obviously remains undetermined. Under $0 < \delta h/h_{\text{th}} \ll 1$ we assume

$$\Phi = \pi/2 + \Phi_1 \, , \quad a = a_0 + a_1 \, , \quad b = b_0 + b_1 \, . \tag{6.5.35}$$

From the integral of motion (6.5.33) we obtain $\Phi_1(z)$:

$$\Phi_1(z) = (SN/2h_{\text{th}}V)\sin(\kappa z)\sin[\kappa(L-z)] \, . \tag{6.5.36}$$

Now for a_1 and b_1 from (6.5.31) we can obtain

$$\hat{L}_0 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} (v\frac{d}{dz}+\gamma)a_1 & -hVb_1 \\ hVa_1 & +(v\frac{d}{dz}+\gamma)b_1 \end{bmatrix} = Y(z)\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \tag{6.5.37}$$

$$Y(z) = hV - (Sa_0b_0)^2/2h_{\text{th}}V \, .$$

Clearly, the zeroth eigenfunctions of the operator $\hat{L}_0^*$ are $(b,-a)$. The condition of the right-hand side of (6.5.37) being orthogonal to $(b,-a)$ gives the dispersion equation $\int_0^L (a_0^2+b_0^2)Y\,dz = 0$, specifying the total number of parametric waves

$$(SN)^2 = 2h\delta hV^2 d \, , \tag{6.5.38}$$

where d is the ration of integrals

$$d = \frac{2N\int_0^L (a_0^2+b_0^2)\,dz}{\int_0^L (a_0^2+b_0^2)a_0^2b_0^2\,dz} .$$

In changing over from the thin sample to the thick one d will change from 12.8 to 32. Note that in the basic S-theory for the unbounded medium SN is given by (5.5.7), which can be reduced to the form of (6.5.38) with the factor $d=1$.

3 Profile $a(z)$ in a thin plate. In thin samples where even when for low supercriticalities the damping can be neglected, the equations (6.5.31) have one more integral of motion

$$N = a^2(z) + b^2(z). \tag{6.5.39}$$

This enables one to solve (6.5.31) by quadratures. The wave distribution is qualitatively similar to the distribution considered in Subsect. 1: $a(z)$ monotonically increases deep into the sample. The transcendental equation for the determination of the total number of waves N has the form

$$hVL = v\int \frac{dy}{\sqrt{[1-(SN/2hV)^2y_2][1-y^2]}} \, . \tag{6.5.40}$$

Under low supercriticalities we can obtain from the above (6.5.16) for the instability threshold (at $\gamma = 0$ and $b = 1/2$) and (6.5.38) for N (at $d = 32$). Under high supercriticalities (i.e. at $hVL \gg v$), SN quickly tends to $2hV$:

$$SN = hV[2 - \exp(-hVL/v)] \, . \tag{6.5.41}$$

In this case, too, the solution profile can easily be calculated

$$2ab = N\tanh(2hVz/v) \, , \text{at} \quad z < L/2 \, . \tag{6.5.42}$$

Hence it is clear that at the depths of the order v/hV the solution becomes practically homogeneous.

4 On a Solution profile in an arbitrary case. The amplitudes for the depth of the sample $a(\infty) = b(\infty)$ (in the case of a single set of pairs) are determined by the value of the integral (6.5.32) $I = 0$, obtained on the boundary of the sample. But according to (5.5.7) the integral

$$I = -SN/2 = -\sqrt{(hV)^2-\gamma^2}/2$$

and is non-zero. Therefore the values of $a(\infty)$, $b(\infty)$ in the depth of the sample obtained as solutions of (6.5.31) will differ from the values (5.5.7)

$$a_S = b_S = [(hV^2-\gamma^2]^{1/4}/\sqrt{2S} \, , \tag{6.5.43}$$

obtained in the basic S-theory for the unbounded sample. As is known, under homogeneity all the solutions different from (5.5.7) are unstable. Therefore it must be expected that under $SN > v/L$ the above obtained solution is unstable, and, consequently, can be realized only within a narrow range with the thickness of the order of v/SN near the boundary. In order to describe the range transient to the values of a, b (6.5.43) of the basic S-theory, (6.5.29) must allow for the time derivatives and dispersion terms proportional to ω''.

How do the amplitudes $a(z), b(z)$ become the asymptotic values a_S, b_S given by the basic S-theory? In order to answer this question, let us linearize (6.5.29) in small deviations $A(\boldsymbol{k}, z)$ and $A(-\boldsymbol{k}, z)$ from the solution of the basic S-theory and let us assume that all these deviations are proportional to $\exp(-\kappa z)$. The value of κ can be determined from the condition of the determinant of the obtained homogeneous system of equations being equal to zero. This yields

$$(\kappa v)^2 = 4(T_2 - T_1)(S_1 + T_1 + T_2)N = \Delta^2 \, . \tag{6.5.44}$$

Therefore at $\Delta^2 > 0$ the amplitudes exponentially become the asymptotic value of the basic S-theory with the characteristic length $L_1 = 1/\kappa = v/\Delta$. If, on the contrary, $\Delta < 0$ then, as it will be seen, the stationary solution of the basic S-theory a, b proves to be unstable which results in spontaneous auto-oscillations.

On the basis of the above-considered examples we shall try to describe qualitatively the profile $a(z)$ for the plate of the arbitrary thickness. First let us consider a narrow plate ($L \ll v/\gamma = l$). Then up to the supercriticality of about unity the solution of Subsect. 3 will hold true. Under high supercriticalities the wave propagating from the boundary increases to the value approximating the value of a_S (6.5.43) at the distance $L_2 \simeq v/hV$. The wave incident on the boundary does not change significantly after the homogeneous solution a is exponentially attained with characteristic size $L_1 \simeq v/\Delta$ (6.5.44).

In a thick sample ($L \gg v/\gamma = l$) the profile $a(z)$ is similar to the one above considered. Under very small supercriticalities when $SN < L/v$ the solution of the Subsect. 2 is realized. In the intermediate case when $hV \gg SN > v/L$ the solution is quite remarkable. Then the intermediate region consists of two parts. First, at the distance from the boundary of about $l = v/\gamma$ the wave increases up to the value close to a and then over a rather long distance (with the size of about value $L_1 = v/\Delta \simeq l(\gamma/SN)$ the homogeneous solution of the basic S-theory is being exponentially attained.

In conclusion I should like to emphasize that Sect. 6.5 is only the introduction to the inhomogeneous S-theory. We obtained the equations of the S-theory for the case of the smooth unidimensional non-homogeneity and analyzed them in the simplest situations. This almost exhausted the modern scientific data on this question. A host of interesting but unsolved problems have been left out.

6.6 Nonlinear Behavior of Parametric Waves from Various Branches. Asymmetrical S-Theory

6.6.1 Derivation of Basic Equations

In this section it will be assumed as before that the field $h(\boldsymbol{r},t) = h\exp(-i\omega t)$, homogeneous in space, serves as pumping so that the wave vectors $\boldsymbol{k}, -\boldsymbol{k}$ and wave frequencies $\omega(\boldsymbol{k})$ and $\Omega(-\boldsymbol{k})$, belonging to various branches of the spectrum, are obtained from the relation [6.20]

$$\omega(\boldsymbol{k}) + \Omega(-\boldsymbol{k}) = \omega_p \ . \tag{6.6.1}$$

The Hamiltonian of the problem

$$\mathcal{H} = \sum_{\boldsymbol{k}} [\omega(\boldsymbol{k})a(\boldsymbol{k})a^*(\boldsymbol{k}) + \Omega(\boldsymbol{k})b(\boldsymbol{k})b^*(\boldsymbol{k})] + \mathcal{H}_p + \mathcal{H}_{int} \tag{6.6.2a}$$

comprises the squared Hamiltonian of the two-wave type, the Hamiltonian of their interaction with the pumping

$$\mathcal{H}_p = \sum_{\boldsymbol{k}} [h(t)V(\boldsymbol{k})a^*(\boldsymbol{k})b^*(-\boldsymbol{k}) + \text{c.c.}] \tag{6.6.2b}$$

and the Hamiltonian of the wave interaction

$$\begin{aligned} \mathcal{H}_{int} = \sum_{1+2=3+4} \Big\{ & \frac{1}{4} T_{LL}(\mathbf{1},\mathbf{2};\mathbf{3},\mathbf{4}) a^*(\boldsymbol{k}_1)a^*(\boldsymbol{k}_2)a(\boldsymbol{k}_3)a(\boldsymbol{k}_4) \\ & + \frac{1}{4} T_{SS}(\mathbf{1},\mathbf{2};\mathbf{3},\mathbf{4}) b^*(\boldsymbol{k}_1)b^*(\boldsymbol{k}_2)b(\boldsymbol{k}_3)b(\boldsymbol{k}_4) \\ & + T_{SL}(\mathbf{1},\mathbf{2};\mathbf{3},\mathbf{4}) a^*(\boldsymbol{k}_1)b^*(\boldsymbol{k}_2)a(\boldsymbol{k}_3)b(\boldsymbol{k}_4) \Big\} \ . \end{aligned} \tag{6.6.2c}$$

The dynamical equations of motion allowing for the wave damping are represented in the following form:

$$\begin{aligned} \partial a(\boldsymbol{k},t)/\partial t + \gamma(\boldsymbol{k})a(\boldsymbol{k},t) &= -i\delta\mathcal{H}/\delta a^*(\boldsymbol{k},t) \ , \\ \partial b(\boldsymbol{k},t)/\partial t + \Gamma(\boldsymbol{k})b(\boldsymbol{k},t) &= -i\delta\mathcal{H}/\delta b(\boldsymbol{k},t) \ . \end{aligned} \tag{6.6.3}$$

As in the basic S-theory the energy flux from the pumping with Hamiltonian $\mathcal{H}_p$ (6.6.2b) can be shown to be proportional to $\sin[\varphi_S(\boldsymbol{k}) + \varphi_L(-\boldsymbol{k}) - \varphi_p]$, where φ_S and φ_L are the phases of the waves $a(\boldsymbol{k})$ and $b(\boldsymbol{k})$ with the respective dispersion laws $\omega(\boldsymbol{k})$ and $\Omega(\boldsymbol{k})$, φ_p denotes the pumping phase. This implies that pumping $\mathcal{H}_p$ leads to the correlation of the sum of phases $\Psi(\boldsymbol{k})$ in the pair $a(\boldsymbol{k})$, $b(-\boldsymbol{k})$:

$$\Psi(\boldsymbol{k}) = \varphi_L(\boldsymbol{k}) + \varphi_S(-\boldsymbol{k}) \ . \tag{6.6.4}$$

Therefore at the nonlinear stage of the parametric instability the anomalous correlator

$$\sigma(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{k}_1) = \langle a(\boldsymbol{k})b(-\boldsymbol{k})\exp(i\omega_{\rm p}t)\rangle \tag{6.6.5}$$

must be allowed for. Let us also determine the normal correlators

$$n_{\rm L}(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{k}_1) = \langle a(\boldsymbol{k})a^*(\boldsymbol{k}_1)\rangle\,,\ n_{\rm S}(\boldsymbol{k})\delta\boldsymbol{k}-\boldsymbol{k}_1) = \langle b(\boldsymbol{k})b^*(\boldsymbol{k}_1)\rangle \tag{6.6.6a}$$

It will subsequently be shown that the state described by these correlators as a result of the nonlinear interaction of waves may prove to be unstable with respect to the wave pair inside each branch. This results in new anomalous correlators

$$\begin{aligned}
\sigma_{\rm L}(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{k}_1) &= \langle a(\boldsymbol{k})a(-\boldsymbol{k})\exp[2i\omega_1(\boldsymbol{k})t]\rangle\,,\\
\sigma_{\rm S}(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{k}_1) &= \langle b(\boldsymbol{k})b(-\boldsymbol{k})\exp[2i\Omega_1(\boldsymbol{k})t]\rangle\,,\\
n_{\rm SL}(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{k}_1) &= \langle a(\boldsymbol{k})b^*(\boldsymbol{k}_1)\exp i[\omega(\boldsymbol{k})-\Omega(\boldsymbol{k})]\,,\\
n_{\rm SL}(\boldsymbol{k}) &= n^*_{\rm SL}(\boldsymbol{k})\,,\quad \omega_1(\boldsymbol{k})\simeq\omega(\boldsymbol{k})\,,\quad \Omega_1(\boldsymbol{k})\simeq\Omega(\boldsymbol{k})\,.
\end{aligned} \tag{6.6.6b}$$

By differentiating these relations with respect to time and making use of (6.6.3) with the Hamiltonian (6.6.2) one can obtain the equations of motion for all correlators (6.6.6). By splitting the fourth-order correlators of the wave amplitudes through various products of the paired correlators similar to (6.6.6) we obtain in the first order of the perturbation theory with respect to $\mathcal{H}_{\rm int}$:

$$\begin{aligned}
&\frac{\partial n_{\rm L}(\boldsymbol{k},t)}{2\partial t} + \gamma(\boldsymbol{k})n_{\rm L}(\boldsymbol{k},t) = {\rm Im}\{P(\boldsymbol{k})\sigma^*(\boldsymbol{k},t)\\
&\qquad + P_{\rm L}(\boldsymbol{k})\sigma^*_{\rm L}(\boldsymbol{k},t) + G(\boldsymbol{k})n^*_{\rm SL}(\boldsymbol{k},t)\}\,,\\
&\frac{\partial n_{\rm S}(\boldsymbol{k},t)}{2\partial t} + \Gamma(\boldsymbol{k})n_{\rm S}(\boldsymbol{k},t) = {\rm Im}\{P(-\boldsymbol{k})\sigma^*(-\boldsymbol{k},t)\\
&\qquad + P_{\rm S}(\boldsymbol{k})\sigma_{\rm S}(\boldsymbol{k},t) + G^*(\boldsymbol{k})n^*_{\rm LS}(\boldsymbol{k},t)\}\,,\\
&\frac{\partial \sigma(\boldsymbol{k},t)}{\partial t} + \{\gamma(\boldsymbol{k})+\Gamma(\boldsymbol{k})+i[\omega_{\rm NL}(\boldsymbol{k})+\Omega_{\rm NL}(\boldsymbol{k})-\omega_{\rm p}]\}\sigma(\boldsymbol{k},t)\\
&\qquad = -\,iP(\boldsymbol{k})[n_{\rm L}(\boldsymbol{k},t)+n_{\rm S}(\boldsymbol{k},t)] - iP_{\rm L}(\boldsymbol{k})n_{\rm SL}(\boldsymbol{k},t)\\
&\qquad\quad - iP_{\rm S}(\boldsymbol{k})n_{\rm LS}(\boldsymbol{k},t) - iG(\boldsymbol{k})\sigma_{\rm S}(\boldsymbol{k},t) - iG^*(\boldsymbol{k})\sigma_{\rm L}(\boldsymbol{k},t)\,,
\end{aligned} \tag{6.6.7}$$

$$\begin{aligned}
&\frac{\partial \sigma_{\rm L}(\boldsymbol{k},t)}{2\partial t} + \{\gamma(\boldsymbol{k})+i[\omega_{\rm NL}(\boldsymbol{k})-\omega_1(\boldsymbol{k})]\}\sigma_{\rm L}(\boldsymbol{k},t)\\
&\qquad + iP(\boldsymbol{k})n_{\rm LS}(\boldsymbol{k},t) + iP_{\rm L}(\boldsymbol{k})n_{\rm L}(\boldsymbol{k},t) = -iG(\boldsymbol{k})\sigma(\boldsymbol{k},t),\\
&\frac{\partial \sigma_{\rm S}(\boldsymbol{k},t)}{2\partial t} + \{\Gamma(\boldsymbol{k})+i[\Omega_{\rm NL}(\boldsymbol{k})-\Omega_1(\boldsymbol{k})]\}\sigma_{\rm L}(\boldsymbol{k},t)\\
&\qquad + iP(\boldsymbol{k})n_{\rm LS}(\boldsymbol{k},t) + iP_{\rm L}(\boldsymbol{k})n_{\rm S}(\boldsymbol{k},t) = -iG^*(\boldsymbol{k})\sigma(-\boldsymbol{k},t),\\
&\frac{\partial n_{\rm LS}(\boldsymbol{k},t)}{\partial t} + [\gamma(\boldsymbol{k})+\Gamma(\boldsymbol{k})]n_{\rm LS}(\boldsymbol{k},t)\\
&\qquad + i[\omega_{\rm NL}(\boldsymbol{k})-\omega_1(\boldsymbol{k})+\Omega_{\rm NL}(\boldsymbol{k})-\Omega_1(\boldsymbol{k})]n_{\rm LS}(\boldsymbol{k},t)\\
&\qquad + iP(\boldsymbol{k})\sigma_{\rm S}(\boldsymbol{k},t) - iP(\boldsymbol{k})\sigma_{\rm L}(\boldsymbol{k},t) + iP_{\rm L}(\boldsymbol{k})\sigma^*(\boldsymbol{k},t)\\
&\qquad - iP_{\rm S}(\boldsymbol{k})\sigma(-\boldsymbol{k},t) + iG(\boldsymbol{k})[n_{\rm S}(\boldsymbol{k},t)-n_{\rm L}(\boldsymbol{k},t)] = 0
\end{aligned} \tag{6.6.8}$$

In these equations we selected the values of the frequencies $\omega_1(\boldsymbol{k})$ and $\Omega_1(\boldsymbol{k})$ entering into the definitions of correlators $\sigma_{\rm L}$ and $\sigma_{\rm S}$ in such a way:

$$\omega_1(\boldsymbol{k}) + \Omega_1(\boldsymbol{k}) = \omega_{\rm p} \tag{6.6.9}$$

and introduced the following designations for the frequencies and self-consistent pumpings renormalized to the interaction:

$$\begin{aligned}
\omega_{\rm NL}(\boldsymbol{k}) &= \omega(\boldsymbol{k}) + 2\sum_1[T_{\rm L}(\boldsymbol{k},\boldsymbol{k}_1)n_{\rm L}(\boldsymbol{k}) + T(\boldsymbol{k},\boldsymbol{k}_1)n_{\rm S}(\boldsymbol{k}_1)]\,,\\
\Omega_{\rm NL}(\boldsymbol{k}) &= \Omega(\boldsymbol{k}) + 2\sum_1[T_{\rm S}(\boldsymbol{k},\boldsymbol{k}_1)n_{\rm S}(\boldsymbol{k}) + T(\boldsymbol{k}_1,\boldsymbol{k})n_{\rm L}(\boldsymbol{k}_1)]\,,\\
P(\boldsymbol{k}) &= hV(\boldsymbol{k}) + \sum_1 S(\boldsymbol{k},\boldsymbol{k}_1)\sigma(\boldsymbol{k}_1),\\
P_{\rm L}(\boldsymbol{k}) &= \sum_1 S_{\rm L}(\boldsymbol{k},\boldsymbol{k}_1)\sigma_{\rm L}(\boldsymbol{k}_1)\,,\ P_{\rm S}(\boldsymbol{k}) = \sum_1 S_{\rm S}(\boldsymbol{k},\boldsymbol{k}_1)\sigma_{\rm S}(\boldsymbol{k}_1)\,,\\
G(\boldsymbol{k}) &= \sum_1 F(\boldsymbol{k},\boldsymbol{k}_1)n_{\rm LS}(\boldsymbol{k}_1),\quad F(\boldsymbol{k},\boldsymbol{k}_1) = T_{\rm LS}(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k}_1,\boldsymbol{k})\,.
\end{aligned} \tag{6.6.11}$$

$$\begin{aligned}
T_{\rm L}(\boldsymbol{k},\boldsymbol{k}_1) &= T_{\rm LL}(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k},\boldsymbol{k}_1)/2\,, & T_{\rm S}(\boldsymbol{k},\boldsymbol{k}_1) &= T_{\rm SS}(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k},\boldsymbol{k}_1)/2\,,\\
S_{\rm L}(\boldsymbol{k},\boldsymbol{k}_1) &= T_{\rm LL}(\boldsymbol{k},-\boldsymbol{k},\boldsymbol{k}_1,-\boldsymbol{k}_1)/2\,, & S_{\rm S}(\boldsymbol{k},\boldsymbol{k}_1) &= T_{\rm SS}(\boldsymbol{k},-\boldsymbol{k},\boldsymbol{k}_1,-\boldsymbol{k}_1)/2\,,\\
T(\boldsymbol{k},\boldsymbol{k}_1) &= T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k},\boldsymbol{k}_1)\,, & S(\boldsymbol{k},\boldsymbol{k}_1) &= T_{\rm LS}(\boldsymbol{k},-\boldsymbol{k};\boldsymbol{k}_1,-\boldsymbol{k}_1)/2\,.
\end{aligned}$$

6.6.2 Stationary States in Isotropic Case

The equations of the asymmetric S-theory (6.6.7–11) are rather cumbersome. Therefore we shall confine ourselves to the consideration of the isotropic case which is realized, for instance, under parametric excitation of magnons of different branches in antiferromagnets. In the isotropic case $\gamma(\boldsymbol{k})$=$\gamma(k)$, $\Gamma(\boldsymbol{k})$ =$\Gamma(k)$ and $V(\boldsymbol{k})$ =$V(k)$ and the interaction amplitudes T in $\mathcal{H}_{\rm int}$ depend only on k, k_1 and $(\boldsymbol{k}\boldsymbol{k}_1)$. Then the solution of the equations will be isotropic and $P(\boldsymbol{k}) = P(k)$, $P_S(\boldsymbol{k}) = P_S(k)$, $P_L(\boldsymbol{k}) = P_L(k)$ and $G(\boldsymbol{k}) = G(k)$. As in the basic S-theory we can neglect all the dependences of all the functions except the frequencies $\omega(\boldsymbol{k})$ and $\Omega(\boldsymbol{k})$ on the module k, substituting into the equations their values on the resonance surface (6.6.1). Thus it can be assumed that

$$\begin{aligned}&\gamma(\boldsymbol{k}) = \gamma\,,\ \Gamma(\boldsymbol{k}) = \Gamma\,,\ V(\boldsymbol{k}) = V\,,\ P(\boldsymbol{k}) = P\,,\\ &P_L(\boldsymbol{k}) = P_L\,,\ P_S(\boldsymbol{k}) = P_S\,,\ G(\boldsymbol{k}) = G\end{aligned} \tag{6.6.12}$$

and in the expressions (6.6.10) we can replace the following functions T_L, T_S, T, S_L, S_S, S and F of the argument $x = \cos\Theta_1 = (\boldsymbol{k}\boldsymbol{k}_1)/kk_1$ for their mean values

$$T_L = \frac{1}{2}\int_{-1}^{1} T_L(x)\,dx\,, \qquad T_S = \frac{1}{2}\int_{-1}^{1} T_S(x)\,dx \ \text{ and so on.} \tag{6.6.13a}$$

As can easily be seen, (6.6.7–11) in the stationary case are a system of linear equations with coefficients γ, Γ, $\omega_{NL}(\boldsymbol{k})$, $\Omega_{NL}(\boldsymbol{k})$, P, P_S, P_L and G. These equations have non-zero solutions only at such values $\omega_{NL}(\boldsymbol{k})$ and $\Omega_{NL}(\boldsymbol{k})$, under which the determinant of the system becomes zero. Therefore the stationary solutions are concentrated in the $\boldsymbol{k}$-space on the spherical surfaces. Because the determinant of the system is a second-order polynomial with respect to $\omega_{NL}(\boldsymbol{k})$ there can be no more than two such surfaces.

To analyze the stationary states and the stability of the solutions with $k = \text{const}$ we obtain from (6.6.7–11) the equations for the following integral values:

$$\begin{aligned}&N_L = \sum_{\boldsymbol{k}} n_L(\boldsymbol{k})\,, \quad N_S = \sum_{\boldsymbol{k}} n_S(\boldsymbol{k})\,, \quad \Sigma = \sum_{\boldsymbol{k}} \sigma(\boldsymbol{k})\,,\\ &\Sigma_L = \sum_{\boldsymbol{k}} \sigma_L(\boldsymbol{k})\,, \quad \Sigma_S = \sum_{\boldsymbol{k}} \sigma_S(\boldsymbol{k})\,, \quad N_{LS} = \sum_{\boldsymbol{k}} n_{LS}(\boldsymbol{k})\,.\end{aligned} \tag{6.6.13b}$$

To this end, the equations (6.6.7, 8) must be summed with respect to $\boldsymbol{k}$, the relations (6.6.12, 13) must be allowed for and one must keep in mind that the wave amplitudes are non-zero only on the sphere. Then we have:

$$\begin{aligned}&\frac{dN_L}{2dt} + \gamma N_L + \operatorname{Im}\{hV\Sigma^*\} = 0\,,\\ &\frac{dN_S}{2dt} + \Gamma N_S + \operatorname{Im}\{hV\Sigma^*\} = 0\,,\\ &\frac{d\Sigma}{dt} + \{\gamma + \Gamma + i[\omega_{NL}(\boldsymbol{k}) + \Omega_{NL}(\boldsymbol{k}) - \omega_p + S(N_L + N_S)]\}\Sigma\\ &\quad + ihV(N_L + N_S) + i(F + S_S)\Sigma_L N_{LS} = 0\,,\end{aligned} \tag{6.6.14}$$

$$\begin{aligned}&\frac{dN_{LS}}{dt} + \{\gamma + \Gamma + i[\omega_{NL}(\boldsymbol{k}) - \Omega_{NL}(\boldsymbol{k})\\ &\quad - \omega_1(\boldsymbol{k}) + \Omega_1(\boldsymbol{k}) + iF(N_S - N_L)]\}N_{LS}\\ &\quad + i[hV + (S - S_S)\Sigma]\Sigma_S^* - i[hV + (S - S_L)\Sigma^*]\Sigma_L = 0\,,\\ &\frac{d\Sigma_L}{2dt} + \{\gamma + i[\omega_{NL}(\boldsymbol{k}) - \omega_1(\boldsymbol{k}) + S_L N_L]\}\Sigma_L\\ &\quad + i[hV + (S + F)\Sigma]N_{LS} = 0\,,\\ &\frac{d\Sigma_S^*}{2dt} + \{\Gamma - i[\Omega_{NL}(\boldsymbol{k}) - \Omega_1(\boldsymbol{k}) + S_S N_S]\}\Sigma_S^*\\ &\quad - i[hV + (S + F)\Sigma^*]N_{LS} = 0\,,\end{aligned} \tag{6.6.15}$$

Proceeding to the analysis of the stationary solutions of these equations note that the group of equations (6.6.15) can be treated as the system of the linear algebraic equations homogeneous in N_{LS}, Σ_L and Σ_S^*. Therefore the solutions of the entire system of equations (6.6.14, 15) can be of two types, i.e. state **A** where $N_{LS} = \Sigma_L = \Sigma_S^* = 0$, and the state **B** where these correlators differ from zero. In the state **A**

$$\begin{aligned}&\Sigma = M\exp(-i\Psi)\,, \quad M^2 = N_L N_S\,,\\ &\gamma N_L = \Gamma N_S = hVM\sin\Psi\,,\end{aligned} \tag{6.6.16a}$$

$$\begin{aligned}&[hV\cos\Psi + SM][\sqrt{\gamma/\Gamma} + \sqrt{\Gamma/\gamma}] = \omega_p - \omega_{NL}(\boldsymbol{k}) - \Omega_{NL}(\boldsymbol{k})\,,\\ &N_{LS} = \Sigma_L = \Sigma_S = 0\,.\end{aligned} \tag{6.6.16b}$$

In the state **B**

$$\begin{aligned}&\Sigma = M\exp(-i\Psi), \quad M^2 = N_L N_S\,,\\ &\Sigma_S = N_S\exp(-i\varphi_S)\,, \quad \Sigma_L = N_L\exp(-i\varphi_L)\,,\\ &N_{LS} = M\exp(-i\varphi)\,, \quad \Psi = \varphi_L - \varphi = \varphi_S + \varphi\end{aligned} \tag{6.6.17a}$$

$$\begin{aligned}&[hV\cos\Psi + SM][\sqrt{\gamma/\Gamma} + \sqrt{\Gamma/\gamma}]\\ &\quad = \omega_p - \omega_{NL}(\boldsymbol{k}) - \Omega_{NL}(\boldsymbol{k}) - \Delta_S - \Delta_L,\\ &\gamma N_L = \Gamma N_S = hV\,M\sin\Psi\,,\\ &\Delta_S = S_S N_S + F N_L \quad , \Delta_L = S_L N_L + F N_S.\end{aligned} \tag{6.6.17b}$$

In both cases (states **A** and **B**) the radius of the resonant surface k is still arbitrary. Like in the basic S-theory it is determined from the condition of the external stability. This will be studied below.

1 Stationary state A. Making use of (6.6.3) with the Hamiltonian (6.6.2) for the pair of the perturbation waves $a(\boldsymbol{k}_1), b(-\boldsymbol{k}_1)$ and assuming that $\omega + \Omega = \omega_p$ and:

$$a(\boldsymbol{k}) \propto \exp(i\omega t + 2\nu(\boldsymbol{k}_1)t]\,, \quad b^*(-\boldsymbol{k}) \propto \exp(-i\Omega t + 2\nu(\boldsymbol{k}_1)t]\,, \tag{6.6.18}$$

we obtain for the increment $\nu(\boldsymbol{k}_1)$:

$$\{\nu(k_1)+\gamma+i[\omega_{\rm NL}(\boldsymbol{k}_1)-\omega]\}\{\nu(k_1)+\Gamma-i[\Omega_{\rm NL}(\boldsymbol{k}_1)-\Omega]\} = |P|^2 \,. \quad (6.6.19)$$

The maximum (with respect to k_1) increment $\nu_{\max}(\boldsymbol{k}_1) = \nu$ corresponds to the wave vector k_1 satisfying the relations

$$\Omega_{\rm NL}(\boldsymbol{k}_1) + \omega_{\rm NL}(\boldsymbol{k}_1) = \omega_{\rm p}/2 \quad (6.6.20)$$

and is determined by the equation:

$$\nu^2 + \nu(\gamma+\Gamma) + \gamma\Gamma - |P|^2 = 0\,. \quad (6.6.21)$$

Therefore the condition of the external stability $\nu \le 0$ has the following form

$$|P|^2 \le \gamma\Gamma\,. \quad (6.6.22)$$

Allowing for this equation, (6.6.16b) enables us to find the radius of the resonant surface for the stable condition (6.6.19). In addition, these relations give:

$$M^2 = \gamma N_{\rm L}^2/\Gamma = \Gamma/N_{\rm S}^2/\gamma = \sqrt{h^2V^2-\gamma\Gamma}\,, \quad hV\sin\Psi = \sqrt{\gamma\Gamma}\,. \quad (6.6.23)$$

These relations explicitly generalize the corresponding results of the basic S-theory (see (5.5.7)). To study the stability of this state with respect to the emergence of new anomalous correlators (6.6.6b) is of great interest. Taking $\Sigma_{\rm L}$, $\Sigma_{\rm S}$ and $N_{\rm NL}$ to be proportional to $\exp[2\mu t]$, we obtain from (6.6.15) the equations for the increment of the *correlation instability* μ :

$$\begin{aligned} 2\mu^3 &+ \mu^2[3(\gamma+\Gamma) + i(\Delta_{\rm L} + S_{\rm L}N_{\rm L} - \Delta_{\rm S} - S_{\rm S}N_{\rm S})] \\ &+ \mu[(\gamma+\Gamma)^2 + \Delta_{\rm L}S_{\rm S}N_{\rm S} + \Delta_{\rm S}S_{\rm L}N_{\rm L} \\ &+ 3i(\Gamma\Delta_{\rm L} - \gamma\Delta_{\rm S}) + 2i\sqrt{\gamma\Gamma N_{\rm L}N_{\rm S}}(S_{\rm L}-S_{\rm S})] \\ &+ (\gamma+\Gamma)[\Delta_{\rm L}(S_{\rm S}N_{\rm S} + i\Gamma) + \Delta_{\rm S}(S_{\rm L}N_{\rm L} - i\gamma)] = 0\,. \end{aligned} \quad (6.6.24)$$

Hence under low supercriticalities we can obtain

$$(\gamma+\Gamma)^3{\rm Re}\mu = -2\gamma\Gamma(S_{\rm L}N_{\rm L}+S_{\rm S}N_{\rm S})^2 - (\gamma+\Gamma)^2F(S_{\rm L}N_{\rm L}^2+S_{\rm S}N_{\rm S}^2)\,. \quad (6.6.25)$$

Hence and from (6.6.16a) we can readily obtain the conditions of correlation stability under low supercriticality

$$\gamma\Gamma(\Gamma S_{\rm L} + \gamma S_{\rm S})^2 + (\gamma+\Gamma)^2F(\Gamma^2S_{\rm L} + \gamma^2S_{\rm S}) > 0 \quad (6.6.26a)$$

Similarly, we can find the condition of the correlation stability under high supercriticality

$$4(\Gamma S_{\rm L} + \gamma S_{\rm S})[(\Gamma S_{\rm L} + \gamma S_{\rm S}) + F(\gamma+\Gamma)] + F^2(\gamma-\Gamma)^2 > 0\,. \quad (6.6.26b)$$

Interestingly, at $\gamma = \Gamma$ the condition of the correlation stability under high and low supercriticalities coincide

$$(S_{\rm L} + S_{\rm S})(S_{\rm L} + S_{\rm S} + 2F) > 0\,. \quad (6.6.26c)$$

Of fundamental importance is the fact that under some relations between the coefficients this relation can fail to be observed. This means that the stationary state **A** is unstable with respect to the pairing inside the branches of the spectrum. It will be natural to assume that the development of this instability results in the transition of the parametric wave system to the state **B** with the complete phase correlation. Let us study the peculiarities of this state

2 Stationary state B. In order to obtain the radius of the resonance surface we must employ the condition of the external stability. Studying it we obtain a coupled system of equations for the following four variables:

$$a(\boldsymbol{k})\,, \quad b^*(-\boldsymbol{k})\,, \quad a^*(-\boldsymbol{k})\,, \quad b(\boldsymbol{k})\,. \quad (6.6.27)$$

The increment $\nu(\boldsymbol{k}_1)$ of the external instability is found from the condition that the determinant of the system of equations equals zero. The determinant is fourth-order polynomial with respect to $\nu(\boldsymbol{k}_1)$. It has four roots. One of them (namely $\nu(\boldsymbol{k}_1)$ at $k_1 = k$) leads to the condition

$$\gamma[\omega_{\rm NL}(\boldsymbol{k}) + \Delta_{\rm L} - \omega] = \Gamma[\Omega_{\rm NL}(\boldsymbol{k}) + \Delta_{\rm S} - \Omega]\,, \quad \omega + \Omega = \omega_{\rm p}\,. \quad (6.6.28)$$

In addition, for the external instability $\partial = {\rm Re}\{\nu(\boldsymbol{k}_1)\}/\partial k$ must be zero at $k = k_1$. This can be shown to result in the following equation

$$v_{\rm L}D_{\rm L} + v_{\rm S}D_S = 0\,, \quad (6.6.29)$$

where $v_{\rm L} = \partial\omega(k)/\partial k$, $v_{\rm S} = \partial\Omega(k)/\partial k$ and

$$\begin{aligned} D_{\rm L} = &\Delta_{\rm L}[\Omega_{\rm NL}(k) + \Delta_{\rm S} - \Omega][\Omega_{\rm NL}(k) + \Delta_{\rm L} - \Omega] \\ &+ \Delta_{\rm S}[\Omega_{\rm NL}(k) + \Delta_{\rm S} - \Omega][\omega_{\rm NL}(k) + \Delta_{\rm L} - \omega] \\ &+ \Gamma\{\gamma[\Omega_{\rm NL}(k) + \Delta_{\rm S} - \Omega] - \Gamma[\omega_{\rm NL}(k) + \Delta_{\rm L} - \omega]\} \\ &+ FN_{\rm L}(S_{\rm L}N_{\rm L}\Delta_{\rm S} + S_{\rm S}N_{\rm S}\Delta_{\rm L})\,. \end{aligned} \quad (6.6.30)$$

We can obtain hence the expression for $D_{\rm S}$ by replacing the values of $\Gamma \leftrightarrow \gamma$, $\Omega \leftrightarrow \omega$ and indices $L \leftrightarrow S$. Equations (6.6.28–30) specify two values $\Omega_{\rm NL}(k)$, $\Omega_{\rm NL,1}(k)$ and $\Omega_{\rm NL,2}(k)$ at which the condition of external stability can be satisfied. Accordingly, there are two types of stationary states **B1** and **B2**. The study of the remaining roots ν_2 , ν_3 and ν_4 (carried out in my doctorate thesis) for the case of $\gamma = \Gamma$ and $v_{\rm L} = v_{\rm S}$ showed that the stability of the states **B1** and **B2** is determined by the relations between the coefficients F, $S_{\rm L}$ and $S_{\rm S}$. For example, at $F = 0$ and $S_{\rm L}S_{\rm S} > 0$ the state **B1** is stable and **B2** is unstable, and when $F = 0$ and $S_{\rm L}S_{\rm S} < 0$ the opposite case takes place; at $F \gg S_{\rm L}, S_{\rm S}$ the state **B1** is stable and **B2** is unstable. Both states cannot be stable at the same time. Sometimes (e.g. at $2F > S_{\rm L} = S_{\rm S} > \sqrt{2}F > 0$) they are both unstable.

In conclusion we shall present the expressions for the numbers of parametric waves in the states **B1,2** at $\gamma = \Gamma$ and $v_L = v_S$. In the state **B1**:

$$N = \sqrt{h^2V^2 - \gamma^2}/|S + F| \, . \tag{6.6.31}$$

In the state **B2**:

$$N = \frac{\sqrt{h^2V^2 - \gamma^2}|2F + S_L + S_S|}{|F(S_L + S_S + 2S) + S(S_L + S_S) + 2S_L S_S|} \, . \tag{6.6.32}$$

Both these dependences differ from the function $N(hV)$ for the state **A** in the numeric factor. A more detailed study carried out in my doctorate thesis shows that the states **B1,2** can prove to be unstable with respect to decrease of the anomalous correlators N_{LS}, Σ_L and Σ_S . The development of this instability can bring the wave system into the state **A** when these correlators are zero. In its turn, the state **A** can prove to be unstable with respect to the increase of these correlators. Since we considered all possible stationary states and showed that they can all (in some range of the parameters F, S_L and S_S) be unstable, the only remaining possibility for the wave system is to perform *the correlation auto-oscillations*, under which the correlators of the wave system (N_{LS}, Σ_L, Σ_S, etc.) are non-stationary. We discovered the correlation auto-oscillations in the computer simulation of (6.6.15). It would be very interesting to observe this phenomenon experimentally.

It must be noted that the very possibility of the appearance of the correlation auto-oscillations is basically connected with the non-equilibrium of the system. In the thermodynamic equilibrium one of the states of the system (the *ground state*) must be absolutely stable and auto-oscillations are impossible. Thus in the problem of the superconductivity at $T > T_{th}$ the normal state of electrons (without pairing) is stable, and at $T < T_{th}$ stable is the superconducting state with anomalous correlators.

One more remark to the whole section 6.6. Everything above must be treated as the introduction to the problem of the parametric excitation of waves from the different spectrum waves. This theory may be developed in more detail as the experimental data are accumulated.

6.7 Parametric Excitation of Waves by Noise Pumping

It is well-known that not only monochromatic, but also noise pumping can lead to the parametric instability. The threshold of this instability depends on the frequency width of the pumping Δ. In the order of magnitude

$$\langle h^2V^2 \rangle \simeq \gamma(\gamma + \Delta) \, . \tag{6.7.1}$$

It is often erroneously assumed that under incoherent pumping the "phase" mechanism for the limitation of the parametric wave amplitude does not

function. However, it can easily be seen that the amplitude of parametric waves and phase relations calculated according to the S-theory are independent of the width of the pumping spectrum Δ at $\Delta < \gamma$. Therefore it must be expected that at $\Delta \simeq \gamma$ these values will not change qualitatively, i.e. the anomalous correlators $\sigma(\boldsymbol{k})$ in the pairs of parametric waves are not small in comparison with the normal correlators $n(\boldsymbol{k})$. Consequently, also under $\Delta > \gamma$ the anomalous correlators behave like some function of Δ/γ which tends to zero when $\Delta/\gamma \to \infty$, and if this function decreases slowly enough as Δ/γ increases, there exists a range $\Delta > \gamma$ where the self-consistent interaction is the most important.

6.7.1 Equations of S-Theory Under Noise Pumping

Nonlinear behavior of parametric waves under incoherent (noise) pumping was studied by *Cherepanov* in the approximation of the S-theory [6.21]. This *"Noise S-theory"* is presented below.

The pumping $h(t)$ will be assumed to be $h(t)=\tilde{h}(t)\exp(-i\omega_p t)$, where $\tilde{h}(t)$ is the random stationary function with the spectrum width Δ such that $\omega_p \gg \Delta \gg \gamma$. The equations of the S-theory for the slow amplitudes of parametric waves in this case can readily be obtained from (5.4.15, 16) by replacing $h \to \tilde{h}(t)$. From them it follows, in particular, that

$$\begin{aligned} &\partial|a(\boldsymbol{k},t)|^2/\partial t + 2\gamma(\boldsymbol{k})|a(\boldsymbol{k},t)|^2 = A(t)\,, \\ &A(t) = 2\mathrm{Im}\{P(\boldsymbol{k},t)a^*(\boldsymbol{k},t)a^*(-\boldsymbol{k},t)\}\,. \end{aligned} \tag{6.7.2}$$

The solution of this equation has the following form:

$$|a(\boldsymbol{k},t)|^2 = \int_0^\infty A(t - t_1)\exp[-2\gamma(\boldsymbol{k})t]\,dt \, . \tag{6.7.3}$$

The function $A(t)$ may be divided into two parts $A(t) = \langle A(t) \rangle + \delta A(t)$ where the angular brackets $\langle\ \rangle$ designate averaging over the random phases of the pumping. The statistical properties of the pumping are stationary, and therefore the function $\langle A \rangle$ is independent of time and in (6.7.3) it can be factored outside the integral sign. Then

$$\mathrm{Im}\langle A \rangle = \gamma(\boldsymbol{k})n(\boldsymbol{k})\,, \quad n(\boldsymbol{k}) = \langle |a(\boldsymbol{k},t)|^2 \rangle \, . \tag{6.7.4}$$

Here the contribution to the integral (6.7.3) of the quickly fluctuating part of $A(t)$ has been neglected. Indeed, the characteristic frequency of these fluctuations is $\Delta \gg \gamma$. Therefore the contribution of the $\delta A(t)$ to the integral (6.7.3) is (γ/Δ) times as little as the corresponding contribution of $\langle A \rangle$. Thus the wave number $n(\boldsymbol{k})$ as well as the renormalized frequency $\omega_{NL}(\boldsymbol{k})$ do not fluctuate under noise the pumping. For the correlators

$$\sigma(\boldsymbol{k},t) = \langle a(\boldsymbol{k},t)a(-\boldsymbol{k},t) \rangle \exp(i\omega_p t) \tag{6.7.5}$$

we can similarly obtain

$$\partial\sigma(\boldsymbol{k},t)/\partial t+\{2\gamma(\boldsymbol{k})+i[2\omega_{\rm NL}(\boldsymbol{k})-\omega_p]\}\sigma(\boldsymbol{k},t_1)=-2iP(\boldsymbol{k},t)n(\boldsymbol{k}).\quad(6.7.6)$$

Seeking the solution of the equation we have

$$\begin{aligned}\sigma(\boldsymbol{k},t)&=-\int_0^\infty \sigma(t-t_1)\exp\{[-2\gamma(k)-i(2\omega_{\rm NL}(k)-\omega_p)]t\}\,,\\ \sigma(t)&=n(\boldsymbol{k})P(\boldsymbol{k},t)\,.\end{aligned}\quad(6.7.7)$$

The functions $\sigma(\boldsymbol{k},t)$ and $P(\boldsymbol{k},t)$ contain the same fast time dependence with the characteristic time $1/\Delta$. The function $P^*\sigma$ in our approximation does not depend on time. In order to calculate this function, (6.7.7) must be multiplied by $P^*(\boldsymbol{k},t)$ and the integral must be calculated taking into account only the fast time dependence. This yields

$$\begin{aligned}&\mathrm{Im}\langle P^*(\boldsymbol{k},t)\sigma(\boldsymbol{k},t)\rangle=-2\pi n(\boldsymbol{k})P^2[k,\omega_{\rm NL}(\boldsymbol{k})],\\ &P(\boldsymbol{k},t)=\int_\infty^{-\infty}P(\boldsymbol{k},)\omega)\exp(-i\omega t)\,d\omega\,,\\ &\langle P(\boldsymbol{k},\omega)P^*(\boldsymbol{k},\omega)\rangle=P^2(\boldsymbol{k},\omega)\delta(\omega-\omega_1)\,.\end{aligned}\quad(6.7.8)$$

The combination of (6.7.2b, 4, 5) and (6.6.8) yields

$$\gamma(\boldsymbol{k})n(\boldsymbol{k})=2\pi n(\boldsymbol{k})P^2[k,\omega_{\rm NL}(\boldsymbol{k})]\,.$$

The non-trivial stationary solution of this equation exists if

$$\gamma(\boldsymbol{k})=2\pi P^2[k,\omega_{\rm NL}(\boldsymbol{k})]\,.\quad(6.7.9a)$$

If for some $\boldsymbol{k}$ the following inequality

$$\gamma(\boldsymbol{k})<2\pi P^2[k,\omega_{\rm NL}(\boldsymbol{k})]\,,\quad(6.7.9b)$$

is satisfied, then the amplitudes of waves with corresponding $\boldsymbol{k}$ exponentially increase, which contradicts the assumptions about the stationarity of the state. Therefore, for all $\boldsymbol{k}$ the condition of the external stability

$$\gamma(\boldsymbol{k})\le 2\pi P^2[k,\omega_{\rm NL}(\boldsymbol{k})]\,,\quad(6.7.10)$$

must be satisfied, whereas $n(\boldsymbol{k})$ is non-zero only when there is an equality sign in (6.7.10). This condition is analogous to the condition of the external stability arising under monochromatic pumping $P(\boldsymbol{k})\le\gamma(\boldsymbol{k})$. From the usual expression for the renormalization of the pumping (e.g. (5.4.12)) and (6.7.6) we obtain

$$\begin{aligned}h^2(\omega)V(\boldsymbol{k})V^*(\boldsymbol{k}')&=P^2(\boldsymbol{k},\boldsymbol{k}',\omega)\\ &-2\int\left\{\frac{S(\boldsymbol{k},\boldsymbol{k}')P(\boldsymbol{k},\boldsymbol{k}',\omega)n(\boldsymbol{k}_1)}{\omega-2\omega_{\rm NL}(\boldsymbol{k})+2i\gamma(\boldsymbol{k})}\right.\\ &\left.+\frac{S(\boldsymbol{k}',\boldsymbol{k}_1)P(\boldsymbol{k},\boldsymbol{k}_1,\omega)n(\boldsymbol{k}_1)}{\omega-2\omega_{\rm NL}(\boldsymbol{k})-2i\gamma(\boldsymbol{k}_1)}\right\}d\boldsymbol{k}_1\\ &+\int\frac{n(\boldsymbol{k}_1)n(\boldsymbol{k}_1)S(\boldsymbol{k},\boldsymbol{k}_1)S^*(\boldsymbol{k}',\boldsymbol{k}_2)P^2(\boldsymbol{k}_1,\boldsymbol{k}_2,\omega)}{[\omega-2\omega_{\rm NL}(\boldsymbol{k}_1)+2i\gamma(\boldsymbol{k}_1)][\omega-2\omega_{\rm NL}(\boldsymbol{k}_1)+2i\gamma(\boldsymbol{k}_2)]}d\boldsymbol{k}_1d\boldsymbol{k}_2\,.\end{aligned}\quad(6.7.11)$$

Solving this equation simultaneously with the condition of the external stability (6.7.10) we can obtain the distribution of parametric waves $n(\boldsymbol{k})$.

As it is known, under parametric excitation of waves by coherent pumping $n(\boldsymbol{k})$ differs from zero in a small region of the $\boldsymbol{k}$-space. It will subsequently be shown that at $\delta<kv$ (v is the group velocity of parametric waves) the region occupied by parametric waves is small, too. This enables us to assume that all the coefficients in (6.7.11) are independent of k. In some important cases, e.g. under spherical and axial distribution symmetry of parametric waves we can eliminate the angular dependence of the interaction amplitudes in the equation. This makes it possible to simplify (6.7.11), passing from the variable $\boldsymbol{k}$ to the variable $\omega_1=\omega_{\rm NL}(\boldsymbol{k})$:

$$\begin{aligned}h^2(2\omega)V^2&=P^2(2\omega)\left\{1-S\int\frac{n(\omega_1)\,d\omega_1}{(\omega_1-\omega+i0)}\right.\\ &\left.+S^2\left[\int\frac{n(\omega_1)\,d\omega_1}{(\omega_1-\omega+i0)}\right]^2+[\pi Sn(\omega)]^2\right\},\\ n[\omega(\boldsymbol{k})]&=\int d\Omega k^2(\Omega)n(k,\Omega)/v(\Omega)\,.\end{aligned}\quad(6.7.12)$$

6.7.2 Distribution of Parametric Waves Above Threshold

Cherepanov [6.21] analyzed (6.7.12) for the specific form of the spectral density of the noise pumping:

$$2\pi h^2(2\omega)V^2=\gamma p(1-\omega^2/\Delta^2)\,,\qquad \Delta\gg\gamma\,.\quad(6.7.13)$$

In the region of low supercriticalities ($p-1<1$) he obtained

$$\begin{aligned}3\pi Sn(\omega)&=(p-1)\sqrt{(\sqrt{3}\omega+\mu)^3(3\mu-\sqrt{3}\omega)}/\Delta^2\\ \mu&=\Delta\sqrt{p-1}\cdot\mathrm{sign}\,S\,,\end{aligned}\quad(6.7.14)$$

It must be noted that this solution is non-zero in the asymmetric range $(-\mu/\sqrt{3}\,,3\mu)$ and is not symmetric in spite of the symmetrical profile of the pumping (6.7.13). In the limiting case of high supercriticalities it follows from (6.8.10, 12) that

$$3\pi S n(\omega) = \sqrt{(p-1)(3-4\omega^2/\Delta^2)^3}\,. \tag{6.7.15}$$

This solution differs from zero in the symmetric region $|\omega| < \omega_1 = \sqrt{3}\Delta/2$, which is narrower than the instability region $|\omega| < \Delta$, where $h(2\omega) > 0$. Such a narrowing is characteristic of the S-theory. Because of this the behavior of $h^2(2\omega)$ at $|\omega| > \omega_1$ is not significant for our problem and the solution of (6.7.10) qualitatively holds true whatever the shape of the spectral plane of the pumping.

Integrating (6.7.14, 15) with respect to ω we can readily obtain the dependence of the total number of parametric waves on the supercriticality

$$N = 4\Delta(p-1)^{3/2}/9\pi\sqrt{3}|S|\,, \text{ at } p-1<1\,, \tag{6.7.16}$$

$$N = \sqrt{3(p-1)}/2\pi|S|\,, \text{ at } p \gg 1\,. \tag{6.7.17}$$

At $\Delta \simeq \gamma$ the expression (6.7.17) is in agreement with (5.5.7) for the case of the coherent pumping. As for the agreement of (6.7.15) and (5.5.7) for low supercriticalities, under $\Delta \simeq \gamma$ the first formula holds true because in this case there is not rigid correlation of the wave phases in the pairs. If the ratio γ/Δ becomes smaller than 1 then the term $\mathrm{Re}\{\langle P^*(\boldsymbol{k})\sigma(\boldsymbol{k})\rangle\}$ is about γ/Δ. Therefore under small γ/Δ the term quadratic in σ must be allowed for.

Now let us obtain the limit of applicability of the mean–field approximation under the parametric wave excitation by the noise pumping. To this end, let us compare the term $\mathrm{Im}\{\langle P^*(\boldsymbol{k})\sigma(\boldsymbol{k})\rangle\}$ with the terms of the kinetic equation (of the order of magnitude $(SN)^2 n(\boldsymbol{k})/(kv)$ omitted in (6.7.11). As a result we find that our approach holds true under

$$\Delta^2(p-1) < \gamma k v\,. \tag{6.7.18}$$

In solids $kv/\gamma \simeq 10^2$ - 10^6 which provides wide enough applicability scope for the S-theory approximation. Note also that the opposite limiting case when the scattering of parametric waves exceeds their self-consistent interaction has been studied by *Levinson* [6.22].

Now let us present the estimation of the angular size of the excited packets. As in the case of the monochromatic pumping the angular sizes of the packet of parametric waves are zero. This can easily be checked taking the example of the simplest model

$$V(\boldsymbol{k}) = V f(\Theta)\,, S(\boldsymbol{k},\boldsymbol{k}') = S f(\Theta) f(\Theta')\,, \gamma(\boldsymbol{k}) = \gamma\,,$$

qualitatively valid for ferromagnets. In this case

$$P(\boldsymbol{k},\boldsymbol{k}',\omega) = P^2(\omega) f(\Theta) f(\Theta')$$

and in (6.7.11) the dependence on the angles can be eliminated, and it is possible to show that parametric waves are concentrated in the angle Θ_0, where $|f(\Theta_0)|$ is at maximum. The weak scattering of parametric waves can be shown to lead to the packet broadening by the angle $\Delta\Theta$:

$$\Delta\Theta \simeq (SN/kv) \simeq \sqrt{p-1}/kv\,. \tag{6.7.19}$$

In conclusion note that the above-developed theory is in good agreement with the specially designed experiment on the parametric excitation of magnons in the ferrimagnetic YIG performed by *Zautkin, Orel* and *Cherepanov* [6.23] (see Sect. 9.10.2). It enables us to assume that the amplitude of parametric waves under noise pumping is limited, as under coherent excitation, by the S-theory phase mechanism.