Victor S. L'vov

Wave Turbulence Under Parametric Excitation

Applications to Magnets

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7 Non-Stationary Behavior of Parametrically Excited Waves

7.1 Spectrum of Collective Oscillations (CO)

Chapters 5, 6 described the stationary state of the system of parametrically excited waves. The present chapter treats the non-stationary processes due to the time dependence of the experimental conditions. First we shall consider small *collective oscillations* (CO) with respect to the stationary state.

7.1.1 Spectrum of Spatially Homogeneous CO in the Non-Dissipation Limit

If the damping of the parametrically excited waves γ is zero, the Hamiltonian of the system $\mathcal{H}_0$ is the integral of motion. Let the value of $\mathcal{H}$ in the perturbed state differ from its value $\mathcal{H}_0$ in the ground state. This implies that the system will never attain the ground state and, since it has no other stable stationary states, its behavior will be essentially non-stationary. To study it, we must obtain the frequencies (spectrum) of the collective oscillations of the parametric wave system. For definiteness, consider the cubical ferromagnet under the supercriticality below the second threshold when parametric waves in the ground state are excited in the equator plane. Let $d(\boldsymbol{k},t)$ denote the deviations from the ground state. Then we shall extract the part $\mathcal{H}_0$ of the S-theory Hamiltonian corresponding to the ground states and the parts $\mathcal{H}_1$ and $\mathcal{H}_2$ containing the linear terms and the terms quadratic in $d(\boldsymbol{k},t)$. Taking into account the equations of motion we can see that the ground state energy is extreme: $\mathcal{H}_1 = 0$. The part of the Hamiltonian quadratic in small perturbations after we pass from summation to integration assumes the following form:

$$\mathcal{H}_2 = \frac{N_1}{4\pi^2}\Big\{[\exp(i\Psi)\int T_{12}d_1d_2\exp[-i(\varphi_1+\varphi_2)] + \text{c.c.}]\,d1d2$$
$$+\,2\int\{S_{12}\exp[-i(\varphi_1-\varphi_2)] + T_{12}\exp[i(\varphi_1-\varphi_2)]d_1d_2^*\}\,d1d2\Big\}\,,$$
$$d_j = d(\varphi_j,t)\,,\quad dj = d\varphi_j\,,\quad T_{12} = T(\varphi_1-\varphi_2) = T(\boldsymbol{k}_1,\boldsymbol{k}_2)\,. \qquad (7.1.1)$$

Here $\Theta = \Theta_1 = \pi/2$, $|\boldsymbol{k}_1| = |\boldsymbol{k}_2| = |\boldsymbol{k}_0|$. Changing over to Fourier components and using the axial symmetry of the problem we obtain

$$\mathcal{H}_2 = N_1 \sum_{m=-\infty}^{\infty} [2(T_m + S_m) d_m d_m^* + (T_m d_m d_{-m} + \text{c.c.})] , \tag{7.1.2}$$

$$2\pi T_m = \int_0^{2\pi} T(\varphi - \varphi_1) \exp[-im(\varphi - \varphi_1)] \, d(\varphi - \varphi_1) ,$$

$$2\pi S_m = \int_0^{2\pi} S(\varphi - \varphi_1) \exp[-i(m-2)(\varphi - \varphi_1)] \, d(\varphi - \varphi_1). \tag{7.1.3}$$

The Hamiltonian (7.1.2) can be diagonalized by the linear canonical transformation which yields

$$\mathcal{H}_2 = \sum_m \Omega_m d_m^* d_m , \tag{7.1.4a}$$

$$\Omega_m = \left\{ (S_m - S_{-m}) + \sqrt{(S_m + S_{-m})(S_m + S_{-m} + 4T_m)} \right\} N_1 . \tag{7.1.4b}$$

This transformation is possible if the frequency of the collective oscillations Ω_m in the system of parametric waves is real. At $(S_m + S_{-m})(S_m + S_{-m} + 4T_m) < 0$ this frequency is imaginary which indicates to the instability of the ground state with respect to the excitation of the exponentially increasing oscillations (the *internal instability* mentioned in Chap.5).

The canonical transformation unambiguously determines the sign before the radical in the expression (7.1.4b) for the frequency of the collective oscillations. But again, this sign can be easily obtained from simple considerations. Under small T_m the sign of Ω_m obviously coincides with the sign of $(S_m + T_m)$. Therefore the positive sign before the radical must be chosen for (7.1.4b). As T_m continuously increases the sign before the radical remains the same because the canonical transformation is continuous. Note that the frequency Ω_m may be negative. This means that the excitation of the collective excitation has resulted in the decreased energy of the system of parametric waves. Their relaxation is accompanied by the increase of the system energy. This does not violate the law of conservation of energy since the system of parametric waves acquires energy from the pumping.

In cubic ferromagnets under symmetric directions of magnetization ($\boldsymbol{M}$||[111]) or [100]) the values S_m and T_m are non-zero only at $m = 0, \pm 1, \pm 2$. The expressions for coefficients S_m, T_m, corresponding to these modes can be obtained from (3.1.25, 26) and (7.1.3):

$$S_0 = 2\pi g^2 (\omega_M/\omega_p)^2 \left\{ \sqrt{[1 + (\omega_p/\omega_M)^2]} + N_{z0,\text{ef}} - 1 \right\} ,$$

$$T_0 = S_0 + 2\pi g^2 (N_{z0,\text{ef}} - 1) , \qquad N_{z0,\text{ef}} = N_z + \beta\omega_p/\omega_M ,$$

$$\beta = \begin{cases} -8 & , \text{for } \boldsymbol{M} \parallel [111] , \\ +9 & , \text{for } \boldsymbol{M} \parallel [100] , \end{cases}$$

$$S_1 = S_{-1} = 0 , \quad T_1 = T_{-1} = \frac{1}{2}(T_2 + T_{-2} + T_0 + S_0 - S_2 - S_{-2}) ,$$

$$T_{\pm 2} = \frac{\pi g^2 \omega_M^2}{2\omega_p^2} \left[(N_{z2,\text{ef}} - 1) u_\pm^2 + \sqrt{1 + (\omega_p/\omega_M)^2} \right] ,$$

$$S_{\pm 2} = 2\pi g^2 \left[(N_{z2,\text{ef}} - 1) u_\pm^2 + u_\pm \omega_M / 2\omega_p \right] ,$$

$$u_\pm = (\sqrt{1 + (\omega_M/\omega_p)^2} \mp 1)/2 , \quad N_{z2,\text{ef}} = \omega_{\text{ex}}(ka)^2 + \omega_a/\omega_p . \tag{7.1.5}$$

The formulae (7.1.5) show the dependence of the CO frequency on the experimental conditions, i.e. the supercriticality, pumping frequency, magnetization, external magnetic field, the shape of the sample and crystallographic anisotropy. For the easy-plane ferromagnet it is clear from (3.2.13) that $S_0 = T_0$, and all the rest S_m and T_m are zero. This leads to the stability of the ground state experimentally observed by *Kotuzhansky* and *Prosorova* [7.1], *Prosorova* and *Smirnov* [7.2] up to very high supercriticality.

7.1.2 Influence of Wave Damping on the CO Spectrum

How does the damping of waves influence the spectrum of collective oscillations? This question is very important especially because the damping rate of parametrically excited waves $\gamma(\boldsymbol{k})$ can be of the same order as the frequency of collective oscillations Ω_m. By linearizing (5.4.15) with respect to the deviations from the ground state (5.5.7) and assuming $d, d^* \simeq \exp(-i\Omega t)$ we obtain the system of algebraic equations homogeneous in d, d^*. The condition of their solvability determines the frequency and damping of collective oscillations:

$$\Omega_m = -i\gamma + (S_m - S_{-m}) N_1 + \sqrt{(S_m + S_{-m})(S_m + S_{-m} + 4T_m) N_1^2 - \gamma^2} . \tag{7.1.6}$$

Hence we can draw an important conclusion that the criterion on the emergence of the inner instability of the collective modes is independent of the damping rate and is, as in the conservative case, given by the inequality

$$(S_m + S_{-m})(S_m + S_{-m} + 4T_m) < 0 . \tag{7.1.7}$$

Within the frame of the elementary theory collective oscillations of the parametric wave system are spatially homogeneous. When the spatial dispersion is allowed for a whole branch $\Omega_m(\boldsymbol{\kappa})$ corresponds to each normal mode, (7.1.6) specifying the gap of this branch. The dependence $\Omega_m(\boldsymbol{\kappa})$ will be calculated in the following section.

7.1.3 Spectrum of Spatially Non-Homogeneous CO

In order to obtain the frequency of non-homogeneous collective oscillations $\Omega_{\rm m}(\boldsymbol{\kappa})$ (*spectrum of CO*) (6.5.9) must be linearized in small deviations from the ground state $A(\boldsymbol{k},z,t) = A(\boldsymbol{k},z) + d(\boldsymbol{k},z,t)$ and the following form of their solution must be sought $d(\boldsymbol{k},z,t)$ and $d^*(\boldsymbol{k},z,t) \propto \exp[i(\kappa z - \Omega t)]$. The results for two simple, but important cases are given below (for more details, see *L'vov* and *Rubenchik* [7.3]).

1 Isotropic model: $S(\boldsymbol{kk}') = S$, $T(\boldsymbol{kk}') = T$, $V(\boldsymbol{k}) = V$. This case is characteristic of the easy-plane antiferromagnets. The dependence of the frequency of collective oscillations on their wave vector is given by their dispersion equation

$$\Delta^2(\Omega + 2i\gamma)\Omega \int_0^{2\pi} \left[(kv)^2\cos^2\varphi + \Omega(\Omega+2i\gamma)\right] d\varphi = 4\pi^2 , \qquad (7.1.8)$$

where $\Delta = \Omega_0$, Ω_0 is given by (7.1.6) at $\gamma = 0$. For $(kv) \ll L$ the solution of (7.1.8) has the form

$$\Omega(\kappa) = -i\gamma \pm \sqrt{\Delta^2 + (kv)^2 - \gamma^2} . \qquad (7.1.9)$$

Under $\kappa = 0$ (7.1.9) passes to the expression for the frequency of the spatially-homogeneous collective oscillations discussed in the previous section. The negative Δ^2 results in the development of instability. The oscillations are the most unstable under $\kappa = 0$, the instability region extends to $\kappa v = \Delta/2$. At $\Delta^2 > 0$ the stationary solution is stable under any κ.

2 Axially symmetrical model: Let us study the spectrum of collective oscillations $\Omega(\boldsymbol{\kappa})$ in the case very important for experiments on ferromagnets when the wave amplitude is non-zero at the latitudes $\Theta = \Theta_0$ and $\Theta = \pi/2 - \Theta_0$. In the cubic ferromagnets under parallel pumping with the supercriticality up to 6 – 8 dB $\Theta_0 = \pi/2$ (equator). The transverse pumping often leads to the realization of the case $\Theta_0 \simeq \pi/4$. The dependence $\Omega(\kappa)$ has the simplest form when $\boldsymbol{\kappa}$ is parallel to the axis of symmetry. For each harmonic with the number m we can obtain

$$\Omega_{\rm m1+}(\kappa) = -i\gamma \pm \sqrt{\delta_{\rm m1}^2 - \gamma^2} , \quad \Omega_{\rm m2+}(k) = -i\gamma \pm \sqrt{\delta_{\rm m2}^2 - \gamma^2} .$$

$$\delta_{\rm m1,2}^2 = (\kappa v)^2 + [\Delta_{\rm m1}^2 + \Delta_{\rm m2}^2]/2$$
$$\pm\sqrt{(\Delta_{\rm m1}^2 - \Delta_{\rm m2}^2)^2/4 + (\kappa v)^2(\Delta_{\rm m1}^2 + \Delta_{\rm m2}^2 + \Delta_m^2)} .$$
$$\Delta_{\rm m}^2 = 4(T_{\rm m}^+ - T_{\rm m}^-)(T_{\rm m}^+ + T_{\rm m}^- + S_{\rm m})N^2 ,$$
$$\Delta_{\rm m1}^2 = [\omega''\kappa^2/2 + (2S_{\rm m} + T_{\rm m}^+ + T_{\rm m}^-)N]^2 - (T_{\rm m}^+ + T_{\rm m}^-)^2N^2 ,$$
$$\Delta_{\rm m2}^2 = [\omega''k^2/2 + (T_{\rm m}^+ - T_{\rm m}^-)N]^2 - (T_{\rm m}^+ - T_{\rm m}^-)^2N^2 . \qquad (7.1.10)$$

Here $T_{\rm m}^{\pm}$, $S_{\rm m}$ - are the Fourier harmonics of the coefficients $T(k_z, k_z', (\varphi - \varphi'))$, $S(k_z, k_z'(\varphi - \varphi'))$ (6.5.9). For simplicity, in (7.1.10) we assumed $S_{\rm m} = S_{-\rm m}$. At $\delta_{\rm m1,2}^2(\kappa) > 0$ the expressions (7.1.10) correspond to collective oscillations with the frequency $\delta_{\rm m1,2}$, damping decrement γ_2 equal to the damping decrement of parametric waves. At $\delta_{\rm m1,2}^2(\kappa) < 0$: ${\rm Im}\Omega > 0$. This corresponds to the exponential increase of the spatially inhomogeneous collective oscillations, i.e. to the instability of the spatially homogeneous distribution of parametric waves.

Let us analyze the expressions obtained (7.1.10) for the frequencies of collective oscillations. At $\kappa = 0$

$$\delta_{\rm m1}^2(0) = \Delta_{\rm m1}^2(0) = 4S_{\rm m}(T_{\rm m}^+ + T_{\rm m}^- + S_{\rm m})N^2 . \qquad (7.1.11)$$

The oscillation $\Omega_{\rm m1}(0)$ is stable if $S_{\rm m}(T_{\rm M}^+ + T_{\rm M}^- + S_{\rm m}) > 0$. The oscillation $\Omega_{\rm m2}$ is neutrally stable. At small κv we have

$$\delta_{\rm m1}^2(\kappa) = \delta_{\rm m1}^2(0) + (\kappa v)^2(2S_{\rm m} + T_{\rm m}^+ - T_{\rm m}^-)/S_{\rm m} ,$$
$$\delta_{\rm m1}^2(\kappa) = (\kappa v)^2(T_{\rm m}^+ - T_{\rm m}^-)/S_{\rm m} . \qquad (7.1.12)$$

Therefore if $S_{\rm m}(T_{\rm m}^+ - T_{\rm m}^-) < 0$ the branch of the collective oscillations $\delta_{\rm m2}(\kappa)$ at $\kappa \neq 0$ becomes unstable. At the same time as κ increases the margin of stability of the branch $\delta_{\rm m2}$ also increases. Assuming $\Delta_{\rm m1}^2 > 0$, $\Delta_{\rm m}^2 > 0$ we find that the branch $\delta_{\rm m1}$ is stable at any κ, and the instability region of the branch $\delta_{\rm m2}$ is between $\kappa = 0$ and $\kappa = \kappa_0$, where $(\kappa_0 v)^2 = \Delta_{\rm m}$. In the cubic ferromagnets κv may prove to be zero. This happens if the waves are excited on the equator and the perturbation is perpendicular to its plane and if because of the anisotropic dispersion law group velocity of the excited waves $v(\Theta)$ turns into zero for $\Theta = \Theta_0$. In this case (7.1.10) is reduced and assumes the following form:

$$\Omega_{\rm m+}(\kappa) = -i\gamma \pm \sqrt{[2(T_{\rm m} + S_{\rm m})N + \omega''\kappa^2/2]^2 - 4T_{\rm m}^2N^2 - \gamma^2} . \qquad (7.1.13)$$

For simplicity, we took $T^+ = T^- = T$. This relation is always satisfied, for example for waves excited on the equator. At the same time the second branch of oscillations $\delta_{\rm m2}$ is always stable. From (7.1.13) it is clear that even the $\delta_{\rm m1}^2(0) > 0$ instability can emerge in the region $\omega''\kappa^2 \simeq 4(T_{\rm m} + S_{\rm m})N^2$ if $\omega''(T_{\rm m} + S_{\rm m}) > 0$. As will be shown subsequently, the development of the instability with $\kappa = 0$ leads to the emergence of self-oscillations of the total number of waves. Under parametric excitation of spin waves in ferromagnets they manifest themselves experimentally as magnetization auto-oscillations. The sign of ω'' in cubic ferromagnets is determined by the value of the external field, and we can easily provide the condition when the spatially-homogeneous oscillations are stable and the instability is localized within the range $\omega''\kappa^2 \simeq SN$. The development of this instability can result in a stationary, spatially-homogeneous picture $A(\boldsymbol{k},z)$ with the scale $\sqrt{SN/\omega''}$, modulation depth of the order of unity and the average number of the order

N. This raises the interesting problem of finding such stationary states and studying their stability within (6.5.9) which has not been solved yet. To observe such phenomenon experimentally would also be interesting.

7.2 Linear Theory of CO Resonance Excitation

Collective oscillations (CO) of the system of parametric waves discussed in the previous section can be excited (like all other types of oscillations) in different ways: by resonance external action, parametrically or by impact. All these ways have been employed in experiments on cubic ferromagnets and easy-plane antiferromagnets. To formulate the theory and to interpret experimental data would be easier for the resonance excitation of collective oscillations. This way was employed in 1972 by *Zautkin, L'vov* and *Starobinets* [7.14] who discovered and studied COs. In these experiments in addition to parallel pumping, another SHF signal with the same polarization $\boldsymbol{h}_z||\boldsymbol{h}||\boldsymbol{M}$, $h_z(t) \simeq \exp(-i\omega t)$ was fed to the sample. Its frequency ω differed from the pumping frequency ω_p by the frequency of collective oscillations Ω. The beatings between two SHF-signals at the resonance frequency served in this case as the resonance external force. Later (in 1975) *Orel* and *Starobinets* [7.5] employed the direct resonance excitation by an alternating magnetic field at the frequency of the COs which is within the single-frequency range (about 1 MHz). Collective oscillations could be excited also by a sound whose frequency and wave vector coincide with the frequency and wave vector of collective oscillations. All these methods of excitation of collective oscillations will be discussed in this section.

7.2.1 Basic Equations and Their Solution

As we are interested only in the behavior of spatially-homogeneous collective oscillations of the zeroth mode ($\kappa = 0$, $m = 0$) we shall average only the equations of motion of the basic S-theory (5.4.15) over all $\boldsymbol{k}$ directions in the package of parametric waves:

$$\{\partial/\partial t+\gamma+i[\omega(\boldsymbol{k})-\omega_p/2+2T_0c^*c]\}c+i(hV_0+S_0c^2)c^* = -if(t), \tag{7.2.1}$$

$$f(t) = f(\Omega)\exp(-i\Omega t)+f(-\Omega)\exp(i\Omega t) . \tag{7.2.2}$$

Here $|c(t)|^2 = N(t)$ is the total number of parametric waves in the package V, T and S are the coefficients of the Hamiltonian of the problem averaged over the package. For cubic ferromagnets they are given by (7.1.5); for easy-plane ferromagnets they are specified by (3.2.13). The periodic force $f(t)$ is added to the first right-hand side of (7.2.1), its nature will be discussed in detail.

A. When an additional weak signal acts on the system of parametric waves at the frequency ω (see Problem 5.3) then

$$f(\Omega) = h_zV_0c_0 , \quad f(-\Omega) = 0 , \quad \Omega = \omega - \omega_p . \tag{7.2.3}$$

B. If the frequency of the weak signal Ω is within the radio frequency range, then (see Problem 5.3)

$$f(\Omega) = f(-\Omega) = U_0c_0h_z , \quad U_0 = g[1+\sqrt{1+\omega_M^2/\omega_p^2}]/2 . \tag{7.2.4}$$

The expression for U_0 corresponds to the cubic ferromagnet [(see (4.3.20)] when parametric waves are concentrated on the equator.

C. If collective oscillations in the easy-plane ferromagnets are excited by the sound, then (see Problem 5.4)

$$\begin{aligned} f(\Omega,\boldsymbol{\kappa}) =& V_{sm}(\boldsymbol{\kappa},\boldsymbol{k})c_0[\beta(\Omega,\boldsymbol{\kappa})+\beta^*(-\Omega,-\boldsymbol{\kappa})] , \\ f(-\Omega,\boldsymbol{\kappa}) =& V_{sm}(-\boldsymbol{\kappa},\boldsymbol{k})c_0[\beta(\Omega,-\boldsymbol{\kappa})+\beta^*(-\Omega,\boldsymbol{\kappa})] . \end{aligned} \tag{7.2.5}$$

Here $\beta(\Omega,\boldsymbol{\kappa})$ denotes the Fourier transform of the canonical amplitude of the sound $\beta(\Omega,t)$, and V_{sm} designates the amplitudes of the Hamiltonian of the sound interaction with magnons. This Hamiltonian has been calculated by *Lutovinov* [7.6]:

$$\mathcal{H}_{sm} = \sum_{\boldsymbol{\kappa},\boldsymbol{k}} V_{sm}(\boldsymbol{\kappa},\boldsymbol{k})[\beta(\boldsymbol{\kappa},t)c(\boldsymbol{k},t)c^*(\boldsymbol{k}+\boldsymbol{\kappa},t)+\text{c.c.}] , \tag{7.2.6}$$

where $V_{sm} \simeq \sqrt{\Omega(\boldsymbol{\kappa})/[\omega(\boldsymbol{k})\omega(\boldsymbol{k}+\boldsymbol{\omega})]}$. Equations (7.2.2) suggest how to seek the solution of (7.2.1) in the approximation linear in f:

$$\begin{aligned} c(t) =& c_0+d(\Omega)\exp(-i\Omega t)+d(-\Omega)\exp(i\Omega t , \\ d(\Omega) =& d_+(\Omega)f(\Omega)+d_-(\Omega)f^*(-\Omega) . \end{aligned} \tag{7.2.7}$$

Simple calculations here yield

$$\begin{aligned} d_+(\Omega) =& -[\Omega+i\gamma+2(T_0+S_0)N]/\Delta(\Omega) , \\ c_0d_+ =& c_0^*d_- , \qquad N = |c_0|^2 . \\ \Delta(\Omega) =& \Delta_0^2-\Omega^2-2i\gamma\Omega , \qquad \Delta_0^2 = 4S_0(2T_0+S_0)N^2 . \end{aligned} \tag{7.2.8}$$

As it should be expected, the susceptibilities d are maximum when the frequency of the external force is close to the eigenfrequency of the collective resonance.

7.2.2 CO Excitation by a Microwave Field

The effectiveness of the excitation of collective oscillations by the magnetic field h_z can be characterized by the susceptibility $\chi(\omega_p+\Omega)$:

$$m_z(\omega_p+\Omega) = \chi(\omega_p+\Omega)h_z(\omega_p+\Omega) . \tag{7.2.9}$$

Here $m_z(\omega_p+\Omega)$ denotes the oscillation amplitude at the frequency $(\omega_p+\Omega)$ of the magnetization component $m_z(t)$ which depends on the number of the parametric magnons and, consequently, changes when collective oscillations are excited. In our approximation and using the notation of this section we can easily obtain $m_z((\omega_p+\Omega) = V_0 c_0 d(\Omega)$. Employing (7.2.7, 8) for c_0, $d(\Omega)$ and $f(\Omega)$ and the definitions of (7.2.6) we obtain the expressions for the imaginary part of the susceptibility to the weak SHF-signal, characterizing the absorption of its energy by the system of parametric magnons:

$$\chi''(\omega_p+\Omega) = \chi''\left(\frac{h}{h_{th}}\right)\frac{2\gamma^2[\Omega_0^2+\Omega^2+4\Omega(T_0+S_0)N}{(\Omega_0^2-\Omega^2)^2+4\gamma^2\Omega^2} . \tag{7.2.10}$$

Here χ'' is the imaginary part of the susceptibility to the pumping field h, given by (5.5.31a). Under high supercriticality when $\Delta_0^2 > \gamma^2$ the line shape (7.2.10) is close to the Lorentz shape and the line width is equal to the damping of magnons γ. At the resonance points the susceptibility is equal to

$$\chi''(\omega_p \pm \Omega) = \chi''\frac{h^2}{h_{th}^2}\left\{1 \pm \sqrt{1+\frac{T_0}{S_0(2T_0+S_0)}}\right\} . \tag{7.2.11}$$

The susceptibility may become negative. This fact is of cardinal importance, and corresponds not to the absorption but to the amplification of the weak signal. It follows from (7.2.11) that the absorption takes place at the frequency $\omega_p+\Omega$ and the amplification occurs at the image frequency $\omega_p-\Omega$. The amplification can be interpreted as the result of the instability with respect to decay of the ground state (with the "slow" frequency equal to zero) into the electromagnetic irradiation (with the slow frequency Ω) and collective oscillations with the eigenfrequency Δ_0. The law of conservation of energy in this process is $\delta_0+\Omega=0$. Therefore the amplification occurs at the frequency $\Omega=-\delta_0$ which corresponds to (7.2.11). The absorption in these terms is the consequence of the decay of the weak signal into the ground state and collective oscillations with the conservation law $\Omega=\Omega_0+0$.

The collective resonance has been experimentally observed by *Zautkin* et al. [7.4] on the YIG monocrystals. A good quantitative agreement with the theory presented here has been observed. In particular, the experimental dependences of the susceptibility to the weak signal and of collective oscillation frequency on the pumping intensity are in good agreement with the dependences given by (7.2.8, 11) up to the second threshold $h/h_2 = 8$ dB. Note also that the collective resonance can be used as a convenient and instructive method for changing the relaxation times of magnons. Measurements show that in accordance with the theory the line width of resonance absorption is practically independent of the pumping intensity and is in good agreement with the value obtained from the threshold value of parallel pumping. A more detailed comparison of theory with experiment will be given in Sect. 9.5.

7.2.3 Direct CO Excitation by a Radio Frequency Field

The pumping method of collective oscillations at the frequency Ω by means of the longitudinal SHF field h_2 at the frequency $\omega_p+\Omega$ described in the previous subsection seems somewhat artificial. A more natural way would be to employ the radio frequency magnetic field at the frequency Ω. However, the first attempts to observe this effect were unsuccessful because of the low susceptibility in the radio frequency range and only in 1975 did *Orel* and *Starobinets* [7.5] succeed in detecting this effect by means of their own sensitive methods.

In order to calculate the susceptibility $\chi(\Omega)$ to the longitudinal field $h_z(\Omega)$ note that in the notation of this section the expression for the alternating part m_z with the frequency Ω has the form

$$m_z(\Omega) = U_0[c_0^* d(\Omega)+c_0 d^*(-\Omega)] = \chi(\Omega)h_z(\Omega) . \tag{7.2.12}$$

From this equation and (7.2.4, 7, 8), we readily obtain

$$\chi = 2U_0^2 S_0 N^2/\Delta^*(\Omega) , \qquad \Delta(\Omega) = \Delta_0^2 - \Omega^2 - 2i\gamma\Omega . \tag{7.2.13}$$

The maximum value $\chi''(\Omega)$ corresponds to the resonance frequency Ω_{res} which, generally speaking, does not coincide with Δ_0 and under $\Delta_0 > 2\gamma$ is given to a good accuracy by

$$\Omega_{res}^2 = \Delta_0^2 - \gamma^2 . \tag{7.2.14}$$

In this approximation the half-width of collective resonance curve measured at the level 1/2 equals to the relaxation of magnons. Let us give the value of the susceptibility in resonance

$$\chi''_{res} = \frac{g^2\sqrt{\Omega_0^2-\gamma^2}\,S_0 N_0^2}{\gamma[\Omega_0^2-(3/4)\gamma^2]} . \tag{7.2.15}$$

Under high supercriticality $\Delta_0^2 \gg \gamma^2$ and (7.2.15) is also reduced. Substituting the dependence of Ω_0 and N_0 on $p=(h/h_{th})^2$ we obtain

$$\chi''_{res} = \frac{S_0 g^2\sqrt{p-1}}{|S_0|2S_0\sqrt{T_0+S_0}} . \tag{7.2.16}$$

From this formula an important conclusion can be drawn: at $S_0>0$, (the case of ferromagnets) $\chi''>0$ and radio-frequency radiation absorbs in the sample; at $S_0<0$ (the case of easy-plane antiferromagnets), $\chi''<0$, which corresponds not to the absorption but to the amplification of the radio-frequency signal. If this amplification exceeds the damping in the oscillatory circuit, it will lead to instability with respect to the excitation at the frequency Ω – a laser effect of a kind.

The theory presented in this section is in good agreement with the experimental data for the ferromagnetic YIG (see Chap. 9).

7.2.4 Coupled Motions of Collective Excitations of Parametric Waves and Sound

Theoretically, this question does not differ from the problem of CO interaction with the radio frequency field, only the expression for the force $f(\Omega)$ (7.2.4) must be substituted for expression (7.2.5). However, to setup an experiment on measuring the susceptibility of the collective oscillations to the sound is rather difficult. Therefore, in studying the interaction of collective oscillations with the sound attention should be rather paid to another statement of the problem aimed at the study of their coupled motions.

Let us calculate the frequencies of these motions [7.6]. From the expressions for the Hamiltonian of the magnetoelastic interaction $\mathcal{H}_{\mathrm{sm}}$ (7.2.6) follow the equations of motion for the complex amplitude of sound $\beta(\Omega, \boldsymbol{\kappa})$:

$$\begin{aligned}[][\Omega - \Omega_{\mathrm{S}}(\boldsymbol{\kappa}) + i\Gamma_{\mathrm{S}}(\boldsymbol{\kappa})]\beta(\Omega, \boldsymbol{\kappa}) &= V^*(\boldsymbol{k}, \boldsymbol{\kappa})[c_0^* d(\Omega) + c_0 d^*(-\Omega)],\\ [\Omega + \Omega_{\mathrm{S}}(\boldsymbol{\kappa}) + i\Gamma_{\mathrm{S}}(\boldsymbol{\kappa})]\beta(-\Omega, -\boldsymbol{\kappa}) &= -V(\boldsymbol{k}, \boldsymbol{\kappa})[c_0^* d(\Omega) + c_0 d^*(-\Omega)].\end{aligned} \tag{7.2.17}$$

Here $\Omega_{\mathrm{S}}(\boldsymbol{\kappa})$ and $\Gamma_{\mathrm{S}}(\boldsymbol{\kappa})$ are the frequency and damping of the sound with the wave vector $\boldsymbol{\kappa}$ in the absence of parametric magnons. Considering these equations simultaneously with (7.2.5, 7, 8), we obtain from the condition of zero determinant of the complete set of equations for $\beta(\Omega, \boldsymbol{\kappa})$, $\beta(-\Omega, -\boldsymbol{\kappa})$, $d(\Omega)$ and $d^*(-\Omega)$ the following:

$$\{[\Omega + i\Gamma_{\mathrm{S}}(\boldsymbol{\kappa})]^2 - \Omega_{\mathrm{S}}^2(\boldsymbol{\kappa})\}[\Omega(\Omega + i\gamma) - \Delta_0^2] = 8\Omega_{\mathrm{S}}(\boldsymbol{\kappa})|V(\boldsymbol{k}, \boldsymbol{\kappa})|^2 \,. \tag{7.2.18}$$

The simplest case for study is when the damping of both magnons and sound can be neglected. Then

$$2\Omega_{\pm}^2(\boldsymbol{\kappa}) = \Omega_{\mathrm{S}}^2(\boldsymbol{\kappa}) + \Omega_0^2 + [(\Omega_{\mathrm{S}}^2(\boldsymbol{\kappa}) - \Omega_0^2) + 32\Omega_{\mathrm{S}}^2(\boldsymbol{\kappa})SN^2|V(\boldsymbol{k}, \boldsymbol{\kappa})|^2] \,. \tag{7.2.19}$$

The interaction amplitude S is negative (see (3.2.13)), therefore near the resonance the coupled oscillation of parametric magnons and the elastic subsystem are unstable. If one takes into account damping of parametric magnons and sound then instability of the coupled oscillations (7.2.18) takes place under the number of parametric magnons N exceeds the threshold number $N_{\mathrm{cr}}(\boldsymbol{k})$. The minimum value of N_{cr} is attained at resonance under $\Omega_0 = \Omega_{\mathrm{S}}(\boldsymbol{\kappa})$:

$$2|SN_{\mathrm{cr}}|^2 = -\gamma\Gamma\Omega_0^2|S|/|V(\boldsymbol{k}, \boldsymbol{\kappa})|^2\Omega_{\mathrm{S}}(\boldsymbol{\kappa}) \,. \tag{7.2.20}$$

The interaction of collective oscillations with the sound is significant also far from the resonance. In this case it follows from (7.2.18) that

$$\begin{aligned}\Omega_1(\boldsymbol{\kappa}) =& \Omega_{\mathrm{S}}(\boldsymbol{\kappa}) + 4SN^2\frac{|V(\boldsymbol{k}, \boldsymbol{\kappa})|^2}{\Omega_{\mathrm{S}}^2(\boldsymbol{\kappa}) - \Omega_0^2}\\ &- i\left\{\Gamma_{\mathrm{S}}(\boldsymbol{\kappa}) + 8\gamma(\boldsymbol{k})\frac{\Omega_{\mathrm{S}}(\boldsymbol{\kappa})SN^2|V(\boldsymbol{k}, \boldsymbol{\kappa})|^2}{[\Omega_{\mathrm{S}}^2(\boldsymbol{\kappa}) - \Omega_0^2]^2}\right\},\\ \Omega_2(\boldsymbol{\kappa}) =& \Omega_0 + 4SN^2\frac{|V(\boldsymbol{k}, \boldsymbol{\kappa})|^2\Omega_{\mathrm{S}}(\boldsymbol{\kappa})}{\Omega_{\mathrm{S}}^2(\boldsymbol{\kappa}) - \Omega_0^2}\Omega_0\\ &- i\left\{\gamma(\boldsymbol{k}) + 8\Gamma_{\mathrm{S}}(\boldsymbol{\kappa})\frac{\Omega_{\mathrm{S}}(\boldsymbol{\kappa})SN^2|V(\boldsymbol{k}, \boldsymbol{\kappa})|^2}{[\Omega_0^2 - \Omega_{\mathrm{S}}^2]^2}\right\},\end{aligned} \tag{7.2.21}$$

In the absence of the magneto-elastic interaction the oscillations with the frequency Ω_1 are purely acoustic and with the frequency Ω_2 they are collective oscillations of parametric magnons. The damping of the long-wave sound $\Gamma(\boldsymbol{\kappa})$ is as a rule small in comparison with the damping of parametric magnons $\gamma(\boldsymbol{k})$. Therefore under negative S the damping of the sound-type oscillations $\Omega_1(\kappa)$ can become negative. This implies instability. Its threshold is attained at a certain number of parametric magnons $N_{\mathrm{cr}}(\kappa)$:

$$N_{\mathrm{cr}} = \Gamma(\boldsymbol{\kappa})[\Omega_{\mathrm{S}}^2(\boldsymbol{\Omega}) - \Omega_0^2]/64|SV^2(\boldsymbol{k}, \boldsymbol{\kappa})|\Omega_{\mathrm{S}}(\boldsymbol{\kappa})\gamma(\boldsymbol{k}) \,. \tag{7.2.22}$$

In 1983 *Smirnov* [7.7] discovered the excitation of the acoustic mode of the sample at a certain pumping intensity. As the sample was finite the frequency of the acoustic oscillations $\Omega_{\mathrm{S}}(\boldsymbol{\kappa})$ significantly exceeded the frequency of collective oscillations Ω_0. In this case the excitation threshold of the sound oscillations equals N_{cr} (7.2.22). According to the S-theory the corresponding value $p_{\mathrm{cr}} = (h_{\mathrm{cr}}/h_{\mathrm{th}})^2$ for Smirnov's experiment is $p = 25$. Experimentally this quantity has the value 50 – 100. Taking into account that we have no exact values of $V(\boldsymbol{k}, \boldsymbol{\kappa})|S|$ and the damping $\gamma(\boldsymbol{k})$, such an agreement of theory and experiment must be considered satisfactory.

7.3 Threshold Under Periodic Modulation of Dispersion Law

Now we must study the large-amplitude collective oscillations. They can be excited in the ferromagnets also by the strong periodic modulation of the magnetic field at the frequency approximating the eigenfrequency of collective oscillations. Clearly, before investigating the above-threshold state of parametric magnons in the presence of a strong RF-field we must study the influence of this field on the threshold of the parametric excitation. This problem was first discussed by *Suhl* [7.8], who showed that under the periodic modulation of the biasing field (or of the pumping frequency) the threshold of the parametric instability increases because the condition of the parametric resonance is violated. According to Suhl such frequency and amplitude of the modulation can be chosen in such a way that the instability disappears as its threshold becomes infinitely high. Suhl's idea has been experimentally checked by *Hartwick, Peressini* and *Weiss* [7.9]. Their conclusions, however were not in full agreement with this theory, in particular they could not increase the threshold by more than 10dB. This lack of agreement will be theoretically explained later. Detailed experimental data corroborating our theory will be given in Chap. 9.

Let us consider the parametric excitation of spin waves in the following magnetic field [7.10, 11]:

$$H = H_M(t) + H_0 + h\cos\omega_p t \ . \tag{7.3.1}$$

Here $\mathcal{H}_0$ is the constant magnetic field, $H_M(t)$ stands for the modulating field, $h\cos\omega t$ denotes the SHF pumping at the frequency ω_p. It is rather difficult to calculate the threshold of parallel pumping in the field (7.3.1) with an arbitrary modulation law $H_M(t)$. Therefore first the simplest case of the modulation will be considered, i.e. the modulation by rectangular repetitive pulses (meander-type mode with the period 2τ). The equations of motion for the normal amplitudes of spin waves taking into account the Hamiltonian of their interaction with the pumping $\mathcal{H}_p$ (4.3.18) and with the modulating field $\mathcal{H}_{p1}$ (4.3.19) have the following form:

$$\left\{\frac{\partial}{\partial t} + \gamma(\boldsymbol{k}) + i[\Delta(\boldsymbol{k}) + U(\boldsymbol{k})H_M]\right\} c(\boldsymbol{k},t) = -ihV(\boldsymbol{k})c^*(-\boldsymbol{k},t)$$

at $0<t<\tau$, $2\tau<t<3\tau$,.. and so on . $\Delta(\boldsymbol{k}) = \omega(\boldsymbol{k}) - \omega_p/2$,

$$\left\{\frac{\partial}{\partial t} + \gamma(\boldsymbol{k}) + i[\Delta(\boldsymbol{k}) - U(\boldsymbol{k})H_M]\right\} c(\boldsymbol{k},t) = -hV(\boldsymbol{k})c^*(-\boldsymbol{k},t) \ ,$$

at $\tau<t<2\tau$, $3\tau<t<4\tau$,.. and so on. (7.3.2)

Taking $c(\boldsymbol{k},t)$,$c^*(-\boldsymbol{k},t) \propto \exp(\nu t)$ we find that the increment ν over different times is equal to:

$$\nu_1 = \nu(0,-\tau) = -\gamma + \sqrt{|hV(\boldsymbol{k})|^2 - [\Delta(\boldsymbol{k}) + U(\boldsymbol{k})H_M]^2]} \ ,$$

$$\nu_2 = \nu(\tau,-2\tau) = -\gamma + \sqrt{|hV(\boldsymbol{k})|^2 - [\Delta(\boldsymbol{k}) - U(\boldsymbol{k})H_M]^2} \ . \tag{7.3.3}$$

Let us consider the case $\gamma\tau > 1$ when the damping solutions corresponding to the negative sign before the square root in (7.3.3) can be neglected. Then the total increment ν during the total pulse period is $\nu_1 + \nu_2$. From the condition Re$\nu = 0$ we obtain two expressions for the instability threshold:

$$|h_{th1}V| = \gamma^2(\boldsymbol{k}) + (U(\boldsymbol{k})H_M)^2 + \Delta^2(\boldsymbol{k})[1 + U^2(\boldsymbol{k})H_M^2/\gamma^2] \ , \tag{7.3.4}$$

$$|h_{th2}V| = 4\gamma^2(\boldsymbol{k}) + [\Delta(\boldsymbol{k}) \mp U(\boldsymbol{k})H_M]^2 \ . \tag{7.3.5}$$

The expression (7.3.4) corresponds to the small amplitudes of modulations when $|hV(\boldsymbol{k})| > |U(\boldsymbol{k})H \pm \Delta(\boldsymbol{k})|$ and minimum threshold of instability is

reached at $\Delta(\boldsymbol{k}) = 0$, i.e. for the spin waves over the resonance surface $\omega(\boldsymbol{k}) = \omega_p/2$. The corresponding minimum threshold is

$$|h_{th}V|^2 = \gamma^2(\boldsymbol{k}) + U^2(\boldsymbol{k})H_M^2 \ , \qquad \Delta(\boldsymbol{k}) = 0 \ . \tag{7.3.6}$$

The large modulation amplitudes result in an additional local minimum of the threshold at $\Delta(\boldsymbol{k}) = \pm U(\boldsymbol{k})H_M$. The value of the threshold on these surfaces is constant and equals $2\gamma(\boldsymbol{k})$. Therefore, the critical amplitude of the modulation $\mathcal{H}_{m,cr} = \sqrt{3\gamma/U(\boldsymbol{k})}$ separates two modes of parallel pumping, i.e. the mode of weak modulation when the minimum threshold is specified by (7.3.6) and the mode of strong modulation when its increase stops and the minimum threshold is equal to 2γ. In the last mode the modulating field leads to a detuning whose value increases proportional to the amplitude of modulation. This characteristic effect of the strong modulation, i.e. the "freezing" of the threshold under $H_M > H_{m.cr}$ is, generally speaking inherent only in the modulation of the rectangular type. Thus it can be readily shown (*Zautkin* and *Orel* [7.11]) that for the sinusoidal and saw-tooth modulation the quantity value h_{th} increases, slowly but without bounds, with the increase of H_M. Under $H_M \gg H_{m,cr}$ for the sinusoidal and saw-tooth modulations respectively they have [7.11]:

$$2|h_{th1,\sin}V| = \gamma^{2/3}(U(\boldsymbol{k})H_M)^{1/3} \ , \qquad |h_{th,saw}V| = \sqrt{\gamma U(\boldsymbol{k})H_M} \ . \tag{7.3.7}$$

Until now we considered the low-frequency modulation with the period less than the relaxation time of spin waves. Now we shall drop this limitation and consider parallel pumping of spin waves in the magnetic field modulated by a sinusoid of arbitrary frequency Ω : $H_M(t) = H_M\cos\Omega t$ [7.10]. The linearized equations of motion have the following form:

$$\left\{\frac{\partial}{\partial t} + \gamma(\boldsymbol{k}) + i[\Delta(\boldsymbol{k}) + U(\boldsymbol{k})H_M\cos(\Omega t)]\right\} c(\boldsymbol{k},t) = -ihV(\boldsymbol{k})c^*(-\boldsymbol{k},t) \ . \tag{7.3.8}$$

Confining ourselves to the small amplitudes of H_M let us write

$$c(\boldsymbol{k},t) = c_0(\boldsymbol{k}) + d(\Omega,\boldsymbol{k})\ \exp(-i\Omega t) + d(-\Omega,\boldsymbol{k})\exp(i\Omega t) \ . \tag{7.3.9}$$

The stationary solution of (7.3.8) corresponds to the instability threshold. Substitution of (7.3.9) in (7.3.8) yields the system of equations for $c_0(\boldsymbol{k})$, $d(\pm\Omega,\boldsymbol{k})$. Hence for the approximation quadratic in d we obtain the relation specifying the threshold

$$|h_{th}V(\boldsymbol{k})|^2 = [\gamma^2(\boldsymbol{k}) + \Delta^2(\boldsymbol{k})]\left[1 + \frac{U^2(\boldsymbol{k})H_M^2}{\Omega^2 + 4\gamma^2}\right]^2 \ . \tag{7.3.10}$$

As should be expected the minimum threshold for the mode of weak modulation is reached at $\Delta(\boldsymbol{k}) = 0$. At low frequencies (7.3.10) at $\Delta(\boldsymbol{k}) = 0$ is equivalent to the exact formulae (7.3.6) obtained above for the rectangular modulation. This apparently means that (7.3.10) derived within perturbation theory is applicable with a good accuracy up to the values $H_{\mathrm{M}} = H_{\mathrm{m,cr}}$. This can be accounted for by the fact that at high frequencies of modulation the small parameter of the theory is $U(\boldsymbol{k})H_{\mathrm{M}}/\Omega$ and at low frequencies it is $\Omega/\gamma(\boldsymbol{k})$.

In the general case of the modulation with arbitrary frequency and arbitrary amplitude we can obtain a comparatively simple expression for the instability threshold only for the rectangular modulation. Let us present without deviation the equation specifying h_{th} (see [7.10]):

$$\cos h[2\gamma(\boldsymbol{k})\tau] = \cos h(r_+\tau)\cos h(r_-\tau) + \sin h(r_+\tau)\sin h(r_-\tau)[|h_{\mathrm{th}}V(\boldsymbol{k})|^2 - \Delta_+\Delta_-]/d_+d_- \ ,$$

$$\Delta_\pm = \Delta(\boldsymbol{k}) + U(\boldsymbol{k})H_{\mathrm{M}} \ , \qquad d_\pm^2 = |h_{\mathrm{th}}V(\boldsymbol{k})|^2 - \Delta_\pm^2 \ . \tag{7.3.11}$$

When the frequencies are low (τ are large) we can easily obtain (7.3.2). At high frequencies ($\tau \to 0$) we have

$$|h_{\mathrm{th}}V(\boldsymbol{k})| = \gamma(\boldsymbol{k})\sqrt{1 + U(\boldsymbol{k})H_{\mathrm{M}}\tau)^2/3} \ . \tag{7.3.12}$$

Finally, it must be noted that the conclusion about the monotonic increase of the threshold by (7.3.7) for the periodic modulation refers to the excitation of waves with the limiting frequency shift $\Delta(\boldsymbol{k}) = \pm U(\boldsymbol{k})H$. It is shown by *Frishman* in his interesting study [7.12] that under large H_{M} the waves with the shift $\Delta_{\mathrm{n}}(\boldsymbol{k}) = n\Omega/2$ (Ω is the frequency of the RF-field) are excited, if the pumping field exceeds the value $h_{\mathrm{th}}/|J_{\mathrm{n}}(z)|$. Here $J_{\mathrm{n}}(z)$-is the Bessel function of the argument $z = -U(\boldsymbol{k})H_{\mathrm{M}}/\Omega$. Hence the threshold of the parametric instability $h_{\mathrm{th}}(H_{\mathrm{M}}, \Omega) = h_{\mathrm{th}}(0)[\max_{\mathrm{n}} |J_{\mathrm{n}}(z)|]^{-1}$ is a complex non-monotonic function of H_{M} and Ω. This non-trivial conclusion of the theory is in good quantitative agreement with the experimental data of *Ozhogin* et al. [7.13] on the parametric excitation of spin waves in ferromagnets.

In Chap. 9 the experimental data on the parametric excitation of magnons under conditions of the periodic RF-modulation of the magnetic field H_{M} supporting the above-developed theory will be discussed. Now we shall only note that this theory is applicable not only to spin waves. It describes also the excitation of waves of different nature whose frequency is periodically changed with the amplitude $A_{\mathrm{M}} = U(\boldsymbol{k})H_{\mathrm{M}}$.

7.4 Large-Amplitude Collective Oscillations and Double Parametric Resonance

7.4.1 Stationary State Under Periodic Modulation

It would be natural to assume that the periodic modulation of the magnetic field does not qualitatively change the processes resulting in the limitation of the parametric instability. They are, however, accompanied by some additional effects that can significantly change the stationary values of the total number N and phase Ψ of parametric wave pairs. Under the influence of the modulating RF-field collective oscillations of parametric waves are excited. They nonlinearly interact with the initial parametric waves and with the RF-field. This results in the increased damping of parametric waves and in the change in their interaction. The interaction processes of the type $H_{\mathrm{M}}cd^*$, (which are actually the confluence of parametric waves with the RF-photon resulting in the excitation of the collective mode d) are responsible for the additional damping. Obviously, these processes lead to the addition to γ proportional to the square of the amplitude of the modulating field. Other interaction processes similar to c^*c^*dd provide an additional coupling between parametric waves via the collective oscillations. These processes can be interpreted as the renormalization of the amplitudes of the four-magnon interaction $S(\boldsymbol{k}, \boldsymbol{k}')$. In order to allow for all these effects let us write an equation of motion of parametric waves taking into account their interaction with each other and the RF-field

$$\{\frac{\partial}{\partial t} + \gamma(\boldsymbol{k}) + i[\Delta(\boldsymbol{k}) + 2T|c(t)|^2 + UH_{\mathrm{M}}(t)\}c(t) = -i[hV + Sc^2(t)]c^*(t) . \tag{7.4.1}$$

It is the result of the averaging of the equations of motion for the amplitudes $c(\boldsymbol{k}, t)$ and $c^*(\boldsymbol{k}, t)$ over all the directions of the wave vector $\boldsymbol{k}$ for which in the ground state $c_0(\boldsymbol{k}) \neq 0$. It can be done without loss of generality since we are interested in the reaction of parametric waves to the homogeneous RF-field and we do not consider here the non-homogeneous modes of the collective oscillations. Assuming $H_{\mathrm{M}}(t) = H_{\mathrm{M}} \cos \Omega t$ we obtain from (7.2.4, 7, 8) the linear reaction of the collective oscillations $d(\pm\Omega)$ to the field H_{M}; substituting the expression for this reaction into (7.4.1) we obtain the equation for the stationary amplitude c_0:

$$\begin{aligned} &\gamma\{1 + [UH_{\mathrm{M}}\Omega/\Delta(\Omega)]^2\} + i\{\Delta(\boldsymbol{k}) + [2T + S + (S\Delta_0^2 + 2T\Omega^2 \\ &\quad + 3S\Omega^2)U^2H_{\mathrm{M}}^2/2\Delta^2(\Omega)]N + hV\exp(i\Psi) = 0 \ . \end{aligned} \tag{7.4.2}$$

Here $\Delta(\Omega)$ and Δ_0 are given by (7.2.8) and $\Delta(\boldsymbol{k})$ denotes frequency shift. The position of the surface in $\boldsymbol{k}$-space where parametric waves in stationary state are excited, i.e. the value $\Delta(\boldsymbol{k})$ must be obtained from the condition of the stability of the solution (7.4.2) with respect to small perturbations δ_{c} outside the stationary surface, i.e. from the condition of the *external* *stability.* The reader can find it independently solving the Problem 7.5. Substituting the answer (7.4.11) into (7.4.2) it is possible to represent the latter in the following form:

$$hV\exp(i\Psi)+\tilde{S}N=i\Gamma\,,\quad \Gamma=\gamma[1+(UH_{\rm M})|/\Delta(\Omega)|^2]\,, \tag{7.4.3}$$

$$\tilde{S}=S\left\{1+2\frac{(UH_{\rm M})^2(3\Omega^2+4\gamma^2)(\Omega^2-4S^2N^2)}{2(\Omega^2+4\gamma^2)|\Delta(\Omega)|^2}\right\}\,. \tag{7.4.4}$$

The form of (7.4.3) coincides with the form of the equation for the stationary state of the basic S-theory (5.5.6) if in the latter equation we substitute $\gamma\to\Gamma$, $S\to\tilde{S}$ according to (7.4.4). Therefore in the presence of the RF-field we must write instead of (5.5.7)

$$|\tilde{S}|N=\sqrt{|hV|^2-\Gamma}\,,\quad hV\sin\Psi=\Gamma\,,\quad hV\cos\Psi+\tilde{S}N=0\,. \tag{7.4.5}$$

These equations enable us to calculate the dependences of the nonlinear susceptibility χ' and χ'' on the supercriticality p and the magnitude of the RF-field $H_{\rm M}$. These dependences, as shown by *Zautkin* et al. [7.10] are in qualitative agreement with the experimental results on the YIG sample.

An interesting work [7.13] by *Ozhogin* et al. theoretically and experimentally studied the parametric excitation of nuclear magnons in the antiferromagnetic $CsMnF_3$ under a modulating magnetic field. A good agreement of the conclusions of the theory linear in $H_{\rm M}$ with the experimental results below the supercriticality 7 dB and $H_{\rm M}\simeq 0.25$ Oe has been shown. In particular, the relaxation parameter of nuclear magnons obtained from the threshold of the parametric excitation by the formula $h_{\rm th}V=\gamma$ coincided with the quantity γ calculated from the frequency dependence using (7.3.10) to the accuracy of experimental corrections. It is significant that the last method of the determination of $\gamma(\boldsymbol{k})$ provides a higher absolute accuracy of measurements.

7.4.2 Parametric Excitation of CO of Parametric Wave System

In addition to the discussed linear interaction of the collective oscillations of parametric waves with the RF-field leading to resonance at the frequency $\Omega=\Omega_0$ there is also nonlinear interaction of the type $H_{\rm M}b^*b^*$ which results in the parametric resonance of the collective modes in the RF-field with the frequency $\Omega=2\Omega_0$. As is easily seen, the instability of the initial mode d with the frequency Ω excited by the RF-field with respect to the decay into two modes b with the frequency $\Omega/2$ also refers to this effect. The action of both mechanisms when some critical amplitude of the RF-field is exceeded brings about the *double parametric resonance of parametric waves*, i.e. the simultaneous excitation of oscillations at SHF with the frequency $\omega_{\rm p}/2$ and at RF with the frequency $\Omega/2$.

To obtain the threshold of instability with respect to the parametric excitation of collective oscillations we seek the solution of the equation of motion (7.4.1) in the following form:

$$\begin{aligned} c(t)=&c+d(\Omega)\exp(-i\Omega t)+d(-\Omega)\exp(i\Omega t)\\ &+b(\Omega/2)\exp(-i\Omega t/2)+b(-\Omega/2)\exp(i\Omega t/2)\,. \end{aligned} \tag{7.4.6}$$

By employing the relation (7.4.5) for the ground state and linearizing (7.4.1) with respect to the small amplitudes b and d we obtain

$$\begin{aligned} &[\partial/\partial t+\gamma+2i(T+S)N]b_+ +[(2iTN-\gamma)\exp(-i\Psi)]b_-^*\\ &\quad +i\omega_+b_- +[iP_+\exp(-i\Psi)]b_+^*=0\,,\qquad b_\pm=b(\pm\Omega/2,t)\,, \end{aligned}$$

$$\begin{aligned} \omega_+=&U^*H_{\rm M}^*/2+2(2T+S)(c_0d_++c_0d_-)\,,\qquad \omega_-=\omega_+^*\,,\\ P_\pm=&2(2T+S)c_0d_\pm\exp(i\Psi)\,,\qquad d_\pm=d(\pm\Omega,t)\,. \end{aligned} \tag{7.4.7}$$

It should be recalled that the oscillation amplitudes $d_\pm$ are specified by (7.2.7, 8). Equation (7.4.7) can be written in a more symmetrical form if we pass from the variables $b_\pm$ to the normal variables a via the following u-v-transformation

$$\begin{aligned} a_+=&ub_+\exp(i\Psi/2)-ub_-^*\exp(-i\Psi/2)\,,\\ a_-^*=&-vb_+\exp(i\Psi/2)+u^*b_-(-i\Psi/2)\,, \end{aligned}$$

$$u=\sqrt{\frac{2(T+S)N+\Omega_0}{2\Omega_0}}\,,\qquad v=-\sqrt{\frac{2(T+S)N-\Omega_0}{2\Omega_0}}\,. \tag{7.4.8}$$

Then for the renormalized $a_\pm$ we have

$$(\partial/\partial t+\gamma+i\Omega_0)a_+ +iA_+a_+^*+iB_+a_- -\gamma a_-^*=0\,,$$

$$\begin{aligned} A_\pm=&2uv\omega_\pm+u^2P_++v^2P_-^*\,,\\ B_\pm=&v^2\omega_\pm+uv(P_++P_-^*)\,,\quad C_\pm=\nu+\gamma+i(\Omega_0\pm\Omega/2) \end{aligned} \tag{7.4.9}$$

Assuming $a_\pm(t)=a_\pm\exp(+i\Omega t/2+\nu t)$ and writing the system of four equations for the amplitudes a_+,a_+^*,a_- and a_-^* we find the instability increment ν from the condition

$$\begin{bmatrix} C_+ & iA_+ & iB_+ & -\gamma\\ -iA_+^* & C_+^* & -\gamma & -iB_+^*\\ -iB_+ & -\gamma & C_- & iA_-\\ -\gamma & -iB_+ & -iA_-^* & C_-^* \end{bmatrix}=0\,. \tag{7.4.10}$$

The instability threshold ($\nu=0$) for the given values Ω and Ω_0 was calculated on a computer using (7.4.10) (Fig 7.1). The minimum threshold has been proved to be reached at the resonance $\Omega=2\Omega_0$ and to be only weakly dependent on Ω: $\min\{H_{\rm M,th}\}\simeq 0.4\gamma/v\simeq\gamma/g$. The second minimum of the threshold at the frequency $2\Omega_0$ manifests itself only at a sufficient Q-factor of collective oscillations when $\gamma \ll \Omega_0$.

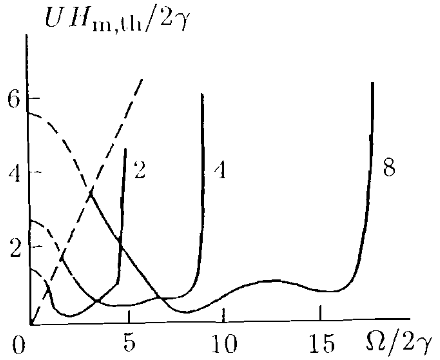


Fig. 7.1. Theoretical dependences (obtained by numerical solution of (7.4.10)) of the threshold of double parametric resonance $H_{\mathrm{m,th}}$ on the RF-frequency. The numbers of the curves (2, 4 and 8) denote the value of dimensionless eigenfrequency of collective oscillations $\Omega_0/2\gamma$. The applicability region of the theory is limited from above by a dashed line

The parametric excitation of collective oscillations in the system of parametrically excited magnons, i.e. the *double parametric resonance* was discovered and experimentally studied by *Zautkin* [7.10, 14]. These results will be described in Chap. 9.

7.5 Transient Processes when Pumping is Turned On

7.5.1 Small Supercriticality Range

This subsection deals with the transition to the S-theory steady stationary state starting from the level of the thermal noise $n(\boldsymbol{k}) = n_0$ under small supercriticality $hV - \gamma \ll \gamma$. For increasing small N up to $N \ll N_1$ (N_1 is the total number of parametric waves in the ground state) the amplitudes of the pairs will increase as predicted by the linear theory, i.e. with the increment $hV - \gamma$. At the same time a narrow packet of parametric waves with the width $\Delta\omega \simeq (hV - \gamma)$ emerges in the $\boldsymbol{k}$-space. Its subsequent behavior is described by (5.4.13) from which it is clear that the relaxation times of the amplitudes and the phases are of the order of $(hV - \gamma)^{-1}$ and γ^{-1}, respectively. As a result the phases $\Psi(\boldsymbol{k})$ can be considered to follow the the amplitudes adiabatically, i.e. $\partial\Psi/\partial t$ in (5.4.13) can be neglected. The fact that the packet is narrow $\Delta\omega \ll \gamma$ enables one to expand the trigonometrical functions in (5.4.13) into a series and to represent these equations in the following form (for more detail, see [7.15]):

$$-\frac{\partial f(x_1,\tau)}{f(x_1,\tau)\partial\tau} = x^2 + x(2r-1)\left(\int f(x_1,\tau)dx - 1\right) + r\int f(x_1,\tau)dx_1$$
$$-r + r^2\left[\int f(x_1,\tau)dx_1 - 1\right]^2 + \int x_1 f(x_1,\tau)dx_1 \,. \qquad (7.5.1)$$

Here $f = n(\boldsymbol{k},\tau)/N_1$, $x = [\omega(\boldsymbol{k}) - \omega(\boldsymbol{k}_0)]/2SN_1$, $\tau = t(2SN_1/\gamma)$ and $r = (2T+S)/S$. Note that the dependence on supercriticality $p = h^2/h_{\mathrm{th}}^2$ in this equation disappeared and the parameters of parametric waves system enter into this equation only via the ratio r. Equation (7.5.1) has a self-similar solution

$$f(x,\tau) = A(\tau)\sqrt{\tau/\pi}\exp\{-[(x - x_0(t))/d(\tau)]^2\} \,, \qquad (7.5.2)$$

where $A(\tau) = \int f(x,\tau)dx$ is the total number of parametric waves, $x_0(\tau) = A^{-1}\int f(x,\tau)x dx$ is the position of the center of gravity and $d(\tau)$ denotes the width of the packet. They satisfy the equations

$$\frac{dA}{Ad\tau} + \frac{1}{2\tau} - \frac{d(\tau x_0^2)}{d\tau} = -Ax_0^2 - rA(A-1)[r(A-1)+1] \,,$$
$$\frac{d(\tau x_0)}{d\tau} = -(A-1)\left(r - \frac{1}{2}\right) \,, \qquad d(\tau) = \frac{1}{\sqrt{\tau}} \,. \qquad (7.5.3)$$

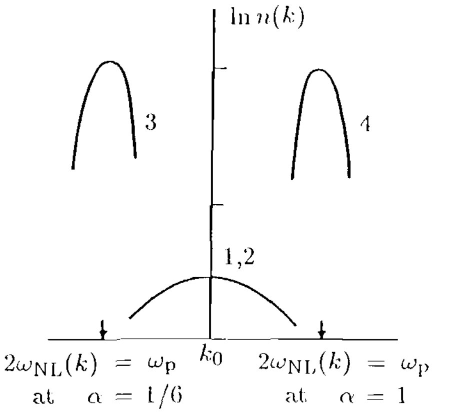
These equations at $\tau \to \infty$ have the following asymptotic behavior

$$x_0 = -1/2\tau \,, \quad A - 1 = -[2(4r-1)\tau^2]^{-1} \,, \quad d(\tau) = 1/\sqrt{\tau} \,. \qquad (7.5.4)$$

Therefore not very high above the threshold the arbitrary distribution function of the pairs $n(\boldsymbol{k})$ relaxes to the stationary state (5.4.13) having the form of the δ-function by the power laws of (7.5.4)

$$N_1 - N \propto N_1/[(hV-\gamma)t]^2 \,,$$
$$[\omega(\boldsymbol{k}) - \omega(\boldsymbol{k}_0)] \propto \sqrt{\gamma/(hV-\gamma)}/t \,, \qquad (7.5.5)$$
$$\omega^2(\boldsymbol{k}) - [\langle\omega(\boldsymbol{k})\rangle]^2 \propto \sqrt{\gamma[(hV-\gamma)}/t \,.$$

To study the non-stationary behavior of the system in detail under the arbitrary hV, S and T Equations (5.4.13) were numerically solved. Fig. 7.2 shows the distribution functions $n(\boldsymbol{k},t)$ for two successive times and values of the parameter $r = 1/6$ and $r = 1$. The homogeneous initial distribution was selected, $hV = 1.4\gamma$. First the waves increase exponentially, the maximum of $n(\boldsymbol{k},T)$ is on the surface $\omega(\boldsymbol{k}) = \omega_{\mathrm{p}}/2$, as follows from the linear theory. Here the form of the function $n(\boldsymbol{k},t)$ is naturally independent of the parameter r. Later, when $N(t)$ is not small in comparison with N_1 the behavior of $n(\boldsymbol{k},t)$ significantly depends on the ratio of S and T describing the interaction of waves. Thus at $T = 0$ the packet $n(\boldsymbol{k},t)$ keeps increasing and converging, still remaining on the surface $\omega(\boldsymbol{k}) = \omega_{\mathrm{p}}/2$. At $T = 0$ the packet increases and moves as a whole, deforming a little, to the surface $\omega(\boldsymbol{k}) + 2TN_1 = \omega_{\mathrm{p}}/2$ (i.e. towards the larger or smaller k depending on the sign of T). When the maximum of the packet $n(\boldsymbol{k},t)$ is near this surface the packet begins to converge asymptotically. Note that the shape of the curve $\ln n(\boldsymbol{k},t)$ (see. Fig. 7.2) at large t is close to a parabola, which confirms that it tends to the self-similar solution (7.5.2), i.e. the Gaussian packet.



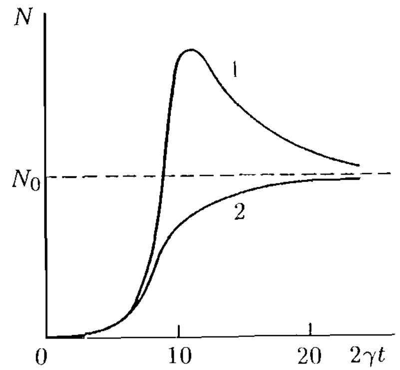


Fig. 7.2. (left) Time evolution of the parametric wave system – data of computer simulation of (5.4.13): distribution of waves $n(k)$ at $p = 2$. Linear stage: (1) at $r = 1/6$, (2) at $r = 1.0$. Asymptotic stage: (3) at $r = 1/6$, (4) at $r = 1.0$

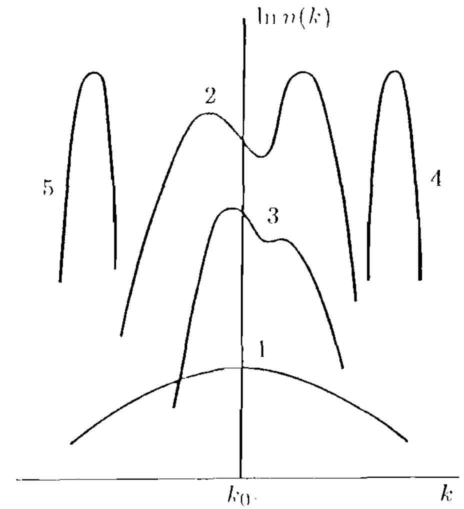
Fig. 7.3. (right) Time dependence of the total number of parametric waves at $p = 2$. (1) at $r = 1/6$, (2) at $r = 1.0$

It can be seen from Fig. 7.3 that there are three transition stages to the steady stationary state from the thermal level when the parametric pumping is turned on. At the first – linear – stage the amplitude exponentially increases and the wave system does not depend on the nonlinear characteristics T and S. At this stage the two curves $N(t)$ (for $r = 1/6$ and r=1) coincide. At the second – nonlinear – stage the relative nonlinear frequency shift and the parametric interaction of wave pairs become significant. At the same time there is no compensation of wave damping $\gamma(\boldsymbol{k})$ by the total pumping $P(\boldsymbol{k})$ typical of the stationary state in the basic S-theory. The behavior of the system is most complex at this stage and can be simulated only on computer. At $T < 0$ the total number of waves N passes over the maximum and under $T > 0$ it monotonically increases. At the third – asymptotic – stage $|P(\boldsymbol{k})| - \gamma(\boldsymbol{k}) \ll \gamma(\boldsymbol{k})$ and the system slowly tends to the stationary state (5.5.7) described by the basic S-theory. Note that the sign of the difference $N - N_1$ in the numerical experiment (see Fig. 7.3) coincides with the sign that follows from the analytical asymptotics (7.5.4).

7.5.2 High Supercriticality Range

The numerical experiment revealed significant qualitative differences in the non-stationary behavior of the parametric wave system at small ($hV - \gamma \ll \gamma$) and $(hV - \gamma) \gg \gamma$ supercriticalities (Fig. 7.4 and 7.5 for $hV = 4\gamma$) at the second – nonlinear – stage of the transition process. The packet $n(\boldsymbol{k}, t)$ does not behave like a single whole when the amplitude $N(t)$ is no longer small in comparison with N_1 and the second maximum of the function $n(\boldsymbol{k}, t)$

"emerges" at the point where $\omega_{NL}(\boldsymbol{k}, t) = \omega_p/2$ at a given time. Then the amplitude of the second maximum increases and the amplitude of the first maximum decreases. On Fig. 7.4 at $\gamma t = 6$ the second maximum is already larger than the first. In addition, as seen from Fig. 7.5, the total amplitude oscillates as it approximates N_0. In the limiting case of the non-dissipative medium when $(\gamma/hV) \to 0$ the parametric wave system does not reach the stationary state at all and it oscillates infinitely long. This is due to the fact that at $\gamma = 0$ there exists the motion integral $\mathcal{H}_S$ (5.4.17). At the initial time $\mathcal{H}_S = 0$ and does not coincide with the value $\mathcal{H}_S = -(2T + S)N_1$ in the ground state which therefore cannot be attainable at $\gamma = 0$.



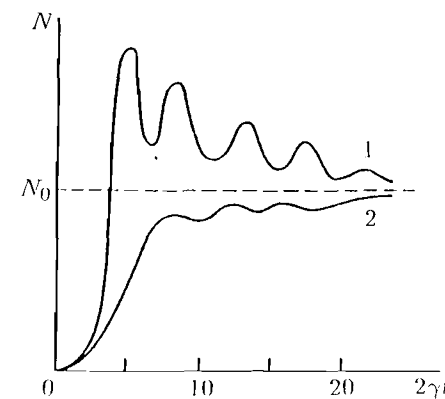


Fig. 7.4. (left) Time evolution of the parametric wave system – data of computer simulation of (5.4.13): distribution of waves $n(k)$ at p=16. Linear stage: (1) at $r = 1/6$, and $r = 1.0$. Nonlinear stage: (2) at $r = 1/6$, (3) at $r = 1.0$. Asymptotic stage: (4) at $r = 1/6$, (5) at $r = 1.0$

Fig. 7.5. (right) Time dependence of the total number of parametric waves at $p = 16$. (1) at $r = 1/6$, (2) at $r = 1.0$

At the last asymptotic stage, as can be seen from Fig. 7.4, the packet $n(\boldsymbol{k}, t)$ is Gaussian. This should be expected because this stage is described by (7.5.1) whose solution (7.5.2) is self-similar and is the Gaussian packet. Indeed the validity criterion (7.5.1) is the smallness of the ratio $S(N - N_1)/\gamma \ll 1$. According to the asymptotics (7.5.4) $N(t)$ comes closer to N_0, from above or from below, as can be seen from Fig. 7.5 depending on the value of T/S. The cause of the oscillations $N(t)$ in the transition mode is obvious: it is the impact excitation of the collective oscillations. Their damping decrement is γ, and the frequency $\Omega_0 = \sqrt{2S(2T+S)N_1} \simeq hV$ at $hV \gg \gamma$. Therefore the number of oscillations is of the order of hV/γ.

7.6 Parametric Excitation Under Sweeping of Wave Frequency

The analysis of the experimental data of *Zhitnik* and *Melkov* [7.16] show that under parametric excitation of magnons in YIG, a sweeping of the eigenfrequency of magnons $\omega(\boldsymbol{k})$ in the $\boldsymbol{k}$-space appears in some value range of the constant field H, i.e. there appears the monotonic dependence $\omega(\boldsymbol{k}, t)$. This sweeping is caused by the accumulation of non-equilibrium magnons resulting from the dissipation of parametric magnons. The accumulation of a large number of parametric magnons in the range of small k was observed long ago by *Le-Hall* et al. [7.17], *Venitsky* et al. [7.18]. It would be natural to assume that over a certain period the number of parametric magnons linearly increases with time, and then attains the stationary mode. *Podivilov* and *Cherepanov* [7.19] (the present section is based on the results of this study), showed that over the time of the order of $10^2/\gamma$ the accumulation of non-equilibrium magnons can take place with the constant rate leading to the constant sweeping of the frequency $\omega(\boldsymbol{k}, t)$ with the dimensionless velocity

$$\eta(\boldsymbol{k}) = \partial\omega(\boldsymbol{k}, t)/2\gamma(\boldsymbol{k})\partial t \simeq 2T\sqrt{p-1}/|S| \ . \tag{7.6.1}$$

Here T denotes the characteristic value of the interacting amplitudes $T(\boldsymbol{k}, \boldsymbol{k}')$ describing the nonlinear shift of the frequency of parametric magnons under the influence of non-equilibrium magnons. Note that since the values T and S, generally speaking, are of the same order, the estimation (7.6.1) reveals the possibility of considerable frequency sweeping, under which $\omega(\boldsymbol{k}, t)$ changes by the value $\gamma(\boldsymbol{k})$ over the time $1/\gamma(\boldsymbol{k})$. Clearly, if this sweeping is taken into account this must lead to a significant change in the results of the basic S-theory, in particular, to the appearance of the finite width of parametric wave packet. The modification of the basic S-theory of parametric excitation under the linear dependence of their frequency on the time (linear sweeping of the frequency) will be considered in the present section.

7.6.1 Qualitative Analysis of the Problem

In this section we shall proceed from the following equations of motion

$$\begin{aligned} &\frac{\partial n(\boldsymbol{k}, t)}{2\partial t} + \gamma(\boldsymbol{k})n(\boldsymbol{k}, t) + \mathrm{Im}\{P^*(\boldsymbol{k}, t)\sigma(\boldsymbol{k}, t)\} = \gamma(\boldsymbol{k})n_0(\boldsymbol{k}) \ , \\ &\frac{\partial \sigma(\boldsymbol{k}, t)}{2\partial t} + \{\gamma(\boldsymbol{k}) + i[\omega_{\mathrm{NL}}(\boldsymbol{k}, t) - \frac{\omega_{\mathrm{p}}}{2}]\}\sigma(\boldsymbol{k}, t) = iP(\boldsymbol{k}, t)n(\boldsymbol{k}, t) \ . \end{aligned} \tag{7.6.2}$$

On the one hand, these equations can be considered to be the generalization of the non-stationary equations of the basic S-theory (5.4.11) (at $n(\boldsymbol{k}, t) = n(-\boldsymbol{k}, t)$) to the case of the non-zero temperature of the thermal bass. Unlike (5.4.11) Eqs. (7.6.2) contain the additional term $\gamma(\boldsymbol{k})n_0(\boldsymbol{k})$ describing the energy flux from the thermal bass to the system of parametric waves. In the absence of the pumping this term leads to the relaxation $n(\boldsymbol{k}, t)$ not to zero but to a thermodynamic equilibrium distribution of $n_0(\boldsymbol{k})$. On the other hand, (6.7.2) are the generalization of the stationary equations (6.4.24) allowing for the interaction with the thermal bass to the non-stationary case. Unlike (6.4.24) Eqs. (6.7.2) contain additional terms with time derivatives $\partial n(\boldsymbol{k}, t)/\partial t$ and $\partial\sigma(\boldsymbol{k}, t)/\partial t$. In principle, using the methods described in Sect. 6.4 equations (7.6.2) can be derived beginning with (6.4.7). We shall not do that here but convince ourselves that the above comments are correct.

Let us proceed to the analysis of the basic equations (7.6.2). To this end, let us go over to the new variables $\varepsilon(\boldsymbol{k}) = [\omega_{\mathrm{NL}}(\boldsymbol{k}) - \omega_{\mathrm{p}}/2]$, i.e. the frequency shift, and $\Omega = (\Theta, \varphi)$, i.e. the polar and azimuthal angles of the wave vector $\boldsymbol{k}$. Using these variables (7.6.2) change over to

$$\begin{aligned} &\frac{\partial n(\varepsilon, \Omega, t)}{2\partial t} + \gamma(\Omega)n(\varepsilon, \Omega, t) + \mathrm{Im}\{P^*(\Omega, t)\sigma(\varepsilon, \Omega, t)\} = \gamma(\Omega)n_0 \ , \\ &\frac{\partial \sigma(\varepsilon, \Omega, t)}{2\partial t} + \{\varepsilon(\Omega) + i[\varepsilon + g(t)]\}\sigma(\varepsilon, \Omega, t) \\ &\qquad + iP(\Omega, t)n(\varepsilon, \Omega, t) = 0 \ . \end{aligned} \tag{7.6.3}$$

$$g(t) = \omega_{\mathrm{NL}}(\boldsymbol{k}, t) - \omega(\boldsymbol{k}) = \alpha t + c \ . \tag{7.6.4}$$

Here $g(t)$ is the stationary sweeping (departure) of the eigenfrequency. In the absence of the sweeping (at $g(t) = 0$) (7.6.3) have the single solution

$$n(\varepsilon, \Omega) = n_0(\varepsilon)\{1 + |P(\Omega)|^2/[\varepsilon^2 + \gamma^2(\Omega) - |P(\Omega)|^2]\} \ . \tag{7.6.5}$$

In the limit $n_0 \to 0$ this solution goes over to the solution of the basic S-theory which satisfies the condition of the external stability (5.5.4). Indeed, $n(\varepsilon, \Omega)$ is at maximum on the resonance surface $\varepsilon = 0$ and in the limit of small n_0 it can differ from zero only at the points where $|P(\Omega)| = \gamma(\Omega)$. On the remaining part of the surface the condition $|P(\Omega)| < \gamma(\Omega)$ must be satisfied, otherwise the contradiction may arise: $n(\Omega) < 0$. At $|P(\Omega)| > \gamma(\Omega)$ the solution of the system (7.6.3) increases exponentially with the increment

$$\nu(\Omega) = -\gamma(\Omega) + \sqrt{|P(\Omega)|^2 - \varepsilon^2} \ . \tag{7.6.6}$$

The instability region with respect to the frequencies $\omega(\boldsymbol{k})$ is determined by the inequality $\nu(\Omega) > 0$. Hence we have

$$\varepsilon < \sqrt{|P(\Omega)|^2 - \gamma^2(\Omega)} \ . \tag{7.6.7}$$

Under linear sweeping of frequency the packet of parametric waves can be such that its frequency varies with the constant velocity and the form of the packet is unchanged. In other words (7.6.3, 4) must have stationary solutions of the following form:

$$n(\varepsilon, \Omega, t) = n(\varepsilon + \alpha t + c, \Omega) \ . \tag{7.6.8}$$

Let us qualitatively analyze this solution. Under sweeping at any time some waves enter the instability range near the resonance surface and others leave it. Incoming waves increase exponentially with the increment $\nu(\Omega, t)$ (7.6.6) beginning with the level of the thermal fluctuations. They pass the instability region (7.6.7) over the time

$$\tau_0 = 2\sqrt{|P(\Omega)|^2 - \gamma^2(\Omega)}/|\alpha| \simeq 2\sqrt{2\gamma(|P| - \gamma)}/|\alpha| \ , \tag{7.6.9}$$

and at the same time their level increases up to

$$\begin{aligned} n_{\max} =& n(\Omega, \tau_0) = n_0 \exp\left[2\int_0^{\tau_0} \nu_0(t)dt\right] \\ \simeq& n_0 \exp[4\sqrt{2\gamma}(|P| - \gamma)^{3/2}/|\alpha|] \ . \end{aligned} \tag{7.6.10}$$

Subsequently, these waves leave the instability region and damp over the time τ_1 with the average decrement $\nu(\tau_1) \simeq (\alpha\tau_1)^2/2\gamma$. Hence and from (7.6.10) we obtain the estimation for τ_1:

$$\tau_1 = \sqrt{\gamma/|\alpha|\sqrt{|P|^2 - \gamma^2}} \ . \tag{7.6.11}$$

Far from the instability region the sweeping can be neglected and according to (7.6.5) $[n(\Omega) - n_0] \propto \varepsilon^{-2}$. The maximum intensity of the packet is given by (7.6.10) and is on the back boundary of the instability region (7.6.7)

$$\Delta\omega(\boldsymbol{k}) \simeq \sqrt{|P|^2 - \gamma^2}\alpha/|\alpha| \ . \tag{7.6.12}$$

The packet width δ is determined by the time over which parametric waves decrease e times passing the boundary of the instability region:

$$\delta^2 \simeq |\eta|\gamma^3/\sqrt{|P|^2 - \gamma^2} \ , \qquad \eta = \alpha/2\gamma^2 \ . \tag{7.6.13}$$

When the condition of the adiabatic slowness of the sweeping is satisfied

$$\eta < (|P|^2 - \gamma^2)^{3/2}/\gamma^3 \ , \tag{7.6.14}$$

under which the above considerations hold true, the packet width (7.6.13) is much less than its shift $\Delta\omega(\boldsymbol{k})$ (7.6.12). The total number of waves $N(\Omega)$ can be readily estimated

$$N(\Omega) \simeq n_{\max}\delta \simeq n_0 \exp\left[\frac{4(|P|^2 - \gamma^2)^{3/2}}{3|\alpha|\gamma}\right] \ . \tag{7.6.15}$$

The pre-exponential factor in (7.6.15) is not defined at our level of rigor. For us it is only important that in the absence of thermal noise ($n_0 = 0$) the number of parametric waves $N(\Omega) = 0$. Note also that (7.6.15) is not the complete solution of the problem, since to determine the number of parametric waves one must self-consistently obtain the renormalized pumping $P(\Omega)$.

In order to qualitatively analyze the number and the phase of parametric waves above the threshold it is convenient to make use of the condition of the equality of the pumped and dissipated energy fluxes

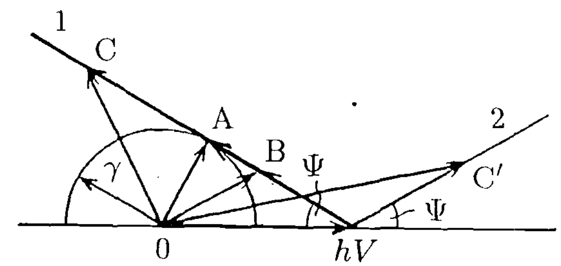


Fig. 7.6. Diagram of the pumping vectors: $\boldsymbol{h}V$ is the external pumping, $S\boldsymbol{\sigma}$ is the pumping arising due to four-wave interaction, vector $\boldsymbol{OA}$ is the self-consistent total pumping without sweeping, vectors $\boldsymbol{AB}$ and $\boldsymbol{AC}$, $\boldsymbol{AC}'$ are the same at positive and negative sweeping, γ is damping and Ψ is the pair phase. Line (1) corresponds to the continuous modification of the basic S-theory, line (2) represents the additional solution at strong negative sweeping

$$hV \sin\Psi = \gamma \ , \tag{7.6.16}$$

where Ψ is the phase difference of the anomalous correlator of parametric waves and the pumping. Therefore on the vector diagram (Fig. 7.6) the vector $S\boldsymbol{\sigma} = \sum_{\boldsymbol{k}'} S(\boldsymbol{k}, \boldsymbol{k}')\sigma(\boldsymbol{k}')$ is inclined at the angle $\Psi = \arcsin(\gamma/hV)$ with they pumping $\boldsymbol{P}$ (for definiteness, we assume $S > 0$). The sum $hV + S\boldsymbol{\sigma}$ is the renormalized pumping. Let us find now the angle between $\boldsymbol{P}$ and $S\boldsymbol{\sigma}$. From (7.6.3) it follows that in the stationary case

$$\begin{aligned} &\sigma(\boldsymbol{k}) = -iP(\boldsymbol{k})n(\boldsymbol{k})/\Delta(\boldsymbol{k}) \ , \qquad \Delta(\boldsymbol{k}) = \gamma(\boldsymbol{k}) + i[\omega_{\mathrm{NL}}(\boldsymbol{k}) - \omega_{\mathrm{p}}/2] \ , \\ &\mathrm{Re}\{P^*(\boldsymbol{k})\sigma(\boldsymbol{k})\} = -|P(\boldsymbol{k})|^2[\omega_{\mathrm{NL}}(\boldsymbol{k}) - \omega_{\mathrm{p}}/2]n(\boldsymbol{k})/|\Delta(\boldsymbol{k})|^2 \ . \end{aligned} \tag{7.6.17}$$

If the frequency sweeping is not too fast the packet is narrow with respect to the frequencies, therefore in the center of the packet

$$|P(\boldsymbol{k})|^2 = \gamma^2(\boldsymbol{k}) + (\Delta\omega)^2 \ , \quad \mathrm{Re}\{P^*(\boldsymbol{k})\sigma(\boldsymbol{k})\} = -\Delta\omega n(\boldsymbol{k}) \ . \tag{7.6.18}$$

Since under positive sweeping ($\Delta\omega > 0$), the angle between $S\boldsymbol{\sigma}$ and $\boldsymbol{P}$ is obtuse. Therefore the quantity is less than its values in the absence of the sweeping, i.e. $|S\sigma| < \sqrt{|hV|^2 - \gamma^2}$. In the case of negative sweeping ($\Delta\omega < 0$, the angle between $\boldsymbol{P}$ and $S\boldsymbol{\sigma}$ is acute and, consequently, $S\sigma > \sqrt{|hV|^2 - \gamma^2}$ at any sweeping velocity. Strictly speaking, under negative sweeping the condition (7.6.16) can be satisfied also when the vector $S\boldsymbol{\sigma}$ is inclined to hV at the angle $(\pi - \mathrm{arctg}\gamma/hV)$ (straight line (2)). For such a solution to exit the sweeping must be sufficiently large. This additional solution is, however, unstable, as it is shown below.

7.6.2 Basic Equations of S-Theory Under Frequency Sweeping

In order to investigate quantitatively the packets of parametric waves under the sweeping with the constant velocity it is convenient to change over from the variable ε to the variable ω, i.e. the current value of the frequency detuning $\omega = \varepsilon + \alpha t$. This is similar to changing over to the reference system moving with the velocity α under the Galilean transformation. In this case (7.6.3) are transformed into

$$\begin{aligned}
&\left[\frac{\alpha\partial}{2\partial\omega} + \gamma(\Omega)\right] n(\omega,\Omega) + \mathrm{Im}\{P^*(\Omega)\sigma(\omega,\Omega)\} = \gamma(\Omega)n_0\,,\\
&\left[\frac{\alpha\partial}{2\partial\omega} + \gamma(\Omega) + i\omega\right] \sigma(\omega,\Omega) + iP(\Omega)n(\omega,\Omega) = 0\,,\\
&\left[\frac{\alpha\partial}{2\partial\omega} + \gamma(\Omega) - i\omega\right] \sigma^*(\omega,\Omega) - iP^*(\Omega)n(\omega,\Omega) = 0\,.
\end{aligned} \tag{7.6.19}$$

The stationary sweeping of the packet is described by the solutions (7.6.19) which do not explicitly depend on time. Apparently they must satisfy the boundary conditions

$$[n(\omega,\Omega) - n_0] \to 0\,, \qquad \sigma(\omega,\Omega) \to 0\,, \qquad \sigma^*(\omega,\Omega) \to 0 \tag{7.6.20}$$

at $\omega \to \pm\infty$. The stationary solutions of the homogeneous system (7.6.19) at $n_0 = 0$,$|\omega| \to \infty$ have the following asymptotic behavior

$$\mathrm{Im}\{P^*(\Omega)\sigma(\omega,\Omega)\} \propto \frac{1}{\omega^3}\exp\left[\frac{(-\gamma\omega \pm i\omega^2)}{\alpha}\right]\,. \tag{7.6.21}$$

None of these solutions tends to zero at $\omega/\alpha \to \pm\infty$, therefore no linear combination of the solutions satisfies the boundary conditions (7.6.20). Physically, this means that in the absence of thermal noise the parametric excitation of waves under frequency sweeping is impossible. Therefore to determine the packet of parametric waves under sweeping we must find the regular stationary solution of the non-homogeneous system (7.6.19) satisfying the boundary conditions (7.6.20). The requirement that the solution should be regular is equivalent to the condition of the external stability in the absence of sweeping.

Under stationary frequency sweeping (7.6.19) can be transformed to the dimensionless form (7.6.24) introducing the following new variables:

$$\begin{aligned}
x(\Omega) &= \omega/\gamma(\Omega)\,, \quad \eta(\Omega) = \alpha/2\gamma^2(\Omega)\,, \quad \Pi(\Omega) = P(\Omega)/\gamma(\Omega)\,,\\
U(x) &= \mathrm{Re}\{\Pi^*(\Omega)\sigma(x)\}\,, \quad V(x) = -\mathrm{Im}\{\Pi^*(\Omega)\sigma(x)\}\,,
\end{aligned} \tag{7.6.22}$$

where $m(x) = n(x) - n_0$. Note that the angle Ω in (7.6.19) is a parameter and the dependence on this angle can be omitted when the eigenfrequency shape of the packet is studied at fixed Ω. It is convenient to perform also the Fourier transform which enables us to reduce the order of the differential operator and to make one more substitution of a variable

$$f(x) = \int f(k)\exp(ikx)dk\,, \quad z = (1 + i\eta k)/i\eta\,. \tag{7.6.23}$$

These allow to express (7.6.19) in the form:

$$\frac{4\partial^2 V}{\partial z^2} - \left(1 - \frac{i|\Pi|^2}{\eta z}\right) V = 2i|\Pi|^2\frac{n_0(2z+i)}{4\eta z}\,. \tag{7.6.24}$$

The solutions of this equation are the Whittaker functions with the index $m = 1/2$ [7.20]: $W_{y,1/2}(z)$ and $W_{-y,1/2}(z)$:

$$W_{y,1/2}(z) = \left[\exp\frac{-z/2}{\Gamma(1-y)}\right]\int [\frac{1+z}{t}]^y \exp(-t)\,dt\,. \tag{7.6.25}$$

where $y = -i|\Pi|^2/4$ and $\Gamma(x)$ is the gamma-function. This solution of (7.6.24) was obtained by *Podivilov* and *Cherepanov* [7.19].

7.6.3 Solution of S-Theory Equations

Note that below the threshold of the parametric instability and under a low rate of sweeping the ordinary below-threshold heating of the wave system takes place. For the adiabatic sweeping when the condition (7.6.14) is satisfied we have

$$n(\Omega) = \frac{\pi|\Pi(\Omega)|^2 n_0}{\sqrt{|\Pi(\Omega)|^2 - 1}}\exp\frac{2[|\Pi(\Omega)|^2 - 1]^{3/2}}{3\eta(\Omega)]}\,. \tag{7.6.26}$$

This case was quantitatively discussed in Sect. 7.6.1 and the solution (7.6.26) here qualitatively coincides with the estimation of (7.6.15) obtained earlier. Under non-adiabatic sweeping when $|\eta\Pi| > (|\Pi|^2 - 1)^{3/2}$ the number of parametric waves is limited by the level close to the thermal level

$$n(\Omega) \simeq \pi|\Pi(\Omega)|^2 n_0(1 + |\eta|^{-1/3})\,. \tag{7.6.27}$$

As the sweeping velocity increases the integrated value $\mathrm{Re}\{P^*\sigma\}$ decreases as $|\eta|^{-1}$. There is one more important case when under fast sweeping ($\eta \gg$

1, $|\Pi|$) the number of parametric waves is large. This is the case of high supercriticality $|\Pi|^2 \gg \eta > 1$, $|\Pi|$. In this case:

$$n(\Omega) = \frac{4k^2(\Omega)\gamma(\Omega)\sqrt{\eta(\Omega)}}{v(\Omega)|\Pi(\Omega)|^2} \exp\frac{|\Pi(\Omega)|^2}{2\eta(\Omega)} ,$$

$$\mu(\Omega) = 2\frac{|\Pi(\Omega)|^2}{\eta(\Omega)} \ln\frac{\eta(\Omega)}{\Pi(\Omega)} . \tag{7.6.28}$$

Now let us proceed to the problem of the shape of the parametric wave packet. In Sect. 7.6.1 it was qualitatively shown that when the condition of the adiabatic sweeping (7.6.14) satisfied the packet width of parametric waves (7.6.13) is much less than its shift from the resonance $\Delta\omega(\boldsymbol{k})$ (7.6.12). The exact expressions for δ and $\Delta\omega(\boldsymbol{k})$ coincide with these estimations with the accuracy up to the constant. When the sweeping velocity is high and the supercriticality is large we can obtain

$$\begin{aligned} \Delta\omega(\boldsymbol{k}) &= -|P|^2L/2\alpha , \quad \delta^2 = |P|^2[1+L/4] , \\ L &= \ln\{\alpha^2/\gamma^2[|P|^2 + 2\alpha\gamma\exp(-CP)]\} . \end{aligned} \tag{7.6.29}$$

Here $C = 0.577...$ is the Euler constant. Clearly, as under adiabatic sweeping, the condition $\Delta\omega|(\boldsymbol{k})| \gg \gamma(\boldsymbol{k})$ is satisfied. Strictly speaking, this is the specific part of the S-theory connected with the sweeping. Further we deal with the standard procedure of the self-consistency of the number of parametric waves with the amplitude of the renormalized pumping P.

7.6.4 Dependence of the Number of Waves on the Pumping Amplitude

The simplest case to study is the one with spherical symmetry (e.g., easy-plane antiferromagnets) when in order to obtain the total number of parametric waves N it is sufficient to multiply (7.6.26) and (7.6.28) by 4π. In the case of the axial symmetry (characteristic of the cubic ferrimagnet YIG used in the experimental study of the frequency sweeping of parametric magnons) Eqs. (7.6.26, 28) can also be rather easily integrated with respect to the angles. Under $|\eta_0| \ll (|\Pi_0|^2 - 1)$ this yields

$$\begin{aligned} N &= N_T\sqrt{\frac{\pi\eta_0}{\beta}}\frac{|\Pi_0|^{3/2}}{[|\Pi_0|^2-1]^{9/16}} \exp\left|\frac{3[|\Pi_0|^2-1]^{3/2}}{\eta_0}\right| , \\ \mu &= \sqrt{|\Pi_0|^2 - 1}\,\mathrm{sign}\, g\eta , \quad N_T = 4\pi^2k^2n_0\gamma/[\partial\omega(k)/\partial k] . \end{aligned} \tag{7.6.30}$$

Here N_T is the number of thermal waves in the layer of the width $\Delta\omega(\boldsymbol{k}) = \pi\gamma(\boldsymbol{k})$ near the resonance surface. If $|\Pi_0|^2 \gg |\eta_0| \gg 1$, $|\Pi_0|$ then

$$N = 2N_T\frac{|\eta_0\Pi_0|}{\sqrt{\pi\beta}}\exp\left(\pi\frac{|\Pi_0|^2}{2|\eta_0|}\right) , \quad \mu = \left(2\frac{|\Pi_0|}{\eta_0}\right)\ln\left(\frac{|\eta_0|}{|\Pi_0|}\right) . \tag{7.6.31}$$

In these expressions Π_0 and η_0 are the values of $\Pi(\Omega)$ and $\eta(\Omega)$ on the resonance surface and the coefficient $\beta \simeq 2$ characterizes the angular dependence $\Pi(\Omega)$:

$$|\Pi(\Omega)|^2 = |\Pi_0| - \beta\cos\Omega , \quad \Omega = \Theta, \Psi . \tag{7.6.32}$$

It is clear that in different cases of symmetry, spherical and axial symmetry, the respective formulae are different only in the pre-exponential factor. Further transformation will lead only to an insignificant difference in the logarithmic factor. Therefore we shall subsequently confine ourselves to the case of axial symmetry experimentally studied by *Zautkin* et al. [7.21].

The next stage of the self-consistency procedure is to make use of the relationship between P and hV. From (5.5.6) we can obtain

$$|hV|^2 = |P|^2 + 2\mu\gamma|S|N + (SN)^2(1+\mu^2)/|P|^2 , \tag{7.6.33}$$

where $\mu = \mathrm{Re}\{P^*\Sigma/\gamma N\}$. The solution of (7.6.33) has the following form:

$$SN_\pm = \frac{|P|^2}{1+\mu^2}\left[-\mu \pm \sqrt{\mu^2 + \frac{1+\mu^2}{|P|^2}(h^2V^2 - |P|^2)}\right] . \tag{7.6.34}$$

If the condition of adiabatic sweeping (7.6.14) is satisfied, the number of parametric waves N is large in comparison with the thermal noise and $\mu = \alpha\sqrt{|P|^2 - \gamma^2}/|\alpha|\gamma$. Then it follows from (7.6.34) that

$$|S|N_\pm = -\frac{\alpha}{|\alpha|}\sqrt{|P|^2-\gamma^2} \pm \gamma\sqrt{p-1} , \quad p = (h/h_{\mathrm{th}})^2 . \tag{7.6.35}$$

The simplest case for the analysis is $S\alpha > 0$. Then in (7.6.35) only the solution N_+ is meaningful. At the low level of the thermal noise when

$$\varepsilon = |S|N_T/\gamma \ll 1 \tag{7.6.36}$$

it follows from (7.6.35) and (7.6.30) that

$$|S|N_+ = \gamma\sqrt{p-1} - \frac{\alpha}{4\gamma^2}(\ln[\frac{(p-1)}{p}])^{1/3} . \tag{7.6.37}$$

The first term here corresponds to the well-known result of the S-theory. As the sweeping velocity increases, the numbers of parametric waves decreases. Under a very small supercriticality when $p - 1 < \eta^2$ (7.6.37) is inapplicable because the condition (7.6.14) is violated. In this case the number of parametric waves is not big in comparison with the number of thermal waves. Under high supercriticality the applicability of condition (7.6.14) is also violated. Then (7.6.31) must be used. The qualitative behavior of the dependence of N on the sweeping rate is retained (Fig. 7.7). The negative sweeping rate results in a much more complicated situation since both solutions (7.6.35) are meaningful. At not too large levels of noise and not too

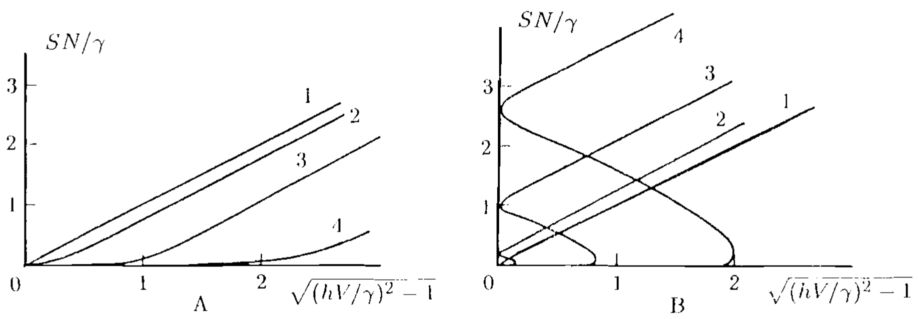


Fig. 7.7. Theoretical dependences of the total number of parametric waves N on the pumping amplitude h at different rates of sweeping: lines (1) correspond to $\eta = 0$, lines (2) to $|\eta| = 10^{-2}$, lines (3) to $|\eta| = 3 \cdot 10^{-2}$, lines (4) to $|\eta| = 0.1$. Fig. A corresponds to the positive sweeping, Fig. B to the negative sweeping

large sweeping rates there exists a single solution corresponding to the root of N_+ in (7.6.35) and described by (7.6.37). At $S\alpha \to 0$ this expression naturally coincides with the basic S-theory result. If the noise level is small and the sweeping rate is sufficiently big, there are three solutions of the wave number of N and the renormalized pumping $|\Pi|$. The smallest solution corresponds to the situation when the noise level is so low and the sweeping rate is so large that the amplitude of parametric waves over the time of passing the instability region has no time to increase significantly. The two other solutions correspond to high N, the intermediate value is unstable. Fig. 7.7 shows a set of dependences of the number of waves N on the pumping power p calculated for different values of the sweeping velocity α and the thermal noise amplitude $\varepsilon = 10^{-3}$. Clearly, there is a hysteresis of the dependence of the number of parametric waves on the pumping intensity under a fixed rate of sweeping. When the pumping intensity is less than a certain value, the critical level of parameter waves is close to the thermal level. Under $p > p_1$ where

$$p_1 \simeq 1 + [(3/4)(|\alpha|/\gamma^2)\ln(|\alpha|/4\gamma^2\varepsilon)]^{3/2} , \qquad (7.6.38)$$

two stable states of parametric waves can exist. As the pumping intensity increases under $p > p_2$ where

$$p_2 = 1 + [(3/4)(|\alpha|/\gamma^2)|\ln\varepsilon|]^{2/3} , \qquad (7.6.39)$$

the single state remains and under $\alpha \to 0$ changes over to the standard solution of the basic S-theory. The number of waves in this solution is close to the well-known expression $|S|N = \gamma\sqrt{p-1}$. It must be emphasized that the number of parametric waves for the solution (7.6.37) corresponding to the basic S-theory in the absence of sweeping indefinitely increases under increasing sweeping velocity and under fixed pumping intensity.

Concluding, the expression for the frequency of the homogeneous collective oscillations for the case of the adiabatic sweeping is presented

$$\Omega_\pm = -i\gamma \pm \sqrt{4[SN + \Delta\omega(\boldsymbol{k})](2T+S)N - \gamma^2} , \qquad (7.6.40)$$

where $\Delta\omega(\boldsymbol{k})$ is the shift of the packet of parametric waves specified by (7.6.12). It must be noted that expressions (7.6.40) were obtained by *Podivilov* and *Cherepanov* by awkward but rather accurate calculations [7.19]. The frequency [7.6.40] can also be obtained from the simple equations of the basic S-theory without frequency sweeping for the narrow packet of parametric waves localized on the surface $\omega_{\rm NL}(\boldsymbol{k}) = \Delta\omega(\boldsymbol{k}) + \omega_{\rm p}/2$. Incidentally, the same is true of the expression for the general number of parametric waves N and their phase Ψ. Thus in the cases when the number of parametric waves N is large in comparison with N_T the role of the frequency sweeping mainly amounts to the modification of the condition of external stability; instead of $\omega_{\rm NL}(\boldsymbol{k}) = \omega_{\rm p}/2$ we obtain $\omega_{\rm NL}(\boldsymbol{k}) = \Delta\omega(\boldsymbol{k}) + \omega_{\rm p}/2$ where the position of the packet center $\Delta\omega(\boldsymbol{k})$ is given by the expressions (7.6.12) and (7.6.29).

Later, in Chap. 9, we shall compare the theory developed in this chapter with the experimental data by *Zautkin* et al. [7.21]. A full qualitative and good quantitative agreement between theory and experiment was established.

7.7 Problems

Problem 7.1. Take into account the influence of the interaction between the resonator and the SHF-magnetic field of the pumping on the frequency of collective oscillations of parametric waves excited by parallel pumping. Investigate the instability region of the coupled oscillations.

Problem 7.2. Obtain the expressions for the frequencies of the coupled collective oscillations of parametric magnons and homogeneous precession under transverse pumping: first, for the processes of the first order far from the resonance of the homogeneous precession and then for the second-order processes at the resonance of the homogeneous precession.

Problem 7.3. By making use of the Hamiltonian of the interaction of magnons with the magnetic field (4.3.18) and (4.3.19) obtain expressions (7.2.4) for the "external force" acting on the system of parametric magnons (7.2.1) in the presence of radio-frequency and SHF magnetic fields.

Problem 7.4. Obtain the expression of the Hamiltonian of magnon–sound interaction in the easy-plane antiferromagnets. (*Lutovinov* [7.6])

Hint. Use the expression for the magnetoelastic energy and the canonical variables for the sound and magnons in easy-plane antiferromagnets.
Answer: See (7.2.6).

Problem 7.5. Obtain the condition of the external stability of the stationary state of parametric magnons (7.4.2) under radio-frequency modulation $H_{\rm M}(t) = H_{\rm M} \cos \Omega t$:
Answer:

$$\begin{aligned}\Delta(k) =& \omega_{\rm NL}(\boldsymbol{k}) - \omega_{\rm p}/2 = \frac{|U(k)H_{\rm M}|^2}{|\Delta(\Omega)|^2}\Big\{T(\Omega^2 + 4S^2N^2) \\ &+4S\Big[\gamma^2 + 2S^2N^2 + \frac{(2\gamma SN)^2}{\Omega^2+4\gamma^2}\Big]\Big\} .\end{aligned} \tag{7.7.1}$$

Problem 7.6. Obtain the expression for the frequencies of the spatially non-homogeneous collective oscillations $\Omega(\boldsymbol{\kappa})$($\boldsymbol{\kappa}$ is the wave vector of collective oscillations) under the parametric excitation of waves in the case of spherical symmetry: $V(\boldsymbol{k}) = V$, $S(\boldsymbol{k}, \boldsymbol{k}') = S$, $T(\boldsymbol{k}, \boldsymbol{k}') = T$. This case is realized in easy-plane antiferromagnets, e.g., $CsMnF_3$, $MnCO_3$, etc.
Hint. For the analysis use (7.1.8)
Answer

$$\begin{aligned}\Omega(\kappa) =& - i\gamma + \sqrt{(v\kappa)^2 \coth^2(v\kappa/\Delta) - \gamma^2} , \\ \Delta =& 4S(2T+S)N^2 , \quad v = \partial\omega(k)/\partial k .\end{aligned} \tag{7.7.2}$$

Problem 7.7. Obtain the expression for the frequencies of the spatially non-homogeneous collective oscillations $\Omega(\boldsymbol{\kappa})$ in the case of axial symmetry when parametric waves are excited only on the equator of the resonance surface and the vector κ is in the plane of the equator.

Problem 7.8. Find the instability increment (frequencies $\Omega_{\pm 1}(\kappa)$) of the spatially non-homogeneous mode of the collective oscillations with the axial number $m = \pm 1$ under the parametric excitation of magnons by parallel pumping in cubic ferromagnets when magnons are excited only on the equator of the resonance surface. Analyze (analytically and on the computer) the nonlinear stage of this instability in some limiting cases. Will this leads to a stationary but spatially homogeneous distribution of parametric magnons? If this is not the case, what type of self-oscillations will arise in the system?

Note. These questions have not yet been answered but are of great interest for the interpretation of the experimental data on the ferrimagnetic YIG when the mode instability $m = \pm 1$ can take place. It is a good challenge. In case of success, please publish the results!

Problem 7.9. Obtain the time dependence of the absorbed intensity at the linear development stage of the parametric instability when the pumping is turned on instantaneously.
Hint. Use (7.6.2) in the linear approximation
Answer:

$$W_+(t) = \pi\omega_{\rm p} \int d\Omega \gamma(\Omega) hV(\Omega) k^2(\Omega) n_0 F[\Omega, 2hV(\Omega)t]/2v(\Omega) ,$$

where $F(\Omega, \tau) = \int_0^\tau I_1(x) \exp[-\gamma(\Omega)x/hV(\Omega)]dx$ and $I_1(x)$ is the Bessel function of the imaginary argument (*Cherepanov*, unpublished).