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CHAPTER 5

Solitons and Nonlinear Phenomena in Parametrically Excited Spin Waves

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Solitons
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Introduction

Physics of nonlinear phenomena is now going through a period of intensive development. Nonlinear optical effects in dielectrics, various nonlinear processes in plasmas, in magnetics, liquids, etc., are investigated. Interesting and important nonlinear objects are those open oscillating systems which are built up so that an external influence changes their parameters in time. The ground state of such systems may be unstable, i.e., a small initial perturbation increases exponentially. This instability is called parametric. It arises under the parametric resonance condition when the frequency ω_{p} of the parameter variation is a multiple of the self-frequency of the oscillator ω_0 (Mandelshtamm 1972, Landau and Lifshitz 1973, Arnold 1974): $n\omega_{\mathrm{p}} = m\omega_0$. The most wide region of parametric instability in the frequency corresponds to the case $n = 1$, $m = 2$. The question of parametric resonance in a continuous medium arose in the 1950s in connection with the experiments of Bloembergen, Damon and Wang (Bloembergen and Damon 1952, Bloembergen and Wang 1954) on ferromagnetic resonance at high power levels. An "additional" (as compared with the case of small amplitudes) absorption of the energy of uniform precession (UP) of magnetization which has a clearly pronounced threshold in the amplitude was observed in those experiments.

Sühl (1957) explained this phenomenon as parametric instability of UP with respect to the excitation of a pair of spin waves (SW) with frequencies $\omega_{k'}$, $\omega_{k''}$ and wave vectors k' and k''. He was the first to formulate the conditions for parametric resonance in a continuous medium, which, as distinct from the parametric resonance conditions for an oscillator, have the form:

$$n\omega_{\mathrm{p}} = \omega_{k'} + \omega_{k''}, \quad k' + k'' = 0, \tag{1}$$

where ω_{p} is the UP frequency. Since $k' = -k''$, the parametric instability results in the creation of a pair of waves with equal and opposite wave vectors. In 1960 Morgenthaler (1960), and independently Schlömann et al. (1960), predicted the phenomenon of "parallel pumping", i.e. the parametric excitation of SW by an alternating magnetic field with the polarization parallel to the direction of magnetization. Schlömann et al. observed experimentally this phenomenon predicted. At the present time parallel excitation is one of the main methods of generation of SW in ferromagnets. Studies of parametric instability in plasmas and in nonlinear optics began in the 1960s. In 1962

Oraevskiĭ and Sagdeev (1962) used an example of Langmuir and ion-acoustic waves in plasma to develop the theory of decay (first-order) instability of a finite-amplitude monochromatic wave in a nonlinear medium. For an initial ω_k-wave this instability leads to the excitation of a pair of waves where frequencies and wave vectors satisfy the conditions

$$\omega_k = \omega_{k'} + \omega_{k''}, \quad k = k' + k''. \tag{2}$$

These conditions are obvious a generalization of eq. (1) for $n = 1$. This instability can be interpreted as a coherent decay of the original ω_k-quanta into pairs of $\omega_{k'}$- and $\omega_{k''}$-quanta, and the relations (2) can be regarded as the conservation laws for this decay. Stimulated Raman scattering and stimulated Mandelstamm–Brillouin scattering have been predicted and experimentally observed in optics in the same years. These scatterings represent the decay instability of an electromagnetic wave and phonons (see, e.g., Bloembergen 1966). In the years following a number of works on parametric instabilities in continuous media (see the review by Silin 1971) have been published. In particular, the parametric instability of the uniform electric field in plasma (Andreev et al. 1969), the second-order decay instability of finite-amplitude waves (Zakharov 1966), and the instability of waves on the surface of a liquid (Zakharov 1968). One can say that the linear theory of parametric instability in homogeneous media has been, in general, accomplished by the beginning of the 1970s (Bengamen and Feir 1967, Witham 1970).

In the case of the nonlinear theory the situation is quite different. Turbulence of the waves appears as a result of the development of the parametric instability in a continuous medium (if the linear dimensions of the system are sufficiently large in comparison with the wavelength of the excited waves). This is the state in which a great number of waves are simultaneously excited and interact intensively. Unfortunately, the character of this turbulence depends on a specific situation: on the form of the dispersion law of the waves, and on the nonlinear and dissipative properties of the medium, which are quite different in different cases. However, there is a simple and at the same time extremely representative case where the development of the general theory proved to be possible. That is when the excitation is produced by a spatially homogeneous field ($k = 0$) or a wave with large wavelength ($k \ll k'$, k'') so that the turbulency may be looked upon as statistically homogeneous. A further requirement is that the dispersion relations for the medium should exclude the first-order decay processes (2) for parametrically excited waves. Exactly this situation occurs in most of the experiments on parametric excitation of spin waves in ferromagnets. It is necessary to point out that these experiments belong to a class of the "purest" experiments in physics of nonlinear waves because of their relative simplicity (in comparison, for example, with experiments on plasma or in nonlinear optics), and because of the high quality of the ferromagnetic single crystals employed. A particularly suitable medium for the experimental study is the YIG crystal, whose garnet possesses in many respect unique properties, including a completely ordered magnetic structure, a high degree of homogeneity, a very large value of the acoustic factor (10^7 at 10^6 hz), and a weak damping of SW. The greater part of the experimental data described in this reviewing chapter deals with YIG. Important experimental data were also observed on the ferromagnets $MnCO_3$ and C_3MnF_3.

Experimental data on spin turbulence have been accumulated since the beginning of the '60s. The first models were introduced at about the same time, and were aiming, first of all, at clarifying the mechanism that restricted the amplitude growth of unstable spin waves. The first step in this direction was made in Sühl's paper (Sühl 1957) where it has been shown that the principal amplitude-restricting mechanism in the case of excitation of spin waves by uniform precession of magnetization was the feed-back reaction of these waves on the pump, which led to "freezing" of its amplitude at the threshold level. However, attempts to explain the phenomena observed in the case of parallel pumping have encountered considerable difficulties. The use of the various methods of restraining parametric instability which are conventional for parametric resonance in systems with a small number of degrees of freedom (nonlinear damping and nonlinear frequency mismatching), proved to be inadequate. In most cases the nonlinear damping is too weak and too sensitive to the effect of a constant magnetic field to explain the observed spin-wave level. The nonlinear frequency detuning does not, in general, restrict the parametric resonance in a continuous medium since it is always possible (whatever the amplitude) to find waves whose renormalized frequencies satisfy the resonance conditions exactly. An important step toward understanding of spin turbulence was made by Schlömann (1959a), who paid attention to the necessity to take into account parametrically excited waves. He has also suggested that the main contribution to this interaction is a result of the nonlinear processes which satisfy the conditions

$$\omega_k + \omega_{-k} = \omega_{k'} + \omega_{-k'}, \tag{3}$$

and which do not take the waves out of the parametric resonance.

In 1969, Zakharov, L'vov and Starobinets have shown in their pioneer works (Zakharov et al. 1969, 1970) that processes of the type described by eq. (3) conserve the phase correlation within each parametrically excited pair of waves with equal and opposite wave vectors and lead to a self-consistent change in the resultant phase of the waves in each pair. This change in the resultant phase leads to a weakening of the coupling between the spin waves and the pump and, ultimately, to the restriction of their amplitude. Excited waves are then precisely those for which the renormalized frequencies exactly satisfy the parametric resonance condition. This "phase" mechanism of the amplitude limitation is specific for continuous media and is only realized in a pure form in systems with very large linear dimensions (in comparison with the wavelength). It is the principal mechanism in limiting the amplitudes of spin waves in the case of parallel pumping. Processess of the type (3) together with the necessary phase relationships are conveniently investigated by diagonalizing the Hamiltonian for the wave interaction, which is analogous to the BCS-approximation in the theory of superconductivity. The theory based on this diagonalization approximation (Zakharov et al. 1970) was subsequently called the "S-theory". This theory and its generalizations resulted in a considerable progress in the study of spin-wave turbulence. In the period 1970–1974 Zakharov, L'vov, Starobinets, their colleagues Musher, Zautkin, Rubenchik and some others gave a qualitative explanation of many observed effects, and obtained a satisfactory quantitative agreement with experimental data (Zakharov et al. 1970, 1972, Zautkin et al. 1970, 1972, L'vov and Starobinets 1971). In particular, giant autooscillations of magnetization during the parametric excitation of spin waves, discovered in 1961 by Hartwick et al. (1961) were later experimentally investigated in detail. In 1974 Zakharov et al. (1974) have published a review (largely based on results obtained with their participation), in which the S-theory was consistently presented to account for experimental results.

In the next five years the rate of theoretical investigations on nonlinear phenomena at the parametric excitation of spin waves decreased. This was mainly due to the fact that by 1974 the developed theory, first, went ahead of experimental works essentially and, second, the principal part of investigators in this field had not yet made the theory their own. In those years theoretical works also appeared. They prejudged the S-theory (see e.g. Tsukernik and Jankelevich 1975, Winikovetsky et al. 1979) or, on the contrary, repeated its results with other methods (e.g. Mikhailov 1979). In some intermediate theoretical works based on the S-theory new but wrong results were connected with a misunderstanding of the framework of applicability (see e.g. Bakai 1978, Mikhailov 1976, Morozov and Mukhay 1982). At the same time Melkov, Prozorova, Ozhogin, Zautkin, and their colleagues obtained interesting and important experimental results on nonlinear behaviour of parametric spin waves, connected with a new understanding of these physical phemomena (Melkov 1975, Melkov and Krutsenko 1977, Krutsenko and Melkov 1979, Kreder et al. 1972, Prozorova and Smirnov 1974, 1975, 1978, Smirnov 1977, Oshogin and Jakubovskii 1974, Jakubovskii 1974, Zautkin et al. 1977, Orel and Starobinets 1975, Zautkin and Orel 1980). The main conclusions of the S-theory were confirmed in details. Interesting aspects of the nonlinear behaviour of SW were confirmed and new nonlinear effects were observed. It was difficult to predict them theoretically. Ultimately towards the end of the '70s the experimental lag was eliminated and the necessary prerequisites were created for further investigations. It became clear that scattering of parametric waves on each other broadens the partition function of PW and this function is actually not singular, as distinct of the "S-theory" prediction. As a rule, in perturbation theory it is enough to take this scattering into account to second order in the matrix element of the T-interaction of the waves. Therefore it is naturally to call this theory the S,T^2-theory because it considers both the S-interaction of pairs and the T^2-scattering of individual waves.

If the S-theory allows to determine the number distribution of PW on the resonance surface and phase correlations, then the S,T^2-theory gives the packet widths of PW in frequencies $\Delta\omega$, modulus $k-\Delta k$, angular dimensions of the packet $\Delta\theta$ and $\Delta\varphi$, and for the same applicability condition as he S-theory. The Wyld diagram technique (DT) (Wyld 1961) can be used as a basis for the development of the S,T^2-theory, the Wyld diagram technique was reformulated by Zakharov and the present author for canonical equations of motion (Zakharov and L'vov 1975). This technique is a classical analogue of the Keldish diagram technique for strongly nonequilibrium systems (Keldish 1964). Neglecting the renormalization of vertices we obtain equations of the S,T^2-theory in terms of the DT which take into account the pairing effect. They were first obtained in 1973 by the present author (L'vov 1975, Krutsenko et al. 1978). At the same time the applicability conditions of the S-theory and the packet width in k-space ($\Delta\omega_k$) were found. Some years ago Melkov and Krutsenko measured the frequency width $\Delta\omega$ and angular dimensions (Bakai et al. 1979, Krutsenko and Melkov 1981a) in several nice experiments. It stimulated Tcherepanov and the present author to develop the S,T^2-theory further. And in a short time we could determine this quantities theoretically, taking into account many important real factors, e.g. weak crystallographic anisotropy of cubic ferromagnets, interaction on inhomogeneities, thermal noise, etc. (Krutsenko et al. 1978, L'vov and Cherpanov 1978, 1979). Ultimately the S,T^2-theory for magnets was nearly so detailed as the S-theory. To date the S,T^2-theory has been represented in the form of separated publications and in this chapter it will be developed consistently for the first time in the second, advanced, part (sections 5–8).

Part I – The S-theory

1. Parametric instability of waves in magnetdielectrics

1.1. The classical Hamiltonian formalism for ferromagnets

Quanta-mechanical occupation numbers for the excitation of parametric waves are much greater than unity for some orders of magnitude. Therefore the classical approach is appropriate to describe this phenomenon. From our point of view the classical Hamiltonian method is more adequate to describe the problem of the interacting of parametrically excited waves. This method allows to eliminate unessential specific properties of a medium, where waves propagate from the equations of motion. Zakharov (1968, 1971) propagandized this method in problems of the nonlinear dynamics of wave actively. He developed it in his works and described it. A review is given in Zakharov (1974). Let us characterize the amplitude of waves by means of complex canonical variables a_k, a_k^* which are classical analogues of the Bose operators of the elimination of waves. Variables a_k, a_k^* satisfy the Hamiltonian equation of motion in the form:

$$\mathrm{i}\frac{\mathrm{d}a_k}{\mathrm{d}t} = \frac{\delta H}{\delta a_k^*}, \tag{1.1}$$

where $\delta/\delta a_k$ is a variational derivative and H a Hamiltonian. We can choose variables a_k, a_k^* so that the quadratic part of the Hamiltonian $H^{(2)} = H_0$ is diagonal:

$$H_0 = \sum_k \omega_k a_k^* a_k. \tag{1.2}$$

These variables are the normal variables of the linear theory, and are particularly convenient for nonlinear problems. Specifically "linear" difficulties associated with this model of the medium are overcome in the course of the search for the variables a_k. In terms of these variables, the linearization of the equations of motion is a fairly trivial task:

$$\dot{a}_k + \mathrm{i}\omega_k a_k = 0.$$

All the "linear" information which is essential for investigating nonlinear problems is contained in the dispersion relation for the waves. All the additional information about the interaction between the waves is contained in the remaining coefficients of the expansion of H in powers of a_k:

$$H = H^{(2)} + H^{(3)} + H^{(4)} + \cdots. \tag{1.3}$$

The Hamiltonian $H^{(3)}$ describes the three-wave processes:

$$\begin{aligned} H^{(3)} = {} & \sum_{1,23} (V_{1,23}^* a_1^* a_2 a_3 + \text{c.c.})\Delta(k_1 - k_2 - k_3) \\ & + \tfrac{1}{3} \sum_{1,2,3} (U_{123}^* a_1 a_2 a_3 + \text{c.c.})\Delta(k_1 + k_2 + k_3), \end{aligned} \tag{1.4}$$

whilst the Hamiltonian $H^{(4)}$ describes the four-wave processes:

$$H^{(4)} = \tfrac{1}{2} \sum_{1,2,3,4} W_{12,34} a_1^* a_2^* a_3 a_4 \Delta(k_1 + k_2 - k_3 - k_4), \tag{1.5}$$

where $a_1 \equiv a_{k_1}$ and so on; $V_{1,23} \equiv V(k_1;\ k_2,\ k_3)$. The physical significance of each of the terms in the hamiltonian can readily be understood by recalling that the canonical variables a_k and a_k^* are the classical analogues of the Bose-operators. For example, the term proportional to $V_{1,23}$ describes, together with its complex conjugate, the interaction of three-waves of the form given by eq. (1.2). Henceforth we shall assume that processes of this type are forbidden by the form of the dispersion law in the region of space in which we are interested.

When an external energy source is present, i.e., the pump field $h(t)$, a number of items occurs in the Hamiltonian. One of them is:

$$H_{\mathrm{p}}^{(2)} = \sum_k \tfrac{1}{2} H_{\mathrm{p}k} = \tfrac{1}{2} \sum_k \left(h(t) V_k a_k a_{-k} + \text{c.c.} \right), \tag{1.6}$$
$$h(t) = h \exp(-\mathrm{i}\omega_{\mathrm{p}} t).$$

It describes the first-order parametric instability with the conservation principle (1.2). Hamiltonian (1.6) corresponds to the Zeeman energy $\boldsymbol{h} \cdot \boldsymbol{M}$. In this case (Morgenthaler 1960, Schlömann et al. 1960)

$$V_{\boldsymbol{k}} = \frac{\boldsymbol{M}}{M} V_{\boldsymbol{k}}, \quad V_{\boldsymbol{k}} = V \sin^2\theta_k \exp(-2\mathrm{i}\phi_k), \tag{1.7}$$

where $\boldsymbol{M}$ is the magnetization, and θ_k and ϕ_k are the polar and azimuthal angles defining the vectors $\boldsymbol{k}$ in the spherical coordinate system whose z-axis lie along $\boldsymbol{M}$. From equation (1.7) we can see that the Hamiltonian(1.6) has only the projection $\boldsymbol{h}$ which is parallel to $\boldsymbol{M}$. Therefore, such a method of the parametric excitation of spin waves (SW) was called the "parallel pumping". The orthogonal part of $\boldsymbol{h}$ to $\boldsymbol{M}$ excites the uniform precession of magnetization (UP). This phemonenon, named ferromagnetic resonance, is described by the Hamiltonian

$$H_{\mathrm{p}}^{(1)} = u\left(h_x + \mathrm{i}h_y\right) a_0 + \text{c.c.} \tag{1.8}$$

On the other hand an UP can parametrically excite the SW in the following processes:

$$\omega_0 = \omega_k + \omega_{-k}, \quad 2\omega_0 = \omega_k + \omega_{-k}, \tag{1.9}$$

which are described by terms $a_0 a_k^* a_k^*$, $a_0^2 a_k^* a_{-k}^*$ contained in Hamiltonians $H^{(3)}$, $H^{(4)}$.

1.2. Equations of motion

The Hamiltonian of the system which takes into account the first-order processes of the parametric excitation (1.6) has the form:

$$H = \sum_k \omega_k a_k a_k^* + \tfrac{1}{2} \sum_k \left(h V_k \exp(-\mathrm{i}\omega_{\mathrm{p}} t) a_k^* a_{-k}^* + \text{c.c.} \right) + H_{\text{int}},$$
$$h \equiv h_z, \tag{1.10}$$

where $H_{\text{int}} = H^{(3)} + H^{(4)}$. Since all parametric waves have close or equal frequencies $\omega_k \approx \omega_{\mathrm{p}}/2$, we can simplify the interaction Hamiltonian H_{int} by retaining in it only the four-wave terms describing the interaction of waves with conservation laws in the form

$$\omega_{k_1} + \omega_{k_2} = \omega_{k_1+\kappa} + \omega_{k_2-\kappa}. \tag{1.11}$$

The latter do not, however, mean that the three-wave Hamiltonian can be ignored at all. Its matrix elements are large in comparison with the four-wave case and, therefore, in general their contribution to the amplitude for the four-wave processes must be taken into account to second order in perturbation theory. This leads to the renormalization of the matrix elements for the four-wave Hamiltonian:

$$W_{12,34} \rightarrow T_{12,34} + \text{terms of the order of } |V|^2/\omega_{\mathrm{p}}$$

(see, for example, Zakharov 1974). For cubic ferromagnets, both contributions to the coefficients $T_{12,34}$ are of the same order of magnitude (Zautkin et al. 1972a). Therefore, the Hamiltonian describing the parametric interaction between the waves must be chosen in the form

$$H_{\mathrm{int}} = \tfrac{1}{2} \sum_{1,2,3,4} T_{12,34} a_1^* a_2^* a_3 a_4 \Delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4). \tag{1.12}$$

It is of course necessary to take the interaction between the parametric waves and the spin waves and phonons into account. This interaction leads to the damping of the parametric waves, which is usually taken into account phenomenologically by introducing a dissipative term into the canonical equations of motion:

$$\dot{a}_k + \gamma_k a_k = -\mathrm{i} \frac{\delta H}{\delta a_k^*}. \tag{1.13}$$

There may be some doubt as to the validity of this procedure for the description of coherent wave systems in which phase relationships are significant. In the second, advanced, part of our review (sections 5–8) we will ground this procedure with the aid of the diagram technique, and show that the damping constant γ_k in eq. (1.13) can be calculated by the usual kinetic equation. This does not apply, however, when the wave damping is due to scattering by inhomogeneities (see Morgenthaler 1960).

Using the explicit form of the Hamiltonian given by eq. (1.10), with H_{int} given by eq. (1.12), we can write the dynamic equations (1.13) in the form

$$\left(\frac{\partial}{\partial t} + \gamma_k + \mathrm{i}\omega_k\right) a_k + \mathrm{i} h V_k \exp(-\mathrm{i}\omega_{\mathrm{p}} t) a_{-k}^* = -\sum_{1,2,3} T_{k1,23} a_1^* a_2 a_3 \Delta(k + k_1 - k_2 - k_3). \tag{1.14}$$

These equations form the starting point for the analysis of the behavior of spin waves beyond the parametric excitation threshold.

1.3. Excitation threshold and amplitude-limiting mechanisms

The parametric excitation threshold can be calculated directly from eq. (1.14). In the linear approximation, these equations split into independent pairs of equations, and when the "fast" dependence on time is eliminated, they have the form

$$\left[\frac{\partial}{\partial t}+\gamma_k+\mathrm{i}(\omega_k-\omega_\mathrm{p}/2)\right]b_k+\mathrm{i}hV_kb^*_{-k}=0,$$

$$-\mathrm{i}(hV_k)^*b_k+\left[\frac{\partial}{\partial t}+\gamma_k-\mathrm{i}(\omega_k-\omega_\mathrm{p}/2)\right]b^*_{-k}=0, \tag{1.15}$$

$$b_k(t)=a_k(t)\exp(\mathrm{i}\omega_\mathrm{p}t/2).$$

Assuming that b_k, $b^*_{-k}\sim\exp(\nu_kt)$, we have

$$\nu_k=-\gamma_k+\left[|hV_k|^2-(\omega_k-\omega_\mathrm{p}/2)\right]^{1/2}. \tag{1.16}$$

The minimum threshold corresponding to the parametric resonance $2\omega_k=\omega_\mathrm{p}$ is determined by the condition $|hV_k|=\gamma_k$, which has the simple interpretation of an energy balance condition. In fact, the energy flux W_+ which is transferred from the pump to the pair of waves $\pm k$ is given by

$$W_+=-\partial H_{\mathrm{p}k}/\partial t=\mathrm{i}\omega_\mathrm{p}(hV_kb^*_kb^*_{-k}-\mathrm{c.c.})$$

$$=2|hV_k|\omega_\mathrm{p}|b_k|^2\sin(\tilde{\psi}_k-\psi_k), \tag{1.17}$$

where $b_k=|b_k|\exp(-\mathrm{i}\phi_k)$, $\psi_k=\phi_k+\phi_{-k}$ is the phase of the pair, and $\tilde{\psi}_\mathrm{k}=\arg(hV_k)$. On the other hand, the W_- energy dissipated by the pair per unit time is given by

$$W_-=2\gamma_k(\omega_k|b_k|^2+\omega_{-k}|b_{-k}|^2)=2\omega_\mathrm{p}\gamma_k|b_k|^2. \tag{1.18}$$

At the threshold $W_+=W_-$. The maximum energy flux and the lowest instability threshold is obtained for the pair with the most convenient phase relationship $\psi_k=\tilde{\psi}_k+\pi/2$. We then again obtain relation $|hV_k|=\gamma_k$ for the threshold. The parametric resonance condition can obviously be simultaneously satisfied for a large number of pairs whose wave vectors lie on the resonance surface. The minimum excitation threshold h_1 is obtained for pairs for which the ratio $\gamma_k/|V_k|$ is a minimum:

$$h_1=\min(\gamma_k/|V_k|). \tag{1.19}$$

For example, in the case of isotropic ferromagnets at parallel pumping, when V_k is determined by condition (1.7), and when $\gamma_k=\gamma$, the first to be excited are the pairs with $\theta=\pi/2$, i.e., those lying on the "equator" of the resonance surface. In antiferromagnets, V_k usually does not depend on the direction k (Ozhogin 1970). Then the excitation of waves for $\gamma_k=\gamma$ must arise on all resonance surface simultaneously. When $|hV_k|>\gamma_k$, the amplitude of the pairs begins to increase exponentially with growth rate (1.17), namely:

$$b_k(t)=b_k\exp(\nu_kt-\mathrm{i}\psi_k/2),$$

$$b^*_{-k}(t)=b^*_{-k}\exp(\nu_kt+\mathrm{i}\psi_k/2),$$

and it follows from eq. (1.16) that

$$\cos(\psi_k - \tilde{\psi}_k) = \frac{2\omega_k - \omega_p}{2\,|\,hV_k\,|}. \tag{1.20}$$

This means that, during the linear stage of parametric instability, a definite relationship is established between the phases of the waves in a pair. The phase correlation of waves with equal and opposite wave vectors can be referred to as "pairing", by analogy with superconductivity. However, in contrast to superconductivity, the physical reason responsible for wave pairing is the presence of the pump which selects pairs of waves out of the initial phase chaos for which the instability growth rate is a maximum. We shall show later that the phase correlation is complete during the nonlinear stage of instability development. This means that although the quantity b_k is random, the quantity $b_k b_{-k}$ is dynamic, and $\langle b_k b_{-k} \rangle = b_k b_{-k}$, $\langle \psi_k \rangle = \psi_k$.

1.4. Nonlinear mechanisms limiting the parametric instability

The simplest mechanism of this kind is nonlinear damping, i.e., the dependence of γ_k on the squares of the amplitudes of the parametric waves $|\,b_k\,|^2$ (Schlömann 1962, Gottlieb and Sühl 1962). The stationary wave amplitudes are given by the well-known energy balance condition $|\,hV_k\,| = \gamma_k$. We will choose the following simple dependence for a qualitative analysis: $\gamma_k = \gamma_0 + \eta \Sigma_{k'} |\,b_{k'}\,|^2$. In that case,

$$\sum_{k'} |\,b_{k'}\,|^2 = \frac{|\,hV\,| - \gamma_0}{\eta} = \frac{|\,V\,|}{\eta}(h - h_1) \tag{1.21}$$

and the phase ψ_k are found from the condition of the parametric resonance, i.e., they are shifted by $\pi/2$ relative to the pump phase.

Zacharov, Starobinets and the present author proposed another restriction "phase" mechanism which plays a principle role by the parametric excitation of waves. In the first part of this review (§§1–3) we present a detailed discussion of it and compare it with the experiments on ferromagnets and antiferromagnets. Here we only notice that it is connected with pairing and the four-wave interaction which leads to the phase shift ($\sin \psi < 1$) between pairs and external pumping, i.e. the energy flow streams into the system slowly.

The third restriction mechanism connected with solutions arising and their collaps is functioning when the maximal Hamiltonian coefficient (1.6) has a maximum in one pair of points and a narrow wave packet will be excited beyond the threshold. We describe this limitation mechanism in §4.

2. The stationary post-threshold state. The S-theory and its comparison with experiment

2.1. Diagonal Hamiltonian and the equations of motion in the S-theory

We must now consider in greater detail the simplification of the Hamiltonian for the wave interaction which was noted in the previous section in connection with the phase mechanism of amplitude limitation. This simplification is analogous to the BCS approximation, and consists of the replacement of the interaction Hamiltonian (1.12) by its diagonal part for the pairs of waves $\pm\boldsymbol{k}$. In terms of "slow variables" (1.15), the diagonal Hamiltonian has the form

$$H_{\mathrm{int}} = \sum_{k,k'} T_{k,k'} b_k^* b_k b_{k'}^* b_{k'} + \tfrac{1}{2} \sum_{k,k'} S_{kk'} b_k^* b_{-k}^* b_{k'} b_{-k'}, \tag{2.1}$$

where

$$T_{kk'} \equiv T_{kk',kk'} = T_{k'k}^*; \quad S_{kk'} \equiv T_{k-k,k'-k'} = S_{k'k}^*. \tag{2.2}$$

In the Hamiltonian H_{int} only terms are retained which are either independent on the phases at all [the first sum in eq. (2.1), or depend only on the resultant phase $\psi_k = \phi_k + \phi_{-k}$ in the pairs. All the other terms which depend on the individual phases (or to be more precisely, on the differences $\phi_k - \phi_{k'}$), are omitted. The physical meaning of the terms retained in eq. (2.1) is clear from the equations of motion for the amplitudes b_k. Substituting the Hamiltonian (2.1) into eqs. (1.13) we obtain

$$\left[\frac{\partial}{\partial t} + \gamma_k + \mathrm{i}(\tilde{\omega}_k - \omega_{\mathrm{p}}/2)\right] b_k + \mathrm{i} P_k b_{-k}^* = 0. \tag{2.3}$$

These equations differ from the linear equations given by eq. (1.16), which describe the parametric instability, only by the frequency renormalization $\omega_k \to \tilde{\omega}_k$ and the pumping $hV_k \to P_k$ due to the first and the second sum in eq. (2.1); accordingly,

$$\tilde{\omega}_k = \omega_k + 2\sum_{k'} T_{kk'} \, |b_{k'}|^2, \quad P_k = hV_k + \sum_{k'} S_{kk'} b_{k'} b_{-k'}. \tag{2.4}$$

Choosing $b_k = |b_k| \exp(-\mathrm{i}\phi_k)$ and calculating $\partial\phi_k/\partial k$, we obtain the relation $\partial(\phi_k - \phi_{-k})/\partial t = 0$, which demonstrates the neutral stability of the phase difference within the framework of the theory using the diagonalized Hamiltonian (2.1). This was, in fact, to be expected because the phase difference between waves propagating in opposite directions defines the space position of the nodes of the resulting standing wave, which is not fixed in any way in an uniform pump field. The neutral equilibrium of the phase differences ensures that they can be randomized by any small perturbation, for example by small random inhomogeneities or imperfections in the crystal shape. The reason why the phase become random may be found in the "residual" interaction, which was not taken into account in the diagonalized Hamiltonian (2.1). Moreover, this interaction leads to a certain correlation between the phase differences in different pairs, but this correlation together with the nondiagonal terms in the Hamiltonian N_{int} stays small.

We shall give a more detailed and stronger substantiation of the approximation of the diagonal Hamiltonian (2.1) in the advanced part of this review (§5). Here we only notice that this approximation can be used right up to the amplitudes $h < h^*$ of the pumping field where $h^* \approx h_1\sqrt{(k\,\partial\omega/\partial k)/\gamma} \gg h_1$.

The theory based on a diagonalization of the interaction Hamiltonian leading to the form given by (2.1), will be called the S-theory. This designation reflects the determining influence of the coefficients $S_{kk'}$ on the nonlinear behavior of a set of parametric waves. Henceforth we will suppose that the individual wave phases are random, and that averaging has been carried out over their ensemble. The turbulence is then described in terms of the correlation functions

$$\langle b_k b_{k'}^* \rangle = n_k \Delta(\boldsymbol{k}-\boldsymbol{k}'), \quad \langle b_k b_{k'} \rangle = \sigma_k \Delta(\boldsymbol{k}+\boldsymbol{k}'). \tag{2.5}$$

The quantities n_k and σ_k have the dimension of action (erg s). The equations for these quantities can readily be obtained by direct averaging of eq. (2.3):

$$\frac{1}{2}\frac{\partial n_k}{\partial t} = \gamma_k n_k - \operatorname{Im}(P_k^* \sigma_k), \tag{2.6}$$

$$\frac{1}{2}\frac{\partial \sigma_k}{\partial t} = -\sigma_k\left[\gamma_k + \mathrm{i}\left(\omega_k - \frac{\omega_{\mathrm{p}}}{2}\right)\right] - \frac{\mathrm{i}}{2}(n_k + n_{-k})P_k.$$

Let us give another derivation of the general equations of the S-theory (2.6). We must start from the dynamic equations (1.14) with the exact Hamiltonian of the problem, eqs. (1.10) and (1.12), and obtain the average equations for correlative functions n_k, σ_k, eq. (2.5). In their right-hand sides appear correlation functions of the fourth order. Splitting $\langle b_1^* b_2^* b_3 b_4 \rangle$ and $\langle b_1^* b_2 b_3 b_4 \rangle$ in the usual way into pair correlation functions we can approximately calculate this functions and obtain equations (2.6) with the aid of

$$\begin{aligned}\langle b_1^* b_2^* b_3 b_4 \rangle &= n_1 n_2 [\Delta(\boldsymbol{k}_1-\boldsymbol{k}_3)\Delta(\boldsymbol{k}_2-\boldsymbol{k}_4) + \Delta(\boldsymbol{k}_1-\boldsymbol{k}_4)\Delta(\boldsymbol{k}_2-\boldsymbol{k}_3)] \\ &\quad + \sigma_1^* \sigma_3 \Delta(\boldsymbol{k}_1+\boldsymbol{k}_2)\Delta(\boldsymbol{k}_3+\boldsymbol{k}_4), \\ \langle b_1^* b_2 b_3 b_4 \rangle &= n_1 [\sigma_3 \Delta(\boldsymbol{k}_1-\boldsymbol{k}_2)\Delta(\boldsymbol{k}_3+\boldsymbol{k}_4) + \sigma_2 \Delta(\boldsymbol{k}_1-\boldsymbol{k}_3)\Delta(\boldsymbol{k}_2+\boldsymbol{k}_4) \\ &\quad + \sigma_2 \Delta(\boldsymbol{k}_1-\boldsymbol{k}_4)\Delta(\boldsymbol{k}_2-\boldsymbol{k}_3)].\end{aligned} \tag{2.7}$$

Of course this procedure is not stronger than a forced reduction of Hamiltonian (1.12) up to the form (2.1). However, it is more traditional, and therefore this derivation of the main equations of the S-theory (2.6) seems to be more understandable.

From the equations (2.6) the next relations follow:

$$\left(\frac{\partial}{\partial t}+4\gamma_k\right)\left(n_k n_{-k}-|\sigma_k|^2\right)=0,$$

$$\left(\frac{\partial}{\partial t}+2\gamma_k\right)\left(n_k-n_{-k}\right)=0,$$

which show that arbitrary initial conditions relax in a time of the order of $1/\gamma_k$ to a state (not necessarily to a stationary state) in which $n_k=n_{-k}=|\sigma_k|$. The condition $|\sigma_k|=n_k$ means that the phases of wave pairs are fully correlated. In this case, we may write $\sigma_k=n_k\exp(-\mathrm{i}\psi_k)$. In terms of these variables, we can write the equations (2.6) and the definitions given by (2.4) in the form

$$\begin{aligned}&\frac{1}{2}\frac{\partial n_k}{\partial t}=n_k\left[-\gamma_k+\mathrm{Im}\left(P_k^*\exp(-\mathrm{i}\psi_k)\right)\right],\\&P_k=hV_k+\sum_{k'}S_{kk'}n_{k'}\exp(-\mathrm{i}\psi_{k'}),\\&\frac{1}{2}\frac{\partial\psi_k}{\partial t}=\tilde{\omega}_k-\frac{\omega_\mathrm{p}}{2}+\mathrm{Re}\,P_k^*\exp(-\mathrm{i}\psi_k),\quad\tilde{\omega}_k=\omega_k+2\sum_{k'}T_{kk'}n_{k'}.\end{aligned}\tag{2.8}$$

2.2. Ground state. Condition of external stability

We now consider the stationary states of a system of pairs in which all the amplitudes n_k and phases ψ_k are time-independent. Assuming that $\dot{n}_k=\dot{\psi}_k=0$ in eq. (2.8), we immediately obtain the following condition for all points in k-space for which $\dot{n}_k=0$:

$$|P_k|^2=\gamma_k^2+\left(\omega_k-\omega_\mathrm{p}/2\right)^2.\tag{2.9}$$

Before we analyze this result, we should note two general points. Firstly, it is clear that the pair amplitudes are nonzero only in a thin layer near the resonance surface $2\omega_k=\omega_\mathrm{p}$. This means that it is convenient to use the following coordinates in k-space: κ, the deviation from this surface in the normal direction, and Ω, the coordinate on the surface. Secondly, the coefficients in eq. (2.8) which have the dimensions of a frequency: γ_k, hV_k, $\sum_{k'}T_{kk'}n_{k'}$, $\sum_{k'}S_{kk'}n_{k'}\exp(-\mathrm{i}\psi_{k'})$, are much smaller than the natural frequency ω_k. It follows that it is sufficient to take a dependence on κ only for $(\omega_k-\omega_\mathrm{p}/2)$ and to replace all the remaining coefficients by their values on the resonance surface, i.e., γ_Ω, hV_Ω, $T_{\Omega\Omega'}$ and $S_{\Omega\Omega'}$, respectively. If we use the above approximations, we can readily find from eq. (2.9) those values of κ for which $n_{\kappa\Omega}\neq 0$:

$$\tilde{\omega}_{\kappa\Omega}=\frac{\omega_\mathrm{p}}{2}\pm\sqrt{|P_\Omega|^2-\gamma_\Omega^2}\,.\tag{2.10}$$

Thus, in the stationary state, the pair-amplitude distribution is singular:

$n_{\kappa\Omega} \neq 0$ only on two surfaces (2.10). However, there is an infinite set of such stationary states which differ both in the form of these surfaces and in the distribution of $n_{\kappa\Omega}$ over them. In fact, the directions of Ω in which $n_{\kappa\Omega}$ is zero can be stimulated arbitrarily. In reality, of all the stationary states only those can be realized that are stable against small perturbations. The requirement of stability imposes a strong restriction on the class of possible stationary states. It is clear that the study of the stability of stationary states within the framework of the diagonalized Hamiltonian divides into two independent problems: the problem of the internal stability with respect to perturbations in the amplitudes and phases of existing pairs, and the problem of external stability against the creation of new pairs.

External stability is the simplest case to investigate. Thus, we can use eq. (2.3) to write down the equation for a pair of perturbation waves b_q, $b_q^* \sim \exp(\nu_q t)$ which is analogous to eq. (1.17). The growth rate ν_Ω which is a maximum in q (with fixed Ω) corresponds to $2\tilde{\omega}_q = \omega_p$, i.e., it lies halfway between the surfaces (2.10):

$$\nu_\Omega = -\gamma_\Omega + |P_\Omega|.$$

The external stability condition, $\nu_\Omega \leqslant 0$, can therefore be written in the form

$$|P_\Omega| \leqslant \gamma_\Omega. \tag{2.11}$$

On the other hand, it follows from eq. (2.10) that $|P_\Omega| \geqslant \gamma_\Omega$ for those directions Ω in which $n_{\kappa\Omega} \neq 0$. Consequently, for these directions the two inequalities are consistent only when $|P_\Omega| = \gamma_\Omega$. The two surfaces (2.10) then merge into one:

$$2\omega_{k_0} = \omega_p. \tag{2.12}$$

Thus, for a given distribution of wave amplitudes over the angles, the condition of external stability completely removes the ambiguity in the choice of the surface on which $n_k \neq 0$. This surface (2.12) will be called the "resonance surface", and the stationary state with external stability will be referred to as the "ground state".

The above result has a simple physical interpretation. It is clear from eq. (1.17) that, during the linear stage, the waves which are most strongly coupled to the pump are those for which the frequency detuning is $\tilde{\omega}_k - \omega_p/2 = 0$. The quantity $\omega_k - \omega_p/2$ is the frequency detuning with nonlinear terms included. When it is not equal to zero at points at which n_k is localized, then there is a region in k-space such that pairs contained by it are more strongly coupled to the pump than the existing pairs, and we have the possibility of parametric excitation of them.

It is interesting to examine in detail how, during the development of parametric instability, the spin-wave state $n_k \sim \delta(k - k_0)$, which is coherent in the modulus of k, arises out of the thermal noise $n_k^0 = T/\omega_k$. We have investigated this problem (Zakharov et al. 1972) analytically as well as numerically on a computer. We found, in particular, that after a certain interval following the introduction of the pump, the distribution of waves over the modulus of k can be described by a Gaussian, the width of which tends to zero asymptotically as $1/\sqrt{t}$.

2.3. The distribution of pairs on the resonance surface

We will introduce the distribution function n_Ω which gives the "number" of pairs per unit solid angle. This function is defined by

$$N = \sum_k n_k = \int n_\Omega \, \mathrm{d}\Omega. \tag{2.13}$$

The stationary equations for n_Ω and ψ_Ω which follow from eqs. (2.18) and (2.12) can be written in the form

$$\begin{gathered}\left(P_\Omega \exp(\mathrm{i}\psi_\Omega) - \mathrm{i}\gamma_\Omega\right) n_\Omega = 0, \\ P_\Omega = hV_\Omega + \int S_{\Omega\Omega'} n_{\Omega'} \exp(-\mathrm{i}\psi_{\Omega'}) \, \mathrm{d}\Omega'.\end{gathered} \tag{2.14}$$

These equations do not yet define unambiguously the distribution of n_Ω and ψ_Ω, because the surfaces on which $n_\Omega \neq 0$ can be chosen arbitrarily. The requirement of external stability will be shown in the next section to reduce substantially the class of possible solutions, and in a number of cases the stable distributions are unique.

The condition of external stability can be usefully interpreted in terms of the following geometric ideas. The expressions $\gamma = \gamma_\Omega$ and $P = P_\Omega$ are the equations of surfaces in k-space. Condition (2.11) means that the surface P_Ω lies wholly inside the surface γ_Ω and touches it at points Ω at which the solution is concentrated. Since $V_k = V_{-k}$ and $S_{kk'} = S_{k-k'}$, both these surfaces have a center of symmetry. The γ_Ω and P_Ω surfaces can touch either at a discrete number of points, or over a continuum, i.e., a line or even a segment of the surface. In the former case we have a finite number of monochromatic pairs, and in the latter a continuous distribution of n_Ω. An intermediate situation is also possible in which the surfaces touch at an isolated pair of points and, in addition, along a certain line. The system then contains a monochromatic pair and a continuous background. We note that the question of the applicability of the S-theory in the presence of a small number of discrete pairs requires special consideration, including the examination of stability within the framework of the exact Hamiltonian (see below).

2.4. The simplest solution within the S-theory

The spherical symmetric case is the simplest case which is realized, for example, in antiferromagnets (neglecting an extremely small magnetic dipole interaction). At $V_\Omega = V$, $\gamma_\Omega = \gamma$ and $S_{\Omega\Omega'} = S$, eqs. (2.16) have an isotropic solution $n_\Omega = N/4\pi$, where

$$N = \sqrt{(hV)^2 - \gamma^2}/S, \quad hV \sin\psi = \gamma. \tag{2.15}$$

These equations have the same solution when $S_{\Omega\Omega'}$ depends only on the angle between k and k'. Then in formula (2.15)

$$S = (4\pi)^{-1} \int S_{\Omega\Omega'}\, \mathrm{d}\Omega'. \tag{2.16}$$

The axially symmetric case is of great interest. It is realized in isotropic and cubic ferromagnets magnetized along the ⟨111⟩ or ⟨100⟩ axis. The coupling coefficient is then given by eq. (1.7), and the coefficients describing the interaction are given by

$$S_{\Omega\Omega'} = S(\theta, \theta'; \phi - \phi'), \quad T_{\Omega\Omega'} = T(\theta, \theta'; \phi - \phi'). \tag{2.17}$$

It is clear from the axial symmetry that, in the stationary state, the amplitude of the pairs $n_\Omega = n_{\theta,\phi}$ is independent of the azimuthal angle ϕ. From the equations for n_k it is clear that the quantity $P_k^*\sigma_k$ is also independent of ϕ. Since $V_k = V_\theta \exp(2\mathrm{i}\psi_\Omega)$ (see eq. 1.7), we obtain

$$\psi_\Omega = \psi_{\theta,\phi} = \psi_\theta + 2\phi. \tag{2.18}$$

These relationships enable us to elminate the dependence on ϕ in eq. (2.14), and write it in the form

$$(P_\theta \exp(\mathrm{i}\psi_\theta) - \mathrm{i}\gamma_\theta)n_\theta = 0, \quad n_\theta \equiv n_\Omega/2\pi, \tag{2.19}$$

$$P_\theta = P_\Omega \exp(-2\mathrm{i}\phi) = hV_\theta + \int S_{\theta\theta'} n_{\theta'} \exp(-\mathrm{i}\psi_{\theta'}) \sin\theta'\, \mathrm{d}\theta',$$

$$S_{\theta\theta'} = \frac{1}{2\pi}\int_0^{2\pi} S(\theta, \theta'; \phi - \phi') \exp[2\mathrm{i}(\phi - \phi')]\, \mathrm{d}(\phi - \phi'). \tag{2.20}$$

To determine the distribution n_θ for small excesses above the threshold, we will use the above geometric interpretation (§2.2) of the external stability condition given by eq. (2.11). For very small excesses above the threshold, when the amplitudes n_θ are small, the surface $|P_\theta|$ is not very different from the hV_θ-surface which has a maximum on the equator at $\theta = \pi/2$. The curvatures of these surfaces (second derivatives with respect to θ) are also very similar. Hence it is clear that the surfaces of $|P_\theta|$ and γ_θ touch one another only along the line $\theta = \pi/2$. This means that, for small excesses above the threshold, the distribution n_θ is nonzero only along this line:

$$n_\theta = n_1 \delta\left(\theta - \frac{\pi}{2}\right). \tag{2.21}$$

For the integral amplitude $n_1 = \Sigma_k |b_k|^2$ and phase $\psi_1 = \psi_{\pi/2}$ from eq. (2.19) it is readily shown that

$$N_1 = \sqrt{h^2V_1^2 - \gamma_1^2}/S_{11}, \quad \sin\psi_1 = \gamma_1/hV_1 \quad V_1 = V_{\pi/2},$$
$$S_{11} = S_{\pi/2,\pi/2}, \quad \gamma_1 = \gamma_{\pi/2}. \tag{2.22}$$

The important characteristic of the obtained solutions is that the pair phase is equal to $\phi = \phi_k + \phi_{-k} - \phi_p$, which differs from the value of $\psi = \pi/2$ (ϕ_p is a pumping phase which is usually chosen to be zero). The energy flow from the external pumping is a maximum, eq. (1.18). The condition $hV \sin\psi = \gamma$ describes the energy balance which is characteristic of the phase limitation mechanism. This important conclusion of the S-theory was confirmed in direct experiments on phase measurement of ψ-pairs at the parametric excitation of spin waves in antiferromagnet $MnCO_3$ by Prozorova and Smirnov (1974) and then by Melkov and Krutsenko (1977) in ferromagnetic $Y_3Fe_5O_{12}$ (yttrium ferrite). Their results are shown in fig. 1 in coordinates $\sin\psi$, h_{cr}/h. We can see that the experimental points are lying along the coordinate angle bisector as follows from the formulas (2.15) for antiferromagnets and (2.22) for ferromagnets. This shows that the S-theory describes general properties of the post-threshold behaviour of parametrically excited waves right. Data deviation for YIG at supercriticality more than 10 dB ($h/h_{cr} > 3$) from the simple formula (2.22) is connected with the excitation of the second pair group with $\theta_k \neq \pi/2$ phenomenon which will be investigated later.

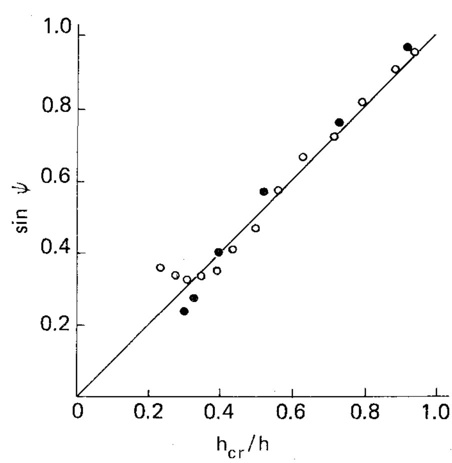

Fig. 1. Dependence of the phase of pairs on the supercriticality. O: results of Melkov and Krutsenko (1977) on YIG; ●: results of Prozorova and Smirnov (1974) on $MnCO_3$; straight line: theoretical dependence.

2.5. Stage-by-stage excitation of waves

Let us investigate the distribution of parametrically excited waves n_Ω in the resonance surface when the external field amplitude h increases. It must be pointed out that the use of the geometric derivation presented in §2.2 may be generalized in case of an arbitrary dependence of V_Ω. The general theorem can be proved, eq. (1.7), that when the excess above the threshold is sufficiently small, n_Ω differs from zero only at those points on the resonance surface where $|V_\Omega|$ is a maximum. In the case of spherical symmetry, these are points of the entire surface, whereas for axial symmetry they are points of one ($\theta = \pi/2$) or of two lines. In the case of lower symmetries, this is one or a few equivalent pairs of points. It is interesting that the integral amplitude is not very sensitive to the degree of symmetry of the problem. It is given by eq. (2.22), in which $V_1 = \max|V_\Omega|$ and S_1 is the mean value of $S_{\Omega\Omega'}$, on a set of points where $n_\Omega \neq 0$.

We will show that the distribution function of pairs, eq. (2.21), localized on the equator, conserves the stability against the pair creation at other latitudes up to sufficiently large excesses above the threshold. To show this, let us consider the function $|P_\theta|$. From eqs. (2.19)–(2.22) we have

$$|P_\theta|^2 = N_1^2\left(S_{11}\frac{V_\theta}{V_1} - S_{\theta 1}\right)^2 + \frac{V_\theta^2\gamma_1^2}{V_1^2}.$$

It is clear that the state described by eq. (2.21) will be stable for $|P_\theta|^2 \leqslant \gamma_\theta^2$ for all θ except $\theta = \pi/2$. The "second threshold" h_2 corresponds to the minimum value of h for which the surfaces $|P_\theta|$ and γ_θ touch for some value $\theta = \theta_2$ not equal to $\pi/2$. Therefore $h_2 = \min h_\theta$ for $\theta = \theta_2$, where h_θ is determined from the condition $|P_\theta| = \gamma_\theta$,

$$h_\theta^2 = h_1^2\left[1 + \frac{S_{11}^2}{\gamma_1^2}\,\frac{\gamma_\theta^2 V_1^2 - \gamma_1^2 V_\theta^2}{(S_{\theta 1}V_1 - S_{11}V_\theta)^2}\right]. \tag{2.23}$$

For the simple assumptions $\gamma_\theta = \gamma$ and $S_{\theta 1} = 0$ we find that $h_2^2/h_1^2 = 2$ for $\theta_2 = 0$. In reality, for cubic ferromagnets the function $S_{\theta 1}$ is very different from a constant, and this leads to a much higher second threshold. Actual calculations of h_2 for ferromagnets are too cumbersome. Estimative considerations (Zakharov et al. 1974) give $(h_2^2/h_1^2) \gtrsim 4$, $\theta_2 \simeq 50°$. However, the fact of the stage-by-stage excitation of singular wave packets in k-space is more important than the numerical values of h_2 and θ_2. This qualitative conclusion of the S-theory was confirmed by a direct experiment designed to establish the presence of the second order yttrium ferrite (Zautkin et al. 1970). The idea was to registrate a radiation in a "transverse" channel with the polarization $h \perp H$. PSW with $\theta_k = \pi/2$ do not contribute to this radiation because of the actual process of a matrix element $\omega_k + \omega_{-k} = \omega_\perp$. The above-mentioned depen-

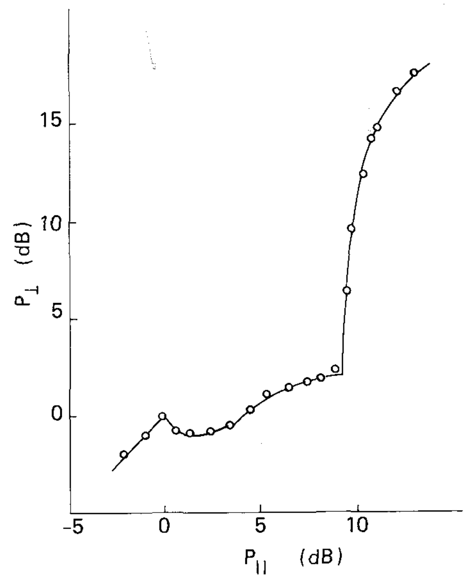

Fig. 2. Radiation into the "transverse" channel as a function of the pump (sphere, YIG, $M \| \langle 100 \rangle$).

dence $h_\perp^2$ on $h_\|^2$ (see fig. 2) confirms that only pairs with $\theta = \pi/2$ are excited in the interval 0–9 dB. It is naturally to connect the radiation $P_\perp$ at large supercriticality with the excitation of the second pair group with $\theta \neq \pi/2$. In a further experiment Krutsenko and Melkov (1979) were lucky not only to registrate the "second threshold" (in their conditions it was equal to 14 dB) but also to define the diffusion angle of the second pair group $\theta_2 \simeq 45°$ which corresponds with the theoretical prediction. In this case the first-group angular smearing is still small: $\Delta\theta_1 \simeq (6 \pm 1)° \ll \theta_2$. We notice that the finite width of packets $\Delta\theta$ arises only beyond the frames of self-consistent field approximation in the more exact S,T^2-theory. An advanced partition of this review will be devoted to the S,T^2-theory.

In conclusion, we must describe, at least qualitatively, the behavior of parametric spin waves for $h > h_2$. We can verify that the state with two groups of pairs at latitudes θ_1 and θ_2 becomes unstable for a particular departure from the threshold h_3, and a third group is greated at the latitude θ_3. The next group θ_4 appears at h_4, and so on. The question as to what happens when h is increased still further is discussed by Zakharov et al. (1970). For large h/h_1 the distribution of n_Ω is very sensitive to the fine structure of the quantities V_Ω and $S_{\Omega\Omega'}$. In some cases, a continuous distribution of pairs over the resonance surface is established, whilst on other cases the limitation mechanism cuts off and an essentially nonstationary state arises.

2.6. Nonlinear susceptibilities

The most widely used method of experimental investigation of the parametric excitation of spin waves is based on the absorption of the pump energy by the waves. We shall define the high-frequency nonlinear susceptibility of a ferromagnet in the usual way, i.e., $M_z(\omega_p) = \chi h$, $\chi = \chi' + i\chi''$. The imaginary part of the susceptibility χ'' determines the absorbed power:

$$W = \omega_p \chi'' h^2/2.$$

If we use eq. (1.18) to determine the energy flowing into the sample from the surrounding pump field, we obtain

$$\chi'' = \frac{2}{h} \sum_k |V_k| \, |b_k|^2 \sin(\tilde{\psi}_k - \psi_k). \tag{2.24}$$

A similar expression is obtained for the real part of the susceptibility:

$$\chi' = -\frac{2}{h} \sum_k |V_k| \, |b_k|^2 \cos(\tilde{\psi}_k - \psi_k). \tag{2.25}$$

The behavior of the susceptibilities χ' and χ'' beyond the threshold is very dependent on the amplitude-limiting mechanism. Thus, for the nonlinear damping mechanism, we have from eq. (1.21)

$$\chi' = 0, \quad \chi'' = \frac{2V^2}{\eta} \frac{h - h_1}{h}, \tag{2.26a}$$

and for the phase-limitation mechanism, we have from eqs. (2.15) and (2.22),

$$\chi' = \frac{2V^2}{S} \frac{h^2 - h_1^2}{h^2}, \quad \chi'' = \frac{2V^2}{S} \frac{h_1\sqrt{h^2 - h_1^2}}{h^2}. \tag{2.26b}$$

Figure 3 shows graphs of the functions $\chi'(h^2)$ and $\chi''(h^2)$ for both cases. The fundamental difference between the dissipative and phase mechanisms can be seen in the behavior of χ' ($\chi' = 0$ for the dissipative, and $\chi' \simeq \chi''$ for the phase mechanisms). The experimental data, which we will review in detail in the next section, show that the real part of the susceptibility χ' is nonzero and can be of the order of or even greater than χ''. These facts tend to support the phase mechanism of amplitude limitation.

The comparison of the theory with experiment is easier for supercritical conditions when only one group is excited and the simple formula (2.26b) can be used. In particular, it follows that

$$\chi''_m \equiv \max \chi'' = \frac{V^2}{|S|}. \tag{2.27}$$

In the case of axial symmetry here $V = V_1 = V_{\pi/2}$, $S = S_{11}$. After substitu-

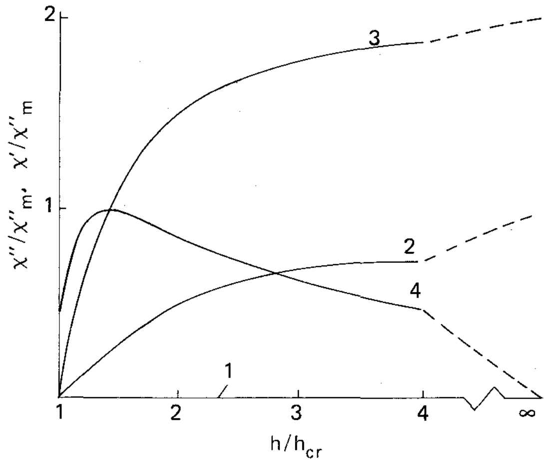

Fig. 3. The dependences of the nonlinear susceptibilities χ' and χ'' on the pump amplitude. (1) and (2): nonlinear damping mechanism; (3) and (4): phase limiting mechanism.

tion of the values of V_1 and S_{11} in eq. (2.27) for cubic ferromagnets it becomes of the form:

$$\chi''_{\mathrm{m}} = \left\{ 8\pi \left[N_z - 1 + \beta \frac{\omega_{\mathrm{a}}}{\omega_M} + \sqrt{\left(\frac{\omega_{\mathrm{p}}}{\omega_M}\right)^2 + 1} \right] \right\}^{-1}, \tag{2.28}$$

$$\beta = -8 \quad \text{for} \quad M \parallel \langle 111 \rangle, \quad \beta = +9 \quad \text{for} \quad M \parallel \langle 100 \rangle.$$

ω_{a} is the anisotropic field, $\omega_M = 4\pi g M$, N_z is a demagnetizing factor which is equal to 1/3 for a sphere and equal to 1 for a normally magnetized disk. In table 1 theoretical (from formula (2.28)] and experimental values of χ''_{m} [obtained by Zakharov et al. (1974)] for different cubic ferromagnets of a

Table 1
Theoretical (eq. 2.28) and experimental (Zakharov et al. 1974) values of the maximum of the susceptibility, χ''_{m}, for spherically shaped cubic ferromagnets.

Crystal	$4\pi M = \omega_M/g$ (Oe)	$H_{\mathrm{a}} = \omega_{\mathrm{a}}/g$ (Oe)	$2\Delta H_k = \gamma_k/g$ (Oe)	χ''_{m} ($\times 10^3$) Experiment	Theory	Orientation
$Y_3Fe_3O_{12}$ (YIG)	1750	84	0.12	24 ± 5	21	100
Y_3Fe	1500	8	0.36	23	22	100
$Bi_{0.2}Co_{2.8}Fe_{3.6}Y_{1.4}O_{1.2}$	650	58	0.45	5	7.5	100
$Li_{0.5}Fe_{2.5}O_4$	3700	580	0.80	80	70	111
$NiFe_2O_4$	3200	490	1.40	25	19	100
				55	84	111

spherical shape are shown. It is clear from it that the simple formula given by eq. (2.28) provides a good description of the absolute post-threshold susceptibility for a wide class of cubic ferromagnets. Some discrepancy between theory and experiment in the case of $NiFe_2O_4$ (in the $\langle 111 \rangle$ orientation) is probably connected with the fact that, in this case, the maximum of the susceptibility lies beyond the threshold of the creation of the second group of pairs, where eq. (2.28) is no longer valid.

The susceptibility χ''_m was measured by Green and Heavy (1963) for uniaxial ferromagnet $Ba_2Zn_2Fe_{12}O_4$ with "easy plane" anisotropy, and the anomalously large value of $\chi''_m = 0.2$ was obtained. Theoretical estimates for χ''_m based on eq. (2.27) using V_1 and S_{11} without taking the dipole–dipole interaction into account (for $\omega_m < \omega_a$, $\omega_p \lesssim \omega_a$) yields

$$\chi''_m = \omega_M \omega_a / 2\pi\omega_p^2. \tag{2.29}$$

If, following Green and Heavy (1963), we assume for $Ba_2Zn_2Fe_{12}C_{14}$ that $4\pi M = 2850$ G, $\omega_a/g = 99000$ Oe, and suppose that the pump frequency is $\omega_p/g = 6300$ Oe, we obtain $\chi''_m = 0.1$, which is in qualitative agreement with experiment. There is also a qualitative correspondence between the theoretical and experimental values of χ''_m for the parametric excitation of SW in AFM (L'vov and Shirokov 1974, Kotyuzhanskii and Prozorova 1972).

To compare the S-theory with experiment in a region of large supercriticalities numerical calculations on a computer were carried out (Zautkin et al. 1972). For this, first of all coefficients $S_{\theta\theta'}$ were calculated of actual experi-

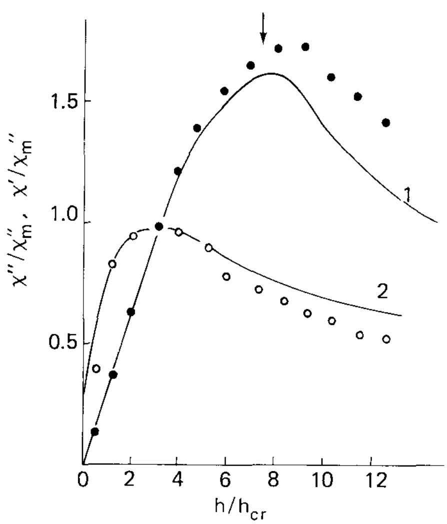

Fig. 4. Result of numerical calculations of the susceptibilities χ' (curve 1) and χ'' (curve 2). Points: results of a laboratory experiment (YIG, sphere, $H = H_c - 50$ Oe, $M \| \langle 100 \rangle$). The arrow shows the threshold for the creation of the second group of pairs.

mental situations for YIG. Here, only known values of general parameters of YIG (magnetization, crystallographic anisotropy field and exchange field) were used. The obtained values of $S_{\theta\theta'}$ were substituted into the nonstationary equations of motion of the S-theory, eq. (2.3), which were calculated on a computer by means of the iteration method on a time of the thermal noise level. The calculated values of stationary amplitudes and phases for different supercriticalities allowed to obtain values χ' and χ'' from formulae (2.24) and (2.25). The result of these calculations together with the experimental results are shown in fig. 4. We see a good qualitative and quantitative agreement of the theory with experiment.

2.7. The role of nonlinear damping

We have already pointed out that the increase of damping with growing of the number of PSW can be used as an additional limitation mechanism of the PSW amplitude. If the n_k are not too large in the dependence $\gamma_k\{n_k\}$, we can, according to eq. (2.20), limit ourselves to the first expanding term:

$$\gamma_k\{n_k\} = \gamma_k + \sum_{k'} \eta_{kk'} n_{k'}. \tag{2.30}$$

Damping in the form (2.30) (and in any other actual form) will relatively simply be included into the calculation scheme n the S-theory. E.g., if function $\eta_{kk'}$ fluently depends on k and k', than it is simple enough to put

$$\gamma(N) = \gamma + \eta N \tag{2.31}$$

for singular spectra of PSW up to the creation of the second pair group. After that, we obtain eqs. (2.32), (2.33), and (2.34), instead of formulae (2.15) and (2.26b) for N, χ' and χ'':

$$N = \frac{\gamma}{|S|} \frac{\sqrt{(1+c^2)P-1} - c}{1+c^2}, \tag{2.32}$$

$$\chi'' = \frac{2\chi''_{\mathrm{m}}}{1+c^2}\left[c + \frac{(1-c^2)\sqrt{P(1+c^2)-1} - 2c}{P(1+c^2)}\right], \tag{2.33}$$

$$\chi' = \frac{2\chi''_{\mathrm{m}} S_{11}}{(1+c^2)|S_{11}|}\left[1 - \frac{2c\sqrt{P(1+c^2)-1} + 1 + c^2}{P(1+c^2)}\right], \tag{2.34}$$

$$P = h^2/h_1^2.$$

Here the coefficient $c = \eta/|S|$ characterizes the relative role of the nonlinear damping mechanism compared with the phase mechanism. Figure 5 shows a plot of the function $\chi''(h^2)$ for different magnitudes and signs of the parameter η/S. Let us consider some of the characteristic features of these curves.

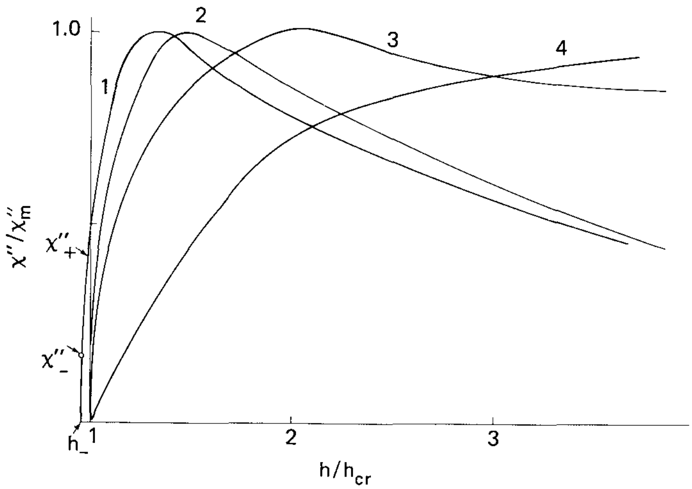

Fig. 5. The dependence of χ'' on the pump amplitude, taking into account nonlinear damping; $\eta/|S| = -0.25$, (curve 1), 0 (curve 2), 0.25 (curve 3) and 1 (curve 4).

When $\eta > 0$, the $\chi''(h^2)$ curve has a finite slope at the threshold point, which is equal to V_1^2/η, and the final value is given by $\chi''(\infty) = \eta V^2(\eta^2 + S_{11}^2)^{-1}$. It is interesting that when $\eta < |S_{11}|$ the maximum value of χ'' is not dependent on η and is defined by eq. (2.27). The nonlinear damping affects only the position h_m of the maximum, which shifts toward greater h as η increases: $h_m = \sqrt{2}\, h_1 S_{11}/(|S_{11}| - \eta)$. When $\eta > |S_{11}|$, the susceptibility χ'' increases monotonically with increasing h. It is interesting to note that even when $|\eta| \ll |S|$, $\eta > 0$, the susceptibility χ' decreases rapidly (as compared with the case $\eta = 0$) in a narrow region near the threshold, and the $\chi'(h)$ curve has a zero tangent at $h = h_1$:

$$\chi' = \frac{2SV^2}{\eta^2}\left(\frac{h - h_1}{h_1}\right)^2, \quad hV - \gamma \lesssim \frac{2\gamma\eta^2}{S^2}.$$

For $\eta < 0$ strong excitation of parametric waves takes place and is accompanied by a hysteresis of the function $\chi''(h^2)$. When $|\eta| \ll S_{11}$, the reverse jump of χ''_- amounts to half the direct jump $\chi''_+ : \chi''_- = \frac{1}{2}\chi''_+ = 2|\eta/S|\chi''_m$. It occurs for the pump amplitude $h_- = h_1 |S_{11}| (S_{11}^2 + \eta^2)^{-1/2}$.

The nonlinear damping of SW in ferromagnets has been experimentally studied in many works (Schlömann 1962, Le Gall et al. 1967, Melkov 1971). The most detailed investigation taking a simultaneous effect of the phase-limitation mechanism into consideration was carried out by Krutsenko and Melkov (1981b). They used formulae of the type (2.32)–(2.34) in the analysis of their experimental data and obtained a relation for the dependence of η/S on the magnetic field (fig. 6). We can see that the nonlinear damping has

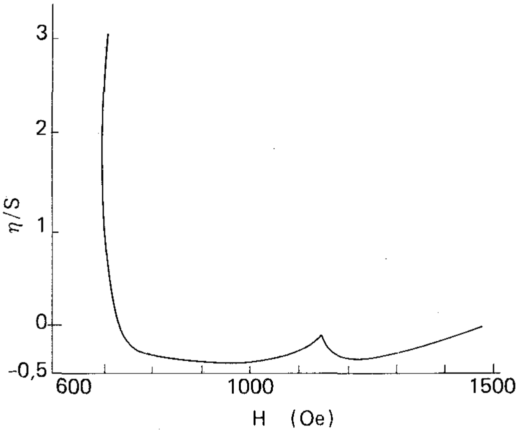

Fig. 6. Dependence of the effective nonlinear damping η/S on the constant magnetic field for $h \to h_1$ (YIG, sphere, $M \parallel \langle 100 \rangle$).

relatively little influence upon the post-threshold behavior of the PSW in a wide interval: $|\eta/S| \lesssim 0.25$.

We will not consider here the different actual mechanisms of nonlinear damping. They strongly depend on the peculiarities of the medium in which the waves propagate. That is, they depend on the dispersion principle of ω_k, on the matrix elements of their interaction between each other and with other types of waves, on the temperature of the medium, etc. Gottlieb and Sühl (1962), Melkov (1971) and others proposed various mechanisms of nonlinear damping in ferromagnets. L'vov and Falk'kovich (1982) accurately calculated the nonlinear damping and it became possible to explain details of the experimental dependence of η/S in fig. 6. Kotyuzhanskii and Prozorova (1972), Lutovinov (1980) and others investigated negative nonlinear damping of spin waves in antiferromagnets which results in the "strong" excitation of the spin waves.

3. Collective excitations and auto-oscillations of the magnetization

3.1. The spectrum of collective oscillations

So far, we have studied stationary states of a system of parametric waves. In this section, we will consider the behavior of the system as it departs from the stationary state. For the sake of simplicity, we will initially neglect dissipation. The Hamiltonian of the system, H, is then a constant of motion. Suppose that the perturbed-state Hamiltonian H differs from the Hamiltonian H_0 in the ground state. This means that the system can never reach the ground state, and since it has no other stable stationary states, its behavior will be essentially nonstationary.

Since we shall eventually have to compare theory with experiment, let us begin by considering the cubic ferromagnet magnetization along the $\langle 111\rangle$ or $\langle 100\rangle$ axes. For excesses above the threshold which lie below the second threshold, the spin waves are excited in the plane of the equator in the ground state. Let α_k be the deviations of the complex amplitudes b_k from the ground state, and let us isolate of Hamiltonian (2.1) of the S-theory the term $H^{(0)}$ that corresponds to the ground state and the terms $H^{(1)}$ and $H^{(2)}$ that contain the linear and quadratic terms in α_k. Using the equations of motion, we can readily verify that the ground state is an extremum: $H^{(1)}=0$. The part of the Hamiltonian that is quadratic in small perturbations, takes the form

$$H^{(2)}=\frac{N_1}{(2\pi)^2}\left\{2\int\left[S_{\phi\phi'}\exp[\mathrm{i}(\phi-\phi')]+T_{\phi\phi'}\exp[\mathrm{i}(\phi-\phi')]\right]\alpha_\phi\alpha_{\phi'}\,\mathrm{d}\phi\,\mathrm{d}\phi'\right.$$

$$\left.+\left[\exp(\mathrm{i}\psi_1)\int T_{\phi\phi'}\exp[-\mathrm{i}(\phi+\phi')]\,\alpha_\phi\alpha_{\phi'}\,\mathrm{d}\phi\,\mathrm{d}\phi'+\mathrm{c.c.}\right]\right\},\tag{3.1}$$

where we have replaced summation by integration and where $T_{\phi\phi'}=T_{kk'}$, $S_{\phi\phi'}=S_{kk'}$, for $\theta=\theta'=\pi/2$, $|k|=|k'|=k_0$. Transforming to the Fourier components

$$\alpha_m=\frac{1}{2\pi}\int_0^{2\pi}\alpha_\phi\exp[\mathrm{i}(m\text{-}1)\phi]\,\mathrm{d}\phi$$

and using the axial symmetry of the situation we obtain

$$H^{(2)}=N_1\sum_{m=-\infty}^{\infty}\left[2(T_m+S_m)\alpha_m\alpha_m^*+(T_m\alpha_m\alpha_{-m}+\mathrm{c.c.})\right],\tag{3.2}$$

where

$$\begin{aligned}T_m&=\frac{1}{2\pi}\int_0^{2\pi}T(\phi-\phi')\exp[-\mathrm{i}m(\phi-\phi')]\,\mathrm{d}(\phi-\phi'),\\ S_m&=\frac{1}{2\pi}\int_0^{2\pi}S(\phi-\phi')\exp[-\mathrm{i}(m-2)(\phi-\phi')]\,\mathrm{d}(\phi-\phi').\end{aligned}\tag{3.3}$$

The Hamiltonian (3.2) can be diagonalized with the aid of the linear canonical transformation. Diagonalization is possible if

$$\Omega_m^2=4N_1^2S_m(2T_m+S_m)>0.\tag{3.4}$$

In this case, the Hamiltonian $H^{(2)}$ can be written in the form

$$H^{(2)}=\sum_m\Omega_m\beta_m\beta_m^*.$$

It is clear that the quantity Ω_m is the frequency of collective oscillations in a system of spin waves. When $(S_m+2T_m)S_m<0$, this frequency is imaginary.

This indicates that the ground state is unstable against the excitation of exponentially growing oscillations (internal instability, see section 3). Thus, the signs of Ω_m^2 and $S_m + T_m$ are always the same. The fact that Ω_m^2 is negative means that the energy of the spin-wave system decreases as a result of the excitation of collective oscillations. The energy of the system increases in the course of their relaxation. This is not inconsistent with the conservation law of energy because the system of parametrically excited waves reveives its energy from the pump.

The quantities S_m and T_m can be calculated for a cubic ferromagnet in the symmetric directions $\langle 111 \rangle$ and $\langle 100 \rangle$. It turns out that they differ from zero only when $m = 0, \pm 2$. The coefficients S_m and T_m corresponding to these modes are presented by L'vov et al. (1973a). They show the dependence of the frequency of collective oscillations on the experimental conditions, namely, the excess above the threshold, pump frequency, magnetization, external magnetic field, shape of the specimen, and crystallographic anisotropy. Calculations for easy-plane antiferromagnets (EP-AFM) show (Lutovinov and Safonov 1980) that $S_0 = T_0$ but other S_m and T_m are equal to zero. Particularly, from this the ground-state stability followed which is observed in experiment (Kotyuzhanskii and Prozorova 1972) right up to very large excesses above the threshold, ≈ 20 dB.

We must now consider the effect of the damping of spin waves on collective oscillations, expecially since the damping γ_k may be of the same order as the frequency Ω_m. Linearizing the equations of motion (2.3) with respect to α, against the background of the ground state (2.22), and assuming that $\alpha, \alpha^* \sim \exp(-\mathrm{i}\Omega t)$, we obtain a set of algebraic equations which are homogeneous in α, α^*. The condition that these equations are consistent, determines the frequency and damping of the collective oscillations:

$$\Omega_m = -\mathrm{i}\gamma \pm \sqrt{4S_m(2T_m + S_m)N_1^2 - \gamma^2}\,. \tag{3.5}$$

This formula leads to the important conclusion that the criterion for internal instability of the mth collective mode is independent of the amount of damping and is determined, just as in the conservative case, by the condition

$$S_m(2T_m + S_m) < 0. \tag{3.6}$$

Within the framework of the S-theory, the collective oscillations of a system of parametric spin waves are spatially homogeneous. When spatial dispersion is taken into account, each normal mode corresponds to a whole branch of $\Omega_m(\kappa)$, where eq. (3.6) defines the gap. Spatially-inhomogeneous collective oscillations δn, $\delta\psi > \exp[\mathrm{i}(\kappa r - \Omega t)]$ have a definite analogy with second sound. In contrast to the usual second sound in a gas of thermal magnons a system of parametric waves exhibits oscillations not only in the number n_k of pairs, but in their phase ψ_k as well.

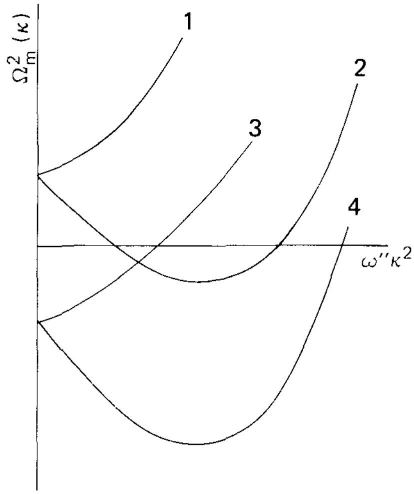

Fig. 7. Possible variants of the spectrum of collective oscillations ($\gamma = 0$). The main state is unstable in regions $\Omega^2 < 0$. Curves 1 and 2 correspond to the case $(T_m + S_m) > 0$, curves 3 and 4 correspond to the case $(T_m + S_m) < 0$.

The spectrum $\Omega_m(\kappa)$ of these waves is investigated by L'vov (1973) and L'vov and Rubenchik (1977). In the simplest case when $\kappa \parallel M$,

$$\Omega_m(\kappa) = -\mathrm{i}\gamma \pm \left\{ \left[2(T_m + S_m)N_1 + \omega'' \frac{\kappa^2}{2} \right]^2 - 4T_m^2 N_1^2 - \gamma^2 \right\}^{1/2},$$

where $\omega'' = \partial^2\omega/\partial^2 k_z$. Possible forms of the function $\Omega_m^2(\kappa)$ (with $\gamma = 0$) are shown in fig. 7. The region of negative values corresponds to the instability of the ground state. We note that, for large κ, collective oscillations are always damped, i.e., $\Omega_m(\kappa) = -\mathrm{i}\gamma + \omega''\kappa^2/2$.

3.2. Resonance excitation of collective oscillations

The collective oscillations discussed above are seen experimentally as low-frequency ($\Omega \approx 10^6\ \mathrm{s}^{-1}$) oscillations of the longitudinal magnetization M_z. The resonance method is the most convenient way of exciting of collective oscillations. The procedure is to apply to the ferromagnet both the pump field $h(t) > \exp(-\mathrm{i}\omega_\mathrm{p} t)$ and a weak signal $h_\mathrm{s} \parallel M$ with frequency close to ω_p, i.e., $\omega_\mathrm{s} = \omega_\mathrm{p} + \Omega$. Calculations reported by Zautkin et al. (1972b) based on the equations of motion (2.3), show that the susceptibility $\chi''_{\omega_\mathrm{p}+\Omega}$ has resonance frequencies, namely, $\Omega = \pm\Omega_0$, which coincide, as expected, with the natural frequency of the zero mode in the absence of damping.

For large *excesses above the threshold*, when $\Omega_0^2 \gg \gamma^2$, the line shape is nearly Lorentzian, with a width equal to the spin-wave damping γ. At resonance, the susceptibility is given by

$$\chi''_{\omega_p+\Omega_0} = \chi''\left(\frac{h}{h_1}\right)^2\left[1 \pm \sqrt{1 + \frac{T_0^2}{S_0(2T_0 + S_0)}}\,\right]. \tag{3.7}$$

The fundamental point is that the susceptibility may turn out to be negative. This corresponds not to absorption but to amplification of the weak signal. It folows from eq. (3.7) that absorption occurs at the frequency $\Omega + \omega_p$, and amplification at the mirror frequency $(\omega_p - \Omega)$.

The appearance of amplification can be regarded as a consequence of decay instability of the ground state (with the "slow" frequency equal to zero) into electromagnetic radiation (with slow frequency Ω) and a collective oscillation with characteristic frequency Ω_0. The law of conservation of energy in this process is $\Omega = -\Omega_0 + 0$. Amplification therefore occurs at the frequency $-\Omega = -\Omega_0$, which corresponds to eq. (3.7). In this terms the absorption is a consequence of the decay of the weak signal into the ground state and CO (collective oscillations) with the conservation law $\Omega = \Omega_0 + 0$.

The collective resonance was observed experimentally by Zautkin et al. (1972b) in single-crystals of YIG and there was good quantitative agreement with theory. In particular, this is the case for the dependence of the susceptibility to a weak signal and the frequency of CO on the pump power (figs. 8, 9), eqs. (3.7) and (3.5).

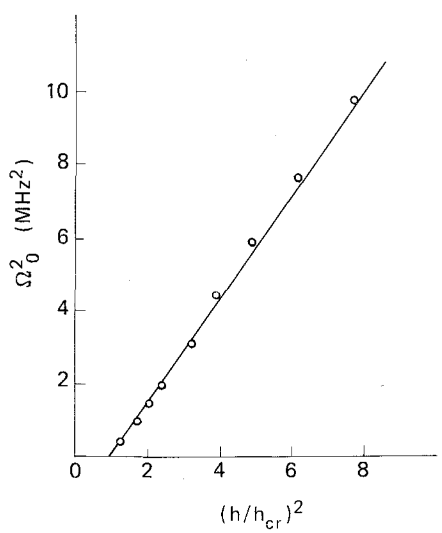

Fig. 8. Dependence of the resonance frequency of collective oscillations on the pump power.

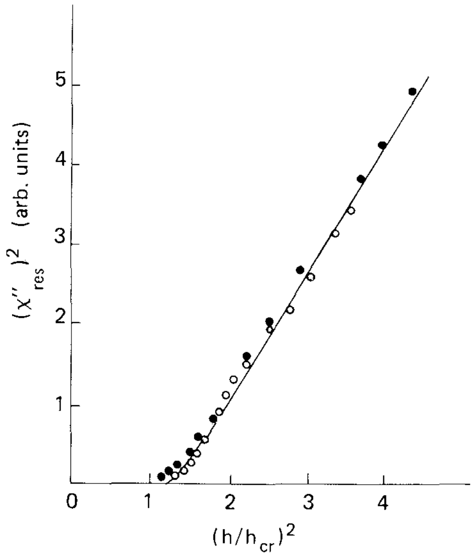

Fig. 9. Square of the resonance susceptibility as a function of the supercriticality. ●: $H = H_c - 10$ Oe; ○: $H = H_c - 50$ Oe.

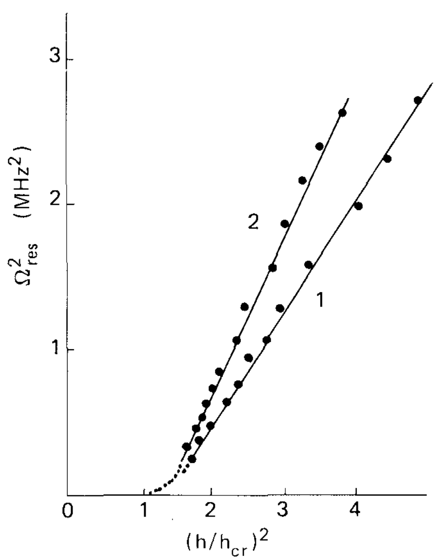

Fig. 10. Dependence of the resonance frequency on the supercriticality; curve 1: $H = H_c - 10$ Oe, curve 2: $H = H_c - 50$ Oe.

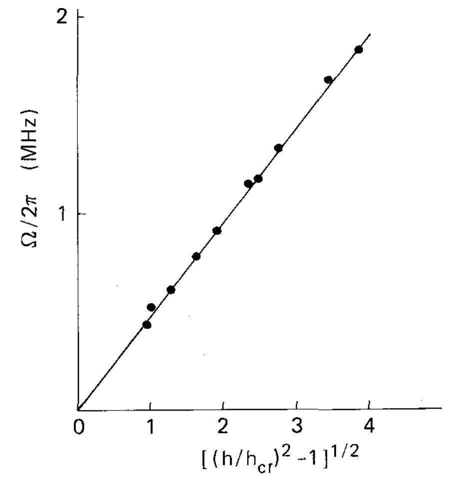

Fig. 11. Dependence of the frequency of collective oscillations on the supercriticality in AFM $MnCO_3$ ($T=1.62$ K, $H=730$ Oe).

Orel and Starobinets (1975) proposed and investigated theoretically and experimentally a direct method of the resonance excitation of CO with the aid of a radio-frequency field parallel to the magnetization. In particular, they calculated the linear susceptibility of the RF signal by equations of the S-theory (eq. 2.3). If the resonance frequency Ω coincides with the frequency of CO Ω_0 (eq. 3.5) we obtain the following equation:

$$\chi''_{\text{res}} = \frac{g^2}{2\sqrt{S_0(2T_0+S_0)}}\left(\frac{h^2}{h_1^2}-1\right)^2 . \tag{3.8}$$

Experimental data correspond completely with this dependence. The observed

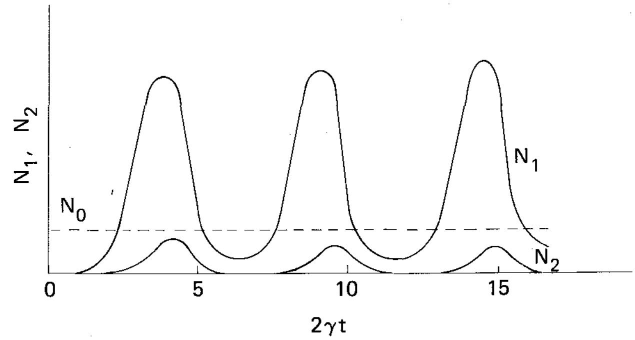

Fig. 12. Time dependence of the amplitude of pairs on beams at the instability of a zero mode. N_1 and N_2 correspond to $\theta=\pi/2$, and $\theta=\pi/4$.

frequency of CO is shown by the points in fig. 10 and also qualitatively coincides with the theory (the solid line in fig. 10, calculated by formula 3.5). Thus the S-theory describes the SWF and the RF of the resonance of CO in good agreement with experiment without any adjusting parametes. It testifies that physical ideas of the nature of CO described in this section are right. A nice method of the percussion excitation of CO, proposed by Prozorova and Smirnov (1974), deals with the resonance methods by means of a sharpe change of the pumping phase. Their data of the self-frequency of PSW in the antiferromagnetic $MnCO_3$ (fig. 11) are also in good accordance with the theoretical dependence (eq. 3.5). It shows once more that the S-theory describes the parametric excitation of the SW right not only in FM but also in AFM.

3.3. *Double parametric resonance and inhomogeneous collective oscillations of magnons*

A new phenomenon, namely the parametric excitation of CO of the frequency Ω, appears at increasing amplitude of excited CO, e.g., by a resonance RF field of fequency $\Omega/2$. This phenomenon is a double parametric resonance. In 1977 it was theoretically predicted and experimentally observed by Zautkin et al. (1977). It is caused by a nonlinear interaction of two types: $H_m\beta^*\beta$ and $\alpha\alpha\beta^*\beta^*$. Here H_m is the amplitude of the resonance frequency field (RFF) of frequency Ω, α is the amplitude of CO of the same frequency and β is the amplitude of CO of frequency $\Omega/2$. First of all the stationary amplitude of CO, α must be found for the calculation of the threshold of the parametric resonance of CO, taking nonlinear terms $\alpha^2\alpha^{*2}$ in the hamiltonian into account; then a set of four linear equations must be solved for $\beta_\pm \sim \exp(\pm i\Omega t/2)$, $\beta^*_\pm \sim \exp(\mp i\Omega t/2)$. These cumbersome calculations, which required the use of a computer, allowed to find a dependence of $H_{m_{cr}}$ on Ω and H in order of the value $gH_{m_{cr}} \approx \gamma$ (Zautkin et al. 1977). It is of fundamental importance that the double parametric resonance allows, generally speaking, to obtain both homogeneous and inhomogeneous CO with $\kappa \neq 0$. However, it is difficult to identify the latter because the signals are the same for both types of CO in a detector which registratesonly integral variations of the magnetization of a sample. Nevertheless, Zautkin and Orjel were lucky not only to registrate the existence of inhomogeneous CO at double parametric resonance but to investigate their spectrum as well, using many indirect data, e.g. the dependence of the amplitude of the frequency of CO on the experimental conditions of H, H_m and h.

3.4. *Auto-oscillations of the magnetization*

It is well-known that, in the case of parametric generation of spin waves, the stationary state is frequently not established, and that the magnetization performs complicated autooscillations (AO) about some mean value.

The main experimental facts on these AO, obtained for high-quality YIG crystals under parallel excitation, are as follows (Courtney and Claricoats 1964, Surin 1969, Monosov 1971):

(1) The AO magnetizations lie in the range between 10^4 and approximately 10^7 Hz (depending on the pump power and the constant magnetic field). For a small excess above the threshold, the AO spectrum consists of a single line. As the power level increases, there is an increase in the number of lines, which also shift toward higher frequencies. For large excesses above the threshold, noise-type spectrum is observed.

(2) The threshold for AO is usually very low: 0.1–1 dB with respect to the parametric excitation threshold, with the exception of small wave vectors ($H > H_c$) where the AO threshold is appreciably higher. It also rises when internal inhomogeneities are introduced into the crystal (Safant'evskii 1971).

(3) Giant crystallographic anisotropy in AO properties, substantially exceeding the anisotropy in the spin-wave spectrum, has been observed. Thus, the AO intensity in YIG when the magnetization lies along the $\langle 111 \rangle$ axis *exceeds* the intensity for the $\langle 100 \rangle$ axis by a factor of roughly 100.

The question of the physical nature of autooscillations is one of the main problems of the parametric excitation of spin waves. Several hypotheses were postulated to explain the nature of AO (Green and Schlömann 1962, Wang et al. 1968, Hsu 1970, Monosov 1971), but, however, failed in experiment.

The autooscillations find a natural explanation within the framework of the S-theory as a result of the instability of the stationary state against the excitation of CO, which was described in §3.1. If the instability conditions $S_m(2T_m + S_m) < 0$ are satisfied for at least one mode (of number m), the system of parametrically excited spin waves has no stable stationary states within the framework of the S-theory. There are then two conceivable possibilities: either the system is brought out of the region within which S-theory is valid (this should be accompanied by a strong increase in the amplitude of the excited waves), or the oscillations should be settled down around the stationary state. These oscillations (if they develop) are observed experimentally as AO of magnetization. General considerations suggest that the AO may be both regular and random. In the latter case they can be looked upon as "secondary" turbulence with a time scale much greater than the scale of the "main" turbulence (the period of the spin waves).

It is clear from eq. (3.5) that the instability of the stationary state is purely aperiodic (Re $\Omega_m = 0$) and, therefore, the secondary turbulence is strong. This means that the analytic solution for the nonlinear stage of the development of collective instability and the description of secondary turbulence (if it appears) are extremely difficult to obtain. Computer simulation of AO would therefore seem to be useful. This would however require an inordinate amount of

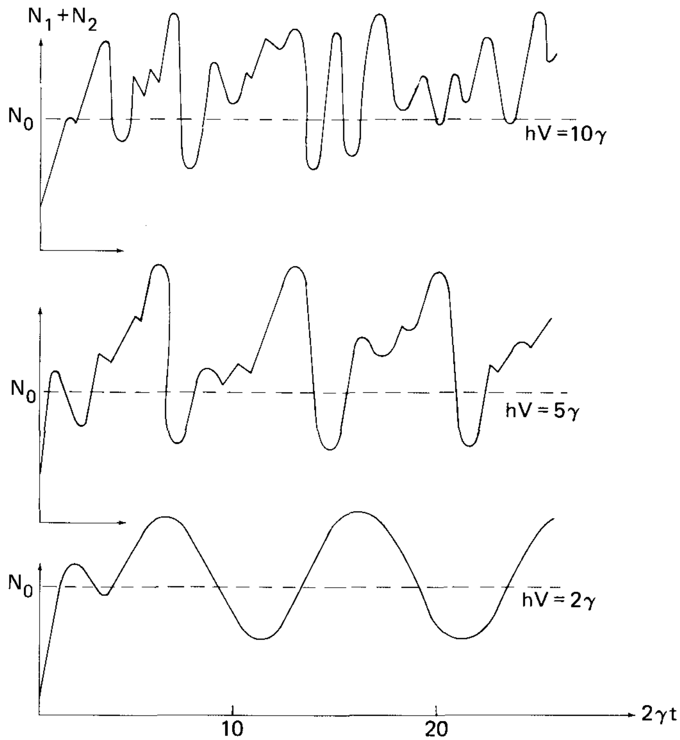

Fig. 13. Time dependence of the summated amplitude of pairs on beams at the instability of mode $m = 2$.

machine time and is hardly practicable in a real situation, for example, in YIG. This makes us turn again to a numerical experiment, using simple models of the ground state.

A numerical experiment on the excitation of AO in a model of two beams was described by L'vov et al. (1973a) and Grankin et al. (1981). It was assumed that the spin waves were localized around two fixed angles $\theta_1 = \pi/2$, $\theta_2 = \pi/4$. The coefficients S_0 and T_0 were chosen to be close to those calculated for YIG, with the magnetic field along the $\langle 111 \rangle$ axis (see below), so that the instability conditions were satisfied for the zero mode. The numerical experiment showed that steady AO of amplitudes and phases of waves on beams were established in this model (fig. 13). The dependence of the frequency in these AO on the pump level is in qualitative agreement with the analogous result observed experimentally. The above model was also used to carry out an experiment simulating the development of collective instability for $m \neq 0$. The behavior of the resultant amplitude during the nonlinear stage of development of the instability was investigated. This is interesting because there is no change in the resultant amplitude during the linear stage. We have chosen beams with $\theta_1 = \theta_2 = \pi/2$ and $\phi_2 = \phi_1 + \pi/2$. The experiment shows (fig. 12) that states were established in which both the sum and the difference of the wave amplitudes underwent oscillations. The oscillations in the summative amplitudes appear as a result of the interaction between collective modes with different m.

The numerical experiment based on the above models thus shows that, within the framework of the S-theory, the development of the internal instability of the ground state leads to AO. The properties of these AO, i.e., the dependence of frequency and spectral composition on the pump power, are comparable with the properties of real AO observed in laboratory experiments. In particular, in both numerical and laboratory experiments AO are periodic for small excesses above the threshold. Such a motion in the phase space is a stable boundary cycle. In both types of experiments transition to stochastic AO with increasing supercriticality is not carried out with the aid of new additional motion types in nondimensional frequencies but is accompanied by broadening of real spectral lines with multiple frequencies and is connected with the loss of instability of the phase trajectory, which leads to their smearing. This means that the creation of secondary turbulence of spin waves arises in accordance with ideas about the strange attractor. Henceforth, both numerical and laboratory experiments show that the development of AO has no substantial effect on the mean level of parametrically excited waves. The numerical experiment shows moreover that, during the development of instability, the zero mode leads to pronounced oscillations in the resultant amplitude. Smaller oscillations in the resultant amplitude which are accompanied by a reduction in the measured χ'' are found for higher-mode instabilities. It is interesting that the appearance of autooscillations in the numerical experiment is usually accompanied by a reduction in the mean value of χ'. This has also been observed in laboratory experiments. The above result can be used to predict situations in which AO should be observed in a real experiment.

Later, L'vov et al. (1973a) calculated coefficients using S_m and T_m for a typical experimental situation: $N_z = 1/3$ (sphere) $\omega_p = 9.4$ Hz, $k = 0$ ($H = H_c$), $\omega_M = 4.9$ Hz, $\omega_a = 0.23$ Hz (room temperature). It is clear that not only the magnitudes but also the signs of the coefficients depend on the direction of magnetization. Substituting the tabulated data in the instability criterion (3.6), we can readily verify that in the "easy" direction $\langle 111 \rangle$ we have an instability for the zero mode, whereas in the "difficult" direction $\langle 100 \rangle$ all the modes are stable in the described situation. In the experment with $H = H_c$ in the difficult direction, the AO are, in fact, absent right up to $h_2 \approx 6$–7 dB, which corresponds to the second threshold. On the other hand, in the "easy", direction there are strong autooscillations, which are observed immediately beyond the threshold h_1 (Monosov 1971).

The condition for the appearance of AO in different experimental situations was analyzed in detail by Zautkin and Starobinets (1974).

They varied coefficients of the Hamiltonian, S_m, T_m, within wide limits (changing the magnetization with by means of the temperature and the concentration of admixtures, changing the share of a sample – sphere, cylinder, disc – and the direction of magnetization). In particular, they showed that intensive AO appear only for the instability of a zero-mode in criterion (3.6), whilst the instability of modes with $m \neq 0$ is always accompanied with the appearance of AO with small amplitude. Other experimentally observed peculiarities of AO may be also naturally explained with the aid of the S-theory.

In conclusion, we must recall that the simple theory of AO discussed in this section predicts that the threshold h_a for the creation of AO coincides with the parametric excitation threshold h_1 and that the frequency of the AO is zero for $h = h_a$. Experiments show, however, a finite threshold for the auto-oscillations and a nonzero initial frequency. In single crystals of YIG, the threshold for the autooscillations is usually 0.1–1 dB, and the initial frequency does not correlate with the size of the threshold and varies in the range 10^4–10^5 Hz (Surin 1969, Monosov 1971) depending on the constant magnetic field. These facts can be explained by the influence of weak nonlinear damping, which has no substantial effects on the magnitudes of χ' and χ'', by the presence of random inhomogeneities in the crystals (L'vov 1973, L'vov and Cherepanov 1981), by the absence of exact axial symmetry, by the reaction of the specimen on the resonator (Shapiro et al. 1970), and so on. Further theoretical studies of auto-correlations will be necessary to establish the relative contributions of these mechanisms.

4. *Strong turbulence and self-focusing of solitons at the parametric excitation of waves*

In previous sections we considered the parametric instability threshold to be a minimum for wave pairs with vectors $\pm k$ which occupy a line or some surface in the k-space. In this case the phase sum in a pair is a dynamic variable and individual wave phases are randomized with a good accuracy. In this section we will describe the situation that the excitation threshold is a minimum for the single pair $\pm k_0$; e.g., at parallel pumping in uniaxial ferromagnets with anisotropy of the "easy-plane" type. Narrowness of wave packets which are excited in the k-space is a principal peculiarity of the problem. On the one hand it does not allow one to use a statistical description as distinct from the S-theory, on the other hand it provides the possibility to simplify the exact interaction Hamiltonian in another parameter, namely, in the packet narrowness. In this case the problem must be formulated in terms of wave envelopes.

4.1. Equations for slowly changing amplitudes

Using canonic equations of motion with the exact Hamiltonian and assuming narrow wave packets to be excited in k-space,

$$a_k = A(k - k_0) + B(k + k_0),$$

we obtain (Zakharov et al. 1968) equations for envelopes $A(r)$, $B(r)$ and the Fourier-components $A(\kappa)$ and $B(\kappa)$ in the usual way,

$$\left[\mathrm{i}\left(\frac{\partial}{\partial t} + V\nabla\right) + \tfrac{1}{2}\hat{L}\right]A$$
$$= -\mathrm{i}\gamma A + hVB^* + \left[\omega_{k_0} - \frac{\omega_p}{2} + T|A|^2 + 2S|B|^2\right]A,$$
$$\left[\mathrm{i}\left(\frac{\partial}{\partial t} - V\nabla\right) + \tfrac{1}{2}\hat{L}\right]B$$
$$= -\mathrm{i}\gamma B + hVA^* + \left[\omega_{k_0} - \frac{\omega_p}{2} + T|B|^2 + 2S|A|^2\right]B, \tag{4.1}$$

where

$$V = \frac{\partial\omega}{\partial k}, \quad \hat{L} = \sum_{\alpha,\beta} \frac{\partial^2\omega}{\partial k_\alpha\, \partial k_\beta} \frac{\partial^2}{\partial x_\alpha\, \partial x_\beta},$$
$$T = T_{k_0 k_0 k_0 k_0}, \quad S = T_{k_0, -k_0, k_0, -k_0}$$

(coefficients of the interaction Hamiltonian of waves).

Equations (4.1) have a trivial solution $A(r) = B(r) = A_0$, where

$$A(r) = |A| \exp(-\mathrm{i}\psi_1), \quad B(r) = |B| \exp(-\mathrm{i}\psi_2),$$
$$2SA_0^2 = \sqrt{(hV)^2 - \gamma^2}, \quad \sin(\psi_1 + \psi_2) = \frac{\gamma}{hV}, \quad \psi_1 = \psi_2 = \frac{\phi}{2}, \tag{4.2}$$

which corresponds to the excitation of the standing wave $\pm k_0$. It can be shown (L'vov and Rubenchik 1972), that the solution (4.2) is practically always instable against the increase of the modulations of amplitudes and phase of the envelopes of waves

$$\delta A(r, t) \sim \exp[\mathrm{i}\kappa r + \nu(\kappa)t], \quad \delta B(r, t) \sim \exp[\mathrm{i}\kappa r + \nu(\kappa)t].$$

The character of the instability development depends essentially on the Hamiltonian parameters (1.1). As an exception for the case $T > 0$, $S > 0$, which we will not discuss further, the increment of the instability is a maximum on the surface $\kappa \perp k_0$ and has the form:

$$[\gamma + \nu(\kappa)]^2 - \gamma^2 = \begin{cases} -\tfrac{1}{4}L\kappa^2\left(L\kappa^2 + 4TA_0^2\right), \\ -8S(2S + T)A_0^4 - \tfrac{1}{4}L\kappa^2\left[L\kappa^2 + 4(4S + T)A_0^2\right], \end{cases} \tag{4.3}$$

and it decreases quickly when moving off from the surface. It leads to the situation that main properties of the development of the nonlinear stage of the instability may be described by means of two-dimensional equations in which A and B depend only on coordinates x and y orthogonal to V. The upper line in expression (4.3) corresponds to a perturbation of type $\delta A(r, t) = +\delta B(r, t)$, the lower line to $\delta A(r, t) = -\delta B(r, t)$. In the case that $T > 0$, $S < 0$, it is clear that the perturbation $\delta A = -\delta B$ has also some additional stability (L'vov and Rubenchik 1972) and that the relation $A(r, t) = -B(r, t)$ is also valid for the nonlinear stage of motion. If we restrict ourselves for the sake of simplicity to the case $\omega'' > 0$, we can modify equation (4.1) by changing the scale into the form

$$\mathrm{i}\left(\frac{\partial}{\partial t} + \gamma\right)A - hVA^* + \tfrac{1}{2}\Delta A = \left[(2S + T)\,|A|^2 - TA_0^2\right]A. \tag{4.4}$$

Here we define $\omega(k_0)$ by the external stability condition (2.12). Thus the most stable standing wave is chosen, for which the region of a positive increment in k-space, as was shown by L'vov (1971), is limited by $\kappa \ll k_0$. The further study of the nonlinear stage of the instability development of a plane standing wave will by carried out within the framework of equations (4.4).

The analysis of the stability of solitons of various types (L'vov and Rubenchik 1972) to assume all the stationary solitons to be unstable. In the same paper the initial stage of this instability has been considered and special attention was payed to the case that the increment is anomalously small. This is of interest for nonlinear interactions, which become important at very small amplitudes. It turned out, however, that it does not restrict the initial perturbation growth but only slows it down strongly. Its amplitude rises in this case as $\sqrt{t}$. As result a narrow packet of waves in the k-space, with $\kappa \sim \kappa_0 \ll k_0$, $\Delta\kappa \sim \kappa_0$, arises. Such a state is strongly turbulent. We shall study it in the next section of this review.

4.2. Average characteristics of strong turbulence

First we estimate the width of the excited region in the k-space at arbitrary excesses above the threshold. It follows from the S-theory that in the case that V_k is a maximum in point k_0, the packet of parametrically excited waves relaxes to a standing monochromatic wave with $k = k_0$ if individual wave phases can be considered random. It is sufficient for the phase randomness that the phases of two waves in a packet diverge by a factor of the order of unity in a time smaller than the characteristic time of nonlinear interaction. This happens in the packet with $(\Delta k)^2 \gg \kappa_0^2 = SA_0^2/\omega_k''$. Therefore, the packet with $\Delta k \gg \kappa_0$ narrows up to a size $\sim \kappa_0$, and its mean amplitude relaxes to a value $\sim A_0$. If $\Delta k \ll \kappa_0$, such a packet is unstable with increment (4.3) with respect to perturbations with $\kappa \sim \kappa_0$, and, therefore, it will be broadening up to $\Delta k \sim \kappa_0 \sim (SA_0^2/\omega_k'')^{1/2}$. Note that the instability increment (4.3) is positive in the narrow layer $\kappa_{\|}V \lesssim SA_0^2$ close to the surface $\kappa \perp V$, i.e., the turbulence, which we are studying, is almost two-dimensional. Namely,

$$\left(\frac{\kappa_{\|}}{\kappa_{\perp}}\right)^2 > \frac{SA_0^2}{\omega_k} \ll 1. \tag{4.5}$$

The mean turbulence level A^2 cannot differ strongly from A_0, which is defined by formula (4.2). In fact, as was pointed out above, L'vov (1971) has shown that the monochromatic plane wave is only stable with respect to short-wave perturbations $\kappa \sim k_0$ when its amplitude is equal to A_0. It is obvious that such an instability is also retaining for a pair which is modulated with $\kappa \ll k_0$. Therefore, if A_0^2 differs appreciably from A^2, the short-wave modulations are excited, in contradiction with the above obtained result of the packet narrowness. Thus the instability development of a plane wave leads to a strong quasi-two-dimensional turbulence of the modulation waves $A(r, t)$ with a mean level

$$\overline{A^2} \simeq \overline{A_0^2} = (h^2V^2 - \gamma^2)/2S, \tag{4.6}$$

a modulation level of the order of unity, the characteristic motion frequency $hV - \gamma$, and with the characteristic scale in the coordinate space $r_{\perp} \sim \kappa_0^{-1} > k_0^{-1}(\omega_k/SA_0^2)^{1/2}$. One can say that the dynamic structure of solitons arises with the coherence length of the order of solitons dimensions, i.e., $A_0^{-1}(\omega''/S)^{1/2}$, which considerably changes in space for the time $(hV - \gamma)^{-1}$.

4.3. Decay and the amplitude-restriction mechanism of solitons

The damping and the pump may be neglected in equations of motion in those regions where the amplitude of solitons in the turbulent motion becomes anomalously large, $A \gg A_0$. This is so, because the system cannot exchange energy with the thermostat and the pumping in the characteristic time of te problem $(SA^2)^{-1}$. In this approximation, eqs. (4.1) describe the nonstationary behavior of a pair of waves in a conservative medium. The behavior of a single nearly monochromatic wave in a nonlinear conservative medium was intensively investigated experimentally and theoretically, and with help of a computer, in connection with problems of nonlinear optics (Akhmanov et al. 1967), plasmas, and hydrodynamics (Talanov 1964, Kadomtsev and Karpman 1970). The phenomenon of self-focusing of light (Zakharov 1967) has been established and later it has been shown that a self-focused light beam is unstable (Zakharov et al. 1971a). In some cases this instability results in the collapse of a light beam in a finite time (Vlasov et al. 1971).

Below we will show that similar phenomena also occur in our case of a pair of waves. By explicit calculation one can verify that eqs. (4.1) (at $\gamma = h = 0$) possess the following integral of motion (L'vov et al. 1973b):

$$I = \frac{\omega''}{2}\int(|\nabla A|^2 + |\nabla B|^2)\,\mathrm{d}r + \frac{T}{2}\int(|A|^4 + |B|^4)\,\mathrm{d}r$$
$$+2S\int|A|^2|B|^2\,\mathrm{d}r. \tag{4.7}$$

Let us show that its sign determines the evolution of a system to a large extent. We consider the second derivative of the essentially positive quantity R,

$$R = \frac{1}{2\omega''}\int r_\perp^2(|A|^2 + |B|^2)\,\mathrm{d}r > 0. \tag{4.8}$$

Due to the small parameter $(SA_0^2/\omega)^{1/2}$ one can neglect second derivatives in z and by explicit calculation with the aid of eq. (4.1) show (L'vov and Rubenchik 1972) that $\mathrm{d}^2R/\mathrm{d}t^2 = 2I$, from which it follows that

$$R(t) = It^2 + 2\alpha t + \beta, \tag{4.9}$$

where α and β are constants of the integration. It is clear that in a finite time $R(t)$ becomes negative when $I < 0$; this fact is at variance with eq. (4.8). This means that the solution of eqs. (4.1) is "disintegrated" in a finite time, i.e., a singularity appears.

It is clear that when studying the evolution of the collapsing soliton we may neglect the influence of pumping and damping. Equation (4.4) turns into a nonlinear parabolic equation whose solution was studied in detail in connection with the problem of self-focusing of light. As was shown by Zakharov (1967), the amplitude in the middle of a soliton increases quickly in a time $A(0, t) \sim (t - t_0)^{-2/3}$. The radius during the collapse decreases so that a certain amount of energy begins to collapse, namely

$$I_c = \omega\int|A|^2\,\mathrm{d}r = \frac{1.86\omega''\omega}{|2S + T|}. \tag{4.10}$$

When the amplitude of waves in the collapsing soliton is large enough,the nonlinear damping becomes important and leads to quick dissipation of the soliton energy. We can estimate the effective nonlinear damping as

$$\gamma_{\mathrm{nl}} \sim I_c\kappa_0^2/(\tau\omega A_0^2) \sim \frac{1}{\tau}, \tag{4.11}$$

where τ is a time between two successive collapses in the region with the size κ_0^2. From a dimensional estimation we find $\tau \sim (hV)^{-1}$ at $h > h_c$.

The same mechanism of energy dissipation leads to a decrease of the mean amplitude of turbulent pulsations, $\bar{A}$ becoming less than A_0; the susceptibility χ'' does not decrease with the amplitude increase h but become a constant coinciding with χ'' in the quantity-order of a maximum. Measuring of the spectral density of the electromagnetic radiation of a ferromagnet with frequencies close to the pump frequency ω_p is a perspective method of the experimental study of strong turbulence of parametrically excited waves. When $h < h_c$, collapses are seldom, the spectral density of noise $(h^2)_\omega$ has a Gaussian width $\gamma(P/P_{nop} - 1)^{1/2}$. When $h > h_c$, collapsed solitons, whose pair phase $\psi(r, t)$ breakes from the pump phase and begins to rotate rapidly, become dominant (Zakharov et al. 1971a). The moment of phase breaking and some first revolutions are well seen in fig. 2. The quick rotation of the phase leads to a considerable broadening of the radiation spectrum $(h^2)\omega$, which helps to registrate collapses. Using results of Zakharov et al. (1971b) it can be shown that $(h^2)\omega \sim |\omega - \omega_p|^{7/4}$. The nonlinear damping which restricts the amplitude of a collapse $A < A_{max}$, must cut off the noise radiation on frequency

$$SA^2_{max} > |\omega - \omega_p|.$$

Part II – The S,T^2-theory

In the first part of this review the nonlinear theory of the parametric excitation of waves was given in the approximation of a self-consistent field. It was called the S-theory, after the matrix element of the interaction of wave pairs, $S_{kk'}$, which plays a decisive role in it. In this approximation the stationary state of the PW is singular in the (k, ω)-space:

$$n_{k\omega},\ \sigma_{k\omega} \sim \delta(\omega - \omega_p/2)\,\delta(\tilde{\omega}_k - \omega_p/2). \tag{1}$$

Actually, singularities in k-states of the PW never occur. To confirm this point it is sufficient to attempt to estimate the effect of the next order in H_{int} with the help of the kinetic equation. Its collision term becomes divergent for the solution (1). This means that the mutual scattering of the PW should broaden their distribution function. Our goal in this part of this review is to describe the consistent S,T^2-theory, which takes not only the S-interaction of pairs into account but also the T^2-scattering of the PW from each other and their interaction with the thermostate. The latter interaction leads to the damping of the PW.

The formalism of the S,T^2-theory is essentially more complicated than that of the S-theory. To keep the review within reasonable limits, the discussion in the below sections presupposes a higher standard of knowledge than in the first part of this review. In particular, the reader is assumed to be familiar with the ideas of the diagram techniques of Feinman (Abrikosov et al. 1962), Keldish (1964) and Wyld (1961).

5. Main equations of the S,T^2-theory

5.1. The solution procedure

From our point of view the most simple and consistent method for the derivation of the equations of the S,T^2-theory is Wyld's diagram technique (Wyld 1961). Its objects are functions of the linear response on a random force f_q, i.e. Green's functions (GP) G_q, L_q,

$$G_q = \left\langle \frac{\delta a_q}{\delta f_q} \right\rangle, \quad L_q = \left\langle \frac{\delta a_q}{\delta f^*_{\bar{q}}} \right\rangle, \quad q = (k, \omega), \quad \bar{q} = (-k_1 \omega_{\mathrm{p}} - \omega), \tag{5.1}$$

and the pair correlation functions of the canonic amplitudes of waves, i.e. the correlators:

$$n_q = \langle a_q a^*_q \rangle, \quad \sigma_q = \langle a_q a_{\bar{q}} \rangle. \tag{5.2}$$

The anomalous GF, L_q, and the correlator σ_q appear due to phase correlation in the pairs of waves. Introducing a Langevin random force f_q, which simulates the interaction of a system of waves with the thermostate, into the right-hand part of eqs. (1.14) for a_q, the diagram technique can be developed. Then one can obtain the formal solution of these equations in the form of a series in degrees of f_q. Using definitions (5.1) and (5.2) one thereupon constructs the quantities G_q, L_q, n_q and σ_q, summarizing weakly coupled diagrams. Ultimately, we obtain the equations of Dyson:

$$\begin{aligned} G_q &= G^0_q \left[1 + \Sigma_{\mathrm{c}}(q) G_q + \Pi_{\mathrm{c}}(q) L^*_q \right], \\ L_q &= G^{0*}_{\bar{q}} \left[\Pi^*_{\mathrm{c}}(\bar{q}) G_q + \Sigma^*_{\mathrm{c}}(\bar{q}) L^*_q \right], \quad G^0_q = (\omega - \omega_k + \mathrm{i}o)^{-1}, \end{aligned} \tag{5.3}$$

and Wyld:

$$\begin{aligned} n_q &= G_q \left[\Sigma_{\mathrm{d}}(q) G^*_q + \Pi_{\mathrm{d}}(q) L^*_{\bar{q}} \right] + L_{\bar{q}} \left[\Pi^*_{\mathrm{d}}(\bar{q}) G^*_q + \Sigma^*_{\mathrm{d}}(\bar{q}) L^*_{\bar{q}} \right], \\ \sigma_q &= G_q \left[\Pi_{\mathrm{d}}(q) G_{\bar{q}} + \Sigma_{\mathrm{d}}(q) L_q \right] + L_{\bar{q}} \left[\Sigma^*_{\mathrm{d}}(\bar{q}) G_{\bar{q}} + \Pi^*_{\mathrm{d}}(\bar{q}) L_q \right]. \end{aligned} \tag{5.4}$$

Here the following notations are introduced for mass operators (MO) of the compact diagram sums:

$\Sigma_{\mathrm{c}}(q)$: normal causal MO,

$\Pi_{\mathrm{c}}(q)$: anomalous causal MO,

$\Sigma_{\mathrm{d}}(q)$: normal distributive MO,

$\Pi_{\mathrm{d}}(q)$: anomalous distributive MO.

Zakharov and L'vov (1975) described the derivation procedure of these equations in detail. To close eqs. (5.3), and (5.4), the MOs must be expressed in terms of the GFs and correlators. These expressions are partially summarized but, in fact, they are infinite series of the perturbation theory. However, it is sufficient to retain first-order diagrams for small amplitudes. Diagrams of second order in the interaction of waves are retained in the S,T^2-theory. In this approximation the simplest distributive MO are those which determine the distribution function of the PW in accordance with eq. (5.4):

$$\Sigma_{\mathrm{d}}(q) = 2\int \left[|T_{k1,23}|^2 n_1 n_2 n_3 + 2T_{k123} T^*_{k\bar{2}\bar{1}3} \sigma_1^* \sigma_2 n_3 \right] \times \delta^4(q+1-2-3)\, \mathrm{d}^4 1\, \mathrm{d}^4 s\, \mathrm{d}^4 3,$$

$$\Pi_{\mathrm{d}}(q) = 2\int \left[T_{k123} T_{\bar{k}\bar{1}\bar{2}\bar{3}} \sigma_1^* \sigma_2 \sigma_3 + 2T_{k123} T_{\bar{k}21\bar{3}} n_1 n_2 \sigma_3 \right] \times \delta^4(q+1-2-3)\, \mathrm{d}^4 1\, \mathrm{d}^4 2\, \mathrm{d}^4 3, \tag{5.5}$$

where $1 \equiv q_1 = (k_1 \omega_1)$, $\mathrm{d}^4 1 = \mathrm{d}^3 k_1\, \mathrm{d}\omega_1$, $n_1 = n_{q_1}$, etc.

Expressions for the causal MO, which defines the causal GF, are, in accordance with eq. (5.3), somewhat more complicated:

$$\begin{aligned} \Sigma_{\mathrm{c}}(q) = {} & 2\int T_{kk'} n_{q'}\, \mathrm{d}^4 q' + 2\int \left\{ |T_{k123}|^2 \left[G_1^* n_2 n_3 + n_1 (G_2 n_3 + n_2 G_3) \right] \right. \\ & \left. + 2T_{k123} T^*_{k\bar{2}\bar{1}3} \left[\sigma_1^* \sigma_2 G_3 + (L_{\bar{1}}^* \sigma_2 + \sigma_1^* L_{\bar{2}}) n_3 \right] \right\} \\ & \times \delta^4(q+1-2-3)\, \mathrm{d}^4 1\, \mathrm{d}^4 2\, \mathrm{d}^4 3, \end{aligned} \tag{5.6}$$

$$\begin{aligned} \Pi_{\mathrm{c}}(q) = {} & hV_k + \int S_{kk'} \sigma_{q'}\, \mathrm{d}^4 q' \\ & + 2\int \left\{ T_{k123} T_{\bar{k}\bar{1}\bar{2}\bar{3}} \left[L_1^* \sigma_2 \sigma_3 + \sigma_1^* (L_2 \sigma_3 + \sigma_2 L_3) \right] \right. \\ & \left. + 2T_{k123} T_{\bar{k}21\bar{3}} \left[n_1 n_2 L_{\bar{3}} + (G_1^* n_2 + n_2 G_2) \sigma_3 \right] \right\} \\ & \times \delta^4(q+1-2-3)\, \mathrm{d}^4 1\, \mathrm{d}^4 2\, \mathrm{d}^4 3. \end{aligned}$$

If there were random inhomogeneities, i.e. defects, pores, deformation fields, in a medium, then the elastic scattering from them leads to an additive contribution to the Hamiltonian,

$$H_{\mathrm{el}} = \int g_{k,k'} \eta_{k-k'} a_k^* a_{k'}\, \mathrm{d}k\, \mathrm{d}k', \tag{5.7}$$

where $g_{k,k'}$ is the matrix element which characterizes the scattering of waves from the inhomogeneoties, and η_k is their amplitude. For point defects we have the following expression:

$$\eta_k = \frac{v_0}{(2\pi)^3} \sum_{r_n} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}_n),$$

where $\boldsymbol{r}_n$ are the coordinates of defects, and v_0 is the volume of the elementary cell. The detailed analysis (Zakharov and L'vov 1972, L'vov 1982) shows that in interesting cases it is enough, as a rule, to retain the lowest diagrams in $g_{k,k'}$ which give an additive contribution into MO:

$$
\begin{aligned}
\Sigma_{\mathrm{c}}^{\mathrm{el}} &= c\int |g_{kk'}|^2 G_{k',\omega}\,\mathrm{d}k', &\quad \Pi_{\mathrm{c}}^{\mathrm{el}} &= c\int g_{kk'} g_{\bar{k}\bar{k}'} L_{k',\omega}\,\mathrm{d}k', \\
\Sigma_{\mathrm{d}}^{\mathrm{el}} &= c\int |g_{kk'}|^2 n_{k'\omega}\,\mathrm{d}k', &\quad \Pi_{\mathrm{d}}^{\mathrm{el}} &= c\int g_{kk'} g_{\bar{k}\bar{k}'} \sigma_{k'\omega}\,\mathrm{d}k',
\end{aligned}
\tag{5.8}
$$

where c is the concentration of defects.

5.2. Form of the Green functions

Let us substitute the solution of the Dyson equations 95.3) in the form

$$
\begin{aligned}
&G_q = -(\epsilon + \tilde{\epsilon}_q + \mathrm{i}\Gamma_q)\Delta_q^{-1}, \quad L_q^* = \Pi_{\mathrm{c}}^*(\bar{q})\Delta_q^{-1}, \\
&\Delta_q = \tilde{\epsilon}_q^2 + \nu_q^2 - 2\mathrm{i}\Gamma_q\epsilon - \epsilon^2,
\end{aligned}
\tag{5.9}
$$

where

$$
\begin{aligned}
&\epsilon = \omega - \omega_{\mathrm{p}}/2, \quad \tilde{\epsilon}_q = \tilde{\omega}_q - \omega_{\mathrm{p}}/2, \quad \tilde{\omega}_q = \omega_k + \mathrm{Re}\ \Sigma_{\mathrm{c}}(q), \\
&\Gamma_q = -\mathrm{Im}\ \Sigma_{\mathrm{c}}(q), \quad \nu_q^2 = \Gamma_q^2 - |\Pi_{\mathrm{c}}(q)|^2,
\end{aligned}
\tag{5.10}
$$

and where small factors in the contributions of ϵ and $\tilde{\epsilon}_q$ to MO, not important for the following, have been neglected. It is obvious that Im G_q and L_q in the middle of the frequency of the packet, i.e., when $\epsilon = 0$, have the Lorentzian form with the width $\Delta k = \nu/V$ (V is the group velocity). This width is essentially less than Γ/V because of the compensation of damping by the pump according to eq. (5.10). The packet width of the frequency $\tilde{\omega}_q$ in the center of the packet, i.e., on the resonance surface, is even smaller: $\Delta\omega \approx \nu^2/\Gamma$. The normal GF, G_q, lookes like a free function far from the resonance surface but the function must have the renormalized frequency and damping

$$
G_q \to (\omega - \omega_q + \mathrm{i}\Gamma_q)^{-1},
\tag{5.11}
$$

and the anomalous GF, L_q, is small: $|L_q| \ll |G_q|$.

5.3. Division procedure of waves into parametric and thermal waves

The Dyson–Wyld equations (5.3) and (5.4) describes waves in the whole (k,ω)-space and have in the absence of the pump a thermodynamic equilibrium solution:

$$
\begin{aligned}
&n_{k\omega}^0 = \Sigma_{\mathrm{d}}^0(k)\left[(\omega - \tilde{\omega}_k)^2 + \gamma_k^2\right]^{-1}, \\
&\Sigma_{\mathrm{d}}^0(k) = \gamma_k n_k^0/\pi, \quad n_k^0 = T/\tilde{\omega}_k,
\end{aligned}
\tag{5.12}
$$

in which γ_k is the equilibrium decrement by wave damping.

The pump not only changes numbers of filling in n_q in the region of the parametric resonance but also in the whole (k, ω)-space. The question is therefore which waves are parametrically excited and which ones belong to the thermal waves. The qualitative answer is: parametric spin waves (PSW) are those pairs of the SW whose phase is correlated with the pump phase; thermal waves are the SW whose n_k-values do not differ much from the equilibrium level. A number n_q^{P} of the PW is mathematically determined by

$$n_q^{\mathrm{P}} = n_q - n_q^{\mathrm{T}}, \tag{5.13}$$

where

$$n_q^{\mathrm{T}} = \Sigma_{\mathrm{d}}(q)\left[(\omega - \tilde{\omega}_q)^2 + \Gamma_q^2\right]^{-1} \tag{5.14}$$

has been defined similar to eq. (5.11). The distinction is the quantity $\Sigma_{\mathrm{d}}(q)$ in (5.14) is not to be calculated in the thermodynamic equilibrium (5.11) but in the real spectrum n_q^{T}, $n_{\mathrm{q}}^{\mathrm{P}}$, σ_q, which was calculated in the presence of the pump. In this definition n_q^{T} is everywhere a fluent function of k but it moves to the equilibrium spectrum (5.11) asymptotically beyond the resonance surface for $|\omega_k - \omega_{\mathrm{p}}/2| \gg \Gamma$. As for the quantities $n_k^{\mathrm{P}} = \int n_q^{\mathrm{P}}{}' \mathrm{d}\omega$ and σ_k, they rapidly decrease beyond the resonance surface. It may be shown that n_k^{P}, $\sigma_k \sim (\omega_k - \omega_{\mathrm{p}}/2)^{-2}$. It must be noted that in above arguments we have nowhere used the actual form of the interaction of waves. The definitions of parametric and thermal waves, eqs. (5.13) and (5.14) are, therefore, valid for each interaction, particularly in cases where three-wave processes and interactions with phonons, etc., are essential.

Now let us use the developed method for studying the packet of the PSW at relatively small amplitudes of the pump, where the scattering of the thermal SW by each other forms the main contribution to the MO. It is necessary to substitute $n_q = n_q^0 + n_q^{\mathrm{P}}$, σ_q into the expressions for Σ_{c}, Π_{c}, Σ_{d} and Π_{d} and to study the obtained expressions. The result obtained in the zero approximation in n^{P}, σ is known from eq. (5.12). Formula $\omega_k^0 = \omega_k + 2\int T_{kk'} n_{q'}^0 \,\mathrm{d}q'$ describes the frequency dependence of the PW on the medium temperature. Let us assume that this dependence has been already enclosed into definition of ω_k, so that

$$\bar{\omega}_k = \omega_k + 2\int T_{kk'} n_{q'}^{\mathrm{P}} \,\mathrm{d}q'. \tag{5.15}$$

Expressions (5.5) and (5.6) for Σ_{c}, Π_{c}, Σ_{d} and Π_{d} are quadratic in coefficients of the interaction Hamiltonian $T_{k1,23}$. At the first stage of the investigation this allows one to calculate them in the zero approximation in the amplitude of the parametric turbulence. The applicability framework of this approximation for the amplitude n^{P}, σ, and the effects appearing for large amplitudes will be considered below.

Substituting eq. (5.12) and $\Pi_{\rm d}=0$ into the Dyson equations (5.3) and (5.4), γ_k, $\Sigma_{\rm d}^0$, we obtain the equations for the self-consistent pump P_Ω on the resonant surface for the quantities n_Ω and σ_Ω:

$$P_\Omega = hV_\Omega + \mathrm{i}\pi n^0 \int S_{\Omega\Omega'} \frac{k_{\Omega'}^2 \gamma_{\Omega'} P_{\Omega'}\, \mathrm{d}\Omega'}{v_{\Omega'}\left(\gamma_{\Omega'}^2 - |P_{\Omega'}|^2\right)^{1/2}},$$
$$\gamma_k n_k + \mathrm{Im}(P_k^* \sigma_k) = \gamma_k n_k^0, \quad (\gamma_k + \mathrm{i}\tilde{\epsilon}_k)\sigma_k + P_k n_k = 0. \tag{5.16}$$

These equations have first been presented intuitively and studied in detail by Zakharov and L'vov (1971). Equations (5.16) differ from the main equations of the S-theory, eqs. (2.8) only with regard to the nonuniform term $\gamma_n n_k^0$, which describes the influence of thermal fluctuations on the system of the PW. It follows that the integral number of the PW, $N_{\rm p}$, slightly differs from the value $N_{\rm p}^{\rm S}$ predicted by the S-theory:

$$\frac{N_{\rm p} - N_{\rm p}^{\rm S}}{N_{\rm p}^{\rm S}} \simeq \xi^2, \quad \xi^2 = (2\pi)^2 k^2 S n^0 / v. \tag{5.17}$$

here $\xi \ll 1$ characterizes the influence of the thermal fluctuations.

But the principal distinction with the S-theory is in the finite width of a packet of the PW in ω and k. These widths $\Delta k = \nu/v$ and $\Delta\omega = \nu^2/2\gamma$ are stimulated by the quantity ν. In the simplest case of spherical symmetry for $h - h_{\rm cr} > \xi^2 h_{\rm cr}$,

$$\frac{\nu}{\gamma} = \xi h_{\rm cr}\left(h^2 - h_{\rm cr}^2\right)^{-1/2}. \tag{5.18}$$

In concluding this section we will discuss how the equations of the S-theory may be obtained from the equations of the S,T^2-theory. We have already answered this question to a significant degree. In fact, to derive equations (5.16), we assume that $\Sigma_{\rm d} = \Sigma_{\rm d}^0$, $\Pi_{\rm d} = 0$, i.e., we neglected the influence of the PSW in diagrams of second order in the vertex T. If the thermal noise is neglected ($\Sigma_{\rm d}^0 \to 0$) but the first-order influence of PW in the interaction is taken into account, then the equations (5.16) will turn into the known equations of the S-theory, eqs. (2.8). It should be recalled that the stability of all solutions of eqs. (2.8) must be tested to find the real stationary solutions of the S-teory. There are no such a difficulties in the equations of the S,T^2-theory, because the introduction of the thermal noise makes the problem fully determined and leads to a regular solution.

6. *Parametric excitation of waves in media with random inhomogeneities*

Except some rare cases, real media possess inhomogeneities of different kinds destroying their ideal translational symmetry. The nature of the inhomogeneities in ferromagnetic crystals and their influence on the ferromagnetic resonance and spin waves were studied in a great number of works: see, e.g., Spark's monography (Sparks 1964), and Schlömann's paper (Schlömann 1969). More detailed information can be found in the book by Gurevich (1973). Many experimental works (Schlömann 1959b, Sühl 1959, Sparks 1964, Schlömann 1969, Melkov and Grankin 1975, Smirnov and Petrov 1981) are devoted to the question of the influence of inhomogeneities on the parallel pump. This section will discuss the theory of the phenonenon and make a comparison with experiment.

6.1. General equations in the S,g²-approximation

We start with the diagram equations (5.4) and (5.9), in which we will take into acount:

(1) interaction of the PSW and TSW which leads to damping of the PSW and to a dependence of the spectrum on the temperature ω_k,

(2) self-consistent S-interaction between the PSW pairs, resulting in renormalization of the pump,

(3) elastic scattering of waves from inhomogeneities in the Born-approximation proportional to $g^2_{kk'}$. In such a S,g^2-approximation we obtain the following equations:

$$\begin{aligned} &\Gamma_k = \gamma_k - \operatorname{Im}\,\Sigma^{\mathrm{el}}_{\mathrm{c}}(q), \qquad && \Pi_{\mathrm{c}}(q) = P_k + \Pi^{\mathrm{el}}_{\mathrm{c}}(q),\\ &\Sigma_{\mathrm{d}}(q) = \Sigma^{\mathrm{el}}_{\mathrm{d}}(q), && \Pi_{\mathrm{d}}(q) = \Pi^{\mathrm{el}}_{\mathrm{d}}(q), \end{aligned} \tag{6.1}$$

where γ_k and P_k are damping and pumping of the PSW in the S-theory, and where the MOs Σ, Π are given by formula (5.8). It is easy to see that, if n_q and $\sigma_q \sim \delta(\omega - \omega_{\mathrm{p}}/2)$, Σ_{d}, and Π_{d} are also $\sim \delta(\omega - \omega_{\mathrm{p}}/2)$. This means that elastic scattering of the PW on inhomogeneities does not disturb the uniform character of the parametric turbulence of waves inherent in the S-theory. With the one-frequency approximation the Wyld equations (5.4) and (5.8) may be integrated over the modulus k. As a result we have:

$$\begin{aligned} &n_\Omega = \frac{\pi k_\Omega^2 \Gamma_\Omega}{v_\Omega \nu_\Omega^3}\left[\Gamma_\Omega \Sigma_{\mathrm{d}}(\Omega) + \operatorname{Im}(\Pi^*_{\mathrm{c}}\Pi_{\mathrm{d}})\right], \quad \Gamma_\Omega \sigma_\Omega + \mathrm{i}\Pi_{\mathrm{c}}(\Omega) n_\Omega = 0,\\ &\nu_\Omega^2 = \Gamma_\Omega^2 - |\Pi_{\mathrm{c}}|^2, \quad \Pi_{\mathrm{c}}(\Omega) = P_\Omega + \pi c \int g_{\Omega\Omega'} g_{\bar\Omega\bar\Omega'} \frac{\Pi_{\Omega'} k_{\Omega'}^2}{\nu_{\Omega'} v_{\Omega'}}\,\mathrm{d}\Omega',\\ &P_\Omega + hV_\Omega + \int S_{\Omega\Omega'}\sigma_{\Omega'}\,\mathrm{d}\Omega', \quad \Gamma_\Omega = \gamma_\Omega + \pi c \int |g_{\Omega\Omega'}|^2 \frac{\Gamma_{\Omega'} k_{\Omega'}^2}{v_{\Omega'}\nu_{\Omega'}}\,\mathrm{d}\Omega',\\ &\Sigma_{\mathrm{d}}(\Omega) = c\int |g_{\Omega\Omega'}|^2 n_{\Omega'}\,\mathrm{d}\Omega', \quad \Pi_{\mathrm{d}}(\Omega) = c\int g_{\Omega\Omega'} g_{\bar\Omega\bar\Omega'} \sigma_{\Omega'}\,\mathrm{d}\Omega'. \end{aligned} \tag{6.2}$$

These formulae represent the S,g^2-theory, a closed system of integral equations which enables one to describe the system of interacting PW in a medium with random inhomogeneities. They were obtained by Zakharov and L'vov (1972) for the first time.

Elastic scattering of waves may be characterized by the damping decrement γ^{el}. From the conventional perturbation theory it follows:

$$\gamma_k^{\mathrm{el}} = \pi c \int |g_{kk'}|^2 \delta(\omega_k - \omega_{k'})\, \mathrm{d}k' \simeq \pi c \frac{\Delta^2 k^2 g^2}{v}, \tag{6.3}$$

where $g \approx g_{kk'}$, and $\Delta^2 \leqslant 4\pi$ is a characteristic scattering space angle (Δ is the scattering angle). Thus, the S,g^2-theory includes three dimensionless parameters: the degree of the homogeneity of the medium $\gamma^{\mathrm{el}}/\gamma$, the scattering angle Δ, and the supercriticality h/h_{cr}.

6.2. *Distribution function of parametric waves*

We note first of all that the integral relation follows from eq. (6.2),

$$\int \gamma_\Omega n_\Omega\, \mathrm{d}\Omega + \mathrm{Im}\left(\int h V_\Omega^* \sigma_\Omega\, \mathrm{d}\Omega \right) = 0, \tag{6.4}$$

which describes the energy balance in the system of PW: a summated dissipation of energy due to the self-mechanisms of the relaxation is equal to integral energy flow into all the pairs. Elastic scattering does not enter into this relation because it occurs with frequency conservation and, hence, does not expel energy out of a system of the PW. Such a scattering results in two effects: isotropization of the distribution functions and destruction of the phase correlations in the pairs, which leads to decrease of a ratio $|\sigma_\Omega|/n_\Omega$. We will briefly discuss them in a more interesting limiting case where $\gamma^{\mathrm{el}} \gg \gamma$. Then it follows from eq. (6.2):

$$n_\Omega = N/4\pi, \quad \sigma_\Omega = -\mathrm{i}(\Pi_{\mathrm{c}}(\Omega)/\Gamma_\Omega) n_\Omega, \tag{6.5}$$

$$\Gamma_\Omega = 4\pi^2 c \left\langle |g'_{\Omega\Omega}|^2 \frac{k_\Omega^{2\prime}}{v'_\Omega} \right\rangle_{\Omega'}, \quad \langle f_\Omega \rangle_\Omega \equiv \frac{1}{4\pi} \int f_\Omega\, \mathrm{d}\Omega, \tag{6.6}$$

$$\begin{gathered} \Pi_{\mathrm{c}}(\Omega) = P_\Omega + \langle K_{\Omega\Omega'} \Pi_{\mathrm{c}}(\Omega') \rangle_{\Omega'}, \\ K_{\Omega\Omega'} \equiv \frac{g_{\Omega\Omega'} g_{\overline{\Omega}\overline{\Omega}'} k_{\Omega'}^2}{v_{\Omega'} \langle |g_{\Omega'\Omega''}|^2 k_{\Omega''}^2 / v_{\Omega''} \rangle}. \end{gathered} \tag{6.7}$$

With the exception of degenerate cases, in which we are not interested, it follows from eq. (6.7) that $\Pi_{\mathrm{c}} \approx \Pi \approx P$. With allowance for eqs. (6.5) and (6.6), this yields

$$|P|^2 \approx \gamma^{\mathrm{el}} \gamma, \quad |\sigma_\Omega| \simeq n_\Omega \sqrt{\gamma/\gamma^{\mathrm{el}}}, \tag{6.8}$$

i.e., the destruction of phase correlations and, accordingly, the increase of the excitation threshold of waves. Instead of the estimate $\langle h_{cr}V \rangle = \langle \gamma \rangle$ which would have held for $|\sigma| = n$, it follows from eq. (6.4) that

$$h_{cr}^2 \langle |V_\Omega|^2 \rangle \simeq \langle \gamma \rangle \gamma^{el}. \tag{6.9}$$

This formula has been qualitatively confirmed in a direct experiment by Smirnov and Petrov (1981), who independently measured the values h_{cr}, γ^{el} and $\langle \gamma_\Omega \rangle$ in the antiferromagnet $CsMnF_3$. Quantitative analysis of the behavior of the PW requires knowledge of the function which is determined by the destroying character of the homogeneity of the medium. After that there are no fundamental difficulties to carry out this analysis.

6.3. Behavior of parametrically excited waves beyond the threshold

In the study of behavior of parametrically excited waves the most interesting case is that of large scattering intensities, where a stronger influence of two-magnon scattering may be expected. In the simple model, with $S_{\theta\theta'} \sim V_\theta V_{\theta'}$, $g_k = g$, and $\gamma_k = \gamma$, it follows from eqs. (6.2):

$$SN = \tfrac{15}{8}\gamma^{el}\sqrt{\left(\frac{h}{h_{cr}}\right)^2 - 1}\,, \quad h_{cr}V = \sqrt{\tfrac{15}{8}\gamma\gamma^{el}}\,. \tag{6.10}$$

In the case of scattering from point defects, when g'_{kk} is proportional to (kk'), one can obtain for the fluctuation of the exchange constant an expression close to eq. (6.10)

$$SN = \tfrac{9}{8}\gamma^{el}\sqrt{(h/h_{cr})^2 - 1}\,. \tag{6.11}$$

For intense small-angle scattering, when $\gamma^{el}\Delta^2 > \gamma$, one can obtain (L'vov 1982):

$$SN = \tfrac{45}{16}\gamma^{el}\Delta^2\sqrt{(h/h_{cr})^2 - 1}\,. \tag{6.12}$$

In all the described cases the excitation level of the PW for strong scattering from inhomogeneities proves to be a factor of $\gamma^{el}\Delta^2/\gamma$ greater (at the same supercriticality) than in an uniform medium. This is caused by artial destruction of the phase correlations, which leads to weakening of the phase restriction mechanism. The specific form of the dependence N on supercriticality in formulae (6.10), (6.11) and (6.12) is not universal, but originates from the specific form of the function $S_{\theta\theta'}$. In other cases (see, e.g., L'vov and Shirokov 1974) the dependence of N on h is more complicated and it reproduces these dependences only qualitatively.

In all the described cases one can obtain from eq. (6.2) for the generalized admittance the following expression:

$$\chi = \frac{2V^2}{|S|}\frac{(h^2 - h_{cr}^2) + \mathrm{i}h_{cr}\sqrt{h^2 - h_{cr}^2}}{h^2}\,. \tag{6.13}$$

This result coincides with the known expressions (2.23) and (2.24) for χ, which were obtained within the S-theory for an uniform medium. It means that the dependences of the nonlinear susceptibilities χ' and χ'' on the value (h/h_{cr}) are not significantly changed by the scattering of waves from inhomogeneities. Of course, the value of the threshold field in a nonuniform medium itself is greater than that in an uniform medium. The literal coincidence of the formulae for χ should not be overestimated. In more complicated situations (L'vov and shirokov 1974) the dependence $\chi(h/h_{cr})$ resembles eq. (6.13) only qualitatively.

7. Spectral density of parametrically excited waves: spectral solitons

As has already been pointed out, the elastic scattering does not change the number and frquency of the PSW but only destroys their coupling to the pump, the process leading to the dephasing of wave pairs, and to the isotropization of the PSW distribution. If the frequency of the elastic distribution is greater than all other relaxation frequencies, the distribution of the PSW is isotropic, and the influence of the parametric pumping and of the PSW scattering from each other can be taken into account as small perturbations. A simple equation appears in this approximation for the distribution function of the PSW in frequencies. One can solve it analytically and isolate the only stable solution from the stationary solutions. We here discuss the results of the S,T^2-theory for this case, looking complicated at first sight. In MO we will take into consideration the contributions of elastic scattering, Σ^{el}, Π^{el}, the contribution of the interaction of the PSW with thermal waves, Σ^0 eq. (5.12), and the contribution of the interaction of the PSW between themselves, eqs. (5.5) and (5.6). In the limit $\gamma^{el} \gg \gamma$ the equations (5.3) and (5.4) have the isotropic solution $n_{k\omega} = n_{k\omega}$. It allows one to integrate them in the general form not only in k but also in the solid angle Ω. Ultimately, after some modifications, we obtain the equation for the spectral density of PSW $n_\omega = \int n_{k\omega}\,\mathrm{d}k$,

$$n_\epsilon \equiv n_{\omega-\omega_p/2}$$

$$= \frac{\Gamma_1}{\epsilon^2+\eta^2}\left\{\frac{4\pi^2k^2n^0}{v} + \frac{T^2}{kv}\int n_{\epsilon_1}n_{\epsilon_2}n_{\epsilon_3}\delta(\epsilon+\epsilon_1-\epsilon_2-\epsilon_3)\mathrm{d}\epsilon_1\mathrm{d}\epsilon_2\mathrm{d}\epsilon_3\right\}. \tag{7.1}$$

Here T^2 is the mean value of the square of T_{1234}. Γ_1 has the order of magnitude $\simeq(\gamma^{el})^2/\gamma$. The second term in eq. (7.1) may be neglected at small supercritcality, when the distribution n_ϵ is a Lorentzian with width $\eta = \eta_T$, which may be determined by integrating eq. (7.1) in ϵ. Taking eq. (6.12) into account we have

$$\eta_T = \frac{\Gamma_1 \gamma}{kv} \frac{4\pi^2 k^3 n^0}{N} \simeq \xi \gamma^{\text{el}} (P-1)^{1/2}, \quad \xi = 4\pi^2 k^2 T n_0 / v. \tag{7.2}$$

In the opposite case of large supercriticality, the thermal term in eq. (7.1) may be neglected. L'vov and Cherepanov (1978) showed that this equation has a one-parametric set of solutions. However, only one of them is stable. It is "spectral soliton":

$$T n_\omega = \frac{kv}{2\Gamma_1} ch^{-1} \frac{\pi \epsilon}{2\eta}, \quad \eta \simeq \frac{(\gamma^{\text{el}})^2}{\gamma} \sqrt{\frac{\gamma}{kv}(P-1)}\,. \tag{7.3}$$

In almost uniform crystals, when $\gamma^{\text{el}} \ll \gamma$, the study of spectral solitons becomes greatly complicated because of the anisotropy of $n_{k\omega}$. Therefore Krutsenko et al. (1978) limited themselves to an axial-symmetric situation which is realized in isotropic and cubic ferromagnets. In the region of supercriticalities $P_1 < P < P_2$, the broadening in ω is determined by the self-noise, and the line shape of n_ω is close to eq. (7.3) with $\eta = \eta_{\text{int}}$ the effective width,

$$\begin{aligned} &\eta_{\text{int}} = \gamma \left(\frac{\gamma^{\text{el}}}{\gamma} \right)^{1/4} \left(\frac{P-1}{P_{\text{s}}} \right)^{1/2}, \quad P_1 - 1 = \xi P_{\text{s}} \left(\frac{\gamma^{\text{el}}}{\gamma} \right)^{3/4}, \\ &P_2 - 1 \simeq P_3 \left(\frac{\gamma^{\text{el}}}{\gamma} \right)^{3/4}, \quad P_{\text{s}} \simeq \frac{kv}{\gamma}. \end{aligned} \tag{7.4}$$

Broadening in angles is determined by two-magnon scattering. Both thermal fluctuations and two-magnon scattering may be neglected at greater supercriticalities $P > P_2$. Then

$$\Delta\theta_k \approx \left(\frac{\eta_{\text{int}}}{\gamma} \right)^{1/2}, \quad \Delta\omega \approx \eta_{\text{int}} \approx \gamma \left[\frac{P-1}{P_{\text{s}}} \right]^{2/3}. \tag{7.5}$$

The line shape of n_ω remains similar to that in eq. (7.3). In the applicability framework of the theory ($P \approx P_{\text{s}}$) $\Delta\theta_{\text{k}} \approx \pi$. However, such supercriticalities are of interest only from an academic point of view since autooscillations arise first, leading to sharp broadening of the spectrum in $\Delta\theta_k$ and $\Delta\omega_k$.

In concluding we point out that the spectral solitons n_ω at the parametric excitation of the SW in YIG have been experimentally investigated in detail by Krutsenko et al. (1978). Good qualitative and quantative agreement with the above described theory has been observed. In particular, the line shape of n_ω, which is observed experimentally, is actually well described by the function of $ch^{-1}(\epsilon/\Delta\omega)$ and does not coincide with the convential shape for spectral lines of Lorentzian or Gaussian functions for any choise of their parameters. To prove this point, the data concerning h_ω are given in the fig. 14 in natural

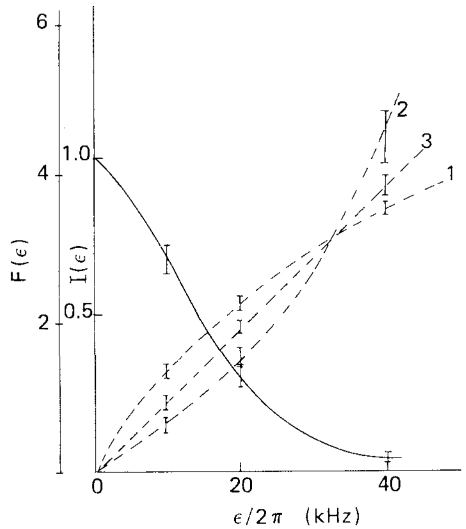

Fig. 14. Solid line: spectral power $I(\epsilon)$ of the PSW radiation (YIG, sphere, $\Delta H_k = 0.15$ Oe, $P = 6$ dB). Dashed curve: build-up of $I(\epsilon)$ in coordinates (1): $F(\epsilon) = 2[\ln(I^{-1}(\epsilon))]^{1/2}$, (2): $F(\epsilon) = [I^{-1}(\epsilon) - 1]^{1/2}$, (3): $F(\epsilon) = \ln[I^{-1}(\epsilon) + (I^{-2}(\epsilon) - 1)^{1/2}]$.

variables in "straightening" coordinates of three types, chosen in such a way that the Gaussian in coordinates 1, the Lorentzian in coordinates 2, and function (7.3) in coordinates 3, will be straight lines. The experimental points lie on a straight line only in coordinates 3. This and other facts are reason to believe that the development theory describes the reality well.

8. Structure of the distribution function in k-space: k-solitons

It has been already pointed out that the T^2-scattering of the PW by each other leads to the finite width of the distribution function of the PW, $n_{k\omega}$, not only in ω but also over the modulus k. There $\Delta\omega \sim \nu^2/2\gamma$, which is much less than $\Delta\omega_k \sim \nu$, when the supercriticality is not large. It gives reason to think that the study of the distribution function structure $n_k = \int n_{k\omega}\, \mathrm{d}\omega$ may be restricted to the "one-frequency turbulence" approximation, i.e. we may assume that

$$n_{k\omega} = n_k \delta\left(\omega - \frac{\omega_\mathrm{p}}{2}\right), \quad \sigma_{k\omega} = \sigma_k \delta\left(\omega - \frac{\omega_\mathrm{p}}{2}\right).$$

Such an approximation allows to analyze equations of the S,T^2-theory effectively in practically all interesting cases, as will be shown in this section. In fact, these equations are already implicit in formula (6.2). Only the expressions

(5.5) and (5.6) for $\Sigma^{\rm el}$, $\Pi^{\rm el}$ have to be added into $\Sigma^{\rm int}$, $\Pi^{\rm int}$. This yields for the one-frequency-approximation:

$$\Sigma_{\rm d}^{\rm int}(\Omega)=2\int\left[\,|T_{\Omega1,23}|^2 n_{\Omega_1}n_{\Omega_2}n_{\Omega_3}+2\Gamma_{\Omega123}T^*_{\Omega\bar{2},\bar{1}3}\sigma^*_{\Omega_1}\sigma_{\Omega_2}n_{\Omega_3}\right]$$
$$\times\delta(k+k_1-k_2-k_3)\,\mathrm{d}\Omega_1\,\mathrm{d}\Omega_2\,\mathrm{d}\Omega_3,$$
$$\Pi_{\rm d}^{\rm int}(\Omega)=2\int\left[T^2_{\Omega1,23}\sigma^*_{\Omega1}\sigma_{\Omega_2}\sigma_{\Omega_3}+2T_{\Omega123}T_{\Omega\bar{2}\bar{1}3}n_{\Omega_1}n_{\Omega_2}\sigma_{\Omega_3}\right]$$
$$\times\delta(k+k_1-k_2-k_3)\,\mathrm{d}\Omega_1\,\mathrm{d}\Omega_2\,\mathrm{d}\Omega_3. \tag{8.1}$$

$\Sigma_{\rm c}^{\rm int}$, $\Pi_{\rm c}^{\rm int}$, which are the renormalizations of the pump and damping (on the PW scattering), are small and will not be taken into consideration.

It is very simple to analyze solutions of one-frequency equations of the S,T^2-theory in a rough form. First of all, assuming that $\Sigma_{\rm d}^{\rm int}=0$, $\Pi_{\rm d}^{\rm int}=0$, we can verify that $\nu_\Omega=0$ for those directions where $n_\Omega\neq 0$. This means that the distribution of the PW in the k-space is singular: $n_k\neq 0$ only on the resonance surface which satisfies the condition of external stability of the S-theory. The distribution of n_Ω on this surface and the integral quantity N are defined by equation (6.11), which reduces to equation (2.8) of the S-theory in the considered approximation. Next, integrating the first of equations (6.11), one gets an estimate for the quantity ν/γ which characterizes the relative part of damping not compensated by the pump:

$$(\nu/\gamma)^3\approx(TN)^2/(\gamma kv)\approx(\gamma/kv)(P-1). \tag{8.2}$$

From the one-frequency equations (6.2) it follows that the distribution $n(k)$ in the modulus k close to the resonant surface is the squared Lorentzian with width $\Delta\omega_k\approx\nu$ in ω_k and a width of the order of ν/γ in θ_k (under conditions of axial symmetry). It may be seen from eqs. (6.2) that the relative difference of their coefficients from those of the S-theory for the parameter $(\nu/\gamma)^2$ is small and, hence, for an integral quantity the difference between the results of the S-theory and the accurate results of the S,T^2-theory is also small for the same parameter. Particularly, from eq. (6.2) it follows that $1-|\sigma|/n\approx\nu^2/2\gamma$, i.e., at $\nu\ll\gamma$ the phase correlations in pairs are retaining almost completely. An analytis of the diagrams which were neglected when solving eq. (6.2) and which are proportional to T^3, T^4, etc., shows that they are arranged in a series with the parameter $\lambda=(\Gamma/\nu)(TN/kv)$ and, consequently, for $\nu\lesssim\gamma$, $\lambda\lesssim\sqrt{\gamma/kv}\ll 1$. This means that the equations of the S,T^2-theory are correct and that the integral quantities N, χ^1 and χ'' are well described by the corresponding formulae of the S-theory right up to the supercritical $h\approx h_{\rm s}$ which is determined by the condition

$$h_{\rm s}=h_1\sqrt{kv/\gamma}\,. \tag{8.3}$$

As a specific example of a solution of the one-frequency equations of the S,T^2-theory, the parallel pump of the SW in a cubic ferromagnet for $M\parallel\langle 100\rangle$

and $\langle 111 \rangle$ was considered by L'vov and Cherepanov (1979). Since the cubic anisotropy is very small, the PSW distribution on the resonant surface comprises a set of long stripes with $\Delta\phi \gg \Delta\theta$ which are stretched along the equator. Analyzing the distribution in θ one can therefore consider the distribution in ϕ to be isotropic. From this assumption it follows:

$$\Delta\theta \approx \frac{\nu_0}{\gamma} \approx \left[\frac{\gamma}{kv}(P-1)\right]^{1/3}, \tag{8.4}$$

and the distribution in ϕ for $M \,|\, \langle 100 \rangle$ has the form of a smeared cross with

$$\Delta\phi = \sqrt{2}\,\nu_0/\nu_1, \quad \nu_\phi^2 = \nu_0^2 + \nu_1^2\left[\sum \sin^2(\phi - \phi_j) - 2\right]. \tag{8.5}$$

However, for $M \,\|\, \langle 111 \rangle$, when the distribution n_ϕ has a shape of a smeared star of six vertices, the estimate for ν_0, and consequently for all the quantities connected with it, is quite different (L'vov and Cherepanov 1979):

$$\frac{\nu_0}{\gamma} \approx \left[\frac{\mu^2}{\gamma kv}(P-1)\right]^{1/5}, \quad \mu = \partial^2\gamma_\phi/\partial\phi^2. \tag{8.6}$$

In some cases (e.g. for real $T_{12,34}$ and for spherical symmetry of the problem) the contribution into the PW scattering proportional to T^2 is almost completely cancelled. Then the T^3 scattering should be taken into account and

$$\frac{\nu}{\gamma} \approx \sqrt{\frac{\gamma}{kv}}\,(P-1)^{3/8}. \tag{8.7}$$

The latter two examples show that the general estimate (8.2) for ν, $\Delta\theta$, $\Delta\omega$ and $\Delta\omega_k$ may prove to be incorrect in some specific situations because of unexpected cancellations. Therefore, in spite of the principal understanding of the basic statements of the S,T^2-theory, the investigation of the parametric excitation of waves in other media may still lead to the discovery of new effects.

Conclusion

The results summarized in this review show that substantial progress has been achieved now in the understanding of the phenomena which occur in ferromagnets in the case of parametric excitation of spin waves. Physical ideas developed here are also valid for a number of other similar problems (Zautkin et al. 1981, Lavrinenko et al. 1981, Thomson and Quate 1970, Govorkov and Tulin 1976, Rosenbluth 1972, Herman et al. 1970, L'vov and Rubenchik 1973).

An interesting and important range of problems is associated with the study of the parametric excitation of waves in media with large scale (in comparison with the wavelength) inhomogeneities. In this case, we have an additional "linear" limiting mechanism because of the loss of energy to the region with higher threshold. This mechanism was investigated in detail in the case of waves in plasmas (Rosenbluth 1972). It is also important to note the interesting phemonenon of the parametric echo which is observed during transverse pumping in weakly inhomogeneous ferromagnets. A qualitative explanation of this phemonenon is given by Herman et al. (1970). It would be interesting to elucidate the role of wave interaction using short intense pump pulses. Studies in this area may be of practical importance in the sense that they may lead to the development of amplifiers, amplifying delay lines, and other devices for pulse shaping.

The importance of the theory outlined in the present review extends beyond the framework of ferromagnetism. It is, essentially, the general theory of parametric excitation of waves in nonlinear media with a nondecay spectrum. In particular, it is valid for certain problems in the physics of plasmas (L'vov and Rubenchik 1973), and can be used to investigate the parametric excitation of waves on the surface of a liquid, the surface spin waves in magnetic fields, sound in dielectrics, etc. It is therefore hoped that this theory reflects many of the essential features of the turbulence established in an unstable continuous medium when the supercriticality is not too large.

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