Nonlinear Dynamics and Kinetics of Magnons

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A review on nonlinear spin waves (magnons) is presented. The corresponding physical models in magnetic dielectrics are analyzed. The parametric dynamics and kinetics of magnons are described in detail.

1. NONLINEAR MAGNONS IN MAGNETIC DIELECTRICS

1.1. Introduction

During the recent two decades, there has been a steady growth of interest in the highly nonequilibrium systems. It concerns above all studies on the behaviour of substances at high power levels of external action: of dielectrics in the powerful laser wave field, magnets in strong microwave fields, plasma heated to thermonuclear temperatures, etc. In fact, a new physical discipline has emerged - *the physics of nonlinear waves*. Its purpose is to study, possibly from a single viewpoint, the phenomena and processes arising upon the excitation, propagation and interaction of limited-amplitude waves in various media. These studies have revealed that some phenomena, such as formation of the "tenth wave" in a stormy sea, self-focusing of light and Langmuir wave collapse in plasma have a common physical reason.

In physics of nonlinear wave processes, dealt with by the Gorky schools, the section on spin waves (magnons) is one of the most advanced ones. This is partly due to the fact that experiments on spin waves in magnetic dielectrics are much simpler than the similar experiments on sound in crystals, on Langmuir and other types of waves in plasma, in nonlinear optics, etc. They may often be carried out at room temperature, in a customary frequency range and on top-quality monocrystals. Spin waves are easily excited by the microwave magnetic field to an essentially nonlinear level, when their behaviour is completely determined by their interaction with each other. There is a great amount of experimental material, as well as well-developed, highly advanced theoretical models. They may certainly prove useful in elaborating other sections of nonlinear physics. Regretfully, these results are almost unknown. The aim of this paper is to draw the attention of researchers engaged in the theoretical and experimental studies on plasma physics, nonlinear optics, hydrodynamics, nonlinear acoustics, etc. to a related field: nonlinear spin waves. We suppose that reduction of a barrier separating these sections of nonlinear physics from each other will be mutually fruitful. For that purpose the author of the present paper has written a book "Nonlinear

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Spin Waves" /1/, which will be published in a considerably revised and updated form by Springer in 1990, under the title "Nonlinear Dynamics and Kinetics of Magnons".

1.2. Magnetically-Ordered Dielectrics

Today, a lot of magnetically-structures materials are known (dielectrics, semiconductors and metals, both crystalline and amorphous). Their structure includes paramagnetic atoms (ions) with uncompensated electron spin magnetic moments $\vec{\mu} = \mu_B \vec{S}$ (μ_B is the Bohr magneton, $1/2 < S < 7/2$ is the atom spin). At low temperatures, these moments are oriented with respect to each other in a definite fashion. In *ferromagnets*, which present the simplest case, magnetic moments of all atoms are parallel. This results in the macroscopic magnetic moment, equal to their sum. The physical reason which causes magnetic ordering is the *exchange interaction*. It has an electrostatic nature and is associated with the Pauli principle, forbidding the existence of two electrons in one quantum-mechanical state. The Hamiltonian of exchange interaction

$$H_{ex} = -J\vec{S}_1\vec{S}_2 \tag{1}$$

is called the *Geisenberg* Hamiltonian, J is the *exchange integral*. In ferromagnets, $J > 0$. In order of magnitude, $S(S+1)J/3 \cong T_C$ is the *Curie temperature*. At $T > T_C$, magnetic ordering disappears. For different substances T_C varies in the range 1-1000K.

1.3. Spin Waves and Equations of Motion

In the excited state, the *magnetic moment density* $\vec{M}(\vec{r},t)$ depends on coordinate $\vec{r}$ and time t. This dependence may be expanded into the series in plane waves, which at $T \ll T_C$ are weakly interacting. These are called *spin waves*, or *magnons*. Similarly, sound waves are called phonons, when one wants to emphasize their quantum-mechanical properties.

The exchange interaction determines the magnitude $M(\vec{r},t)$ with high accuracy at $T \ll T_C$. Thus, spin waves represent the magnetic moment precession waves. Magnitude $M(r,t)$ obeys the phenomenological *Bloch equation* /1/,

$$\partial M/\partial t = g\left[\delta W/\delta M \times M\right] , \tag{2}$$

which defines this precession, conserving both the full energy of a magnet W and value $|\vec{M}|$. Here W is the functional of $M(r,t)$; $g \cong \mu_B/\hbar$ is the magnetomechanical electron factor.

1.4. Classical Hamiltonian Formalism and Spin Waves in Magnets

We see that the Bloch equation (2) has a specific form which is very distinctive, e.g. from the Maxwell equations in a nonlinear dielectric. The latter equations radically differ from the Euler equations for compressible fluid. As a matter of fact, spin, electromagnetic and sound waves are, first of all, waves, i.e. medium oscillations transferred in a relay fashion from one point to another. If we are interested only in some propagation characteristics of the small (but limited) amplitude waves, such as diffraction or self-focusing, then it is absolutely unnecessary to know what is it that oscillates, whether it is the magnetic moment, electric field, or density. All information on the type of a medium, which is necessary and sufficient for investigating the propagation of noninteracting waves in this medium, is given by their dispersion law $\omega(\vec{k})$. Likewise, the existence of other universal functions which describe some of the properties of the medium may be assumed, the knowledge of which is sufficient to describe wave interactions. Such functions appear on passing to the Hamiltonian method of describing motion. This method is applicable to a wide class of weakly interacting and weakly dissipative wave systems. It reveals their common properties.

1.4.1. Transition to the Canonical Description of Magnons. Let us write the Bloch equation (2) in the Hamiltonian form /1/:

$$i\partial a/\partial t = \delta H/\delta a^* \ . \quad (3)$$

Here H is the *Hamiltonian function* representing in this case the energy W defined through the *canonical variables* $a(\vec{r},t)$ and $a^*(\vec{r},t)$. These are connected with magnetization by the ansatz found in 1969 by ZAKHAROV, L'VOV and STAROBINETS /1/:

$$M_x + iM_y = a\left[2gM_0(1-gaa^*/2M_0)\right]^{1/2} ,$$
$$M_x - iM_y = a^*\left[2gM_0(1-gaa^*/2M_0)\right]^{1/2}, \quad M_z = M_0 - gaa^* \ . \quad (4)$$

This is a classical analogy of the Godstein-Primakov ansatz, defining the spin operators $\hat{S}$ through the Bose operators $\hat{a}$ and $\hat{a}^+$. The canonical variables a and a* represent the classical limit $\hat{a}$ and $\hat{a}^+$.

1.4.2. The Hamiltonian Function at Small Nonlinearity. At a small nonlinearity, when $gaa^* << M$, the Hamiltonian function H may be expanded into the series in $a(\vec{r},t)$, $a^*(\vec{r},t)$. In the k-representation the expansion is as follows:

$$H = H_0 + H_1 + H_{int} , \quad H_{int} = H_3 + H_4 + \dots ,$$
$$H_0 = \text{const} , \quad H_2 = \sum_{\vec{k}} \omega_{\vec{k}} a_{\vec{k}} a_{\vec{k}}^* \ . \quad (5)$$

Constant H , which is independent of a, a*, does not arise in equation (3) and may be omitted. The first-order terms in a, a*, H_1 are absent, because we have assumed that in the absence of waves, when a = 0, a medium is in equilibrium. Among the second-order terms in a, a*, the expansion contains no terms of $a_k a_{-k}$ and $a_k^* a_{-k}^*$. If they do appear, they may be eliminated using the linear canonical (U-V) transformation

$$b_{\vec{k}} = U_{\vec{k}} a_{\vec{k}} + V_{\vec{k}} a_{-\vec{k}}^* \quad (6)$$

with an appropriate choice of U and V. From now on, we shall consider this transformation fulfilled, with Hamiltonian H having the form (5). Then in variables a, a* the linearized equations of motion become trivial:

$$\partial a_{\vec{k}}/\partial t + i\omega_{\vec{k}} a_{\vec{k}} = 0 \ . \tag{7}$$

They describe the propagation of noninteracting spin waves having the dispersion law $\omega(\vec{k})$. All information on wave interactions is given by the coefficient of H_{int} expansion to the series of a, a*:

$$H_3 = (1/2) \sum_{1,23} \left[V(1,23)a_1^* a_2 a_3 + H.c.\right]\delta(1-2-3) +$$

$$+ (1/6) \sum_{123} \left[U(123)a_1 a_2 a_3 + H.c.\right]\delta(1+2+3) \ , \tag{8}$$

$$H_4 = (1/2) \sum_{12,34} T(12,34)a_1^* a_2^* a_3 a_4 + \ldots \ .$$

Here and below $a_j = a(k_j)$, $V(1,23) = V(k_1,k_2k_3)$, etc., $\sum_{1,23} = \sum_{k_1k_2k_3} \delta(\vec{k}_1-\vec{k}_2-\vec{k}_3)$, etc. The physical sense of H_3, H_4 could be easily understood by analogy with quantum mechanics: H_3 describes three-magnon processes of the type $1 \leftrightarrow 2$ and $0 \leftrightarrow 3$ (transformation of one wave into two, and vice versa, generation of three waves from vacuum and vice versa). The terms of H_4-expansion describe the four-magnon scattering processes of the type $2 \rightarrow 2$. In the presence of external magnetic pumping field, there appears an additional term in the Hamiltonian H_{int}

$$H_p = (1/2)\sum_k \left[h \exp(-i\omega_p t)V_k a_k^* a_{-k}^* + H.c.\right] \ , \tag{9}$$

describing the *parametric excitation* of spin waves, i.e. the induced process of decay of a photon (with frequency ω_p and zero wavevector) into two magnons with wavevectors k, -k and frequencies $\omega_k = \omega_p/2$.

1.4.3. Equations of Motion. We must note that equations (5) conserve the energy of magnon system H. As a matter of fact, in reality the interactions such as magnon-phonon interactions, interactions of magnons with lattice defects and those of other types always exist, which leads to a small dissipation of their energy. This process may be taken into consideration phenomenologically: by adding the imaginary part γ_k to frequency ω_k. As a result we obtain

$$\partial a_k/\partial t + \gamma_k a_k + i\omega_k a_k = -i\delta H_{int}/\delta a_k^* \ . \tag{10}$$

The interaction Hamiltonian (5) - (9) together with the equations of motion (10), give the canonical formulation of the problem of nonlinear behaviour of magnons, understandable for the physicists unfamiliar with magnetism. The whole specific character of magnetics is given by function $\omega(k)$ and coefficients of the interaction Hamiltonian.

2. PARAMETRIC DYNAMICS OF MAGNONS

A wide class of nonlinear wave processes is described by the dynamic equations of motion for the complex wave amplitudes (10), where phase correlations play an essential role. It would therefore be reasonable to call them *dynamic processes* as distinct from the processes of another class which are described by the kinetic equations for occupation numbers, and which may be called *kinetic processes*, one can mention confluence of two waves into one, generation of the second harmonic, decay of one wave into two waves, and various types of four-magnon processes, including self-focusing and collapse /1-3/. We shall consider here only one dynamic process, viz. the parametric excitation of magnons. This phenomenon was discovered in 1957 by SUHL /4/ (*transverse pumping*, when $\vec{h} \perp \vec{M}$) and in 1960 by MOGRENTHALER /5/ (*parallel pumping*, when $\vec{h} \parallel \vec{M}$).

2.1. Introduction into the S-Theory

2.1.1. Parametric Instability. In order to calculate the threshold of parallel pumping, we shall substitute $H_{int} = H_p$ into equation (10) from formula (9). As a result, we have linear equations of motion for *slow amplitudes* $b_k(t)$:

$$b_{\vec{k}}(t) = a_{\vec{k}}(t)\exp(i\omega_p t/2), \quad b_{\vec{k}}^* = a_{-\vec{k}}^*(t)\exp(-i\omega_p t/2) \ , \tag{11}$$

$$\partial b_{\vec{k}}/\partial t + \gamma_{\vec{k}} + i(\omega_{\vec{k}}-\omega_p/2)\, b_{\vec{k}} + ihV_{\vec{k}} b_{-\vec{k}}^* = 0 \ . \tag{12}$$

A solution to this equation will be as follows:

$$\begin{aligned} b_k(t) &= |b_k(0)|\exp(\nu_k t - i\phi_k) \ , \\ b_{-k}^*(t) &= |b_{-k}(0)|\exp(\nu_k t + i\phi_{-k}) \ . \end{aligned} \tag{13}$$

Then, for the increment of parametric instability ν_k, we have:

$$\nu_k = -\gamma_k \pm \left[|hV_k|^2 - (\omega_k-\omega_p/2)^2\right]^{1/2} \ . \tag{14}$$

The minimal threshold of excitation h_1 (corresponding to max $\nu_k = 0$) is determined from the condition

$$h_1 = \min(\gamma_k/|V_k|) \ . \tag{15}$$

At h h_1, the exponential increase (13) of the amplitude of pairs with increment (14) begins. It follows from (12) that

$$\cos(\phi_k-\phi_{-k}-\phi_p) = (\omega_k-\omega_p/2)/|hV_k| \ . \tag{16}$$

This means that at the linear stage of parametric instability, a definite correlation between the phases of waves in a pair is reached. The phase correlator of waves with equal and oppositely directed wavevectors may be called, by analogy with superconductitivity, *pairing*.

2.1.2. Diagonal Hamiltonian of the S-Theory. The growth of wave amplitude continues until the interactions of waves with each other become essential. As all the parametrically excited waves have almost equal frequencies (close to $\omega_p/2$), the free magnon interaction proves to be nonresonant. In the Hamiltonian H_4 (8), the following terms are most essential:

$$H_S = \sum_{\vec{k},\vec{k}'} T_{\vec{k}\vec{k}'}|a_{\vec{k}}|^2|a_{\vec{k}'}|^2 + (1/2)\sum_{\vec{k},\vec{k}'} S_{\vec{k}\vec{k}'}a_{\vec{k}}^*a_{-\vec{k}}^*a_{\vec{k}'}a_{-\vec{k}'},$$
$$T_{\vec{k}\vec{k}'} = T_{\vec{k}\vec{k}',\vec{k}\vec{k}'}, \quad S_{\vec{k}\vec{k}'} = T_{\vec{k},-\vec{k},\vec{k}',-\vec{k}'}. \tag{17}$$

They either do not depend on wave phases at all, or depend only on the sum of phases $\phi_{\vec{k}} = \phi_{\vec{k}} + \phi_{-\vec{k}}$ in pairs. All other terms in H_4 become zero averaging over chaotic wave phases and make contribution to the equation of motion only in the second order of the perturbation theory in H_4. The reduction of H_4 to the form H_S (17), which is the diagonal in the pairs of waves, was suggested by ZAKHAROV, L'VOV and STAROBINETS /6/. This resulted in the creation of a simple and efficient "S-theory", which in 1970-74 promoted the studies on the above-threshold behaviour of magnons. In particular, it allowed to give qualitative explanation to many experimentally observed effects and to obtain in most cases a good qualitative agreement with the experiment /7/. Later on, the S-theory was adopted by researchers (mostly the Soviet ones), who obtained interesting and important experimental results on the nonlinear behaviour of parametric magnons, associated with a new insight into the physical sense of the phenomena. The most comprehensive description of the S-theory and the pertinent experiments are given in monograph /1/ and reviews /7-9/. We are sure that the significance of these results goes beyond the limits of the physics of magnetic dielectrics. They have played, and will certainly keep on playing, an important role in the development of the physics of nonlinear waves in other media. A brief account of the fundamentals of the S-theory will be given here.

2.1.3. Basic Equations of the S-Theory. Introducing the interaction Hamiltonian into equation (10), we obtain the basic equations for the S-theory:

$$\partial b_{\vec{k}}/\partial t + \left[\gamma_{\vec{k}} + i(\omega_{NL}(\vec{k})-\omega_p/2)\right]b_{\vec{k}} + iP_{\vec{k}}b_{-\vec{k}}^* = 0 ,$$
$$\partial b_{-\vec{k}}^*/\partial t + \left[\gamma_{\vec{k}} - i(\omega_{NL}(\vec{k})-\omega_p/2)\right]b_{-\vec{k}}^* - iP_{\vec{k}}b_{\vec{k}} = 0 . \tag{18}$$

They differ from the linear equations (12), describing the parametric instability, only by renormalization of the frequency $\omega(\vec{k}) \rightarrow \omega_{NL}(\vec{k})$ and of the pumping $hV(\vec{k}) \rightarrow P(\vec{k})$ due to the first and second term in equation (17), respectively:

$$\omega_{NL}(\vec{k}) = \omega(\vec{k}) + 2\sum_{\vec{k}'} T_{\vec{k}\vec{k}'}|b_{\vec{k}'}{}^2| , \quad P_{\vec{k}} = hV_{\vec{k}} + \sum_{\vec{k}'} S_{\vec{k}\vec{k}'}b_{\vec{k}'}b_{-\vec{k}'} . \tag{19}$$

It is evident that the approximation of the diagonal Hamiltonian (17) is essentially the *approximation of the self-consistent field.* Classical examples of this approximation are the Curie-Weiss theory of molecular field, the Landau theory of second-order phase transitions, the Landau theory of weakly supercritical flows in hydrodynamics, and the BCS theory of superconductivity.

2.2. The Ground State in the S-Theory

2.2.1. Stability of the Ground State. Assuming in (18) $\partial b/\partial t = 0$, let us consider the stationary solutions of this equation. It is readily evident that in the points of the k-space where $b_k = 0$, the determinant of this system is equal to zero:

$$|P_{\vec{k}}|^2 = \frac{2}{k} + \left[\omega_{NL}(k) - \omega_p/2\right]^2 . \tag{20}$$

This is the equation of two surfaces, and on their arbitrary part we may assume $b_k =$ $= 0$. Thus we have a great number of stationary states. The requirement of their stability with respect to the growth of waves in the points where in the stationary state $b_k = 0$ strongly reduces the class of possible stationary states. Firstly, the solution is stable if two surfaces (20) coalesce into one

$$\omega_{NL}(\vec{k}) = \omega_p/2 . \tag{21}$$

Secondly, in the points of *resonance surface* (21) where $b_k \neq 0$

$$|P_{\vec{k}}| = \gamma_{\vec{k}}, \quad i\gamma_{\vec{k}} = P_{\vec{k}} \exp(i\psi_{\vec{k}}) . \tag{22}$$

In the residual parts of this surface

$$|P_{\vec{k}}| < \gamma_{\vec{k}} . \tag{23}$$

It should be noted that the ambiguity of solutions of stationary equations and elimination (complete or partial) of this arbitrariness using the stability conditions is not an inherent property of the S-theory. This is a general feature of the approximation of self-consistent field in the theory of nonlinear waves.

2.2.2. The Simplest Solution to the S-Theory. In the simplest, isotropic case, when $V_k = V$, $\gamma_k = \gamma$, and $S_{kk'} = S$, there must obviously be isotropic solution $N(\Omega) = N/4\pi$ ($\Omega = \theta, \phi$, are the polar vectorial and azimuthal angles). In this case, it follows from (19) and (22) that

$$N = \sum_{\vec{k}} |b_k|^2 = \left[(hV)^2 - \gamma^2\right]^{1/2}/|S| , \quad hV \sin\psi = \gamma . \tag{24}$$

The second equation represents a condition of energy balance: $W_+ = W_-$, where $W_+ =$ $= \omega_p hVN \sin\psi$ is the energy influx from pumping, and $W_- = \gamma(\omega_k + \omega_{-k})N$ is the energy consumed. The limitation of amplitude N is achieved due to the fact that the phases of pairs ψ_k differ from the optimal value $\pi/2$, at which the energy flux into the system is maximal. Equation (24b) is fundamentally important in the S-theory. It was verified in the direct experiments involving of the pair phase in the parametric excitation of magnons in the ferromagnet $Y_3Fe_5O_{12}$ /10,11/. It was shown that, within experimental measurements error, the points lie on the bisector of the quadrantal angle, in complete agreement with (24b). This indicates that the S-theory correctly describes the essential features of the above-threshold behaviour of parametric magnons.

2.2.3. Reshaping the Distribution Function with Increase in Supercriticality. It is reasonable to give the geometric interpretation of the stability condition (23) $|P(\Omega)| < \gamma(\Omega)$, which defines the angular distribution function $N(\Omega)$ of the parametric magnons: surface $|P(\Omega)|$ lies entirely inside surface $\gamma(\Omega)$ and touches it in the points where $N(\Omega) \neq 0$. In the case of axial symmetry, characteristic for many ferromagnets (including $Y_3Fe_5O_{12}$), this touching occurs along the lines Θ = const, and $0 < \phi < \pi$ (along the resonant surface parallels). The first touching occurs on the equator ($\Theta = \pi/2$), and the state with one group of pairs $N(\Theta,\phi) = N_1\delta(\Theta-\pi/2)$ exists in a wide range of supercriticalities $h_1 < h < h_2$. For $Y_3Fe_5O_{12}$, $h_2 \approx (3\text{-}4)h_1$. For magnetic fields H > 1500 Oe at $h = h_2$, the second touching is on the parallel $\Theta \approx 50°$ and the second group of pairs is generated. At higher h the third group is generated, etc. At H < 1500 Oe in $Y_3Fe_5O_{12}$ the distribution function evolution takes place in quite a different manner at $h > h_2$ (depending on H), surface $|P(\Omega)|$ coalesces with surface $\gamma(\Omega)$ on the band $|\Theta-\pi/2| < \delta$, the width of which is $\delta \sim (h-h_2)$. As a result, there appears a continuous distribution of pairs near equator with the width 2δ. The theoretical conclusions described here are in a quantitative agreement with the result of the experiment on $Y_3Fe_5O_{12}$ /12/.

2.3. Nonstationary Self-Consistent Dynamics of Parametric Magnons

2.3.1. The Spectrum of Collective Oscillations of Parametric Magnons and Methods of Their Excitation. Linearizing the nonstationary equations of the S-theory (18) with respect to the ground state of the system and assuming the amplitudes of excitation α_k, $\alpha_k^* \sim \exp(-i\Omega t)$, we obtain a system of algebraic equations homogeneous with respect to α and α^*. The condition for their solvability determines the frequency Re Ω and damping Im Ω of collective oscillations. In the case of axial symmetry, one can obtain /1/:

$$\Omega_m = -i\gamma \pm \left[4S_m(2T_m+S_m)N_1^2 - \gamma^2\right] . \qquad (25)$$

Here m is the number of axial mode, T_m and S_m are the corresponding axial-angle Fourier harmonics of function (17) $T_{kk'}$ and $S_{kk'}$, respectively. For the sake of simplicity, it is assumed that $T_m = T_{-m}$ and $S_m = S_{-m}$. It should be noted that collective oscillations may be spatially nonuniform. In this case, Ω_m depends on their wavefactor $\vec{\kappa}$, with Ω_m in Eq.(25) being $\Omega_m(0)$. For the simplest case the dispersion law $\Omega_m(\vec{\kappa})$ has been given in /1/.

The above-described collective oscillations of a system of parametric magnons may be excited (similarly to oscillations of any other nature) in various ways: with the aid of external resonant influence, parametrically or by hit. All these methods of their excitation have been used in the experiments on ferro- and antiferromagnets. It is simple to devise a theory and to interpret experiments for the resonance method of excitation of collective oscillations. It was exactly by this method that these oscillations were experimentally discovered and studied by ZAUTKIN, L'VOV and STAROBINETS in 1972 /19/. In addition to parallel pumping, they applied to the $Y_3Fe_5O_{12}$ sample another microwave signal, the frequency of which differed from the pump frequency ω_p by the frequency of collective oscillations Ω_0. The role of external resonant force was played in this case by the beats between two microwave signals. Later on, in 1975 /14/, OREL and STAROBINETS implemented the direct resonance method of excitation using the alternate magnetic field with a middle-wave frequency (of the order of 1 MHz). In principle, collective oscillations may also be excited by the sound the frequency and wavevector of which coincide with those of the oscillations. Closely related to the resonance methods of exciting collective oscillations is a simple and pleasant method of hit excitation by means of a drastic change of the pump frequency or phase, suggested in 1974 by PROZOROVA and SMIRNOV /10/. Their data on eigenfrequency of collective oscillations in antiferromagnet MnCO as well as the data presented in /13/ for $Y_3Fe_5O_{12}$ are in a quantitative agreement with the formula

$$\Omega_0^2 = 4\gamma^2(2T_0+S_0)S_0^{-1}\left[(h^2/h_1^2) - 1\right]^{1/2} , \tag{26}$$

which follows from the theory (see (24) and (25) at $\gamma << hV$). This indicates that the S-theory adequately describes the parametric excitation of magnons in ferro- and antiferromagnets.

Apart from the above-considered linear interaction ($H_m\alpha^*$ +H.c.) of the collective oscillations of parametric magnons with the external radiofrequency field H_m, which leads to resonance at the frequency $\Omega = \Omega_0$, the S-theory predicts the nonlinear interaction of the type ($H_m\beta^*\beta^*$+H.c.). It should lead to the parametric resonance of collective modes β in the radiofrequency field with a frequency $\Omega = 2\Omega_0$. It is evident that the same effect also involves the instability of the original mode α with a frequency Ω, excited with the radiofrequency field, with respect to its decay into two modes β with the frequency $\Omega/2$. The action of both mechanisms above a certain critical amplitude of the radiofrequency field H give rise to the *double parametric resonance of magnons* - i.e. the parametric excitation of collective oscillations in the system of parametrically excited magnons. This phenomenon was discovered and experimentally studied by ZAUTKIN et al. in 1977 /15/.

2.3.2. Self-Oscillations of Magnetization in the Parametric Excitation of Magnons. In the very first experiments on the parametric excitation of magnons in 1961, HARTWICK, PERESSINI and WEISS /16/ found that the steady-state condition is often established and magnetization performs complex self-oscillation around an average value. Since then, physical origin of self-oscillations has been one of the principal problems of parametric excitation. Various hypotheses were put forward (see, for example, /17/) but all of them are of purely historical interest. In terms of the S-theory, self-oscillations were explained as the result of the generated instability of collective oscillations considered above. Indeed, if

$$S_m(2T_m+S_m) > 0 \ , \tag{27}$$

then it follows from (25) that Im $\Omega_m > 0$. As evident from the same formula, in this case Re $\Omega_m = 0$, i.e. the ground-state instability is purely periodical. Therefore, the interactions of different modes of self-oscillations are extremely strong, and the problem of self-oscillation behaviour at the nonlinear stage requires computer analysis. This was given in /18,19/. An essential result of these numerical experiments was the proof of the fact that at supercriticalities $p = h/h_1$ smaller than $p_c \cong (1\div1,5)$dB, the system of parametric magnons sets to a stable limiting cycle, the region of attraction of which is the whole phase space. At $p > p_c$ the trajectories close to this cycle become exponentially unstable, with the medium-cycle increment of divergence (the Lyapunov factor) growing in proportion to $p - p_c$. At small values of $(p-p_c)$ a narrow layer filled with exponentially unstable trajectories is formed near the limiting cycle and a transition to chaotic self-oscillations occurs. These and other features of self-oscillations in the numerical experiment with the non-stationary equations of the S-theory (18) are in a qualitative agreement with the result of laboratory studies /18,19/. As far as the conditions for the generation of self-oscillations are concerned, the detailed experimental studies carried out by ZAUTKIN and STAROBINETS on the ferromagnet $Y_3Fe_5O_{12}$ in a wide temperature and field range, using samples of different forms, showed a good qualitative agreement with the instability condition (27).

3. NONLINEAR KINETICS OF MAGNONS

3.1. The Kinetic Equation, Thermodynamic Equilibrium and Relaxation

As stated above, the interaction of magnon packets which are wide in the k-space may be described with the kinetic equation for the occupation numbers $n_k(t) = \langle|b_k|^2\rangle/\hbar$

$$dn_k/2dt = St_k\{n_{k'}\} \ . \tag{28}$$

The collision term St is a functional of occupation numbers in a definite region of the k-space. If the principal interaction is the three-magnon interaction H_3(8), then

$$St_k^{(3)}\{n_{k'}\} = (\pi/\hbar^2)\sum_{\vec{k}=\vec{1}+\vec{2}} |V_{\vec{k},\vec{1}\vec{2}}|^2 \ *$$

$$* \left[n_1n_2(n_k+1)-n_k(n_1+1)(n_2+1)\right]\delta(\omega_k-\omega_1-\omega_2)\delta(\vec{k}-\vec{k}_1-\vec{k}_2) \ +$$

$$+ \ (2\pi/\hbar^2)\sum_{2=k+1}|V_{2,k1}|^2\left[n_2(n_k+1)(n_1+1) \ - \ n_kn_1(n_2+1)\right] \ *$$

$$* \ \delta(\omega_2-\omega_k-\omega_1)\delta(\vec{k}_2-\vec{k}-\vec{k}_1) \ . \tag{29}$$

If the three-magnon interaction is absent or forbidden by the laws of conservation of energy and momentum, the leading interaction is the four-magnon one, and

$$St_k = St_k^{(4)}\{n_k,\} = (2\pi/\hbar^2) \sum_{k+1=2+3} |T_{k1,23}|^2 *$$

$$* \left[(n_k+1)(n_1+1)n_2n_3 - n_kn_1(n_2+1)(n_3+1)\right] *$$

$$* \, \delta(\omega_k+\omega_1-\omega_2-\omega_3)\delta(\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3) \ . \tag{30}$$

Equations (29) and (30) are easily obtained with the help of the "golden rule of quantum mechanics" in the perturbation theory, and are given in the text-books on theoretical physics, nonlinear acoustics, etc.

A stationary solution of the kinetic equations (29)-(30) is the Bose-Einstein distribution:

$$n_k = n_0(k) = \left[\exp(\hbar\omega_k/T) - i\right]^{-1} \ , \tag{31}$$

which describes the thermodynamic equilibrium with temperature T. If at some $k = k_0$, the value $n(k_0)$ is made slightly deviating from the equilibrium (31), then

$$\left[n(k_0) - n_0(k_0)\right] \sim \exp(-2\gamma_0(k_0)t) \ . \tag{32}$$

Here $\gamma_0(k)$ is the magnon *damping decrement*, i.e. the factor of proportionality at $n(k)$ in the collision term. For the three-magnon decay processes,

$$\gamma_d(\vec{k}) = (\pi/\hbar^2) \sum_{\vec{k}=\vec{1}+\vec{2}} |V_{\vec{k},\vec{1}\vec{2}}|^2 \left[n_1+n_2+1\right] \delta(\omega_{\vec{k}}-\omega_{\vec{1}}-\omega_{\vec{2}}) \ . \tag{33a}$$

For the three-magnon confluence processes,

$$\gamma_{sp}(\vec{k}) = (2/\hbar^2) \sum_{\vec{2}=\vec{k}+\vec{1}} |V_{\vec{2}.\vec{k}\vec{1}}|^2 \left[n_1-n_2\right] \delta(\omega_2-\omega_k-\omega_1) \ . \tag{33b}$$

For the four-magnon scattering processes,

$$\gamma_{sc}(\vec{k}) = (2\pi/\hbar^2) \sum_{\vec{k}+\vec{1}=\vec{2}+\vec{3}} |T_{\vec{k}\vec{1},\vec{2}\vec{3}}|^2 * \left[n_1(n_2+n_3+1)-n_2n_3\right] \delta(\omega_k+\omega_1-\omega_2-\omega_3) \ . \tag{33c}$$

Near the equilibrium we can substitute the Bose-Einstein distribution (31) for $n_0(k_0)$. Definitely, of greatest interest are the kinetic effects, which are far away from the thermodynamic equation and which just represent the subject of nonlinear kinetics. Of the great variety of nonlinear kinetic effects in magnets, we shall consider only two effects which seem to be instructive.

3.2. Damping of the Monochromatic Wave in the Nonlinear Medium /21/

In the above calculation of the magnon damping decrement $\gamma(k_0)$, we regarded all the residual reservoir of magnons (with $k = k_0$) to be in the thermodynamic equilibrium. This may be so if the number of magnons in the packet N under study is far smaller than the equilibrium number of thermal magnons N_T. However, the energy and momentum conservation laws allow only small part of the entire reservoir of thermal waves ΔN_T to participate in the relaxation of a narrow ($\Delta\omega_k << \omega_k$) packet. Therefore, at a relatively low intensity of the narrow packet N, comparable to N_T, the energy dissipated by it may lead to a substantial deviation of the occupation numbers of thermal magnons in this region from the equilibrium values. For this reason, the relaxation time of the packet will depend on it amplitude, i.e. the relaxation becomes nonlinear.

This effect must be the strongest in the relaxation of the monochromatic wave, which can interact, in accordance with the kinetic equation, with the waves with vectors $\vec{k}$ lying on a definite surface. For example, in decaying the wave with $\vec{k} = \vec{k}_0$, this surface is defined as

$$\omega(\vec{k}_0) = \omega(\vec{k}) + \omega(\vec{k}_0-\vec{k}) \ . \tag{34}$$

Formally, the number of waves ΔN_T on this surface is zero. If, however, we take into account that in reality the conservation law (34) is implemented with an accuracy of damping $\gamma(\vec{k})$, then

$$\Delta N_T{}^0 \cong 4\pi n_0(k)k^2\gamma(k)/(\partial\omega/\partial k) \ . \tag{35}$$

For this packet, we can schematically write the kinetic equation as follows

$$d\Delta N_T/dt = -\gamma(k)(\Delta N_T-\Delta N_T{}^0) + \Phi \ , \tag{36}$$

where $\Phi \cong \gamma(k_0)|A|$ is the in-term, which shows that in every relaxation of the monochromatic wave with amplitude A the number of waves in the packet increases by 1. Therefore the in-term of Eq.(36) should coincide (except for alternating the sign) with the out-term for the monochromatic packet. At $\Phi = 0$, Eq.(36) describes the relaxation of the number of waves N_T to the thermodynamically equilibrium value $N_T{}^0$ (35). From Eqs.(35) and (36), an estimate for the relative variation of the number of waves in a packet follows:

$$a = \frac{\delta n(k)}{n_0(k)} \cong \frac{\Delta N_T - \Delta N_T{}^0}{N_T{}^0} \cong \frac{\gamma(k_0)\,|A|^2}{\gamma^2(k)} \cdot \frac{\partial\omega/\partial k}{n_0(k)k^2} \quad .$$

If we substitute here the estimate for

$$\gamma(k_0) \cong |V|^2k^2(k_0)/(\partial\omega/\partial k)$$

following from (29), then

$$a \cong |V|^2|A|^2/\gamma^2(k) \quad . \tag{37}$$

It is evident that at a <<1

$$\gamma(k_0,|A|^2) - \gamma(k_0,0) \cong a\gamma(k_0,0) \ . \tag{38}$$

But if a ≅ 1, the damping of the monochromatic wave should substantially differ from that of the equilibrium. Accurate calculation /21/ gives for the decay processes,

$$\gamma_d(\vec{k},|A|^2) = \frac{\pi}{\hbar^2} \sum_{\vec{k}=\vec{1}+\vec{2}} \frac{|V_{\vec{k},\vec{1}\vec{2}}|^2(n_1+n_2+1)\delta(\omega_k-\omega_1-\omega_2)}{\left[1-|V_{\vec{k},\vec{1}\vec{2}}|^2|A|^2/\gamma_1\gamma_2\right]^{1/2}} \quad , \tag{39a}$$

and for the confluence processes

$$\gamma_{sp}(\vec{k},|A|^2) = \frac{2\pi}{\hbar^2} \sum_{\vec{2}=\vec{k}+\vec{1}} \frac{|V_{\vec{2},\vec{k}\vec{1}}|^2(n_1-n_2)\delta(\omega_2-\omega_{\vec{k}}-\omega_1)}{\left[1+|V_{2,k1}|^2|A|^2/\gamma_1\gamma_2\right]^{1/2}} \quad . \tag{39b}$$

At A = 0 these equations coincide with Eqs.(33). At small A values estimate (38) is confirmed. It is important that in the confluence process the damping decreases with the increase in A (the effect of medium clarification) and in the decay processes it increases. At A = A, where

$$|V_{k,12}|^2 A_1^2 = \gamma_1\gamma_2 \quad , \tag{40}$$

the damping (39a) formally becomes infinite. Recall that formula (40) defines the decay instability threshold of the monochromatic wave. At $A > A_1$ secondary waves with $k = k_1$ and $k = k_2$ grow exponentially with time, and formula (39a) is inapplicable.

3.3. The Kinetic Instability of Magnons.

Let us assume that magnons are parametrically excited in a magnet, on the resonance surface $2\omega(\vec{k}_p) = \omega_p$. Function $n_p(\vec{k})$ is their distribution function. In the isotropic case,

$$n_p(k) = N_p k_p^2 \delta(\omega_k-\omega_p/2) 2\pi^2 \partial\omega/\partial k \ . \tag{41}$$

The general distribution function n(k) includes the thermodynamically equilibrium term $n_p(k)$

$$n(k) = n_0(k) + n_p(k) \ . \tag{42}$$

The divergence of the distribution function (42) form the equilibrium alters wave damping in the whole k-space. Let us consider first the case when the decay processes are allowed for the parametric magnons

$$\omega_p/2 = \omega(\vec{k}_p) = \omega(\vec{k}_1) + \omega(\vec{k}_2) \ , \quad \vec{k}_p = \vec{k}_1 + \vec{k}_2 \ . \tag{43}$$

Now let us consider the damping of magnons with $\vec{k} = \vec{k}_1$. For them, (43) are the confluence processes. According to (33b) and (42):

$$\gamma_{sp}(\vec{k}_1) = \gamma_{sp}^0(\vec{k}_1) - (2\pi/\hbar^2) \sum_{\vec{k}_p=\vec{k}_1+\vec{k}_2} |V(\vec{k}_p,\vec{k}_1\vec{k}_2)|^2 \ *$$

$$* \ n_p(\vec{k}_p)\delta\left[(\omega_p/2) - \omega_1 - \omega_2\right] \ . \tag{44}$$

The first term in (44) arises from the equilibrium part of the distribution function

(42), and the second term from the parametric magnons. It is important that this term is negative, and at a reasonably high n_p the general damping $\gamma_S(k_1)$ may also become negative. As a consequence, the number of magnons with $k = k_1$ will exponentially grow. This phenomenon may be called the *first-order kinetic instability*. The kinetic instability threshold may be estimated from (44) by introducing distribution (41) there. As a result, we have:

$$\gamma_{sp}(\vec{k}_1) - \gamma_{sp}{}^0(\vec{k}_1) \cong |V|^2 N_p/k_p(\partial\omega/\partial k) \ , \tag{45a}$$

$$|V|^2 N_{cr} \cong k(\partial\omega/\partial k)\gamma_k \cong \omega_k\gamma_k \ . \tag{45b}$$

Comparing this estimate with formula (40), we see that the first-order kinetic instability threshold (in the number of waves N in a packet) is ω_k/γ_k times higher than the threshold of decay instability (of the monochromatic wave), which is dynamic in its nature.

The theory of kinetic instability resulting from three-magnon processes has been most comprehensively developed for the antiferromagnets, in which the decay of parametric magnons to magnons and sound is allowed /22/. Experimental evidence of this instability has been found in the antiferromagnet FeBO /23/. Quite recently, the first-order instability has been discovered in $Y_3Fe_5O_{12}$ /27/.

In the non-decay region of the spectrum, the first-order instability is impossible. However, with the decreased divergence of magnon distribution (42) from the equilibrium (with increase in N_p), there arises the *second-order kinetic instability*, in which *two* parametric magnons confluence to give two secondary magnons. By contrast with (43), the conservation laws have the form

$$\omega_p = \omega(\vec{k}_{p1}) + \omega(\vec{k}_{p2}) = \omega(\vec{k}_1) + \omega(\vec{k}_2) \ , \quad \vec{k}_{p1} + \vec{k}_{p2} = \vec{k}_1 + \vec{k}_2 \ . \tag{46}$$

Substituting the distribution function (42) into formula (33c), we obtain the following equation for the damping of secondary magnons

$$\gamma_{sc}(\vec{k}_1) = \gamma^0{}_{sc}(k_1) - (2\pi/\hbar^2)\cdot$$

$$\cdot \sum_{\vec{k}_{p1}+\vec{k}_{p2}=\vec{k}_1+\vec{k}_2} * \ |T(\vec{k}_{p1},\vec{k}_{p2},\vec{k}_1,\vec{k}_2)|^2 n_p(\vec{k}_{p1}) n_p(\vec{k}_{p2}) \delta(\omega_p - \omega(\vec{k}_1) - \omega(\vec{k}_2)). \tag{47a}$$

Using (41), we obtain the estimate:

$$\gamma_{sc}(k_1) - \gamma_{sc}{}^0(k_2) \cong \pi|TN_p|^2/k_p(\partial\omega/\partial k_p) \ . \tag{47b}$$

It is seen that at a reasonably high $N_p = N_{cr}$, secondary magnon damping can become negative. For N_{cr}, it follows from (47b) that:

$$|TN_{cr}|^2 \cong \gamma(k_1) k_p(\partial\omega/\partial k_p) \ . \tag{48}$$

Here into damping γ all essential relaxation processes were included.

A more comprehensive analysis carried out in /24/ has shown that in ferromagnets, the instability (48) has a minimal threshold for magnons at the bottom of the spin-wave spectrum ($k_1 || M$, $k_p >> k_1$ $(10^2 \div 10^3) cm^{-1}$), for which damping $\gamma(k_1)$ is minimal. Experimentally the second-order kinetic instability was found in 1981 /24/. Its brightest effect is the electromagnetic radiation caused by secondary magnons, which onsets with increased pumping amplitude. This radiation is fairly monochromatic $\Delta\omega_{rad}/\omega_{rad}$ $(10^{-3} \div 10^{-2})$, its frequency

$$\omega_{rad} = \omega(\vec{k}_1) + \omega(-\vec{k}_1) \cong 2\omega_0 \qquad (49)$$

does practically not differ from the doubled frequency of the spin-wave spectrum bottom ω_0 and linearly depends on the external magnetic field H. For the spherical samples

$$\omega_0 = h(H-4\pi M_0/3) \ , \qquad (50)$$

where $g = 2\pi * 2.8$ Ghz/kOe is the magnetomechanical electron ratio. The emission mechanism is clear from (49): two secondary magnons with $\vec{k} = \vec{k}_1$ and $\vec{k} = -\vec{k}_1$ confluence to give a photon with $\vec{k} = 0$. The direct radiation caused by secondary magnons at a frequency ω_0 is strongly suppressed due to the fact that $k_1 \cong 10^2 cm^{-1}$ by far exceeds the wavevector of the photon at a frequency $\omega_0 (k_{ph} \cong 10^{-1} cm^{-1})$. In /25/ and /26/ the nonlinear theory of kinetic instability in ferrites has been worked out, being in good agreement with the experiment on $Y_3Fe_5O_{12}$.

There are many other kinetic effects stimulated by the increasing divergence of magnons from the equilibrium. Their investigation has just begun. The author hopes that he will be able to deliver a lecture on them at one of the next Gorky schools on nonlinear waves.

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