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The equations governing fluid mechanics were finalized in 1845, but solutions for turbulent flows are only just beginning to emerge

Turbulence: a universal problem

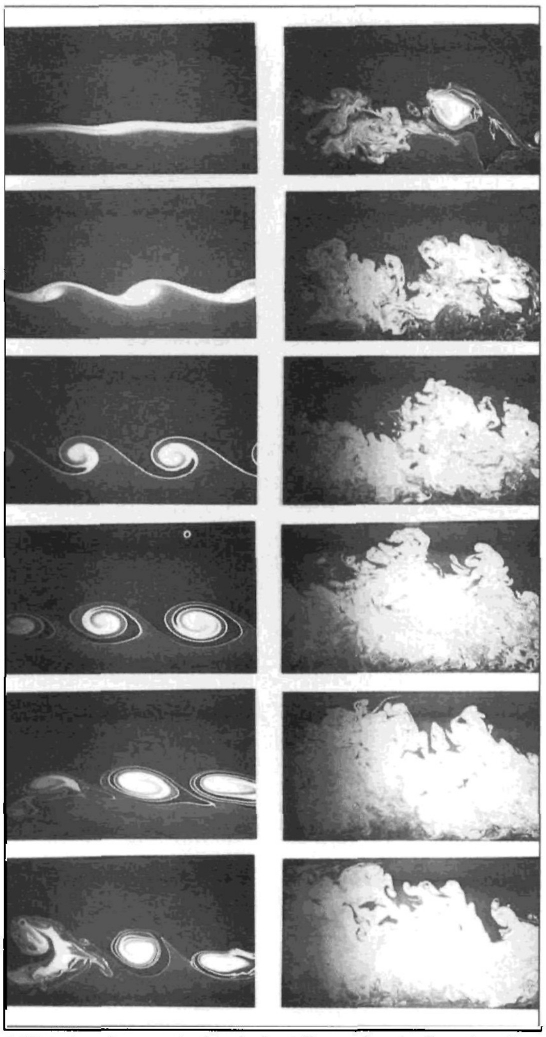
VICTOR L'VOV AND ITAMAR PROCACCIA

SIR Horace Lamb, the 19th-century mathematician and geophysicist, once said: "I am an old man now, and when I die and go to Heaven there are two matters on which I hope enlightenment. One is quantum electrodynamics (QED) and the other is turbulence of fluids. About the former, I am really rather optimistic." Lamb was right to be optimistic about QED, which was clarified in the 1950s, but his pessimism about turbulence may have been short-sighted. Indeed, there are even signs that QED and turbulence are not unrelated. The connections, which have been developing for some time, promise renewed vigour in the quest to understand this long-standing problem.

This new-found optimism is not due to a particular breakthrough in the theoretical or experimental study of turbulence *per se*, but rather to developments in neighbouring fields, such as chaos, critical phenomena and nonlinear systems. The convergence of quantum field theory and condensed matter physics in these areas has led to renewed optimism in the search to understand the origins and effects of turbulence in fluids fully. Indeed, it seems that the renaissance in turbulence will continue well into the next century.

Turbulence means different things to different researchers. All agree, however, that hydrodynamic turbulence arises in fluids that are highly stressed, or stirred, such that there exist significant fluid velocities on the largest scales of motion, and highly erratic and complex velocities on smaller scales (figure 1). It is a problem of immense interest to both physicists and engineers.

The physicist is interested in the small-scale structure of turbulence away from any boundary, where the action of fluid mechanics effectively smoothes out the large-scale characteristics of the flow and where universal phenomena may be sought. Turbulence is "universal" in the sense that its small-scale structure is independent of the nature of the fluid (water, oil and honey, for example, can all display turbulence), independent of the mechanism stirring the flow and independent of the shape of the container through which the fluid is flowing. These are inherent features of fluid mechanics as a classical field theory.



1 The development of turbulent flows due to the shearing motion between two streams. The images show the flow at equally spaced intervals of time, beginning top left. The region of shear between the streams is made visible by placing a thin layer of dye between the two streams and illuminating it with a pulsed laser (K R Sreenivasan, R Ramshankar and C Menevau 1988 *Proc. R. Soc. London* **A421** 79).

The engineer may find such characteristics irrelevant because they cannot be manipulated. The engineer seeks control, and control means a ready response to perturbations. Universal phenomena are immune to perturbations. The engineer is typically interested in the flow characteristics near the boundaries of the fluid (boundary layers, aeroplane wings, pipes, turbines and so on). By understanding how to manipulate the boundary region, the engineer may be able to reduce drag and improve the performance of technological devices.

To many people, turbulence research seems orthogonal to the two main lines of progress in modern physics. Particle physicists seek to study matter on ever-smaller scales (by building bigger and bigger accelerators), while astrophysicists probe the universe at ever-increasing distances from our galaxy. But the physics of phenomena on the human scale is considered to be of lesser importance. Of course, problems related to health are widely studied in biology and medicine. However, physical phenomena that can be observed simply by looking out of the window are dismissed by many as "non-fundamental", little more than relics of 19th-century research.

Attitudes are changing, however. We and many other researchers believe that physics on the human scale offers tremendously rewarding intellectual challenges. Witness, for example, chaos and the subject that is vaguely termed the "physics of complex systems". Fluid turbulence is a paramount example of such phenomena, and one that is immensely challenging to the physicist and the mathematician alike.

Turbulent beginnings

The mathematical history of fluid mechanics begins in 1741 when, it is said, Leonhard Euler was invited to Potsdam by Frederick the Great to engineer a water fountain. As a true theorist, Euler started by writing Newton's laws for a fluid of constant density

$$\frac{\partial \boldsymbol{u}(\boldsymbol{r},t)}{\partial t} + \boldsymbol{u}(\boldsymbol{r},t)\cdot\nabla\boldsymbol{u}(\boldsymbol{r},t) = -\nabla p(\boldsymbol{r},t)$$

where $\boldsymbol{u}(\boldsymbol{r},t)$ is the fluid velocity at position $\boldsymbol{r}$ and time t, and $p(\boldsymbol{r},t)$ is the pressure. The left-hand side is the time derivative of the momentum and the right-hand side is the force. However, this equation predicts velocities that are much higher than anything observed. In 1827 Navier realized that the viscosity (or internal friction) of the liquid was missing. His work was extended by Stokes in 1845 to give the Navier–Stokes equations, a set of five coupled partial differential equations that represent the conservation of mass, energy and the three components of momentum. The equations describing the conservation of momentum can be written in terms of the fluid velocity

$$\frac{\partial \boldsymbol{u}(\boldsymbol{r},t)}{\partial t} + \boldsymbol{u}(\boldsymbol{r},t)\cdot\nabla\boldsymbol{u}(\boldsymbol{r},t) = -\nabla p(\boldsymbol{r},t) + \nu\nabla^2\boldsymbol{u}(\boldsymbol{r},t)$$

where ν, the kinematic viscosity, is $\sim 10^{-6}\ \mathrm{m^2\,s^{-1}}$ for water. The last term, $\nu\nabla^2\boldsymbol{u}$, describes the amount of kinetic energy that has been converted into heat. Without this term the kinetic energy, $\boldsymbol{u}^2/2$, is conserved.

Straightforward attempts to apply this equation may still be non-realistic. For example, we could estimate the velocity, u, in a mighty river like the Nile or the Volga. These rivers drop hundreds of metres over about a thousand kilometres: if we equate the acceleration due to gravity at this angle with the viscous drag, $\nu d^2u/dz^2 \sim \nu u/L^2$, where L is the depth of the river (~10 m), we predict u to be about $10^5\ \mathrm{m\,s^{-1}}$. Much to the regret of the white-water-rafting industry, the observed value is only about $1\ \mathrm{m\,s^{-1}}$.

The resolution of this discrepancy was suggested in 1894 by Reynolds, who stressed the importance of the ratio of the nonlinear term, $\boldsymbol{u}(\boldsymbol{r},t)\cdot\nabla\boldsymbol{u}(\boldsymbol{r},t)$, to the viscous term, $\nu\nabla^2\boldsymbol{u}(\boldsymbol{r},t)$. This ratio is now known as the Reynolds number. For a velocity drop of U on a scale of L, the Reynolds number is UL/ν. For Reynolds numbers much less than unity we can neglect the nonlinearity, which allows closed-form solutions of the Navier–Stokes equations in many instances.

In natural phenomena, however, the Reynolds number is very large. In a river, for example, it is about 10^7. Reynolds understood that there is no stable, stationary solution for the equations of motion at large Reynolds numbers. The solutions are strongly affected by the nonlinearity, and the actual flow is turbulent.

Modern concepts about high Reynolds number turbulence started to evolve in 1922 when Richardson, taking his inspiration from Jonathan Swift, wrote: "Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity – in the molecular sense." Richardson conveyed an image of turbulence in which large-scale forcing set up a cascade of energy transfers to smaller and smaller scales through nonlinearities of fluid motion. Eventually, at the smallest scales, viscosity caused the energy to be dissipated as heat. This picture led in time to innumerable "cascade models" that tried to capture the statistical physics of turbulence by making assumptions about the cascade process.

How do we solve the Navier–Stokes equations for large Reynolds numbers? First, no-one in their right mind is interested in the full solution of the turbulent velocity field at all points in space-time. It is the statistical properties of the flow that are important (figure 2). However, the statistics of the velocity field are too heavily dependent on the particular boundary conditions of the flow. Richardson understood that universal properties may be found in the statistics of velocity *differences*, $\delta\boldsymbol{u}(\boldsymbol{r}_1,\boldsymbol{r}_2) \equiv \boldsymbol{u}(\boldsymbol{r}_2)-\boldsymbol{u}(\boldsymbol{r}_1)$, in which non-universal large-scale motions, such as the "wind" in atmospheric flows, have been subtracted. Experiments measure quantities related to these differences known as "structure functions"

$$S_n(R) \equiv \langle[\delta\boldsymbol{u}(\boldsymbol{r}_1,\boldsymbol{r}_2)\cdot\boldsymbol{R}/R]^n\rangle$$

where $\langle\ldots\rangle$ is an average over time. In some systems the second moment, $S_2(R)$, would tell us everything about the flow. In turbulent flow, however, the higher-order structure functions – that is, $S_3(R)$, $S_4(R)$..., – also contain important information.

Possibly the most ingenious attempt to understand the statistics of turbulence is due to Andrei Nikolayevich Kolmogorov, who in 1941 proposed the idea of universality based on the notion of the "inertial range". This moved the study of small-scale turbulence from the realm of mechanics to fundamental physics. Kolmogorov's idea was that for very large Reynolds numbers there is a wide separation between the scale of energy input, L, and the typical scale at which viscous friction becomes important, η. In the stationary situation, when the statistical characteristics of the turbulent flow are independent of time, the rate of energy input at large scales is balanced by the rate of energy dissipation at small scales. Moreover, this rate must be the same as the flux of energy from larger to smaller scales measured at any scale, R, between η and L. This is the "inertial interval".

Kolmogorov proposed that the only relevant parameter in the inertial interval is the energy flux per unit time per unit mass, $\bar{\varepsilon}$, and because L and η are irrelevant, R is the only length available for dimensional analysis. Therefore we have three parameters – R (which has dimensions L), $\bar{\varepsilon}$ ($\mathrm{L^2T^{-3}}$) and ρ ($\mathrm{ML^{-3}}$) – which we can combine to form any dimensionality that we want. This leads to detailed predictions about the statistical physics of turbulence. For example, to predict $S_n(R)$ we note that the only combination of $\bar{\varepsilon}$ and R that gives the right dimension ($\mathrm{L^nT^{-n}}$) is $(\bar{\varepsilon}R)^{n/3}$. For $n=2$ this leads to the famous Kolmogorov "2/3" law

$$S_2(R) \sim (\bar{\varepsilon}R)^{2/3}$$

The idea that one extracts universal properties by focusing on statistical quantities can be applied to other correlations. An important example is $\varepsilon(\boldsymbol{r},t)$, the rate at which energy is dissipated into heat due to viscous damping. We are interested in how this rate fluctuates about its mean, $\bar{\varepsilon}$, and how these fluctuations are correlated in space. This is described by the correlation function, $K_{\varepsilon\varepsilon}(R)$. If the fluctuations at different points are not correlated, $K_{\varepsilon\varepsilon}(R)$ is zero for all non-zero values of R.

Experimental measurements show that Kolmogorov was remarkably close to the truth. The statistical quantities are related to R by power laws, but the exponents do not agree. For example, dimensional analysis predicted that $K_{\varepsilon\varepsilon}(R) \sim R^{-8/3}$, whereas experiments find that the exponent, μ, is between –0.2 and –0.3.

Something fundamental seems to be missing. The uninitiated reader might think that the numerical value of an exponent is not a fundamental issue. However, one needs to understand that the Kolmogorov theory is based on the assumption that $\bar{\varepsilon}$ is the only relevant parameter. Therefore, if the theory does not predict the correct exponents, we need to add an extra variable to our dimensional analysis. Indeed, experiments indicate that the energy-input scale, L, appears as a normalization scale for deviations from Kolmogorov's predictions: $S_n(R) \simeq \varepsilon^{n/3} R^{\zeta_n} L^{\delta_n}$. Moreover, the exponent measured in experiments, ζ_n, differs from the $n/3$ predicted by Kolmogorov (figure 3). Scaling that deviates from the predictions of dimensional analysis is referred to as "anomalous scaling".

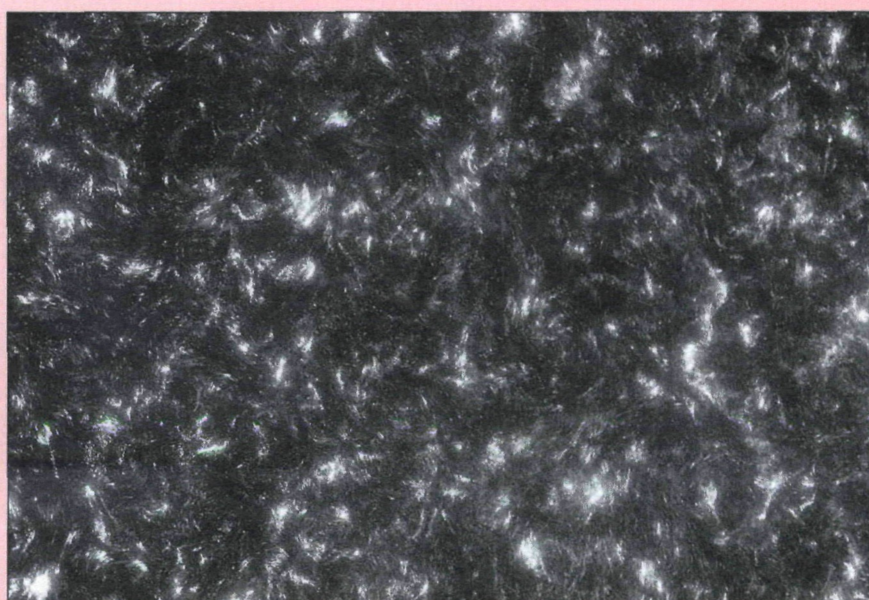
The experimental realization that the structure functions were consistent with L as the normalization scale, rather than η, developed over a long time and involved many experiments. It was Katepalli Sreenivasan and co-workers at Yale University in the US who demonstrated the appearance of L in the dissipation correlation in 1993. The first accurate measurements of the exponents, ζ_n, were made by Roberto Benzi of the University of Rome, Sergio Ciliberto and co-workers at the Ecole Normale Supérieure in Lyon in 1993, and by Alexander Praskovskii and Stephen Oncley of the National Center for Atmospheric Research in Boulder, Colorado, in 1994.

Not a normal problem

Theoretical studies of the universal small-scale structure of turbulence can be classified into two main classes. First, there is a large body of phenomenological models that, by attempting to achieve agreement with experiments, have reached important insights on the nature of the cascade and the statistics of turbulent fields (see Frisch in Further reading). In particular influential ideas appeared, following work by Benoit Mandelbrot of IBM Research in 1974, about the fractal geometry of highly turbulent fields. These allow scaling properties that are sufficiently complicated to include non-Kolmogorov scaling as well. In the early 1980s Giorgio Parisi of the University of Rome and Uriel Frisch of the Observatoire de la Côte d'Azur in France showed that by introducing multifractals one can accommodate the nonlinear dependence of ζ_n as a function of n. However, these models are not derived from the equations of fluid mechanics and one is always left with uncertainties about their validity and relevance.

The second class of approaches accepts that fluid mechanics is a (classical) field theory and uses field-theory techniques to compute statistical quantities. However, despite continuous effort lasting almost 50 years, the analytical derivation of the scaling laws for $K_{\varepsilon\varepsilon}(R)$ and $S_n(R)$ from the Navier–Stokes equations and the calculation of the numerical value of the scaling exponents, μ and ζ_n,

a



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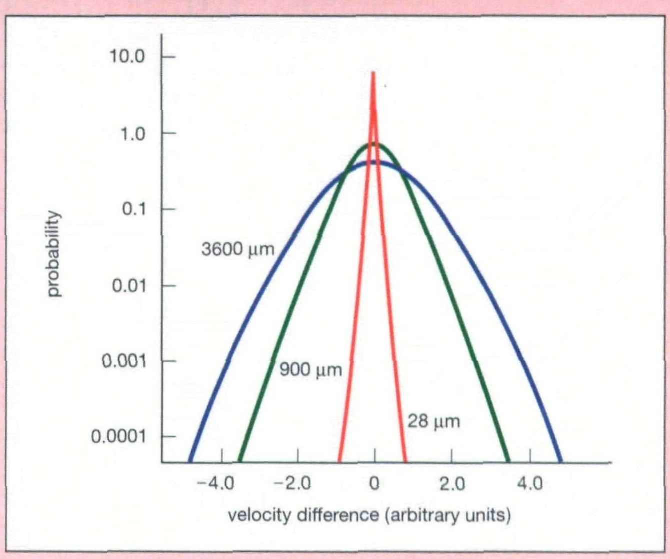


2 The aim of theorists is not to predict the flow at every point in a turbulent system (top), but the statistical properties of the flow (bottom). (a) The small-scale structure of turbulence. A grid of round bars (mesh size 1.27 × 1.27 cm) is swept through a tank of still water at a constant speed of 120 cm s^{-1}. The mesh Reynolds number is about 15 000. The turbulence is made visible by reflecting laser light from disc-like particles (nominally 30 × 20 × 6 μm) in the water onto a photographic film. To a first approximation the particles follow the flow. As time increases, the small scales die out by viscous action, leaving the large scales intact (K R Sreenivasan, Yale University). (*b*) Typical probability distribution functions of velocity differences across a scale R, for three values of R. The smaller the value of R, the larger the deviation from Gaussian distribution functions. The structure functions, $S_n(R)$, can be derived from such plots (A Noullez, R Miles, G Wallace and W Lempert, Princeton University).

have been among the most elusive goals of theoretical research. Why did it turn out to be so difficult?

To understand the difficulties, we need to recall some field theory. Suppose that we want to calculate the average response of a turbulent fluid at some point $\boldsymbol{r}_0$ to forcing at point $\boldsymbol{r}_1$. In field theory we consider this response to be an infinite sum of all the following processes: the direct response at $\boldsymbol{r}_0$ due to the forcing at $\boldsymbol{r}_1$ (this response is instantaneous if we assume that the fluid is incompressible and therefore that the speed of sound is infinite); an indirect or second-order response at $\boldsymbol{r}_0$ due to forcing via an intermediate point $\boldsymbol{r}_2$ (this intermediate process can take time and we need to integrate over all $\boldsymbol{r}_2$ and all times); a third-order effect due to forcing via two intermediate points, $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$ (which has to be integrated over all possible intermediate positions and all intermediate times); and so on. The actual response at $\boldsymbol{r}_0$ is the infinite sum of all these contributions.

There are three main difficulties when applying the field theory method:

(i) There is no small parameter that can be used in perturbation theory. Indeed, the obvious parameter to use has the same magnitude as the Reynolds number which, for the problems we are interested in, is much greater than one.

(ii) The theory exhibits two types of nonlinear interactions, both hidden in the term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$. The larger effect is known to anyone who has watched how a small object floating in a river becomes entrained in the eddies of the river and is swept along a complicated path with the turbulent flow. Similarly, any small-scale fluctuation is swept along by all the larger eddies, which causes the integrals in the perturbation theory to diverge as $L\to\infty$. Such divergences are common in other field theories, where they can be "renormalized" by introducing finite constants to the theory (like the charge and mass of the electron in quantum electrodynamics). The problem with hydrodynamics is that an infinite number of constants is needed to renormalize the theory. But luckily sweeping does not redistribute energy between different length-scales, so it might be possible to derive a theory of hydrodynamics that can be renormalized.

However, the second type of interaction, due to shear and torsion effects, can redistribute energy and might well lead to a scale-invariant theory. The problem with this second type of interaction is that it is much smaller in magnitude than the first, and therefore can easily be masked by it.

(iii) The pressure at any given point is determined by the velocity field everywhere. Physically this stems from the fact that in the incompressible limit of the Navier–Stokes equations, the speed of sound is infinite and the velocity fluctuations at all distant points are instantaneously coupled. This non-locality makes the equations extremely difficult to solve.

Indeed, such were these difficulties that many researchers doubted if field-theoretic methods could be applied to turbulence, even though they had proved so fruitful in other areas. In the past few years, however, a growing number of authors, including us, began to think differently.

Tricks of the trade

The first task of a successful theory of turbulence is to overcome the difficulty described in (ii) before applying the standard field-theoretical methods. The removal of the effects of sweeping is based on Richardson's remark that universality in turbulence is expected for the statistics of velocity *differences* rather than the velocity itself. The veloc-

Refining the theory

How do we calculate the anomalous exponents, δ_n, that describe the difference between Kolmogorov's predictions and the full theory? This box describes briefly some of the advances made by us in the past few years. First, using the global balance between energy input and dissipation as a boundary condition, it is possible to derive a set of equations for the structure functions, $S_n(R,t)$, from the Navier–Stokes equations

$$\frac{\partial S_n(R,t)}{\partial t}+D_n(R,t)=\nu J_n(R,t)$$

where the expressions D_n and J_n stem from the nonlinear and the viscous terms, respectively. For $n=2$ we note the following: $S_2(R)$ is the mean kinetic energy of motions of size R; $D_2(R)$ is the rate of energy flux through the scale R and is known exactly; and $\nu J_2(R)$ is the rate of energy dissipation due to viscous effects. The higher-order equations for $n>2$ are direct generalizations of this to higher-order moments.

In the stationary state, the time derivative vanishes and the resulting equation, $D_n(R)=\nu J_n(R)$, reflects the balance between energy flux and energy dissipation (for $n=2$). The authors have shown that $D_n(R)$ is of the order dS_{n+1}/dR. On the other hand, the evaluation of $J_n(R)$ raises a number of very interesting issues, and the resolution of these lies at the heart of the universal scaling properties of turbulence. Although not all of these issues have been resolved, we briefly mention some areas where progress has been made.

It turns out that $J_n(R)$ consists of a correlation of $\nabla^2 u$ with $n-1$ velocity differences across a scale R. How do we evaluate such a quantity in terms of the usual structure functions, $S_n(R)$? Recall that a gradient of a field such as $\nabla^2 u$ is the difference in the field values at two points divided by the separation as the separation goes to zero. This obviously involves crossing the dissipative scale, η. Therefore, in addition to rules for calculating many-point correlation functions, we need rules for when pairs of co-ordinates get very close to each other. These rules are known in the theoretical jargon as "fusion rules" (see the work by L'vov and Procaccia in Further reading).

It turns out that there are many cross-over scales from the regime of smooth behaviour at the shortest length-scales to scaling behaviour at intermediate scales. Indeed, for an n-point correlation function the dissipative scale depends on n itself, on the number of co-ordinates separated by a small distance and, in particular, on R. This dependence must be known when one attempts to determine the scaling exponents, ζ_n. □

ity fields are dominated by the large-scale motions, which are not universal since they are produced directly by the agent that forces the flow (e.g. the atmosphere, wind tunnels, channel flow, etc).

Richardson's insight was developed by Robert Kraichnan at Los Alamos, who in 1965 attempted to cast the field-theoretic approach in terms of Lagrangian paths, meaning a description of the fluid flow that follows the paths of every individual fluid particle. Such a description automatically removes the large-scale contributions. Kraichnan's approach was fundamentally correct, and gave rise to important and influential insights in the description of turbulence, but it did not deal with the higher orders in the perturbation series.

Another way to overcome this difficulty was suggested by Victor Belinicher of the Institute of Automation and Electrometry in Novosibirsk, Russia, and one of us (VL) in 1987. They introduced a novel, nonlinear transformation that eliminated the sweeping that leads to infrared divergences. The transformation also led to simple rules for writing down any arbitrary order in the perturbation theory for the structure functions. The essential idea in this transformation is the use of a co-ordinate frame in which velocities are measured relative to the velocity of one fluid particle. Of course, the perturbation series still diverges rapidly for large Reynolds numbers, however standard field theory methods can now be used to reformulate the perturbation expansion such that the viscosity is changed by an effective "eddy viscosity".

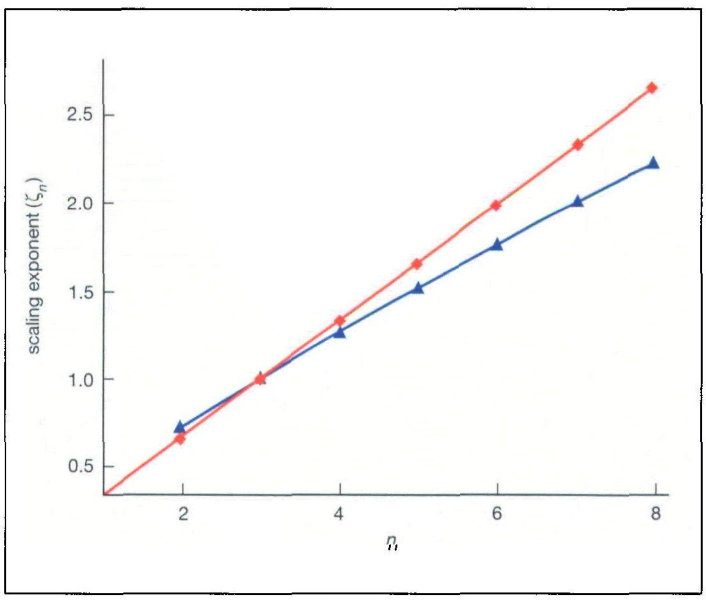
The end result is that the effective expansion parameter is no longer the Reynolds number but a parameter of order unity. Moreover, each term in the perturbation expansion for a given structure function is finite. Of course, such a perturbation series might still diverge, so it is crucial to examine the terms in the series order-by-order. Such an examination leads to an unwelcome surprise, however: every term remains finite when the energy-input scale, L, goes to infinity and the viscous-dissipation scale, η, goes to zero. In other words, perturbation theory does not indicate the existence of any typical length-scale. This is obviously wrong because such a length is needed to explain the deviations from Kolmogorov's predictions.

This is another example of how hydrodynamics differs from other field theories, where it is common for perturbation series to reflect the presence of anomalous scaling. In many cases this is seen in the appearance of logarithmic divergences that must be tamed by truncating the integrals at some renormalization length. Hydrodynamic turbulence seems different and the big question remains: how does a renormalization scale appear in the statistical theory of turbulence?

It turns out that there are two different mechanisms that furnish a renormalization scale, and that finally both L and η appear in the theory. The viscous scale, η, appears when we consider the statistics of gradient fields – that is, ∇u – rather than velocity differences. Consider, for example, the perturbation series for $K_{\varepsilon\varepsilon}(R)$, the correlation function of the rate of energy dissipation. We find logarithmic ultraviolet divergences in every order of the perturbation theory. These divergences can be controlled by truncating the integrals at η, and when we add up the infinite series the sum includes a factor $(R/\eta)^{2\Delta}$, where the anomalous exponent, Δ, is generally about one. The net result is that the exponent μ changes from 8/3, as predicted by dimensional arguments, to $(8/3)-2\Delta$.

Physically, this means that the correlation between the energy dissipation at points separated by R is much larger than the naive prediction of dimensional analysis, when R is much larger than η. This is due to the multi-step interaction of two small eddies of scale η (separated by R) with a large eddy of scale R via an infinite set of eddies of intermediate scales.

At this point it is important to calculate the numerical value of Δ. In 1995 we found that $\Delta=2-\zeta_2$. This simple relation links the scaling exponents for $K_{\varepsilon\varepsilon}(R)$, the correlation function for the rate of energy dissipation and $S_n(R)$, the structure function. Such a relation is known as a "scaling relation" or a "bridge relation". Physically it is a consequence of the existence of a universal non-equilibrium stationary state that supports an energy flux from large to small scales. The scaling relation for Δ has far-reaching implications for the theory of the structure functions. We have shown that with this value of Δ, the series for the structure functions diverge as $(L/R)^{\delta_n}$ as the energy-input scale, L, approaches infinity. The anomalous exponents, δ_n, are the deviations of the exponents of $S_n(R)$ from their Kolmogorov value.



3 The scaling exponents, ζ_n, as a function of n. The exponents describe the scaling of the structure function, $S_n(R)$, with the length-scale R; $S_n(R) \approx R^{\zeta_n}$. The predictions of Kolmogorov's dimensional analysis are shown in red; the results of experiments and computer simulation are shown in blue. The deviations from the Kolmogorov predictions increase with n.

The next step in the theoretical development is to understand how to compute the anomalous exponents. This involves careful consideration of correlation functions involving more than just two points in the fluid, and the calculation of velocity differences across very small scales (see box). The end result is another "bridge relation" $\mu=2-\zeta_6$. This result, which is in close agreement with experiment, had been conjectured before but had never been derived rigorously. The full analysis also shows that although the exponents, ζ_n, differ from those predicted by Kolmogorov, they are nevertheless linear in n for large n (figure 3).

Where now?

So what have we learned so far? It appears that there are four conceptual steps in constructing a theory of the universal anomalous statistics of turbulence on the basis of the Navier–Stokes equations. First, one needs to take care of the sweeping interactions that mask the scale-invariant theory. After doing so, the perturbation expansion converges order by order, and the Kolmogorov scaling of the velocity structure functions is found as a perturbative solution.

Second, one understands the appearance of the viscous-dissipation scale, η, as the natural normalization scale in the theory of the correlation functions of the gradient fields. This step is similar to critical phenomena and it leads to a similarly rich theory of anomalous behaviour of the gradient fields. Only the tip of the iceberg was considered above. In fact, when one considers the correlations of tensor fields, such as the tensor field derived from the vorticity vector, one finds that every field with a different transformation property under the rotation of the coordinates has its own independent scaling exponent.

Third, is the understanding of the divergence of the perturbation series for the structure functions, which sheds light on the emergence of the energy-input scale, L, as a normalization length in the theory of turbulence. This means that Kolmogorov's basic assertion that there is no typical scale in the expressions for statistical quantities in the inertial interval is wrong on two counts. In general both L and the viscous-dissipation length, η, appear in dimensionless combinations and change the exponents from the predictions of Kolmogorov's dimensional analysis.

Last, but not least, is the formulation of the so-called fusion rules and the realization that there exist many dissipative length-scales. This knowledge should eventually result in a satisfactory description of all the scaling properties.

The road ahead is not fully charted, but it seems that some of the conceptual difficulties have been surmounted. We believe that the crucial building blocks of the theory are now available, and they begin to delineate the structure of the theory. We hope that the remaining four years of this century will suffice to achieve a proper understanding of the anomalous scaling exponents in turbulence. Considerable work, however, is still needed to clarify many aspects of the problem fully, and most of these are proving to be as exciting and important as the scaling properties.

There are universal aspects that go beyond exponents, such as distribution functions and the eddy viscosity, and there are important non-universal aspects like the role of inhomogeneities, the effect of boundaries and so on. Progress on these issues will bring the theory closer to the concern of the engineers. This marriage of physics and engineering will be the challenge ahead in the 21st century.

Further reading

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