
LECTURE 3

SPIN DYNAMICS

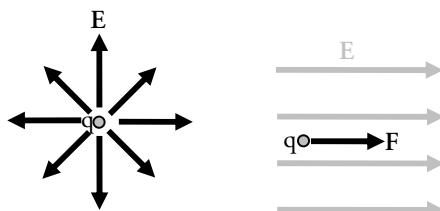
Lecture Notes by Assaf Tal

THE MAGNETIC MOMENT: THE "BASIC UNIT" OF MAGNETISM

The Magnetic Dipole/Moment

Before talking about magnetic resonance, we need to recount a few basic facts about magnetism.

Electrodynamics is the field of study that deals with **magnetic fields** (**B**) and electric fields (**E**), and their interactions with matter. The basic entity that creates electric fields is the electric charge. For example, the electron has a charge, q , and it creates an electric field about it, $\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$, where \mathbf{r} is a vector extending from the electron to the point of observation. The electric field, in turn, can act on another electron or charged particle by applying a force $\mathbf{F}=q\mathbf{E}$.



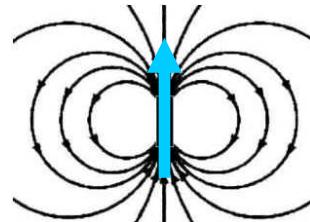
Left: a (stationary) electric charge q will create a radial electric field about it. Right: a charge q in an electric field will experience a force $\mathbf{F}=q\mathbf{E}$.

There is, however, no magnetic charge. The “elementary unit of magnetism” is the **magnetic moment**, also called the **magnetic dipole**. It is more complicated than charge because it is a vector, meaning it has both magnitude and direction. We will ask ourselves two basic questions:

1. What sort of magnetic fields does a magnetic moment create?
2. How does an external magnetic field affect the magnetic moment (apply force/torque, etc)?

We begin by answering the first question: the magnetic moment creates magnetic field lines (to

which **B** is parallel) which resemble in shape of an apple:

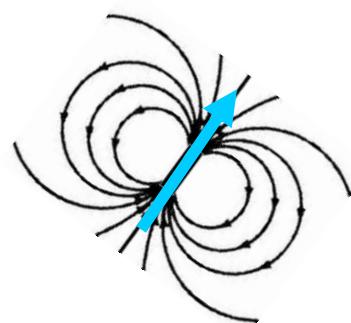


Mathematically, if we have a magnetic moment \mathbf{m} at the origin, and if \mathbf{r} is a vector pointing from the origin to the point of observation, then it will give off a **dipolar field** described by:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}$$

Number Time. The earth’s magnetic field is about $0.5 \text{ G} = 0.5 \cdot 10^{-4} \text{ T}$. Clinical MRI scanners operate at $1.5 \text{ T} - 3.0 \text{ T}$, and the highest human MRI scanner as of early 2015 is the 11.75 Tesla human magnet being built in the University of Freiburg, Germany.

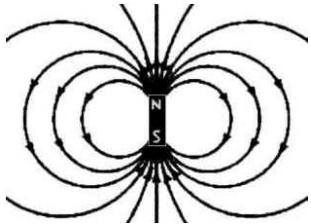
The magnitude of the generated magnetic field **B** is proportional to the size of the magnetic charge¹. The direction of the magnetic moment determines the direction of the field lines. For example, if we tilt the moment, we tilt the lines with it:



The simplest example of a magnetic moment is the refrigerator magnet. We’ll soon meet other, much

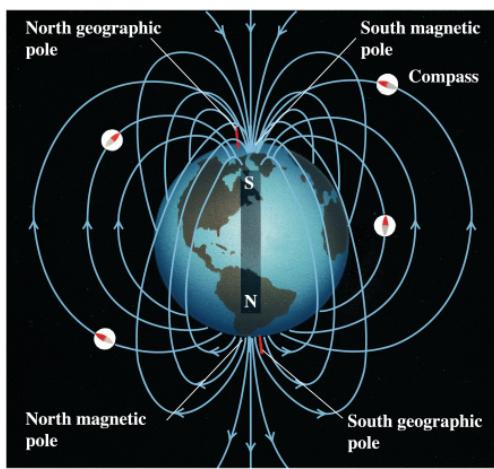
¹ Magnetic fields are measured in Tesla (T) in the SI system of units. Other systems use the Gauss (G). The conversion is straightforward: $1 \text{ T} = 10^4 \text{ G}$

smaller and weaker magnetic moments, when we discuss the atomic nucleus.



Your refrigerator magnet has a permanent magnetic moment

Another interesting example is the Earth itself, which behaves as if it had a giant magnetic moment stuck in its core:



Magnetic moments are measured in units of Joule/Tesla or (equivalently) in Ampere·meter² ($J/T = 1 \text{ A}\cdot\text{m}^2$).

Number Time. A typical refrigerator magnet might have a macroscopic magnetic moment of about 0.1 J/T . The tiny proton has an intrinsic magnetic moment equal to about $1.4 \cdot 10^{-26} \text{ J/T}$.

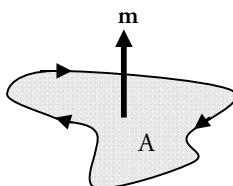
Magnetic Moments Are Either Intrinsic Or Induced

Magnetic moments are divided into two groups: current-induced and intrinsic.

Induced Moments: Basic electromagnetism tells us that a current flowing in a closed loop will give off a magnetic field. The loop can be *macroscopic*, like a wire, or *microscopic*, like an electron orbiting the nucleus. Far away from the current loop the field will look as if it were being generated by a magnetic dipole. If the magnetic loop is assumed to be planar, the magnetic dipole will be perpendicular to the loop, and have a magnitude given by

$$\mathbf{m} = I\mathbf{A}$$

where I is the current in the loop and A is the area enclosed by the loop:



For a general (non-planar) current loop, the expression for \mathbf{m} is somewhat more complicated, but the principle is the same.

Intrinsic Moments: It also appears that the fundamental particles - the proton, neutron and electron – carry **intrinsic** magnetic moments. That is, they “give off” a magnetic field as if a magnetic dipole were fixed to them, without having any current associated with them.

The angular momentum of elementary particles is measured in units of a fundamental constant known as Planck’s constant (divided by 2π), $\hbar \approx 1.05 \times 10^{-34} \text{ J}\cdot\text{sec.}$

	Electron	Neutron	Proton
Charge (Coulombs)	-1.6×10^{-19}	0	1.6×10^{-19}
Mass (kg)	9.1×10^{-31}	1.6×10^{-27}	1.6×10^{-27}
Magnetic moment (J/T , $2\pi\gamma S$)	9.26×10^{-24}	-0.96×10^{-26}	1.4×10^{-26}
Magnetic moment (μ_B)	-1.0	Irrelevant	Irrelevant
Magnetic moment (μ_N)	Irrelevant	-1.91	2.79
Spin, S (in units of \hbar)	1/2	1/2	1/2
Gyromagnetic ratio, γ (rad·Hz/T)	$2.8 \cdot 10^{10}$	-2.91×10^7	4.257×10^7

The **Bohr magneton**, μ_B , is just a quantity that makes it easy to talk about electron magnetism. It's not used often in nuclear magnetism, though:

$$\mu_B = \frac{e\hbar}{2m_e} = 9.27 \times 10^{-24} \frac{J}{T}.$$

A similar quantity, the **nuclear magneton**, μ_N , is used more often in nuclear magnetism, although we won't be making direct use of it in these lecture notes:

$$\mu_N = \frac{e\hbar}{2m_p} = 5.05 \times 10^{-27} \frac{J}{T}.$$

The phenomenon of intrinsic magnetic moments is directly related to another fundamental property of these particles called **spin**, and one speaks of a "nuclear spin" or an "electron spin". This is intrinsic angular momentum possessed by all electrons, protons and neutrons. Semi-classically, we can think of the proton or electron as a rotating ball of charge. The rotating charge can be thought of as loops of current, which give off a magnetic moment. In reality this picture is wrong, and you should always keep in mind spin is an intrinsic, somewhat weird quantum mechanical property; for example, the neutron has no charge and yet has a spin magnetic moment.

The semi-classical picture gets one thing right: the angular momentum and magnetic moment of the spinning sphere are parallel:

$$\mathbf{m} = \gamma \mathbf{S}.$$

The constant of proportionality is known as the **gyromagnetic ratio**, and is given in units of

$$[\gamma] = \frac{\text{Coulomb}}{\text{kg}} = \frac{\text{Hz}}{\text{Tesla}}.$$

A word of caution about units: some books or tables quote γ in units of rad·MHz/T. For example, $\gamma = 2\pi \cdot 42.576$ rad·MHz/T for the hydrogen nucleus. Always be mindful of the units being used. Remember that, if we multiply γ by 2π , we will sometimes need to divide another quantity by 2π along the way. A simple example is that of the magnetic moment of the proton:

$$\mathbf{m} = \frac{\gamma}{42.576 \frac{\text{MHz}}{\text{T}}} \times \frac{\mathbf{S}}{(no \ 2\pi)} = \frac{(1/2)\hbar}{(has \ 2\pi)} \mathbf{S}.$$

Equivalently,

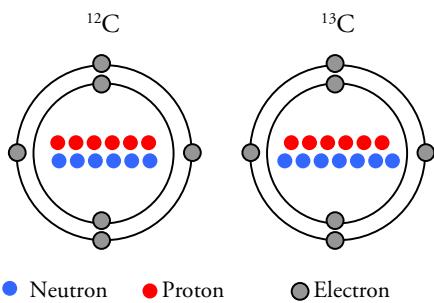
$$\mathbf{m} = \frac{\gamma}{2\pi \cdot 42.576 \frac{\text{MHz}}{\text{T}}} \times \frac{\mathbf{S}}{(has \ 2\pi)} = \frac{(1/2)\hbar}{(no \ 2\pi)} \mathbf{S}.$$

In the second form, I moved the 2π factor from \hbar to γ . The end result is the same, but now we must remember to specify the angular momentum in units without radians.

All electrons have an intrinsic magnetic moment, but that is not true for all nuclei, as we will see in the next section.

The Nuclear Magnetic Moment Is Determined By The Nucleus's Composition (Protons + Neutrons)

The nucleus is made up of protons and neutrons. The chemical name of an atom – carbon, hydrogen, phosphorous and so on – is determined by the number of protons it has. This will ultimately determine how many electrons it has and, therefore, its "chemistry". However, since neutrons are electrically neutral, their number might vary without changing the atom's "chemistry". Two such atoms are called **isotopes**. For example, shown here are two isotopes of carbon:



Two cartoon representations of ^{12}C (left), which has no nuclear spin, and ^{13}C (right), which has a nuclear spin of $1/2$.

Proton and neutron spins tend to pair up anti-parallel due to the **Pauli exclusion principle**, in a manner similar to that of the electronic model of the atom, where levels fill up from lowest energy and up. This is quite surprising when you consider how strongly coupled the nucleons are, but it works. This reasoning works fairly well. For example, it predicts that **nuclei with an equal number of protons and neutrons should have 0 nuclear spin**. This works well for ^{12}C , ^{16}O , but not for ^2H , as shown by the next table:

Number of protons	Number of neutrons	Spin quantum number	Examples
Even	Even	0	^{12}C , ^{16}O , ^{32}S
Odd	Even	$1/2$	^1H , ^{19}F , ^{31}P
"	"	$3/2$	^{11}B , ^{35}Cl , ^{79}Br
Even	Odd	$1/2$	^{13}C
"	"	$3/2$	^{127}I
"	"	$5/2$	^{17}O
Odd	Odd	1	^2H , ^{14}N

It also predicts nuclei with an “extra” neutron or proton should have spin- $1/2$. This works for ^{13}C , ^1H , ^{31}P , ^{19}F , but not for ^{17}O . The breakdown of the pairing occurs before some nuclei have asymmetric nuclear charge distributions. These lead in some cases to favorable energy configurations with non-paired nucleons. Nuclei with spin- $1/2$ have asymmetric charge distribution and are known as **quadrupolar nuclei**, which we won’t discuss in this course.

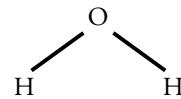
Nuclei With Low Natural Abundance Have “Low MRI Visibility”

It is very important to take into account the **natural abundance** of each isotope in determining

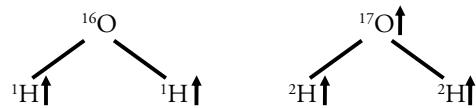
how large its signal will be. The natural abundance tells us if we take N atoms of an element then, on average, what percentage of each isotope we will get.

Nuclei with low or very low natural abundance will be difficult to detect, simply because there are very few such nuclei around. For example, ^{13}C has a natural abundance of about 1% and ^{12}C has a natural abundance of about² 99%. In a sample containing 100 carbon atoms, only about 1 will be a ^{13}C nucleus and the rest will be ^{12}C . Since only ^{13}C has a nuclear spin it will be the only one giving off a signal.

Natural abundance should be kept in mind also on the molecular level. Molecules are made out of atoms, connected between them by chemical bonds. The most important molecule in MRI is without a doubt water:



A “typical” water molecule actually comes in many isotopic flavors. Here are two examples:



Two isotopes of H_2O . The left is the most commonly found in nature. The one on the right is much more rare.

On the left is the most common variant by far. Oxygen-16 has no spin (its 8 protons pair up destructively, as do its 8 neutrons), and ^1H has spin $1/2$. Because of symmetry, the two hydrogen atoms are equivalent, in the sense that they behave as **one** spin- $1/2$ entity with **double** the magnetic moment. The variant on the right is very rare, and has markedly different NMR properties (^{17}O has spin $5/2$, and Deuterium has spin 1). Deviations from the “regular” H_2O are so rare, that their contribution to any experiment are negligible, as shown in the following table. Natural abundances are calculated by multiplying the natural abundances of the individual components

² Carbon has other isotopes but they do not occur naturally in nature and have zero natural abundance.

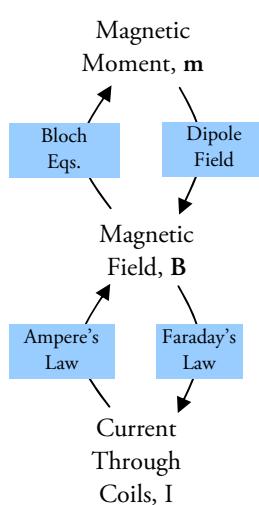
(assuming statistical independence, which is an excellent assumption):

Oxygen	Hydrogen	Hydrogen	Nat. Ab. (%)
^{16}O	^1H	^1H	99.74
^{16}O	^1H	^2H	$9.97 \cdot 10^{-3}$
^{16}O	^2H	^1H	$9.97 \cdot 10^{-3}$
^{16}O	^2H	^2H	$9.98 \cdot 10^{-7}$
^{17}O	^1H	^1H	$3.99 \cdot 10^{-2}$
^{17}O	^1H	^2H	$3.99 \cdot 10^{-6}$
^{17}O	^2H	^1H	$3.99 \cdot 10^{-6}$
^{17}O	^2H	^2H	$4 \cdot 10^{-10}$
^{18}O	^1H	^1H	$1.99 \cdot 10^{-1}$
^{18}O	^1H	^2H	$1.99 \cdot 10^{-5}$
^{18}O	^2H	^1H	$1.99 \cdot 10^{-5}$
^{18}O	^2H	^2H	$2 \cdot 10^{-9}$

Thus, when we speak of water we're really neglecting all isotopic variants except for $^{16}\text{O}-^1\text{H}-^1\text{H}$.

MRI USES THE INTERACTION OF MAGNETIC MOMENTS WITH MAGNETIC FIELDS

Just as electric charges give off electric fields and are affected by them, magnetic moments give off magnetic fields and are affected by them. This will turn out to be important since, as we'll see, we ourselves can create magnetic fields and pick them up using suitably constructed coils.



We've already noted that a moment will give off a dipole field. We therefore have three additional question we'd like to address in this lecture:

- How do magnetic fields affect magnetic moments? The answer to that will come in the form of a set of equations known as the **Bloch Equations**, which will have a surprisingly simple solution.
- How can we pick up magnetic fields using coils? Here, the answer will be by a process known as **induction**, by which time changing magnetic fields induce a voltage – and hence a current – in a coil of wire. The basic law of induction is known as **Faraday's law**.
- How can we generate magnetic fields, thereby affecting the evolution of magnetic moments? The answer here will come in the form of **Ampere's Law**: current passed through a piece of wire or a coil will generate a magnetic field. The spatial distribution of the field will depend on the wire's shape, while its time characteristics will depend on the current as a function of time.

Magnetic Fields Cause Magnetic Moments To Precess: The Bloch Equations

How do magnetic fields affect magnetic moments? This is a question in basic electromagnetism, from which we will merely borrow the answer: as long as the wavelengths involved are long enough, which is the case for MRI, then:

- \mathbf{m} feels a force given by $\mathbf{F} = (\mathbf{m} \cdot \nabla) \mathbf{B}$
- \mathbf{m} feels a torque given by $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$

The force \mathbf{F} turns out to be completely negligible in-vivo. As for the torque,

$$\frac{d\mathbf{m}}{dt} = \gamma \frac{d\mathbf{S}}{dt} = \gamma \boldsymbol{\tau} = \gamma (\mathbf{m} \times \mathbf{B})$$

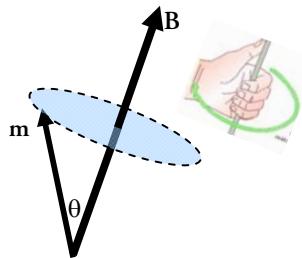
This equation is known as the **Bloch Equation** (BE). It is actually three separate equations:

$$\begin{cases} \dot{m}_x = \gamma (m_y B_z - m_z B_y) \\ \dot{m}_y = \gamma (m_z B_x - m_x B_z) \\ \dot{m}_z = \gamma (m_x B_y - m_y B_x) \end{cases}$$

These are three coupled first order linear differential equations. As far as differential equations they are considered very easy from a numerical point of view, but for a general magnetic field they have no analytical solution. However, if the magnetic field is constant, their solution is quite straightforward, and I will quote here without proof. It is so important and fundamental that I'll put it in a textbox:

A spin \mathbf{m} in a time-constant magnetic field \mathbf{B} will *precess* around the field \mathbf{B} at an angular velocity $\omega = \gamma|\mathbf{B}|$ according to the left hand rule.

Let's break this down slowly. First, a **precession** is a motion by which \mathbf{m} traces out a cone around \mathbf{B} , while keeping their angle θ fixed:



In precessional motion, the tip of \mathbf{m} traces out the dashed circle around \mathbf{B} , while keeping θ fixed.

The sense of the rotation is determined using the left hand rule: take your left hand and curl it with the thumb pointing along the field \mathbf{B} . The way your fingers curl will tell you in which sense the magnetization is executing its precession. Finally, the angular velocity of the precession is fixed and given by $\omega = \gamma|\mathbf{B}|$ (a negative γ will reverse the sense of the rotation).

Since precession is really just a rotation of \mathbf{m} about \mathbf{B} , we can describe it mathematically using rotations. For example, if \mathbf{B} is pointing along the z -axis, then \mathbf{m} will simply rotate about the z -axis. A left-handed rotation matrix about z by an angle α is:

$$R(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For a constant field, $\alpha = \omega t = \gamma B t$. If at time $t=0$ \mathbf{m} points along the x -axis, so

$$\mathbf{m}(t=0) \equiv \mathbf{m}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

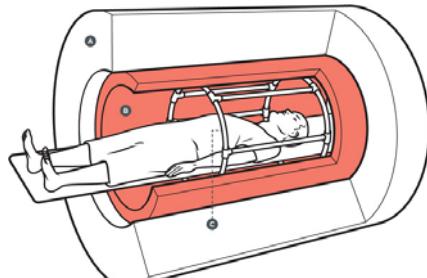
then, for times $t \geq 0$,

$$\begin{aligned} \mathbf{m}(t) &= R(\gamma B t) \mathbf{m}_0 \\ &= \begin{pmatrix} \cos(\gamma B t) & \sin(\gamma B t) & 0 \\ -\sin(\gamma B t) & \cos(\gamma B t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\gamma B t) \\ -\sin(\gamma B t) \\ 0 \end{pmatrix} \end{aligned}$$

Conceptually, a non-constant magnetic field $\mathbf{B}(t)$ can be broken down into very short time segments, δt . For short enough segments, \mathbf{B} will be constant in each segment and we can predict its effect as a precession by some small amount around a fixed axis (which might change its orientation between time segments). Practically this might prove difficult for most cases, and will require a numerical solution.

Spins Can Be Manipulated With Magnetic Fields: Ampere's Law

An MRI machine is basically just a collection of coils. We current is passed through a coil it generates a magnetic field, and it is through these magnetic fields that we control the nuclear magnetic moments and produce an image. There are three major coil groups in the magnet:



Main magnet coil (A), gradient coil (B) and body (RF) coil (C) inside a typical MRI scanner.

MRI-GENERATED FIELDS

The Main Field

A large cylindrical coil is wound along the patient's body. This coil is cooled with liquid helium and is superconducting, and can therefore carry large amounts of current without melting. Clinical scanners go up to 3 **Tesla**, which is about 60,000 times the Earth's magnetic field, which is 0.5 **Gauss** ($1 \text{ T} = 10^4 \text{ G}$). However, research scanners have already surpassed 10 T, although these are very expensive to build. The main field is usually called \mathbf{B}_0 and its direction is taken to coincide with the z-axis:

$$\mathbf{B}_0 = \begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix}.$$

Number Time. For a clinical MRI scanner, $B_0=3\text{T}$. A proton nucleus ($\gamma=2\pi\cdot42.57 \text{ kHz/mT}$) will precess at a frequency of $\nu = \gamma B_0 / 2\pi = 127 \text{ MHz}$, while a carbon nucleus ($\gamma=2\pi\cdot10.705 \text{ kHz/mT}$) will precess at about $\nu = \gamma B_0 / 2\pi = 32 \text{ MHz}$ about the main B_0 field. This precession frequency is called the **Larmor Frequency**.

The RF coils

The **radiofrequency (RF) coils** are capable of generating arbitrarily shaped, albeit weak (around $10 \mu\text{T}$ at most) field at the radiofrequency range. More precisely:

$$\mathbf{B}_{RF}(t) = \begin{pmatrix} B_{RF}(t)\cos(\phi_{RF}(t)) \\ B_{RF}(t)\sin(\phi_{RF}(t)) \\ 0 \end{pmatrix}.$$

We can shape the amplitude, $B_{RF}(t)$, and the phase, $\phi_{RF}(t)$, and create in theory any shape, although modern hardware limits our abilities somewhat (as noted earlier, peak $B_{RF}(t)$ is around $10 \mu\text{T}$, and $\frac{d\phi_{RF}}{dt} \sim \text{radiofrequency range}$, usually tens or hundreds of MHz).

The Gradient Coils

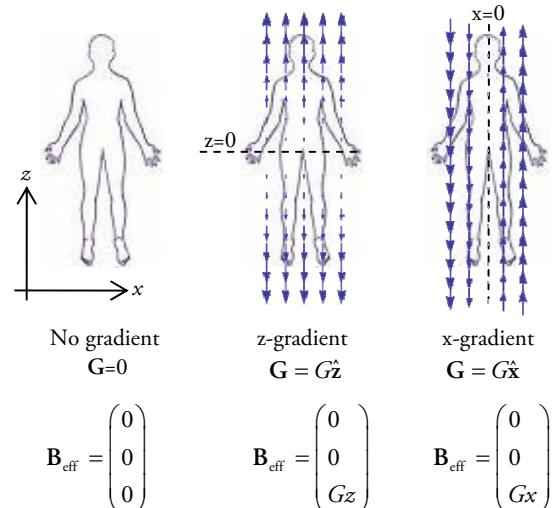
The **gradient coils** generate a linear, spatially varying magnetic field. So far, the RF and main fields have been spatially homogeneous, at least ideally. It is the gradient field that will enable us to image the sample. How precisely that will happen remains to be seen. For now, it suffices that we write down the general shape of the gradient field:

$$\mathbf{B}_{grad}(\mathbf{r}, t) = \begin{pmatrix} 0 \\ 0 \\ \mathbf{G}(t) \cdot \mathbf{r} \end{pmatrix}.$$

Note we can "shape" the gradient field by shaping $\mathbf{G}(t)$, by shaping the current passing through the gradient coils. However, they are built to always be linear in position, \mathbf{r} .

Number Time. The maximal gradient field strength is on the order of 10 mT/m , meaning over the human head ($\sim 0.2 \text{ m}$) one can create an additional z-field of about $10\cdot0.2 \text{ mT} \sim 1 \text{ mT}$.

It is important to understand visually what sort of fields the different gradient coils generate. The following illustration focuses on the case of a constant gradient:



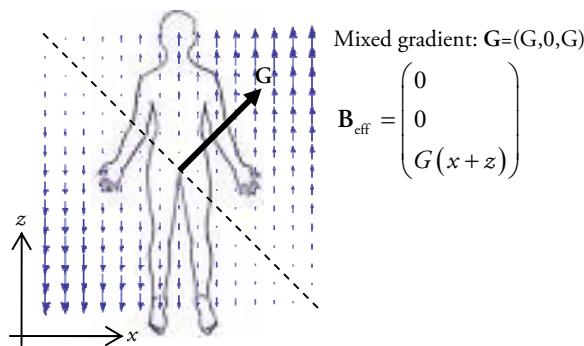
Effective field in the rotating frame for the cases of no gradient (left), z-gradient (middle) and x-gradient (right).

In all cases the gradient field superimposes a field pointing along the z-axis!

We can also turn on several gradient coils at once, generating a field which is a linear combination of the individual fields. For example, if we turn on both the x- and z-gradient fields at equal magnitude, the field will become

$$\mathbf{G} = \begin{pmatrix} G \\ 0 \\ G \end{pmatrix}, \quad \mathbf{B}_{\text{eff}} = \begin{pmatrix} 0 \\ 0 \\ G(x+z) \end{pmatrix}.$$

This is a linearly increasing field along an axis pointing along the direction of \mathbf{G} :



Putting It All Together

The general, combined laboratory-generated magnetic field felt by a microscopic spin is therefore:

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 + \mathbf{B}_{RF} + \mathbf{B}_{\text{grad}} = \begin{pmatrix} B_{RF}(t) \cos(\phi_{RF}(t)) \\ B_{RF}(t) \sin(\phi_{RF}(t)) \\ B_0 + \mathbf{G}(t) \cdot \mathbf{r} \end{pmatrix}$$

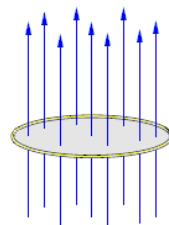
Microscopic Fields

The magnetic moments themselves create magnetic fields which affect each other. These will be treated in a short while.

SIGNAL RECEPTION

Time Varying Magnetic Fields Can Be Picked Up With A Coil: Faraday's Law

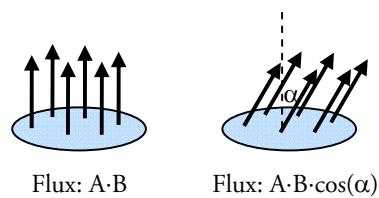
The **magnetic flux** through a coil equals the integral of the normal component of the magnetic field through the surface of a coil:



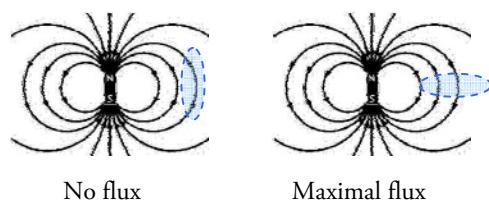
Mathematically, this amounts to a surface integral over the surface enclosed by the loop:

$$\phi = \int \mathbf{B} \cdot d\mathbf{S}.$$

Intuitively, this is the “amount of magnetic field lines crossing the coil.” For example, if we had a constant magnetic field \mathbf{B} normal to the coil, and the coil had area A , the magnetic flux through it would be $A \cdot B$. If \mathbf{B} were to make an angle α with the normal to the coil’s surface, the flux would be reduced to $A \cdot B \cdot \cos(\alpha)$:



Another example: consider placing a coil around a magnetic moment. In one orientation there would be no flux through the coil, while if we were to rotate the coil by 90° the flux would be maximal:



The importance of flux comes from **Faraday's law**:

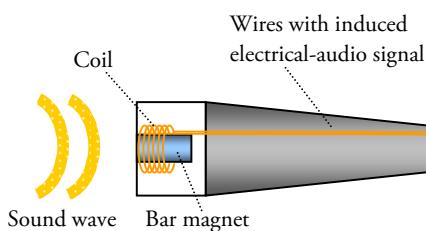
A time varying flux $\phi(t)$ through a coil will generate a voltage given by:

$$v = -\frac{d\phi}{dt}$$

(Faraday's Law)

Note that the amount of flux (ϕ) itself has no direct bearing on the generated voltage, and even if ϕ is large it might not generate any current if it is static.

This law underlies much of modern electricity and electronics, since it provides a mechanism for turning one type of energy into another. An example is the microphone: some microphones, known as dynamic microphones, are comprised of a diaphragm connected to a bar magnet, around which a coil is tightly wound. As sound waves oscillate the diaphragm they also physically move the magnet which changes the magnetic field's flux through the coil as a function of time. These oscillations are therefore reproduced in the electrical signal induced in the coil and recorded on tape (or, in modern hardware, on the computer):

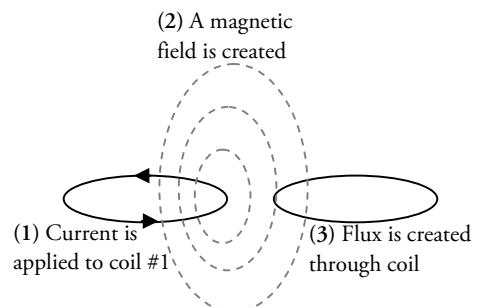


In our case, a precessing magnetic moment will create a precessing dipolar field around it – that is, a time-varying magnetic field. The dipolar field will rotate at the same angular velocity as the spin. A current will then be generated in a suitably-positioned coil, known as a **receiver coil**. Any receiver coil can also create a magnetic RF field by putting an oscillating current through it, making it a **transmitter coil**. Thus, any coil can be used for both reception and transmission (but not simultaneously).

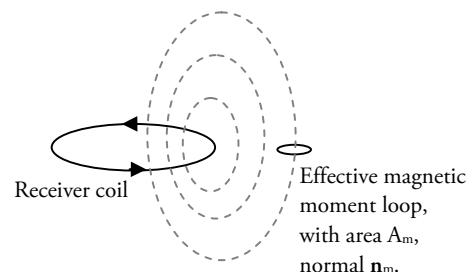
The Law Of Reciprocity: A Good Transmitter Is A Good Receiver

Calculating the signal explicitly using Faraday's law is tricky, so we will make use of a very neat trick known as the **principle of reciprocity**, by which the efficiency of a coil as a receiver is proportional to its efficiency as a transmitter.

When two coils are put next to each other, they will not only induce fluxes through themselves, but also in each other in what's known as **mutual inductance**



When you think about it, this is similar to the problem of signal reception in MRI. First, we can model the microscopic nuclear magnetic moment using an infinitesimal loop of current, since we remarked such a loop will create a magnetic moment $\mathbf{m}=(\text{area})\times(\text{current})$:



There is no requirement for the moment's loop to be co-planar with the receiver coil, nor do we assume the receiver coil is planar (it's just easier for me to draw a planar one!). Any current I_m through the moment's coil will create a magnetic moment given by

$$\mathbf{m} = A_m I_m \hat{\mathbf{n}}_m$$

The signal reception question can be formulated as follows:

Given a time dependent magnetic moment $\mathbf{m}(t)$, what voltage v_{rec} will be induced in the receiver coil?

We already know the answer (Faraday's law), but what we're going to prove now is that there is a simple way to calculate it that depends on the field created by the receiver coil itself

$$v_{rec} = -\mathbf{B}_{rec}(\mathbf{r}) \cdot \frac{d\mathbf{m}}{dt},$$

where $\mathbf{B}_{rec}(\mathbf{r})$ means the field created by the receiver coil at the position of the magnetic moment, \mathbf{r} , when we pass a unit current through the receiver (which is the opposite of what happens in reception!). What this says is that the voltage induced in the receiver coil is proportional to the strength of the field created by the receiver coil *if we use it as a transmitter*. In other words, a good transmitter is a good receiver! We prove this assertion below, although you can skip the proof. The above expression can be extended to a spatial distribution of moments by integrating over space:

$$v_{rec} = - \int_{body} \mathbf{B}_{rec}(\mathbf{r}) \cdot \frac{dM(\mathbf{r}, t)}{dt} dV$$

Proof. Let's go back to the two coils in the first diagram. The voltage induced in coil #2 is, by Faraday's law:

$$v_2 = -\frac{d\phi_{21}}{dt} = -\frac{d}{dt} \int_{coil 2} \mathbf{B}_1(\mathbf{r}, t) \cdot d\mathbf{S}_2.$$

The field $\mathbf{B}_1(\mathbf{r}, t)$ is the field created by passing a current $I_1(t)$ through coil #1. Since it is always proportional to the current, we can write

$$\mathbf{B}_1(\mathbf{r}, t) = I_1(t) \mathbf{B}_1(\mathbf{r})$$

$\mathbf{B}_1(\mathbf{r})$ is the magnetic field created through coil #2 when unit current is passed through coil #1. So:

$$v_2 = -\frac{d\phi_{21}}{dt} = -\frac{dI_1(t)}{dt} \int_{coil 2} \mathbf{B}_1(\mathbf{r}) \cdot d\mathbf{S}_2.$$

The quantity

$$L_{21} \equiv \int_{coil 2} \mathbf{B}_1(\mathbf{r}) \cdot d\mathbf{S}_2$$

is called the mutual inductance and depends only on the geometry of the coils, so we can write:

$$v_2 = -L_{21} \frac{dI_1}{dt}.$$

One can reverse the situation and pass a current through coil 2, inducing a voltage in coil 1:

$$\begin{aligned} v_1 &= -\frac{d\phi_{21}}{dt} = -\frac{d}{dt} \int_{coil 1} \mathbf{B}_2(\mathbf{r}, t) \cdot d\mathbf{S}_1 \\ &= -\frac{dI_2}{dt} \underbrace{\int_{coil 1} \mathbf{B}_2(\mathbf{r}) \cdot d\mathbf{S}_1}_{L_{12}} \equiv -L_{12} \frac{dI_2}{dt} \end{aligned}$$

The principle of reciprocity states the mutual inductances are the same:

$$L_{12} = L_{21} \equiv L_{mutual}.$$

This can be proved from first principles using Maxwell's equations which govern electromagnetism, although we won't try to prove it here.

Based on the previous discussion, the voltage in the receiver coil can be written as:

$$v_{rec} = -L_{mutual} \frac{dI_m}{dt}.$$

It's difficult to calculate L_{12} because this means we need to know the field created by the moment at each point through the coil. However, it's easy (well, easier ...) to calculate L_{21} – that is, the field induced at the position of the moment by passing a current through the receiver coil:

$$\begin{aligned} L_{mutual} &= \int_{moment coil} \mathbf{B}_{rec}(\mathbf{r}) \cdot d\mathbf{S}_m \\ &\approx \mathbf{B}_{rec}(\mathbf{r}) \cdot \int_{moment coil} d\mathbf{S}_m = \mathbf{B}_{rec}(\mathbf{r}) \cdot (A_m \hat{\mathbf{n}}_m) \end{aligned}$$

Thus:

$$\begin{aligned} v_{rec} &= -\mathbf{B}_{rec}(\mathbf{r}) \cdot (\mathbf{A}_m \hat{\mathbf{n}}_m) \frac{dI_m}{dt} \\ &= -\mathbf{B}_{rec}(\mathbf{r}) \cdot \frac{d(\mathbf{A}_m I_m \hat{\mathbf{n}}_m)}{dt} \\ &= -\mathbf{B}_{rec}(\mathbf{r}) \cdot \frac{d\mathbf{m}}{dt} \end{aligned}$$

MRI Happens In The Near Field

It is very important to keep in mind that almost all of the phenomena we will discuss in this course happen in the **near field**. This is a term used to describe distances that are small compared to the wavelengths involved. In general, any oscillating moment in free space with an angular frequency $\omega = 2\pi\nu$ would create electromagnetic waves with a wavelength

$$\lambda = \frac{c}{\nu}$$

In a vacuum we have $c \approx 3 \cdot 10^8 \frac{\text{m}}{\text{sec}}$, and for a hydrogen at 3T we have $\nu = \gamma B_0 \approx 127 \text{ MHz}$, implying

$$\lambda \approx 2.4 \text{ m}.$$

Detection at distances $\ll \lambda$ are said to be in the near field, which is precisely the case with MRI, in which the coils are placed as closely as possible to the subject.

The consequences of operating in the near field are subtle and we've made some hidden assumptions along the way, some of which you might have spotted:

1. We've assumed a magnetic moment creates a dipolar magnetic field $\mathbf{B}(\mathbf{r})$ which changes immediately when we rotate the moment. This neglects the fact that field changes propagate at the speed of light (in a vacuum), which is permissible in the near field.

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{3(\mathbf{m}(t) \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}(t)}{r^3}$$

2. Detection is driven by Faraday's law by magnetic flux through receiver coils. In the far field, detection occurs usually by having electromagnetic radiation picked up by causing electrons in an antenna to oscillate.

3. We've assumed our detector picks up a signal from the entire body (as far as $\mathbf{B}_{rec}(\mathbf{r}) \neq 0$).

$$v_{rec} = - \int_{\text{body}} \mathbf{B}_{rec}(\mathbf{r}) \cdot \frac{d\mathbf{M}(\mathbf{r}, t)}{dt} dV$$

The speed of light through a medium such as human tissue differs from that in vacuum, and is given by

$$\nu = \frac{c}{n} = \frac{c}{\sqrt{\epsilon_r \mu_r}}$$

where c is the speed of light in vacuum, n the index of refraction, and ϵ_r , μ_r the (frequency dependent) relative permittivity and permeability of the medium. This makes wavelengths shorter and the near-field criterion more difficult to fulfill:

$$\lambda = \frac{c}{n\nu} = \frac{c}{\nu\sqrt{\epsilon_r \mu_r}}.$$

For clinical field strengths (1.5 T and 3 T) this remains a reasonable-to-excellent approximation, depending on tissue type, but for ultra high field imaging (7 T and above) this assumption breaks down and correspondingly artifacts can be seen in the image. The following table shows some approximate values for these quantities at 1.5, 3 and 7 Tesla:

Material	ϵ_r	μ_r	Field (T)	λ (m)
Vacuum	1	1	1.5	4.7
	1	1	3	2.3
	1	1	7	1.0
	97	1	1.5	0.48
	74	1	3	0.27
	60	1	7	0.13
White matter	68	1	1.5	0.57
	53	1	3	0.32
	44	1	7.0	0.15
	86	1	1.5	0.51
	73	1	3	0.27
	65	1	7	0.12
Fat	6	1	1.5	1.92
	5.9	1	3	0.97
	5.6	1	7	0.43

It should be noted that μ_r is not truly unity but very close, such that $\sqrt{\mu_r} = 1$ for all practical purposes. The true value of μ_r , however cannot be neglected when calculating susceptibility artifacts (which we will not take upon ourselves in this course), since magnetic resonance is very sensitive to even small distortions in the main magnetic field.

Another effect that must be taken into account is the conductance of the body's tissues: an electromagnetic field with frequency ν will get absorbed in any conductor with conductance σ (in ohms·meter) after traveling for a distance given by the **skin depth**:

$$\delta = \sqrt{\frac{1}{\pi \sigma \nu \mu_r \mu_0}}$$

A CONSTANT MAGNETIC FIELD POLARIZES THE SPINS: NUCLEAR PARAMAGNETISM

The Energy Of A Magnetic Moment In A Magnetic Field Is Minimal When Parallel To The Field

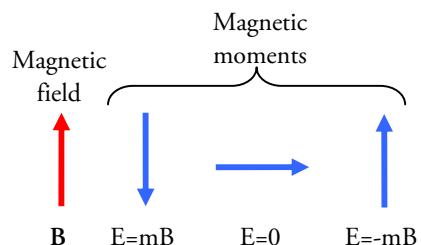
We have so far looked at single isolated spins. We now move on to describing large statistical ensembles of spins.

If you take a compass, which is nothing more than a magnetized iron needle, having a magnetic moment itself, it will align itself along the earth's magnetic field. This illustrates an important point of interest which we'll make use of: magnetic moments tend to align themselves along the magnetic field they are in when in equilibrium, in which they minimize the moment's energy:

$$E = -\mathbf{m} \cdot \mathbf{B} = -mB \cos(\theta).$$

where θ is the angle between \mathbf{m} and \mathbf{B} . This phenomena is known as **paramagnetism**.

The energy E is at its minimum when \mathbf{m} and \mathbf{B} are parallel, and maximal when they are antiparallel:



A fundamental principle of statistical mechanics states that systems tend to minimize their energy, which explains why the compass needle aligns along \mathbf{B} . However, one should be mindful that whether or not a macroscopic magnetic moment will actually align is dependent on competing interactions. For example, thermal motion might tend to randomize a magnetic moment's direction. **Question:** why do microscopic spins precess about the magnetic field, instead of align along it?

Answer: Aligning along the field is a macroscopic behavior that is brought about by friction (in the cases of a needle, the friction is caused by the

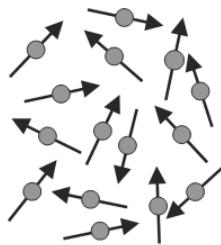
needle's mount). An idealized microscopic spin in free space does not experience frictional sources.

Bulk Magnetization: We Always Think In Terms Of Large Groups Of Spins

In an MRI machine one cannot study single spins or single molecules, due to the low sensitivity of magnetic resonance. A typical voxel is $\sim \text{mm}^3$, and it often contains many many spins. MRI therefore studies the properties of nuclear spins *in bulk*.

Suppose you have N molecules in a volume V , each having a magnetic moment \mathbf{m}_i . Recall that the moments are all vectors, so we can imagine a vector "attached" to each atom. In general, without the large external field of the MRI machine, they would all point in different directions:

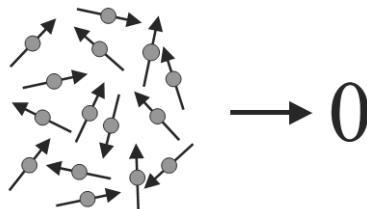
Schematic representation of an ensemble of microscopic magnetic moments. Each circle represents the magnetic moment of, say, a water molecule



The **bulk magnetization** \mathbf{M} of the volume V is defined as the (vector!) sum over all elements in the volume:

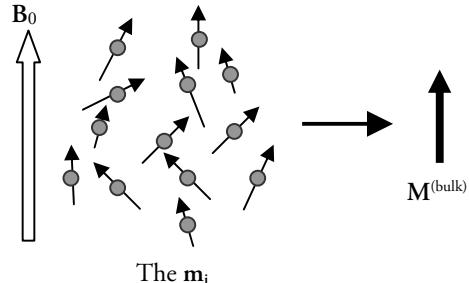
$$\mathbf{M}^{(\text{bulk})} = \sum_{i=1}^N \mathbf{m}_i$$

It is $\mathbf{M}^{(\text{bulk})}$ that MRI studies. In the above example, $\mathbf{M}^{(\text{bulk})} = 0$ because the spins cancel out each other:



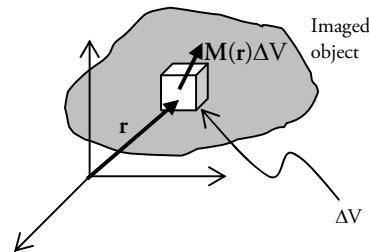
Upon the application of an external field, the spins tend to align along the field – although thermal motion will prevent them from doing so

completely. A "snapshot" of the spins in the presence of an external field might look like this:



The \mathbf{m}_i

Let us define the **bulk magnetization per unit volume**, $\mathbf{M}(\mathbf{r})$, such that if we take a small volume ΔV around the point \mathbf{r} then $\mathbf{M}(\mathbf{r})\Delta V = \mathbf{M}^{(\text{bulk})}$:



We will use the capital letters \mathbf{M} , $\mathbf{M}^{(\text{bulk})}$ to denote the macroscopic bulk magnetism properties, as opposed to \mathbf{m} which we will reserve for microscopic moments.

What volume ΔV should we use? On the one hand, we want enough spins in ΔV to make it statistically meaningful – that is, we want the variance of our fluctuations to be small as possible. Put another way, we want \mathbf{M} to vary smoothly if we start shifting our volume of interest around.

Volume of Water	Number of spins
Liter = 10^3 cm^3	10^{25}
cm^3	10^{22}
mm^3	10^{19}
μm^3	10^{10}
$(10 \text{ nm})^3$	10^4
nm^3	10

So it seems smaller than a box with sides 10 nm is shaky.

On the other hand, is there an upper limit on ΔV ? A natural choice might be a voxel ($\sim \text{mm}^3$), but there is a lot of variance inside a voxel.

Sometimes to understand the signal originating from a single voxel we need to think in terms of what happens inside the voxel because there is a lot happening inside that mm^3 . So, in general, we stick to the smallest ΔV we can take (say, $(10 \text{ nm})^3$).

From now on when we talk about the magnetization vector we will take it to mean the bulk (macroscopic) magnetization vector per unit volume, unless specifically noted otherwise. At times I will remark how the macroscopic picture ties in with the microscopic one.

Equilibrium Magnetization

We now come to the very important problem of calculating the bulk magnetic moment of a sample placed in a constant magnetic field (such as the 3 Tesla field of an MRI scanner) at thermal equilibrium.

At thermal equilibrium, the probability of the system being in a state with energy E is given by Boltzmann's distribution:

$$\Pr(E) = \frac{1}{Z} e^{-E/kT}$$

where Z is a constant number independent of the energy or kT , given by:

$$Z = e^{-E_1/kT} + \dots + e^{-E_N/kT},$$

where the system has N states having energies E_1, \dots, E_N . The probability of being in state i is:

$$\Pr(i) = \frac{1}{Z} e^{-E_i/kT}$$

Note that our definition of Z implies that:

$$\Pr(1) + \Pr(2) + \dots + \Pr(N) = 1$$

or

$$\frac{1}{Z} e^{-E_1/kT} + \dots + \frac{1}{Z} e^{-E_N/kT} = 1.$$

This is the only point in our lectures where we'll use the quantum-mechanical nature of spin. Quantum mechanics tells us a spin S in a field B_0 has $2S+1$ energy levels:

$$E_n = n\hbar\gamma B_0, \quad n = -S, -S+1, \dots, S.$$

We now ask: *what is the average magnetic moment of (one) such spin at thermal equilibrium?* By symmetry, we expect $\langle m_x \rangle = \langle m_y \rangle = 0$ in thermal equilibrium. For the z -component,

$$\langle m_z \rangle = \sum_{n=-S}^S m_n \Pr(E_n) = \sum_{n=-S}^S \left(-\frac{n\gamma\hbar}{Z} \right) \exp\left(-\frac{n\hbar\gamma B_0}{kT}\right)$$

If we knew that, then for N non-interacting spins at equilibrium,

$$\mathbf{M}^{(\text{bulk})} = N \langle m_z \rangle \hat{\mathbf{z}},$$

which would be our **equilibrium magnetic moment**. So we really just need to compute $\langle m_z \rangle$. Our assumption of non-interacting spins is a bit suspect, since the nuclear spins "talk" via dipolar coupling, but one can prove using quantum mechanics this holds even in the presence of dipolar and other interactions.

The expression for $\langle m_z \rangle$ can be simplified considerably if we remember that, for $a \ll 1$:

$$e^a \approx 1 + a \\ (\text{e.g. } e^{-0.01} \approx 0.99)$$

In our case, at room temperature (homework!),

$$\frac{n\hbar\gamma B_0}{kT} \ll 1 \quad (\text{for all } n = -S, \dots, S),$$

so we can simplify:

$$Z = \sum_{n=-S}^S \exp\left(-\frac{n\hbar\gamma B_0}{kT}\right) \approx \sum_{n=-S}^S \left(1 - \frac{n\hbar\gamma B_0}{kT}\right) = (2S+1) \\ \langle m_z \rangle \approx \sum_{n=-S}^S \frac{-n\gamma\hbar}{(2S+1)} \left(1 - \frac{n\hbar\gamma B_0}{kT}\right) = \sum_{n=-S}^S \frac{n\gamma\hbar}{(2S+1)} \frac{n\hbar\gamma B_0}{kT}$$

Using the algebraic identity

$$\sum_{n=-S}^S n^2 = \frac{1}{3} S(S+1)(2S+1)$$

yields:

$$\langle m_z \rangle \approx \sum_{n=-S}^S \frac{n\gamma\hbar}{(2S+1)} \left(1 - \frac{n\hbar\gamma B_0}{kT}\right) = -\frac{\gamma^2 \hbar^2 S(S+1)}{3kT} B_0.$$

and, for N spins,

$$M_0^{(\text{bulk})} = \frac{N(\gamma\hbar)^2 S(S+1)}{3kT} B_0$$

Equilibrium macroscopic magnetic moment

Number Time. 1 mL of water will weigh about 1 gram and have about $5 \cdot 10^{-9}$ moles of water (which has a molecular weight of about 18 gr/mol), or about $N=6 \cdot 10^{22}$ atoms. At room temperature ($kT \sim 4 \cdot 10^{-21} \text{ m}^2 \cdot \text{kg} \cdot \text{s}^{-1}$) and a field of $B_0=3$ Tesla, we have ($\gamma=2\pi \cdot 42.576$ kHz/mT, $S=1/2$), $M_0 \approx 10^{-8}$ J/T.

The above equation gives the macroscopic magnetic moment for N spins, but one could equally talk about magnetic moment per unit volume by dividing both sides by the volume, V. Then N would turn into the number density of spins, also known as the proton density when dealing with protons (number of nuclei per unit volume):

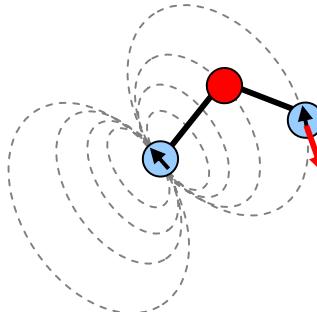
$$M_0 = \frac{PD \cdot (\gamma\hbar)^2 S(S+1)}{3kT} B_0$$

SPIN INTERACTIONS LEAD TO RELAXATION PHENOMENA

Spins Are Subjected To Microscopic Fluctuating Magnetic Fields Due To Their Thermal Motion

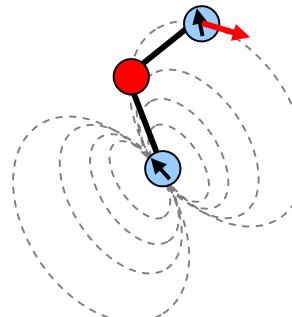
Each microscopic nuclear magnetic moment m "sees" a magnetic field made up of two components: the macroscopic field generated by the coils in the lab, and the microscopic fields given off by its surroundings. For example, the dipolar field generated by one nuclear spin in a

molecule will be felt by other nuclear spins in the same molecule.



Shown here is the magnetic field (red arrow) felt by one spin due to the dipolar field of the other spin in an H₂O molecule.

It's very important to realize that the orientation of the nuclear magnetic moment has nothing to do with the molecular orientation: if you rotate the molecule by 90°, the nuclear moment will not change, since it's not related to the nuclear charge or mass distribution; it "lives" in its own space and "talks" to the environment only through the magnetic fields it feels and emits:



Upon rotation of the molecule, the spins (black arrows) do not change their orientation. Consequently, the spin feels a different magnetic field, in both magnitude and direction.

Since most of the water molecules in the body are in the liquid state in the extra and intracellular matrices³ –All molecules rotate and tumble around very rapidly. A small water molecule might

³ This is actually not entirely correct, since water molecules often get "stuck" to cell membranes or confined in tight spaces. We will look more into this in later lectures.

Tissue Type	Nuc.	Mol.	1.5 T		3 T		7 T	
			T ₁	T ₂	T ₁	T ₂	T ₁	T ₂
Gray Matter ^b	¹ H	H ₂ O	1188 ± 69	95 ± 8	1820 ± 114	99 ± 7	2132 ± 103	
White Matter ^b	¹ H	H ₂ O	656 ± 16	72 ± 4	1084 ± 45	69 ± 3	1220 ± 36	
Cerebrospinal Fluid ^b	¹ H	H ₂ O	4070 ± 65				4425 ± 137	
Blood ^b	¹ H	H ₂ O	1540 ± 23	290 ± 30	1932 ± 85	275 ± 50	2587 ± 283	
Kidney Cortex ^a	¹ H	H ₂ O	966 ± 58	87 ± 4	1142 ± 154	76 ± 4		
Kidney Medula ^a	¹ H	H ₂ O	1412 ± 58	85 ± 11	1545 ± 142	81 ± 8		
Liver ^a	¹ H	H ₂ O	586 ± 39	46 ± 6	809 ± 71	34 ± 4		
Cartilage, 0° ^d	¹ H	H ₂ O	1024 ± 70	30 ± 4	1168 ± 18	27 ± 3		
Cartilage, 55° ^d	¹ H	H ₂ O	1038 ± 67	44 ± 5	1156 ± 10	43 ± 2		
Bone marrow (L4 vertebra) ^a	¹ H	H ₂ O	549 ± 52	49 ± 8	586 ± 73	49 ± 4		
Prostate ^a	¹ H	H ₂ O	1317 ± 85	88 ± 0	1597 ± 42	74 ± 4		
Subcutaneous fat ^a	¹ H	Fat	343 ± 37	58 ± 4	382 ± 13	68 ± 4		
NAA CH ₃ (GM) ^c	¹ H	NAA	1270 ± 50		1470 ± 80	269 ± 7		
NAA CH ₃ (WM) ^c	¹ H	NAA	1360 ± 60		1400 ± 150	374 ± 9		

Typical T₁ and T₂ relaxation times from the literature, in milliseconds, in humans. The ± sign indicates standard deviation of the cohort examined. Note that variations may occur within a particular tissue (e.g. cortical vs. deep gray matter), and that numbers provided from different papers might originate from different regions within the same tissue. Also, some skepticism should be practiced when using values obtained for flowing/pulsating media, such as the cerebrospinal fluid.

^a From: Bazelaire et. al., Radiology 230(3):652-659 (2004)

^b T₁ values at 1.5 T and 7 T taken from Rooney et. al., Magn. Reson. Med. 57:308-318 (2007).

T₁, T₂ values at 3 T taken from Rooney et. al., Magn. Reson. Med. 57:308-318 (2007) and Stanisz et. al., Magn. Reson. Med 54:507-512 (2005).

^c T₂ values at 3T taken from Kirov et. al., Magn. Reson. Med. 60:790-795 (2008).

T₁ values at 1.5T and 3T from Ethofer et. al., Magn Reson Med 50:1296-1301 (2003)

^d From: Stanisz et. al., Magn Reson Med 54:507-512 (2005).

perform a rotation on picosecond timescales, while larger molecules would rotate more slowly. This molecular rotation leads, by the arguments just laid out, to fluctuating microscopic fields.

Fluctuating Microscopic Fields Lead To Decoherence (T₂) And Return to Thermal Equilibrium (T₁)

The magnetic field felt by a microscopic nuclear magnetic moment can be subdivided into two parts, macroscopic and microscopic:

$$\mathbf{B}(t) = \mathbf{B}_{macro}(t) + \mathbf{B}_{micro}(t),$$

where the macroscopic fields are those generated by the laboratory coils and controlled by the

scientist, and the microscopic fields are those fluctuating fields created by other spins in the molecule, electrons, and so forth. Consequently, the Bloch equations which describe the spin's precession become:

$$\frac{d\mathbf{m}}{dt} = \gamma \mathbf{m} \times \mathbf{B} = \gamma \mathbf{m} \times \mathbf{B}_{macro}(t) + \gamma \mathbf{m} \times \mathbf{B}_{micro}(t).$$

Now assume we have N magnetic moments, \mathbf{m}_1 , \mathbf{m}_2 , ..., \mathbf{m}_N , each experiencing its own unique microscopic field, but all experiencing the same macroscopic one:

$$\begin{aligned}\frac{d\mathbf{m}_1}{dt} &= \gamma \mathbf{m}_1 \times \mathbf{B}_{macro}(t) + \gamma \mathbf{m}_1 \times \mathbf{B}_{micro}^{(1)}(t) \\ \frac{d\mathbf{m}_2}{dt} &= \gamma \mathbf{m}_2 \times \mathbf{B}_{macro}(t) + \gamma \mathbf{m}_2 \times \mathbf{B}_{micro}^{(2)}(t) \\ &\vdots \\ \frac{d\mathbf{m}_N}{dt} &= \gamma \mathbf{m}_N \times \mathbf{B}_{macro}(t) + \gamma \mathbf{m}_N \times \mathbf{B}_{micro}^{(N)}(t)\end{aligned}$$

We now sum over multiple microscopic spins:

$$\sum_{n=1}^N \frac{d\mathbf{m}_n}{dt} = \gamma \sum_{n=1}^N \mathbf{m}_n \times \mathbf{B}_{macro}(t) + \gamma \sum_{n=1}^N \mathbf{m}_n \times \mathbf{B}_{micro}^{(n)}(t)$$

Since \mathbf{B}_{macro} is common to all summed terms, and since the derivative of the sum equals the sum of the derivatives, we can substitute the microscopic moments by the macroscopic one, $\mathbf{M} = \sum_{n=1}^N \mathbf{m}_n$ and obtain:

$$\frac{d\mathbf{M}}{dt} = \gamma \mathbf{M} \times \mathbf{B}_{macro}(t) + \gamma \sum_{n=1}^N \mathbf{m}_n \times \mathbf{B}_{micro}^{(n)}(t)$$

The last term on the RHS represents the effects of the fluctuating fields and is intractable really. Physically speaking, these fluctuating magnetic fields are the source of (1) **decoherence** (i.e. loss of signal) and (2) **thermalization** (return to thermal equilibrium). Luckily, phenomenologically these effects can be respectively embodied by two constants, T_2 and T_1 , respectively, which can be integrated into the Bloch equations using simple terms:

$$\begin{cases} \dot{M}_x = \gamma(M_y B_{macro,z} - M_z B_{macro,y}) - \frac{M_x}{T_2} \\ \dot{M}_y = \gamma(M_z B_{macro,x} - M_x B_{macro,z}) - \frac{M_y}{T_2} \\ \dot{M}_z = \gamma(M_x B_{macro,y} - M_y B_{macro,x}) - \frac{M_z - M_0}{T_1} \end{cases}$$

We will omit the subscript *macro* and take the magnetic field appearing in the Bloch equations to signify only the macroscopic (lab-generated) magnetic field.

M_0 is the thermal equilibrium value of the magnetization, as can be seen by turning “off” the macroscopic RF and gradient fields, setting the time derivatives to 0 and solving:

$$\begin{cases} 0 = \gamma M_y B_0 - \frac{M_x}{T_2} \\ 0 = -\gamma M_x B_0 - \frac{M_y}{T_2} \\ 0 = -\frac{M_z - M_0}{T_1} \end{cases} \xrightarrow{\text{solve}} \begin{cases} M_x = 0 \\ M_y = 0 \\ M_z = M_0 \end{cases}$$

A table of some T_1 and T_2 values has been compiled above. We note that for most tissues, T_1 is on the order of a second, while T_2 is on the order of 100 ms. Furthermore, T_1 tends to increase with increasing field strength, while T_2 tends to decrease. The field-dependence of T_1 and T_2 will await a further chapter which will discuss T_1 and T_2 as sources of contrast.

T₂ Leads To Decoherence

To gain a better understanding of the sort of effect T_2 has on the spins, let us set the macroscopic laboratory field to 0 and examine the time evolution of the magnetization.

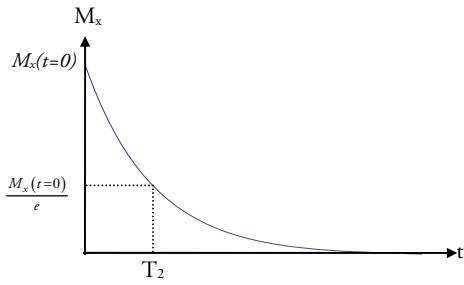
$$\begin{cases} \dot{M}_x = -\frac{M_x}{T_2} \\ \dot{M}_y = -\frac{M_y}{T_2} \\ \dot{M}_z = -\frac{M_z - M_0}{T_1} \end{cases}$$

One interesting this is that the transverse (x, y) and longitudinal (z) components of the magnetization become decoupled: M_z does not feature in the equations for M_x and M_y , and M_x and M_y do not appear in the equation for M_z .

The equations for M_x and M_y have simple solutions:

$$\begin{aligned} M_x(t) &= M_x(t=0) e^{-t/T_2} \\ M_y(t) &= M_y(t=0) e^{-t/T_2} \end{aligned}$$

This means that, whatever magnetization we start out with, it will decay with a time constant T_2 to zero:



This is called **decoherence**, and represents the physical fact that, unless something specific is done, the spins will point in all possible directions perpendicular to the MRI's static B_0 field, since there is no reason – energetic preference – for them to align in any single particular direction. The time T_2 can be thought of the time it takes M_x (or M_y) to drop to $1/e \sim 37\%$ of its initial value.

T₁ Leads To Thermal Equilibrium

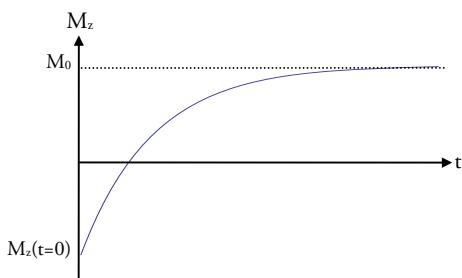
At thermal equilibrium the spins align themselves along the external B_0 field. This is brought about by T_1 relaxation. The solution to the equation involving M_z is:

$$M_z(t) = M_z(t=0)e^{-t/T_1} + (1 - e^{-t/T_1})M_0.$$

We see that, for $t \gg T_1$,

$$M_z(t \gg T_1) \approx M_0.$$

Thus, whatever longitudinal magnetization we start out from at $t=0$, it will converge back to its thermal equilibrium value M_0 :



B_0 Inhomogeneity Leads To Additional Transverse Decay (T_2')

An MRI magnet is built to yield a homogeneous field over a volume roughly the size of the human head. The ability of NMR and MRI to discern changes of \sim Hz to the proton frequency means severe constraints are placed on the homogeneity. Indeed, a change of a single Hz would correspond to a change in the main field given by

$$\gamma \Delta B = 1 \text{ Hz}$$

or

$$\Delta B \approx 0.02 \mu T.$$

This is

$$\frac{\Delta B}{B_0} \approx 6 \cdot 10^{-9}.$$

This is an incredibly difficult demand on the hardware: we need it to be homogeneous to about 0.01 ppm over a head-sized volume! The main coil's imperfections make it impossible to achieve. To approximate this requirement, special **passive shims** – pieces of iron – are added to the magnet to "shape" the main field.

Even if a perfect magnet is constructed, once we put in a sample, be it a human or an inanimate object, the main field will get distorted and its homogeneity would get ruined. Microscopically, human tissue is **diamagnetic**. This means an external field such as B_0 will induce magnetic moments in matter (of course the moments will induce a magnetic field which will create further moments which will induce further fields ... so the full solution must be self-consistent). The additional moment-induced field distorts the main field. The phenomenon is known as **magnetic susceptibility**.

The effect of these inhomogeneities can be quite intricate, but they always lead to a decay of the signal. We'll delve into specific models later on, but for now we'll just state that on a small scale – on the order of a voxel or smaller – it can usually be modeled by swapping T_2 by a shorter time, T_2^* :

$$\frac{1}{T_2} \rightarrow \frac{1}{T_2^*} .$$

In almost all realistic cases, T_2^* can be written down as a sum of two contributions: the microscopic decay and decay effects due to inhomogeneity:

$$\frac{1}{T_2} \rightarrow \frac{1}{T_2^*} = \underbrace{\frac{1}{T_2}}_{\text{microscopic, due to thermal motion of water molecules}} + \underbrace{\frac{1}{T_2'}}_{\text{meso/macrosopic, due to spatial variations in the magnetic field}}$$