**Non-Equilibrium Continuum Physics** **HW set #1**

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# **Mathematical Preliminaries** - Solution

## A general note

The purpose of the HW exercises is to give you hands-on experience with the course material. We try hard to ask questions that requite a conceptual process of understanding, rather than technical computation. Whenever some complicated calculations are required, please remember that it is only in order to convey the mathematical structure of the physical problems that we tackle, a structure that might elude the "passive listener" in the classroom. Accordingly, in the answers you hand in we do not require detailed calculations, unless they are crucial for the understanding.

## Tensor integration - Archimedes law

1. Fluids exert forces on bodies that are submerged in them. At each point on the body's surface, denote the local normal by $\hat{\boldsymbol{n}}$. The force per unit area exerted by the fluid is given by $f_i = \sigma_{ij} n_j$. $\sigma_{ij}$ is called the *stress tensor* of the fluid, and we'll deal with it extensively in the course.

   Consider a stationary (hydro-static), isotropic fluid that occupies the bottom half-space $z < 0$. The fluid is subject to a constant gravitational field $-g\hat{\boldsymbol{z}}$. At $z = 0$, we have $\sigma_{ij} = 0$.

   (a) The off-diagonal elements of $\sigma_{ij}$ are called *shear stresses*. Almost by definition, in a stationary fluid the shear stresses must vanish. Therefore, for $i \neq j$ we must have $\sigma_{ij} = 0$ *for every choice of coordinate system*. Prove that this implies $\sigma_{ij} = -p(\boldsymbol{r})\,\delta_{ij}$, where $p(\boldsymbol{r})$ is a scalar field. Note: $p = -\frac{1}{3}\,\mathrm{tr}(\sigma)$ is called *pressure*.

Solution

Choose any coordinate system. Since the shear stresses vanish, we can write

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}$$

Let's rotate around the $z$ axis, by:

$$\boldsymbol{R} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{R}^T \boldsymbol{\sigma} \boldsymbol{R} = \begin{pmatrix} \sigma_{11}\cos\alpha^2 + \sigma_{22}\sin\alpha^2 & (\sigma_{22}-\sigma_{11})\cos\alpha\sin\alpha & 0 \\ (\sigma_{11}-\sigma_{22})\cos\alpha\sin\alpha & \sigma_{22}\cos\alpha^2 + \sigma_{11}\sin\alpha^2 & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}.$$

Since the off-diagonal terms must vanish in this basis too, we have $\sigma_{11} = \sigma_{22}$. The same works for $\sigma_{33}$, and we see that $\boldsymbol{\sigma}$ is proportional to $\delta_{ij}$.
A different way, in the spirit of our discussion in the TA session: differentiate $\boldsymbol{\sigma}(\alpha)$ with respect to $\alpha$ and plug in $\alpha = 0$:

$$\partial_\alpha \boldsymbol{\sigma}|_{\alpha=0} = \boldsymbol{L}^z\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{L}^z = [\boldsymbol{\sigma}, \boldsymbol{L}^z] = \begin{pmatrix} 0 & \sigma_{yy} - \sigma_{xx} & 0 \\ -\sigma_{yy} + \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

And since $\boldsymbol{\sigma}$ is always be diagonal, its derivative cannot have off-diagonal elements, so we must have $\sigma_{yy} = \sigma_{xx}$.

(b) By considering the force balance on a small cube of fluid and the translational symmetries of the system, show that the stress field satisfies the equation

$$\partial_z \sigma_{zz}(x, y, z) = -\rho g$$

where $\rho$ is the fluid's density. Together with the results of (1a), conclude that the stress tensor is given by $\sigma_{ij} = -\rho g z \delta_{ij}$.

Solution

Due to symmetry, $\partial_x \sigma_{ij} = \partial_y \sigma_{ij} = 0$. The total force on a small cube of fluid must vanish, because the situation is static. The gravitational force is $\rho g \cdot dx \cdot dy \cdot dz$. The vertical force exerted on this cube from the lower side is $\sigma_{zz}(x, y, z) dx \cdot dy$, and the force exerted by the upper side is $-\sigma_{zz}(x, y, z + dz)$. Therefore, force balance tells us that

$$[-\sigma_{zz}(x, y, z + dz) + \sigma_{zz}(x, y, z)]\, dx \cdot dy = \rho g\, dx \cdot dy \cdot dz$$
$$\frac{\sigma_{zz}(x, y, z + dz) - \sigma_{zz}(x, y, z)}{dz} = -\rho g$$

Taking the limit $dz \to 0$, one gets the differential equation $\partial_z \sigma_{zz} = -\rho g$, the solution of which is clearly $\sigma_{zz} = -\rho g z$. Since we know already that $\boldsymbol{\sigma} \propto \delta_{ij}$ we immediately have $\sigma_{ij} = -\rho g z \delta_{ij}$.

(c) Consider a body of arbitrary shape and of volume $V$, which is submerged in the fluid. Calculate the magnitude and direction of the total force exerted by the fluid on the body by integrating $\sigma_{ij} n_j$ over the surface of the body. This force is called the *Buoyancy force.*

Solution

Denote the space occupied by the body by $\Omega$ and its surface by $\partial\Omega$. The total force is

$$\vec{F} = -\int_{\partial\Omega} \boldsymbol{\sigma}\hat{\boldsymbol{n}} dS \ .$$

Note the minus sign, because $\boldsymbol{\sigma}\hat{\boldsymbol{n}}$ (for outwards-pointing normal) is the force exerted by the body on the fluid, and we want the force exerted by the fluid on the body. By Gauss' theorem this is equal to

$$\vec{F} = -\int_{\Omega} \operatorname{div} \boldsymbol{\sigma} dV = \int_{\Omega} \rho g \vec{e}_z dV = \rho g \vec{e}_z \int_{\Omega} dV = \rho g V \vec{e}_z$$

where $\vec{e}_z$ is the unit vector in the $z$ direction.

Food for thought: how do you justify the fact that we integrate $\operatorname{div} \boldsymbol{\sigma}$ over the volume $V$? Strictly speaking, we do not know what is $\boldsymbol{\sigma}$ inside $V$ since there is no fluid there. I mean, inside $V$ there is a body which we didn't say anything about, and we only know $\operatorname{div} \boldsymbol{\sigma}$ outside $V$. Inside it might be a crazy function, right?

(d) Imagine that we remove the body, so the volume that was previously occupied by it is now filled with fluid. Give a one-line derivation of the result you obtained in (1c) by analyzing this new setting (hint: the situation is *static*).

Solution

The force exerted on the fluid must exactly balance its weight. Therefore

$$\vec{F} = mg\vec{e}_z = \rho g V \vec{e}_z \ .$$

(e) *Bonus*: Stand up and shout out loud "Eurika!!" (*Note: only filmed evidence will be considered for bonus purposes*).

Solution

There are many possible solutions to this question. One of them is shown here: `https://www.youtube.com/watch?v=JUxJBgJ5FJ4`. It's not the best one, it's simply the only one that was uploaded to YouTube.

# Tensor differentiation - Stress tensor of a fluid

2. In the TA session we derived the Navier-Stokes equation

$$\rho\left(\partial_t + \vec{v}\cdot\vec{\nabla}\right)\vec{v} = \vec{\nabla}P + \eta\nabla^2\vec{v} + \mu\vec{\nabla}\left(\nabla\cdot\vec{v}\right)$$

In class, we will derive the formula for the material derivative

$$\frac{DA}{Dt} = \frac{\partial A}{\partial t} + \vec{v}\cdot\nabla A\ ,$$

where $A$ is a tensor. So we see that the left-hand-side of the Navier-Stokes equation is nothing but the material derivative of the velocity field, that is, *inertia*. Find a tensor $\boldsymbol{\sigma}$ such that its divergence is the (minus of) right-hand-side of the NS equation. Thus the equation will take the form of a continuity equation

$$\rho\frac{D\vec{v}}{Dt} + \operatorname{div}\boldsymbol{\sigma} = 0\ .$$

Like in Question 1, the tensor $\boldsymbol{\sigma}$ is called the stress tensor, and its divergence is the total force exerted on a fluid element. Thus, the NS equation is nothing but a fancy way to write $F = ma$.

*Bonus:* There is more than one tensor whose divergence is the RHS of the NS equation, that is, the solution to this question is not unique. However, there's an extra physical requirement, which is that $\boldsymbol{\sigma}$ will be symmetric $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$. Can you find a symmetric $\boldsymbol{\sigma}$?

Solution

An immediate solution is

$$\begin{aligned}
\vec{\nabla}P + \eta\nabla^2\vec{v} + \mu\vec{\nabla}\left(\nabla\cdot\vec{v}\right) &= \partial_i P + \eta\,\partial_j\partial_j v_i + \mu\,\partial_i\partial_j v_j && (\star)\\
&= \partial_j\,\delta_{ij}\,P + \eta\,\partial_j\partial_j v_i + \mu\,\partial_i\partial_j v_j\\
&= \partial_j\left(\delta_{ij}\,P + \eta\,\partial_j v_i + \mu\,\partial_i v_j\right)\\
&= \operatorname{div}\left(P\,\boldsymbol{I} + \eta\,\nabla\vec{v} + \mu\left(\nabla\vec{v}\right)^T\right) && (\star\star)
\end{aligned}$$

This, of course, is not symmetric.

Here's how one gets a symmetric tensor from physical considerations: If we forget for a moment about the $\nabla P$ term, we see from the structure of Eq. $(\star)$ that $\boldsymbol{\sigma}$ should be a linear function of the velocity gradient $\nabla\vec{v}$. This is not surprising because the NS equation was derived by expansion in small powers of $\vec{v}$ and assuming isotropy. So we know $\boldsymbol{\sigma}$ should be of the form $\boldsymbol{\sigma} = C_{ijkl}\partial_k v_l$, where $\boldsymbol{C}$ is a 4-rank isotropic tensor. We therefore write $\boldsymbol{C}$ as a linear combination of the rank-4 isotropic tensors that we found in the TA session:

$$\begin{aligned}\boldsymbol{\sigma} &= (\alpha\,\delta_{ij}\delta_{kl} + \beta\,\delta_{ik}\delta_{jl} + \gamma\,\delta_{il}\delta_{kj})\,\partial_k v_l \\ &= \alpha\,\partial_i(\partial_k v_k) + \beta\,\partial_i v_j + \gamma\,\partial_j v_i\end{aligned}$$

Since we know that $\boldsymbol{\sigma}$ should be symmetric, we demand $\beta = \gamma$. We then choose $\alpha$ and $\beta$ so that $\mathrm{div}(\boldsymbol{\sigma})$ will be equal what it should, and the result reads

$$\boldsymbol{\sigma} = P\boldsymbol{I} + (\mu - \eta)\,(\mathrm{div}\,\vec{v})\,\boldsymbol{I} + \eta\,\boldsymbol{d}\ ,$$

where $\boldsymbol{d}$ is the deformation rate tensor $\boldsymbol{d} = \nabla\vec{v} + (\nabla\vec{v})^T$ (the spatial form of the material tensor defined right after Eq. (3.31) in the lecture notes).

Basically, what we did is adding the term $(\mu-\eta)\left[(\mathrm{div}\,\vec{v})\delta_{ij} - \nabla\vec{v}^T\right]$ to the asymmetric tensor $(\star\star)$. This addition does not change the divergence because what we added is divergence free (check!).

Last comment: As we said, apart from the $\nabla P$ term the stress is of the form $C_{ijkl}\partial_k v_l$. $\boldsymbol{C}$ is a 4-rank isotropic tensor, that relates between the velocity gradient and the stress. Next week we'll see the analogue for solids (Hooke's law) where we'll also have a 4-rank isotropic tensor that relates the strain ($\approx$displacement gradient) and the stress. You'll see that the equation will have exactly the same form, as these are both linear-response isotropic theories.

Many of you wrote in their solutions that we can simply assume $\mu = \eta$ to get a symmetric tensor. This is indeed true in the sense that you get a symmetric tensor when you that, but it is not as general as the above solution. As the "Last comment" clearly shows, there are physically two independent scalars that quantify the relation between the velocity gradient and the stress. In the notations used above it is readily seen that $\eta$ quantifies "shear viscosity" and $\mu - \eta$ "compressional viscosity" (neither these names nor the notations $\mu, \eta$ are standard in the literature).

# Symmetries

3. In the second TA session we've shown that in 3 dimensions a $2^{nd}$ rank isotropic tensor must be proportional to $\delta_{ij}$, (in fact, this is true for all dimensions $\geq 3$). However, in 2D this does not hold. Find the general form of an isotropic two-dimensional $2^{nd}$ rank tensor. What kind of symmetry do these tensors violate (those not proportional to the identity)?

Solution

In 2D, there's only one rotation, so $A$ is isotropic iff

$$\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} a_{12}+a_{21} & a_{22}-a_{11} \\ a_{22}-a_{11} & -a_{12}-a_{21} \end{pmatrix} = 0$$

That is, we $A$ is isotropic iff $a_{12} = -a_{21}$ and $a_{11} = a_{22}$. Thus, the general form of an isotropic 2D tensor is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and is itself proportional to a rotation, because

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

for $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$. So it is not surprising that it commutes with other rotations.

Another way to look at the same thing: We can write the general isotropic tensor as

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a\,\boldsymbol{I} + b\,\boldsymbol{L}^z \ .$$

So we actually proved that there are exactly two independent isotropic tensors in 2D: $\boldsymbol{I}$ and $\boldsymbol{L}^z$. Both can be generalized to higher dimensions and ranks. In general, one can always construct an isotropic tensor with an even number of indices from copies of $\boldsymbol{I}$ (in our case, one copy), and this is not surprising. About $\boldsymbol{L}^z$, you may look at it as the *completely antisymmetric tensor of rank 2*, i.e. the 2-rank analogue of $\boldsymbol{\mathcal{E}}$. This is because $L^z_{ij}$ is zero when $i = j$, 1 when $(ij)$ is an even permutation of (12) and -1 when it's an odd permutation. In general, the completely antisymmetric tensor of rank $k$ in $k$ dimensions is isotropic. It's geometrical meaning is that

$$\mathcal{E}_{i_1,i_2,i_3,\ldots,i_k}\, v^1_{i_1}\, v^2_{i_2} \ldots v^k_{i_k} = \det \begin{pmatrix} - & v^1 & - \\ - & v^2 & - \\ & \vdots & \\ - & v^k & - \end{pmatrix} \ ,$$

and in our case

$$L^z_{ij} v_i u_j = v_1 u_2 - v_2 u_1 = \det \begin{pmatrix} v_1 & v_2 \\ u_1 & u_2 \end{pmatrix} \ .$$

Last comment: This is out of the scope of this course, but for those of you who are interested in this kinda stuff: The fact that there are non-trivial isotropic 2D matrices is closely related to the fact that $SO_2$ is a one-parameter Lie group, and hence abelian. $SO_n$ for $n > 2$ is not abelian.

*Bonus:* Can you think of an example of an isotropic 2D tensor that is not diagonal, from a real physical system?

Solution

Three generic (and closely related) examples would be:

(i) The conductivity tensor in a 2D plate when a perpendicular magnetic field is present,

(ii) The matrix that relates the velocity of a charged particle in 2D with the Lorentz force,

(iii) The tensor that relates the velocity to the Coriolis force in a rotating disc.

These tensors violate reflection symmetry in the $3^{rd}$ dimension, which is also related to time-reversal symmetry.

4. **Invariants**: A scalar function of a tensor $f(\boldsymbol{A}) = f(A_{ij})$ or of a vector $g(\vec{v}) = g(v_i)$ is called invariant if its value is independent of the choice of basis. That is, if it has a proper geometric meaning which is independent of the particular basis that one happens to choose. Later in this course, we will be interested in scalar invariants of tensors. For example, the elastic energy is a scalar invariant of the strain tensor.

(a) Show that the trace is the only linear invariant scalar of a $2^{nd}$ rank tensor $\boldsymbol{A}$. That is, show that if $f(\boldsymbol{A})$ is an invariant function that is linear in $\boldsymbol{A}$'s entries, it can be written as $f(\boldsymbol{A}) = \lambda \operatorname{tr} \boldsymbol{A}$ for some constant $\lambda$. Assume the dimension is $\geq 3$.

Solution

Let $f(A_{ij})$ be a scalar invariant that is linear in the entries of $\boldsymbol{A}$. We know $f$ is linear means that we can write it as a linear combination of the entries of $\boldsymbol{A}$, i.e.

$$f(A_{ij}) = C_{ij}A_{ij} = \operatorname{tr}\left(\boldsymbol{C}\boldsymbol{A}^T\right) = \boldsymbol{C} : \boldsymbol{A}$$

where $C_{ij}$ is some matrix. The invariance of $f$ means that $f(A)$ should be unchanged when applying a rotation $\boldsymbol{Q}$:

$$\operatorname{tr}\left(\boldsymbol{C}\tilde{\boldsymbol{A}}^T\right) = \operatorname{tr}\left(\boldsymbol{C}\boldsymbol{Q}^T\boldsymbol{A}^T\boldsymbol{Q}\right) = \operatorname{tr}\left(\boldsymbol{Q}\boldsymbol{C}\boldsymbol{Q}^T\boldsymbol{A}^T\right)$$

This should be true for all $\boldsymbol{A}$, which means $\boldsymbol{Q}\boldsymbol{C}\boldsymbol{Q}^T = \boldsymbol{C}$. Or in other words, this means that $\boldsymbol{C}$ rotates like a proper tensor, and is isotropic. As we've seen, the only $2^{nd}$ rank isotropic tensor is $\delta_{ij}$, up to a multiplicative constant. Therefore

$$f(\boldsymbol{A}) \propto \delta_{ij}A_{ij} = A_{ii} = \operatorname{tr}(\boldsymbol{A})$$

As you know by now, this only works for $d \geq 3$ unless we demand that $f$ should also be reflection invariant. Otherwise, $\boldsymbol{L}^z : \boldsymbol{A}$ is also invariant.

(b) Show that the only quadratic invariants of a $2^{nd}$ rank tensor $\boldsymbol{A}$ are $\operatorname{tr}(\boldsymbol{A}^2)$, $(\operatorname{tr}\boldsymbol{A})^2$, and $\operatorname{tr}\left(\boldsymbol{A}\boldsymbol{A}^T\right)$. That is, show that if $f(\boldsymbol{A})$ is invariant and quadratic in $\boldsymbol{A}$'s entries, it can be written as $f(\boldsymbol{A}) = \lambda_1 \operatorname{tr}(\boldsymbol{A}^2) + \lambda_2 (\operatorname{tr}\boldsymbol{A})^2 + \lambda_3 \operatorname{tr}\left(\boldsymbol{A}\boldsymbol{A}^T\right)$.

*hint for the last two questions:* Think about the discussion in the TA session regarding isotropic tensors, i.e. Section 2 in TA #2 lecture notes.

Solution

Exactly like before, if $f$ is a quadratic function in the entries of $\boldsymbol{A}$, it must be of the form

$$f(\boldsymbol{A}) = C_{ijkl} A_{ij} A_{kl}$$

for some tensor $\boldsymbol{C}$, which must be isotropic. There are exactly 3 options for such a tensor, as we saw in class:

$$\begin{aligned} A_{ij}A_{kl}\delta_{ij}\,\delta_{kl} &= A_{ii}A_{kk} = (\operatorname{tr}\boldsymbol{A})^2 \\ A_{ij}A_{kl}\delta_{il}\,\delta_{jk} &= A_{ij}A_{ji} = \operatorname{tr}\left(\boldsymbol{A}^2\right) \\ A_{ij}A_{kl}\delta_{ik}\,\delta_{kj} &= A_{ij}A_{ij} = \operatorname{tr}\left(\boldsymbol{A}\boldsymbol{A}^T\right) \end{aligned}$$

All quadratic invariant functions are linear combination of these.

# Angular velocity

5. In this short and nice exercise, we'll prove a theorem that you used without knowing it in your undergrad: we'll show that a rigid body that rotates has a well-defined angular velocity. That is, we'll show that a the velocity of a particle located at $\vec{r}_i$ can be written as

$$\vec{v}_i(t) = \frac{\partial \vec{r}_i(t)}{\partial t} = \vec{\omega} \times \vec{r}_i(t) \tag{1}$$

for some time-dependent vector $\vec{\omega}(t)$. (If you think this is trivial, try proving it without reading further!)

The meaning of the assumption that a body is rigid is that it can only move by global rotations (and translations, which we'll assume are not present). Therefore, the trajectory of the $i$-th particle must be given by

$$\vec{r}_i(t) = \boldsymbol{R}(t)\vec{r}_i(0)\ , \tag{2}$$

where $\vec{r}_i(0)$ is the particle's position at $t = 0$, and $\boldsymbol{R}(t)$ is a (time-dependent) rotation matrix.

(a) Show that every anti-symmetric matrix can be written as $\mathcal{E}_{ijk}\,\omega_k$ for some vector $\vec{\omega}$ ($\boldsymbol{\mathcal{E}}$ is the Levi-Civita tensor defined in TA #2).

Solution

The general form of an anti-symmetric matrix is

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} .$$

Explicit calculation shows that this is equal to $\mathcal{E}_{ijk}\,\omega_k$ if we choose $\vec{\omega} = (c, -b, a)$.
Another (better?) way: the space of antisymmetric $3 \times 3$ matrices is 3 dimensional. For any $\omega$ the matrix $\mathcal{E}_{ijk}\,\omega_k$ is antisymmetric, and the space spanned by all $\omega$'s is also 3 dimensional, so $\mathcal{E}_{ijk}\,\omega_k$ span all must antisymmetric matrices.

(b) Show that $\dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T$ is anti-symmetric at all $t$'s (hint: orthogonality).

Solution

$\boldsymbol{R}$ is orthogonal at all times, i.e. $\boldsymbol{R}(t)\boldsymbol{R}(t)^T = \boldsymbol{I}$. Differentiating this gives

$$0 = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}\dot{\boldsymbol{R}}^T = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \left(\dot{\boldsymbol{R}}\boldsymbol{R}^T\right)^T ,$$

as needed.

(c) Write $\vec{v} = \dot{\vec{r}}$ as a function of $\vec{r}(t)$ (and *not* as a function of $\vec{r}(0)$!!).

Solution

$$\vec{v}(t) = \dot{\vec{r}}(t) = \dot{\boldsymbol{R}}\,\vec{r}(0) = \dot{\boldsymbol{R}}\,\boldsymbol{R}^{-1}\,\vec{r}(t) = \dot{\boldsymbol{R}}\,\boldsymbol{R}^T\,\vec{r}(t)$$

(d) Finish it off. Show that $\vec{v}(t) = \omega(t) \times \vec{r}(t)$ for some vector $\vec{\omega}$.

Solution

We've shown that $\vec{v} = \dot{\boldsymbol{R}}\,\boldsymbol{R}^T\,\vec{r}$. According to (b), $\dot{\boldsymbol{R}}\,\boldsymbol{R}^T$ is anti-symmetric.
According to (a), we can write

$$v_i = \mathcal{E}_{ijk}\,\omega_j\,r_k = (\vec{\omega} \times \vec{r})_i$$

for some vector $\vec{\omega}$.