Comment on "Symmetry of Kelvin-wave dynamics and the Kelvin-wave cascade in the $T = 0$ superfluid turbulence"

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We comment on the paper by Sonin [Phys. Rev. B **85**, 104516 (2012)] with most statements of which we disagree. We use this option to shed light on some important issues of a theory of Kelvin-wave turbulence, touched on in Sonin's paper, in particular, on the relation between the Vinen spectrum of strong and the L'vov-Nazarenko spectrum of weak turbulence of Kelvin waves. We also discuss the role of explicit calculation of the Kelvin-wave interaction Hamiltonian and "symmetry arguments" that have to resolve a contradiction between the Kozik-Svistunov and the L'vov-Nazarenko spectrum of weak turbulence of Kelvin waves.

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I. INTRODUCTION

Because of its importance in superfluid turbulence and the growing experimental capabilities in this field, there has recently been a renewed interest in the statistical physics of Kelvin waves propagating on a vortex line. A complete understanding of the statistical behavior of Kelvin waves is, therefore, crucial in order to develop a theory of superfluid turbulence. There were various attempts to find the energy spectrum of Kelvin-wave turbulence. In historical ordering, they are

$$E_{\rm V}(k) \propto \epsilon^0 k^{-1}, \quad \text{Vinen[1]} \tag{1a}$$

$$E_{\rm KS}(k) \propto \epsilon^{1/5} k^{-7/5}, \quad \text{Kosik-Svistunov,[2]} \tag{1b}$$

$$E_{\rm LN}(k) \propto \varepsilon^{1/3} k^{-5/3}, \quad \text{L'vov-Nazarenko[3] (LN).} \tag{1c}$$

Here, ϵ is the energy flux in the "inertial interval" of wave vectors k located between the energy-pumping scale $k_{\rm in}$ and the dissipation scale $k_{\rm dis}$: $k_{\rm in} \ll k \ll k_{\rm dis}$.

Vinen spectrum (1a) describes *strong wave turbulence* when the inclination angle φ of the vortex line from the straight line is not small $\varphi \sim 1$. As is well known in the theory of wave turbulence,[4–6] in this case, a loss-free step-by-step cascading of the energy is absent, energy flux over scales ϵ becomes an irrelevant quantity, and the power spectrum (as a rule) is determined by the structure of singularities in physical space. Well-known examples are surface waves on deep water: When acceleration on the top of water waves exceeds the gravity acceleration g, there are discontinuities of the first derivative of the wave profile (creation of "white caps"). In the k representation, this corresponds to the universal,

$$E(k) \simeq \frac{\rho g}{k^3}, \quad \text{Phillips spectrum of gravity waves,} \tag{2a}$$

$$E(k) \simeq \frac{\sigma}{k}, \quad \text{Hix spectrum of capillary waves.} \tag{2b}$$

Here, ρ is the fluid density, and σ is the surface tension. As expected, all the spectra (1a) and (2) are independent of the ϵ. They can be found from dimensional reasoning using the facts that, for gravity waves, the the only remaining parameter in the problem is the gravity acceleration g, for the capillary waves—surface tension σ, and for the Kelvin waves, as Vinen realized, this is the circulation quantum κ. Physically speaking, the spectrum (1a) is a consequence of the vortex reconnections, that (presumably) happens for all scales and leads to the creation of discontinuity in the vortex directions.

Philips, Hix, and Vinen spectra, being independent of ϵ, belong to the same class of so-called *critical balance* states in which the linear and the nonlinear time scales are balanced for each k. In Ref. 5, it was explained that the critical balance states arise due to a wave-strength-limiting process, e.g., wave breaking of water waves or reconnections of Kelvin waves.

KS and LN spectra (1b) and (1c) are related to the *weak wave turbulence* of Kelvin waves in which angle φ is assumed to be small. In the theory of weak wave turbulence, the Hamiltonian of the wave interaction can be expanded in a series of small wave amplitudes (inclination angle for the Kelvin, gravity, and capillary surface waves), and only the first nontrivial term in this expansion, describing the interaction of $p \geqslant 3$ waves, determines the turbulent energy spectra. For the surface capillary waves, $p = 3$, for the surface gravity waves (in which three-wave processes are forbidden by the conservation laws[4]), $p = 4$, and for the Kelvin waves, $p = 6$ as correctly found by KS.[2]

Within the framework of wave turbulence,[4,5] the energy flux over scales ϵ is proportional to the wave-collision integral $\mathrm{St}_p(k)$, which, in turn, is proportional to the energies of $p-1$ waves $E(k'), E(k''), \ldots, E(k^{(p-1)})$ participating in p-wave collision processes. Under the assumption of the locality of the energy transfer, when the leading contribution to $\mathrm{St}_p(k)$ comes from $k', k'', \ldots, k^{(p-1)} \sim k$, one immediately concludes that $\epsilon \propto [E(k)]^{p-1}$ or

$$E(k) \propto \epsilon^{1/(p-1)}. \tag{3}$$

These simple arguments can be found in Refs. 4 and 5 or, e.g., in Ref. 6.

The KS spectrum (1b) was found in Ref. 2 as a result of $3 \Leftrightarrow 3$ Kelvin-wave scattering ($p = 6$) under the assumption of the interaction locality. However, in Ref. 7, the locality assumption used in Ref. 2 was checked by explicit calculations and was shown to be violated for $3 \Leftrightarrow 3$ Kelvin-wave interactions. This invalidated the local theory, and a nonlocal theory was proposed,[3] resulting in $1 \Leftrightarrow 3$ Kelvin-wave interactions (with $p = 4$). This has prompted a lively debate about the correct spectrum of Kelvin waves in Refs. 8–11, which was summarized in two Abu Dhabi workshops on superfluid turbulence in May 2011 and June 2012.

The bottom line of these discussions is very simple: The basic assumptions and the calculation schemes were the same in both KS (Ref. 2) and LN (Ref. 3) approaches. Namely, the initial Hamiltonian formulation of the Biot-Savart equation of the vortex line motion was the same, the Hamiltonian expansion approach up to the six-order terms (under the assumption of smallness of the Kelvin-wave amplitudes) was identical, the canonical transformation technique aiming at elimination of the four-order terms was the same, and only the results were different. In Ref. 7, we presented an explicit infrared (IR) asymptotic form of the effective six-wave interaction amplitude,

$$\mathcal{W}_{1,2,3}^{4,5,6} = -\frac{3}{4\pi} k_1 k_2 k_3 k_4 k_5 k_6, \tag{4}$$

which directly leads to LN spectrum (1c).

To encourage our colleagues to check our derivation, we have made it publicly available in the form of a line-by-line commented MATHEMATICA code.[12] On the other hand, KS have not calculated the IR asymptote of $\mathcal{W}_{1,2,3}^{4,5,6}$. Instead, the presented symmetry argument, which was aimed at showing that $\mathcal{W}_{1,2,3}^{4,5,6}$ cannot have linear IR asymptotics, therefore, cannot have form (4). In other words, KS claimed that *our derivation of Eq.* (4) *contains algebraic mistakes*. The symmetry argument of KS was refuted in Refs. 9 and 11 and further discussions at the 2011 and 2012 Abu Dhabi workshops: It was shown that the presence of the tilt symmetry does not imply the absence of linear IR asymptotics in the nonlinear interaction coefficients. Thus, the resolution of the controversy must be performed by a careful rigorous derivation rather than by further hand waving, avoiding the direct check. This was summed up in the 2011 Abu Dhabi workshop by LN by the call "put your Hamiltonian on the table!"

Unfortunately, we cannot say that the paper by Sonin[13] adds much to this discussion and clarifies the issue. Besides more or less simple statements with which we agree, it has a set of unclear, questionable, and sometimes even incorrect hand-waving arguments. These arguments are related to two main issues in the theory of Kelvin-wave turbulence in $T = 0$ superfluids:

(1) The role of the tilt symmetry in Kelvin-wave turbulence,
(2) properties of strong Kelvin-wave turbulence and Vinen spectrum (1a).

Because of their importance, we found it important to further clarify our position to the superfluid physics community and to comment on, at least, some of Sonin's statements. In particular, we clarify several issues and past results, which appear to be misinterpreted in Ref. 13.

Sonin[13] has also suggested an alternative scenario of crossover between classical and quantum regions of energy spectra without bottleneck energy accumulation near the intervortex scale ℓ. This problem is closely related to the temperature dependence of the effective Vinen viscosity $\nu'(T)$ and, in particular, to the small value ν' in the zero-temperature limit. These problems are not addressed in the Sonin paper.[13] We disagree with the Sonin scenario and are going to return to this question soon suggesting a mechanism of temperature suppression of the bottleneck energy accumulation, that allows rationalizing the observed T dependence of $\nu'(T)$.[14]

II. ROLE OF THE TILT SYMMETRY IN KELVIN-WAVE TURBULENCE

It is known that the tilt symmetry of the Kelvin-wave Hamiltonian is broken when the nonlinearity is truncated at a finite order of the wave amplitude. Sonin, in Sec. II of Ref. 13, illustrates his points by considering a tilt transformation of an exact fully nonlinear solution for the Kelvin wave within the local induction approximation, the frequency of which is

$$\omega = \frac{\kappa \Lambda k^2}{4\pi \sqrt{1 + a^2 k^2}}. \tag{5}$$

Here, a is the wave amplitude, $\Lambda = \ln(\ell/a_0)$ in which a_0 is the vortex core radius.

A. First Sonin objection

"Since the main contribution to Kelvin-wave dynamics comes from the sixth-order terms, the Hamiltonian used by L'vov et al. violates tilt symmetry."

Here, we see a lack of understanding that the Hamiltonian, required for describing Kelvin-wave turbulence, must not (and could never be!) tilt invariant. Indeed, the whole notion of the Kelvin wave is only defined as an evolving deviation from a vortex filament of a fixed shape—in our (together with KS) case from an infinite straight line. The choice of such a background state immediately breaks the original tilt symmetry in the Hamiltonian in *any finite order* of truncation of the series in the deviation. Thus, to demand preservation of the tilt symmetry amounts to simultaneously considering *all orders* of the series, which only make sense when the deviation angles are of order 1, and the resulting motion, in general, cannot even be described as a set of waves (there predominately is a nonoscillatory motion with many reconnections, etc.). Nobody before, including L'vov *et al.* and Kozik and Svistunov, has considered such an untruncated fully nonlinear motion when developing the wave turbulence description! Instead, wave turbulence deals with *weakly nonlinear* waves whose evolution is fully described by a Hamiltonian, which is *truncated* at the first nonvanishing interaction order, which, for Kelvin waves, is the sixth order. But such a truncated Hamiltonian cannot and must not be tilt invariant. To put it in stronger terms, the Hamiltonian for Kelvin waves must never be tilt invariant—else they cannot be called waves because they would have no vortex line to propagate on. There is no smile without a cat unless we are in a wonderland!

Here, it is worth reminding and clarifying: What Lebedev, L'vov, and Nazarenko actually showed (in responses[9,11] to Kozik and Svistunov's objection) is that the tilt symmetry does not prevent the interaction coefficients *of any order* to be linear in k asymptotics. This is completely different from the claim that the truncated system is tilt symmetric, which is, of course, wrong but which has never been made. Moreover, the fact that all the higher-order terms are needed for preserving the tilt invariance does not contradict the fact that only the leading order (six-wave) nonlinear terms are important for the wave turbulence spectrum when the forcing is weak.

B. Second Sonin objection

"The mechanism of L'vov et al.[3] *is absent in the coordinate frame, with the axis coinciding with the average position of the vortex line in which average vortex displacement and tilt are absent."*

And later:

"The mechanism of L'vov et al. originates from the quasistatic Kelvin mode, which in the limit of small k is equivalent to a tilt $\varphi \sim ka$ of the z axis. The tilt can be removed by transformation to another coordinate frame, in which the mechanism disappears."

This is incorrect. The scenario of L'vov *et al.* is not eliminated by introducing the frame with the axis coinciding with the average position of the vortex line because it involves the *rms* tilt at and near the forcing scales and not the straight average of the line tilt. The forcing scale is the largest scale for the direct cascade setup, but it is, of course, much less than the total system size. Thus, the forcing scale motions make many oscillations within the containing "box," and they cannot be eliminated by the rotation by any angle. For simplicity, one can consider an idealized system where the scales in between the box size and the forcing scale are damped (suppressed inverse cascade setup). This is precisely the frame with zero-mean tilt that, if it exists, is relevant for the wave turbulence construction. When the average tilt does not stabilize at a finite value in the infinite box limit, then the average should simply be taken over finite distances, which are considerably greater than the forcing scale.

C. One more Sonin objection

"In summary, tilting of the axis affects distribution in k space.

Notice that Lebedev et al.[9] *revealed evidence of this effect in the series expansion in k space and called it the nonlinear shift of the Kelvin-wave frequency with the wave of small k. They argued that this was a nontrivial observable physical effect, which supported their position. Without arguing the observability of the effect, I would prefer to call it a visual rather than a physical effect, which has nothing to do with the global symmetry at the border."*

This is also incorrect. The nonlinear frequency shift has nothing to do with redistribution in the k space, but rather, it is a change in frequency at a fixed k with respect to the frequency of the linear waves. It is clearly present in Sonin's example of the monochromatic Kelvin wave with the frequency given by formula (5) above. Note that, in the leading order, the frequency correction is $-\frac{1}{2}k^4a^2$, which confirms the linear asymptotics of the four-mode coupling coefficient with respect to each of the four coupled wave vectors (which are all equal to k in this example). This is a real and observable effect!

Because of the space (in *Phys. Rev. B*) and the time limitation (ours and the readers), in this Comment, we decided not to respond to other similar objections by Sonin and to postpone their discussion (if it is really required) to a later time. Right now, we can just repeat our May 2011 call "*put your Hamiltonian on the table!*" Up to now, we have no response from KS. That is why we may only hope that Sonin could finalize the discussion either by presenting his own detailed calculation of the explicit form of the interaction amplitude $\mathcal{W}$ or by stating in which line of our calculation,[12] in his opinion, there is a mistake.

III. STRONG KELVIN-WAVE TURBULENCE AND VINEN SPECTRUM

In Sec. IV of Ref. 13, Sonin considers the case when nonlinearity parameter φ is not small and we are dealing with the *strong wave turbulence* (without saying this explicitly). As we explained above, in this case, one cannot expand the interaction Hamiltonian, an inertial step-by-step cascading of the energy is absent, energy flux over scales ϵ becomes irrelevant (together with the hypothesis of the interaction locality), and the power spectrum (as a rule) is determined by the structure of singularities in the physical space.

Sonin's discussion of this problem presents an example of how wrong arguments can give correct answers. First, he made a strange statement that, in the formal expansion of the interaction Hamiltonian (without small parameter $\varphi \sim 1$), "*the higher-order terms can be important as well, or even more important.*" Next, he assumed the interaction locality, which is highly questionable and (as we know) even wrong for $p = 6$. Then, he applied Eq. (3) from the theory of weak wave turbulence, which is not valid in the case of strong turbulence. After that, he set $p \to \infty$ in Eq. (3) and came to the correct conclusion that the energy spectra of strong wave turbulence are independent of (meaningless) parameter ε. This is the Sonin way to *rediscover the Vinen 2003 spectrum of strong Kelvin-wave turbulence.*[1]

The positive content of Sec. IV in Sonin's paper[13] touches on, rather inconclusively, an important question of the relations of the strong- and weak-wave turbulence regimes of Kelvin-wave turbulence, i.e., as we believe, between the Vinen and the L'vov-Nazarenko spectra (1a) and (1c). We will return to this issue elsewhere.

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