Analytic Solution of the Approach of Quantum Vortices Towards Reconnection

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Experimental and simulational studies of the dynamics of vortex reconnections in quantum fluids showed that the distance d between the reconnecting vortices is close to a universal time dependence $d = D[\kappa|t_0 - t|]^\alpha$ with α fluctuating around $1/2$ and $\kappa = h/m$ is the quantum of circulation. Dimensional analysis, based on the assumption that the quantum of circulation $\kappa = h/m$ is the only relevant parameter in the problem, predicts $\alpha = 1/2$. The theoretical calculation of the dimensionless coefficient D in this formula remained an open problem. In this Letter we present an analytic calculation of D in terms of the given geometry of the reconnecting vortices. We start from the numerically observed generic geometry on the way to vortex reconnection and demonstrate that the dynamics is well described by a self-similar analytic solution which provides the wanted information.

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The study of vortex reconnections in quantum fluids received a huge boost by the experimental visualization of this process in liquid helium [1]. Of special interest is the dynamics of the approach of vortices towards reconnection as displayed by the minimal distance between them, denoted as $d(t)$. When $d(t)$ is much larger than the vortex core size one can neglect the core size; then dimensional considerations based on the assumption that the quantum circulation $\kappa = h/m$ (with h being Planck's constant and m the mass of the ^{4}He atom) is the only relevant parameter of the problem predict that [2,3]

$$d(t) = D[\kappa|t_0 - t|]^\alpha, \qquad \alpha = 1/2, \tag{1}$$

where D is a dimensionless parameter. Experiments and simulations exhibited a range of exponents near $\alpha = 1/2$ [1,2], but the question of the coefficient D and how to compute it theoretically remained an open question. The aim of this Letter is to solve this open question.

We first note that the core size δ in superfluid ^{4}He is about one Angström, $\delta \approx 10^{-8}$ cm. We can assume that $d(t) \gg \delta$, and describe the evolution of quantized vortex lines by the Biot-Savart equation according to which each point of the vortex line is swept by the velocity field produced by the other existing vortices

$$\boldsymbol{V}(\boldsymbol{s}) = \frac{\kappa}{4\pi}\int_C \frac{(\tilde{\boldsymbol{s}} - \boldsymbol{s}) \times d\tilde{\boldsymbol{s}}}{|\tilde{\boldsymbol{s}} - \boldsymbol{s}|^3}. \tag{2}$$

Here the vortex line is represented in a parametric form $\boldsymbol{s}(\xi, t)$, where ξ is an arc length, t is the time, and the integral is taken over the entire vortex tangle configuration. The core size δ appears implicitly in this equation as the cutoff length that protects the logarithmic singularity that is embodied in Eq. (2). Finally the core radius δ appears in the present problem as a logarithmic term of the form $\Lambda = \ln(d(t)/\delta)$.

The strategy employed here is to start from the remarkable numerical discovery that independently of initial conditions, near the reconnection point vortices arrange themselves to become antiparallel, forming an evolving structure that appears like a self-similar solution [4,5]. The whole reconnection process is made of the dynamics of the approach of the vortices towards reconnection, the process of reconnection itself, and finally the receding of the reconnected vortices [6–9]. In this Letter we construct an approximate self-similar evolution of the vortex lines *towards* their reconnection, ignoring the weak logarithmic dependence on δ. Thus our starting point is the vortex configuration found in [4] and reproduced in Fig. 1 as a pyramidal construction with A, B, C, D, E, and F points on

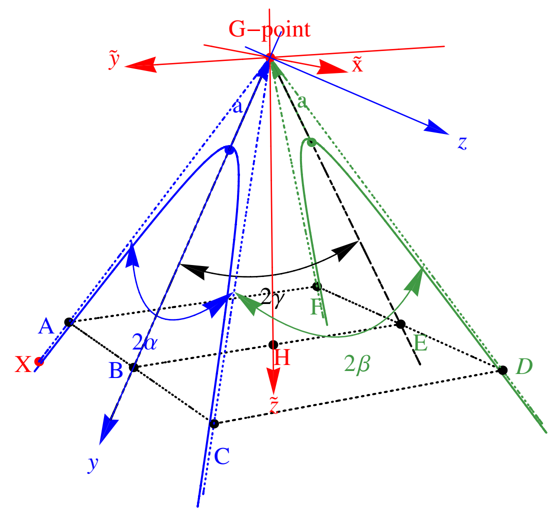

FIG. 1 (color online). The proposed "generic" pyramidal configuration of two vortex lines on the way to collision and reconnection [4]. The two vortex lines (blue and green) are hyperbolas on the two opposite side edges ACG and DFG of the $ACDFG$ pyramid. The conjecture of Ref. [4] is that the vortex lines proceed to collide at the G point. In this example $\alpha = 60°$, $\beta = 12.5°$. We show in this Letter that some serious modifications are necessary to reach a true self-similar solution.

the base and the G point on the top. Two vortices occupy two side edges as shown by the solid (A–C blue and D–F green) lines. The vortex lines are separated (at a given moment of time t) from the reconnection G point by a distance $a(t) \to 0$ at $t \to t_0$. Far away from the G point the lines are almost straight, approaching four semi-infinite straight lines shown in Fig. 1 as dashed AG and GC lines (blue vortex) and DG with GF lines (green vortex).

In our analysis we will use two coordinate systems. The first (basic) $(\tilde{x}, \tilde{y}, \tilde{z})$-coordinate system (shown in red) has an origin at the G point and the $\tilde{x}$ axis is parallel to the straight ABC- and DEF-straight lines. By 2α we denote the angle between the AG and CG lines, equal to the angle between the DG and FG lines. By 2β we denote the angle between the CG and DG lines, equal to the angle between the AG and FG lines. The $\tilde{z}$ axis is directed down from G to the point H and the $\tilde{y}$ axis is parallel to the BE line. In addition we will use a (blue) $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$-coordinate system, in which $\boldsymbol{x} = \tilde{\boldsymbol{x}}$, the $(\boldsymbol{x}, \boldsymbol{y})$ plane coincides with the ACG plane such that the $\boldsymbol{y}$ axis is obtained by turning the $\tilde{\boldsymbol{y}}$ by an angle $(\pi/2) - \gamma$ around the $\boldsymbol{x}$ axis. Here 2γ is the angle between the ACG and the DEG planes (or between BG and EG lines), related to α and β as follows: $\cos\alpha \sin\gamma = \sin\beta$.

The construction of the self-similar evolution will be achieved in three steps. In the first step we will disregard the tip region and approximate the vortex configuration by the four straight lines: AG, CG, DG, and FG. Then we compute the velocity induced on a given vortex line, e.g., AG, by the other three. Clearly the contribution from one straight vortex line on itself is zero, since a straight vortex line is an obvious null solution of Eq. (2). For this goal we recall the fundamental result of the Biot-Savart equation for the velocity field $\boldsymbol{V}(r, \varphi)$ produced by a semi-infinite vortex line

$$\boldsymbol{V}(r, \varphi) = \frac{\kappa}{4\pi r}(\cos\varphi + 1). \tag{3}$$

Here r is a distance from the $\boldsymbol{X}$ point shown in Fig. 1 where the velocity is measured to the semi-infinite vortex line; φ is the angle between the semi-infinite vortex line and the line between the end point of the vortex line and the measurement point $\boldsymbol{X}$. Using Eq. (3) we can find the velocity $\boldsymbol{V}(\boldsymbol{X})$ induced by the CG, DG, and FG lines on the $\boldsymbol{X}$ point. A straightforward but quite cumbersome calculation that involves lots of trigonometry yields

$$\boldsymbol{V}^{\mathrm{BS}}(\boldsymbol{X}) = \frac{\kappa}{4\pi x}\boldsymbol{V}, \qquad \boldsymbol{V} = \{\mathcal{V}_\perp, \mathcal{V}_+, 0\}, \tag{4a}$$

$$\mathcal{V}_\perp = -\cos\alpha + \frac{\sin\alpha\tan\alpha}{\sin^2\alpha + \sin^2\beta}, \tag{4b}$$

$$\mathcal{V}_+ = \frac{\cos\gamma\sin^3\alpha}{(\sin^2\alpha + \sin^2\beta)\sin\beta}. \tag{4c}$$

Here $\boldsymbol{V}$ is a vector of dimensionless numbers defined entirely by the geometry: $\mathcal{V}_\perp$ is the component orthogonal to the ACG plane, $\mathcal{V}_+$ is an in-plane component, normal to the AG line, and $\mathcal{V}_\| = 0$ is the component along the AG line. The crucial observation of the first step of the calculation is that for large x the velocity field decays like $1/x$. This asymptotic statement is independent of the near-tip structure, and will be reproduced by the solution discussed below. Note that improving the accuracy of the solution below can achieved by going to the next order expansion [terms of $O(1/x^3)$] at the price of increasing complexity.

In the second step of the calculation we will adopt for a start the proposition of Ref. [4] that the reconnecting vortex lines have a form of two identical hyperbolae AC (blue) and FD (green) lying in the corresponding planes, see Fig. 1. The AC hyperbola in the (x, y) plane is

$$y_0^2(x, a(t)) = a^2(t) + x^2\cot^2\alpha. \tag{5a}$$

We will see below that this proposition is too restrictive and it cannot be satisfied accurately by the a self-similar solution. We thus go beyond the guess of Ref. [4]; we find it advantageous to use a less restrictive form which we refer to as "quasihyperbola" and write as

$$y_1^2(x, a(t), \varepsilon) = y_0^2(x, a(t)) + \frac{\varepsilon a^2(t)x^2\cot^2\alpha}{y_0^2(x, a(t))}. \tag{5b}$$

Clearly, $y_1(x, a(t)) \to y_0(x, t)$ for $\varepsilon \to 0$ and/or $x \to \infty$, while for $x \lesssim a$ the curvature of $y_1(x, t)$ depends on ε.

The $y(x, t)$ lines (5) are translated in time due to the t dependence of $a(t)$ such that $a(t) \to 0$ for $t \to t_0$. Naturally there are infinitely many mappings of a given $y(x, t)$ line to a future $y(x, t + \delta t)$ line. We remember, however, that these are quantized vortex lines and therefore vortex stretching does not affect the circulation. Therefore the tangential component of the velocity is not relevant for the present construction. We thus seek self-similar kinematics by requiring that each point of the vortex line should move perpendicularly to the vortex line with the in-plane, normal to the line velocity V_+ which is

$$V_+^M(x, t) = \frac{da}{dt}\left(\frac{\partial y}{\partial a}\right)_x \Bigg/ \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)_a^2}. \tag{6}$$

From Eq. (6) we see that the asymptomatic behavior of $V_+^M(x, t)$ for $x \gg a(t)$ is determined by $(\partial y/\partial a)_x$. But in step 1 we found that the asymptotic is $1/x$. Consistency requires the following condition on the function y: $\lim_{x\to\infty}(\partial y/\partial a) \propto (1/x)$. It is obvious that both lines (5) indeed satisfy this condition. In addition, a direct calculation using Eqs. (6) and (5) yields

$$V_+^{\mathrm{M}}(x, a, \alpha, \varepsilon) = \frac{\frac{da}{dt}a[y_0^4(x, a) + \varepsilon x^4\cot^4\alpha]}{\sqrt{y_0^8(x, a)(a^2 + x^2\cot^2\alpha\csc^2\alpha) + \varepsilon x^2a^2\cot^2\alpha y_0^4(x, a)[y_0^2(x, a) + 2a^2\cot^2\alpha] + \varepsilon^2a^8x^2\cot^4\alpha}}. \tag{7}$$

Of course, the leading asymptotics of V_+^M (for $x \gg a$) must be consistent with Eqs. (4) which resulted from Eq. (2). The conditions for consistency are

$$\frac{da^2}{dt} = -\frac{\kappa \mathcal{V}_+ \cot\alpha}{2\pi \sin\alpha(1+\varepsilon)}. \tag{8}$$

Solving this equation one finds $a^2(t) = A^2\kappa(t_0 - t)$, where

$$A^2 = \frac{\sqrt{\cos(2\alpha) + \cos(2\beta)}\,\sin\alpha}{2\sqrt{2}\pi(\sin^2\alpha + \sin^2\beta)\sin\beta(1+\varepsilon)}. \tag{9}$$

Although we showed that the self-similar solutions (5) are asymptotically consistent with the Biot-Savart result for the normal component of the velocity in the plane, we should note that the Biot-Savart result contains also a $V_\perp^{\rm BS}$ component of the velocity, orthogonal to the plane. In the considered geometry this component vanishes, $V_\perp^M = 0$, because the lines (5) are moving in the plane. Such a component would obviously destroy the self-similarity of the chosen configuration of two flat quasihyperbolae lying each in its own plane. This is another point where we have to go beyond the guessed solution of Ref. [4]. In the third step of the analysis we fix this discrepancy by choosing quasihyperbolae that lie not on the planes but on quasihyperbolic surfaces as shown in Fig. 2. Explicitly, we choose the quasihyperbolic surface using the relation

$$\tilde{z}(\tilde{y}, t) = y_1(\tilde{y}, b(t), \tilde{\varepsilon}), \tag{10a}$$

with the same quasihyperbola (5b) but defined in the different(tilde) coordinate system, see Figs. 1 and 2.

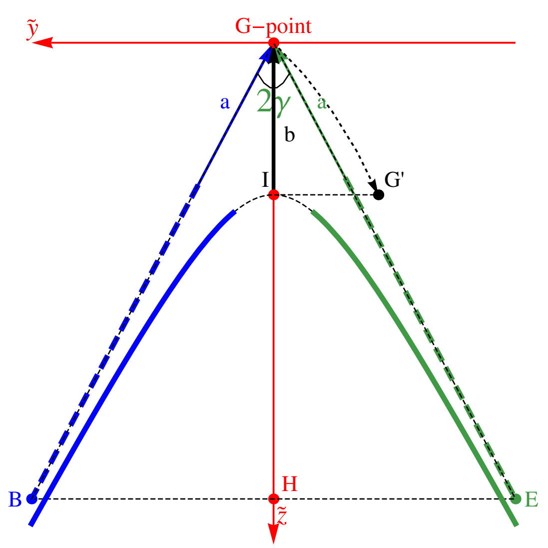

FIG. 2 (color online). The third step of the analysis including the second modification to Ref. [4]: now the quasihyperbolas are embedded in a surface which is also a quasihyperbola. Consistency with the asymptotic solution (large X) requires moving the collision G point to a new G' point. Note that the two vortex lines are here but only one branch in each is shown.

Such a construction automatically provides the desirable asymptotics because the $V_\perp^M$ component of the velocity is defined now by the same type of expression as Eq. (7), replacing $\alpha \Rightarrow \gamma$ and $a \Rightarrow b$:

$$V_\perp^{\rm M} = V_+^{\rm M}(\tilde{y}, b, \gamma, \tilde{\varepsilon}). \tag{10b}$$

In the new coordinates we have asymptotically $\lim_{x\to\infty}\tilde{y} = x\cot\alpha\sin\gamma$. In its turn, the $V_\perp^M$ component of velocity will be asymptotically

$$\lim_{x\to\infty} V_\perp^{\rm M} = \frac{b^2(1+\tilde{\varepsilon})}{ax\cot(\gamma)\cot\alpha}\frac{da}{dt}. \tag{11}$$

Now the distance b is defined by the condition

$$\lim_{x\to\infty} V_\perp^{\rm M} = V_\perp^{\rm BS}, \qquad \frac{b^2(1+\tilde{\varepsilon})}{ax\cot(\gamma)\cot\alpha}\frac{da}{dt} = \frac{\kappa\mathcal{V}_\perp}{4\pi x}. \tag{12}$$

After a direct calculation we find

$$\frac{b}{a} = \sqrt{\frac{1+\varepsilon}{1+\tilde{\varepsilon}}(\sin^2\alpha - \cot^2\alpha\sin^2\beta)}. \tag{13}$$

Finally, we note that we defined the quasihyperbolic surface instead of the two (blue and green) planes to embed the vortex lines in them, but did not specify explicitly how this is done. To achieve the correct embedding we employ the condition that far from the tip the vortex lines and the measurement point $\boldsymbol{X}$ should coincide for a large value of x. This condition will move the point G to a new point G', see Fig. 2. The distance IG' between the collision point and the point G' is determined by the requirement that the arc length in the bent plane (quasihyperbolic surface) coincides with the distance in the flat plane:

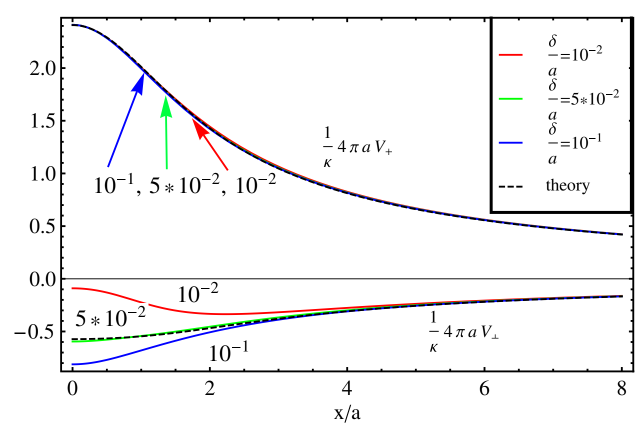

FIG. 3 (color online). Comparison of the the normalized vortex velocity profiles $4a(t)\pi V_+(x,t)/\kappa$ (in plane) and $4a(t)\pi V_\perp(x,t)/\kappa$ (normal to the plane), computed from the Biot-Savart equation (solid lines) and different value of δ/a with the model prediction, shown by a dashed black line. The parameters used in the theory are $\alpha = 60°$, $\beta = 12.5°$, $\epsilon = -0.06$ and $\tilde{\epsilon} = 1$.

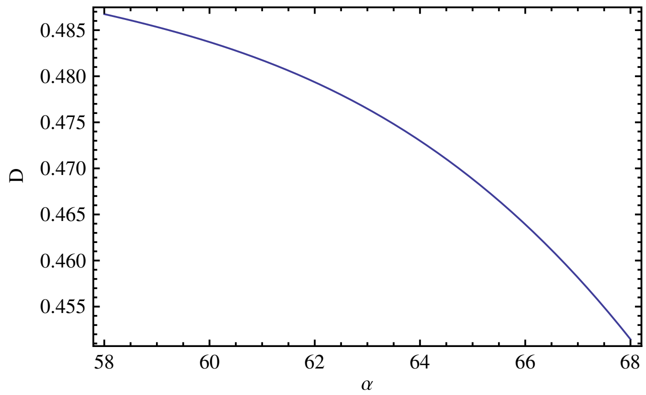

FIG. 4 (color online). The coefficient D as a function of the angle α (in °) in the range of angles considered in Ref. [4] with $\beta = 12.5°$.

$$IG' = \lim_{\tilde{y}\to\infty}\left[\frac{\tilde{y}}{\sin(\gamma)} - L(\tilde{y})\right], \tag{14}$$

where L is the arc length of the quasihyperbola which is defined by

$$L(\tilde{y}) = \int_0^{\tilde{y}} \sqrt{1 + \left(\frac{d\tilde{z}}{d\tilde{y}'}\right)^2} d\tilde{y}'. \tag{15}$$

Now we can write the parametric equation for the configuration of the vortex line in the tilde coordinate system:

$$\tilde{x}(x) = x, \qquad \tilde{y}(x) = L^{-1}[y_1(x, a, \varepsilon) - IG'], \tag{16}$$

where L^{-1} is the inverse function of L, defined in Eq. (15), and $\tilde{z}(x) = y_1(\tilde{y}(x), b, \tilde{\varepsilon})$.

This finalizes the self-similar solution, given by Eqs. (5b), (10), and (13). While we expect this solution to be quite accurate, it cannot be exact, since we neglected the logarithmic correction involved with the inner cutoff of the Biot-Savart integral (2). To assess the accuracy we compare the vortex velocity found from the time evolution of the suggested self-similar solution [(5b), (10), (13), and (16)], denoted as "theory" in Fig. 3, with direct numerical calculations of the vortex velocity from the Biot-Savart Eq. (2) with the vortex configuration [(5b), (10), (13), and (16)]. The calculations were performed at different values of δ/a. At this point we can fit the parameters ε and $\tilde{\varepsilon}$ as detailed in the caption of Fig. 3 where the results are presented as a function of x/a. We conclude that the Biot-Savart velocity always agrees with the theory far from the tip (for $x > 4a$). The agreement is almost perfect for the V_+ component for all values of x/a and δ/a. On the other hand the $V_\perp$ component is more sensitive, near the tip, to the local contributions that exist in the Biot-Savart integral. It differs from the self-similar solution in the tip region and obviously cannot be fitted for all values of δ/a. Nevertheless, even near the tip the agreement is still reasonable, especially taking into account the fact that $V_\perp$ is less than a $1/3$ of V_+ for all values of δ/a.

Finally, we return to the question of computing the coefficient D in Eq. (1). The numerical simulations in Ref. [4] were used to determine an approximate value of D in the vicinity of 0.4. In light of our analysis it is obvious that D depends on the angle α and is a function rather than a number, as is indeed found in the experimental work, cf. [1]. Since Ref. [4] estimated the value of D only in the range $58° \le \alpha \le 68°$ we calculate D in the same range (with the angle $\beta = 12.5°$). The result is shown in Fig. 4. It is clear that the analytic prediction is in quite close agreement with the numerical estimate.

In summary, we presented an analytic self-similar solution for the dynamics of vortex lines approaching a reconnection. The solution is in good agreement with numerical simulations. In future work one needs to understand how to continue and extend this work to include the reconnection event itself and the change in topology. There are reasons to believe that this process includes the release of energy in the form of sound waves [10], leading to an asymmetry between the dynamics before and after the reconnection. This cannot be done with Biot-Savart dynamics and calls for a fully quantum mechanical model.

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