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Law of space-decorrelation for developed hydrodynamic turbulence

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Laws of decay of simultaneous many-point correlators of turbulent-velocity differences are derived in the asymptotic region where either one space point or a group of points is far away from another group of points. An asymptotic decomposition rule of $(n+m)$-point correlators in terms of $(n+1)$-, $(m+1)$-, and two-point correlators is presented. These results may be directly applied or easily extended to the turbulence of cold electron plasma, convective turbulence, some problems of surface roughening at crystal growth, etc.

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I. INTRODUCTION

Consider the simultaneous $(n+m)$-point correlation function of velocity in the case where the distance R between the "left" group of n points and the "right" group of m points becomes much larger than the characteristic distance r within each group, see Fig. 1.

Clearly these correlators have to decay as the distance R increases. In the present Rapid Communication this law of decay is derived in the limit $R \gg r$ for two types of $(n+m)$-point correlators, which are averaged over chains of velocity differences $v_{(l,m)} = |\mathbf{v}(\mathbf{r}_l) - \mathbf{v}(\mathbf{r}_m)|$. The first type is $\langle \Psi_{n,m} \rangle$, where $\Psi_{n,m} \equiv v_{(1,2)} v_{(2,3)} \cdots v_{(n,n+1)} \cdots v_{(n+m-1,n+m)} v_{(n+m,1)}$ denotes a cyclic chain running though n points of the "left" and m points of the "right" group. There are only two intergroup links in the chain. In terms of the double correlator $D(R) = \langle |\mathbf{v}(\mathbf{r}_1) - \mathbf{v}(\mathbf{r}_1 + \mathbf{R})|^2 \rangle$, we obtain for $m \geq 2$,

$$v_{(1,2)} \Psi_{0,m} \propto \frac{\partial D(R)}{\partial R}\,, \quad \langle \Psi_{n,m} \rangle \propto \left(\frac{\partial D(R)}{\partial R} \right)^2 . \qquad (1)$$

Equation (1) for $\langle \Psi_{1,2} \rangle$ was found by L'vov and Falkovich [1]. Correlators of the second type describe correlations between two disconnected cyclic chains $\Psi_{n,0}$ and $\Psi_{0,m}$ formed within each of the separated groups:

$$\langle \Psi_{n,0} \Psi_{0,m} \rangle - \langle \Psi_{n,0} \rangle \langle \Psi_{0,m} \rangle \propto \frac{\partial^2 D(R)}{\partial R^2} . \qquad (2)$$

Equations (1) and (2) represent the asymptotic decorrelation laws for different types of correlation functions.

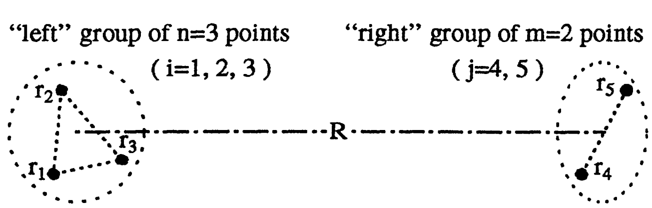

FIG. 1. Positions of points placed in two ("left" and "right") compact groups which are separated by a large distance R.

The physical meaning of these relations is rather simple. First of all the correlation of the turbulent velocity field between two points (or close group of points) at relative distance R is mostly carried by eddies of characteristic size R. That is why the correlators (1) and (2) contain the double correlator $D(R)$ describing the level of excitation of those R eddies. However, the velocity field of R eddies is almost homogeneous on scales $r \ll R$. It is clear that a homogeneous velocity field does not effect the behavior of small r eddies, which determine the correlations within each group of closely spaced points. For $r \ll R$ the leading-order effect on r eddies is due to the velocity gradient of R eddies. This explains the appearance of $\partial/\partial R$ in Eq. (1). As we are interested in correlations between two groups of points $\mathbf{r}_i$ and $\mathbf{r}_j$ the effect of the velocity gradient occurs twice, once for each group. Therefore one has the factor $[\partial/\partial R]^2$ in (2). Unlike $\Psi_{n,0}\Psi_{0,m}$ the function $\Psi_{m,n}$ contains the factor $|\mathbf{v}(\mathbf{r}_1) - \mathbf{v}(\mathbf{r}_1 + \mathbf{R})|^2$. Therefore $\langle \Psi_{n,m} \rangle$ (1) contains the additional factor $D(R)$ with respect to (2).

Consider the above relation in $\mathbf{k}$ representation, expressing the correlators $\langle \Psi_{n,m} \rangle$, $\langle \Psi_{n,0} \Psi_{0,m} \rangle$ (with the help of corresponding $\mathbf{k}$ integrations) in terms of

$$F_{N;\alpha,\beta,\ldots,\mu}(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N)\delta(\mathbf{k}_1 + \mathbf{k}_2 + \cdots + \mathbf{k}_N) = \langle v_\alpha(\mathbf{k}_1, t) v_\beta(\mathbf{k}_2, t) \cdots v_\mu(\mathbf{k}_N, t) \rangle \,, \qquad (3)$$

where $N = n + m$. The main contribution to the correlator $\langle \Psi_{n,m} \rangle$ comes from the region of integration $k_1 \simeq (1/R) \ll k_2 \simeq k_3 \simeq \cdots \simeq k_N \simeq (1/r)$ for $n = 1, m \geq 2$ and from the region $k_1 \simeq k_2 \simeq (1/R) \ll k_3 \simeq \cdots k_N \simeq (1/r)$ for $n \geq 2, m \geq 2$, while the correlator $\langle \Psi_{n,0} \Psi_{0,m} \rangle$ is determined by the region $|\mathbf{k}_1 - \mathbf{k}_2| \simeq (1/R) \ll k_1, k_2, \ldots, k_N \simeq (1/r)$. Actually relations (1) and (2) are a consequence of the following asymptotic relations derived in this Rapid Communication:

$$pF_{N;\alpha,\beta,\ldots,\mu}(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \ldots, \boldsymbol{\kappa}_\ell,\ \mathbf{k}_{\ell+1}, \ldots,\ \mathbf{k}_N)$$

$$\propto \Big[\boldsymbol{\kappa}_1 F(\kappa_1) \Big] \qquad (4)$$

in the limit $\kappa_1 \ll k_j$,

$$F_{N;\alpha,\beta,\ldots,\mu}(\mathbf{k}_1,\mathbf{k}_2,\ldots,\ \mathbf{k}_n\ \mathbf{k}_{n+1},\ldots,\ \mathbf{k}_N) \propto \left[\kappa^2 F(\kappa)\right] \quad (5)$$

in the limit

$$|\mathbf{k}_1+\mathbf{k}_2+\cdots+\mathbf{k}_\ell| = \kappa \ll k_n\,,\quad \ell < n\,. \quad (6)$$

Results (1) and (2) may be easily generalized to describe other types of correlators, as, e.g., the case where a single closed chain crosses the gap an even number of times, etc. To do this one has to represent a given correlator in the $\mathbf{k}$ representation and then to utilize relations (4) and (6).

Note that in the limit (6) one may explicitly express the correlator F_N ($N = n + m$) for isotropic turbulence in terms of lower-order correlators $F_{(n+1)}$, $F_{(1+m)}$, F_2:

$$\begin{aligned}&F_{N;\alpha_1,\alpha_2,\ldots,\alpha_n,\beta_1,\beta_2,\ldots,\beta_m}(\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_n,\ \mathbf{k}_{n+1},\ldots,\mathbf{k}_N)\\ &\quad = F_{n+1;\alpha_1,\alpha_2\ldots,\alpha_n,\gamma}(\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_n,-\boldsymbol{\kappa})\\ &\qquad \times F_{1+m;\gamma\beta_1,\beta_2,\ldots,\beta_m}(\boldsymbol{\kappa},\mathbf{k}_{n+1},\ldots,\mathbf{k}_N)/F_{2;\delta,\delta}(\kappa)\,. \quad (7)\end{aligned}$$

This is the *law of correlator decomposition.*

The proof of Eqs. (4)–(7) given below is based on the Navier-Stokes equation and contains no approximations, such as truncation of perturbation series, etc. The basic idea is that in the limit $R \gg r$ the correlation between the "left" point (or "left" group of points) $\mathbf{r}_i$ and the "right" group of points $\mathbf{r}_j$ is described by a very simple sequence of diagrams *if one uses the quasi-Lagrangian diagrammatic approach* [2,3], which allows us to eliminate the sweeping of small eddies by the velocity field of the larger ones. The principal sequence of such diagrams describes *one-eddy exchange* and contains the "left" and "right" parts of the diagrams connected to each other only via the line of the double velocity correlator—see Figs. 2 and 3. These diagrams exceed all others by a factor $\mathcal{K} \simeq (R/r)^p$, with p either $\simeq 8/3$ or $\simeq 5/3$, as will be detailed below. It is important to stress that such an order-by-order analysis of the diagrammatic expansion becomes possible only in quasi-Lagrangian variables preserving the Galilean invariance of the problem in each diagram. In the framework of Wyld's initial diagrammatic technique [4] any truncation of the series breaks this symmetry and leads to qualitatively wrong results.

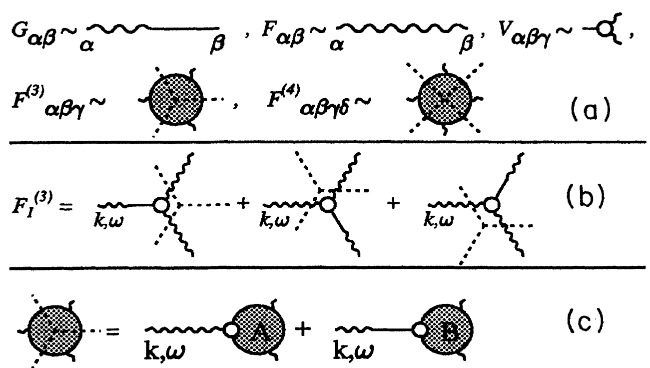

FIG. 2. Diagrammatic representation for triple correlator of QL-velocity. *A*, graphical notation; *B*, diagrams for first-order contribution; *C*, classification of diagrams with respect to type of $\mathbf{k},\omega$ leg.

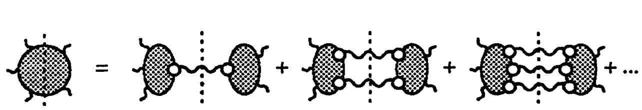

FIG. 3. Classification of diagrams for five-order correlator with respect to the number of double correlators in the principal intersection.

The exact asymptotic predictions (1)–(7) are interesting in themselves. They may be checked in physical or computer experiments. Expression (7) gives also an important theoretical relation between scaling exponents of n-order correlators (10). We proceed now to discuss the different types of turbulence scaling and constraints for scaling exponents following from some exact relations derived in this paper as well as in the paper of L'vov and Lebedev [5].

Note that the method of deriving the relations (1)–(7) is based on the topological properties of diagrams and on the fact of locality of interaction and does not use more detailed properties of the system under consideration. Therefore our results may be directly applied or easily extended to various systems of hydrodynamic type with local interaction like the turbulence of cold electron plasma, turbulent diffusion of passive scalar, convective turbulence, some problem of surface roughening at crystal growths and flame propagation, etc.

II. TYPES OF TURBULENT SCALING AND CONSTRAINTS FOR SCALING EXPONENTS

One may distinguish three levels of description of velocity correlation functions. In the simplest case we are interested in *simultaneous two-point correlators of velocity differences* of order n $D_n(r_{1,2}) = \langle v^n_{1,2}\rangle$ which are functions of only one argument $r_{1,2}$, the distance between two points. At the next level is the description in terms of *simultaneous n-point correlators of velocity differences* such as (1)–(3).

These are functions of n arguments. It is known that consistent analytical theories of turbulence deal with different time objects reflecting the dynamics of the system. Therefore the third level of turbulence description includes n-time, n-point correlators of order n, which are functions of $2n$ arguments.

The simplest scaling assumption made on the first level of turbulence description is that in the inertial interval $D_n(r)$ is a uniform function of r and consequently

$$D_n(\lambda r) = \lambda^{\zeta_n} D_n(r)\,, \quad (8)$$

with ζ_n being static scaling exponents. We will call this assumption *weak* or *two-point scaling.* Different phenomenological models of turbulence predict different behavior of the function ζ_n. In particular, the famous Kolmogorov 1941 (KO-41) model [6–8] results in $\zeta_n = n/3$, with the dynamic exponent $z = 2/3$ describing the scaling of the turnover time $\tau(r)$ (or lifetime) of eddies of scale r as $\tau(r) \propto r^z$. Such a scaling is called *global KO-41 scaling.* The β model [9] leads to *one-exponent scaling*

$$\zeta_n = n/3 + \nu(n-3) \quad (9)$$

with some value of the exponent ν. In the multifractal models [10,11] the static exponents ζ_n are in fact phenomenological parameters. So, the *assumption of weak scaling leaves scaling exponents undefined.*

In describing turbulence in terms of simultaneous n-point correlators of velocity differences one may assume that these objects are homogeneous functions of degree x_n in the inertial interval of scales, i.e., $F_N(\lambda \mathbf{k}_\ell) = \lambda^{-x_n} F_N(\mathbf{k}_\ell)$ with arbitrary values of x_n. We will call this assumption *many-point scaling.* Obviously many-point scaling is a stronger assumption than two-point scaling. It is important that the derived law of decomposition (7) provides constraints between scaling exponents ζ_n:

$$\zeta_{n+m} + \zeta_2 = \zeta_{n+1} + \zeta_{m+1} \ . \tag{10}$$

Together with the well known constraint $\zeta_3 = 1$ [6], this yields the relation of the β model (9). Thus, many-point scaling implies one-exponent scaling (9).

In [5] we discussed many-time, many-point scaling, which is the assumption that different-time two-point, three-point, etc. objects of the theory of turbulence are homogeneous functions in the inertial interval and may be characterized by some scaling exponents. We showed that this strongest scaling is consistent with the exact relation deduced if scaling exponents are related according to the KO-41 model with $\zeta_n = n/3$. So, many-time, many-point scaling gives birth to global KO-41 scaling.

Apart from many-time, many-point scaling leading to global scaling, one may expect solutions of greater complexity consistent with the multifractal models of turbulence [10,11]. We cannot reject this possibility, but postpone the question of the relation between multifractal models of turbulence and the Navier–Stokes equations to the future.

III. PROOF OF ASYMPTOTIC RELATIONS

As usual [6] in the inertial interval of scales we shall start with the Euler equation in an unbounded region:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} + \boldsymbol{\nabla} P = \mathbf{0}\,, \quad \boldsymbol{\nabla}\cdot\mathbf{v} = \mathbf{0}\ . \tag{11}$$

Here $\mathbf{v}(t,\mathbf{r})$ is the velocity field of an incompressible fluid, P is the pressure, and we have set the density $\rho = 1$. In order to eliminate the sweeping from the theory let us use the *quasi-Lagrangian* (QL) velocity $\mathbf{u}(\mathbf{r}_0|t,\mathbf{r})$ (see [2,3])

$$\mathbf{v}(t,\mathbf{r}) = \mathbf{u}\left(\mathbf{r}_0|t,\mathbf{r} - \int^t \mathbf{u}(\mathbf{r}_0|\tau,\mathbf{r}_0)d\tau\right)\ . \tag{12}$$

Here the function $\mathbf{u}$ has an additional argument, the coordinate of the marked reference point $\mathbf{r}_0$. Substituting (12) in (11) one obtains the equation of motion for the QL velocity. In the $\mathbf{k}$ representation it takes the form

$$i\partial u_\alpha(\mathbf{r}_0|t,\mathbf{k})/\partial t = \tfrac{1}{2}(2\pi)^{-6}\int d^3q\, d^3p V_{\alpha\beta\gamma}(\mathbf{k};\mathbf{q},\mathbf{p}) \times u_\beta(\mathbf{r}_0|t,-\mathbf{q})u_\gamma(\mathbf{r}_0|t,-\mathbf{p})\ . \tag{13}$$

The expression for the *dynamic vertex* V is given for example in [5]. We do not need it here. The only property of $V_{\alpha\beta\gamma}(\mathbf{k};\mathbf{q},\mathbf{p})$ which is now important is its *locality* in k space: in asymptotic regimes where one of the wave vectors (k, q or p) is much smaller than the other two, the vertex V tends to zero as the smallest wave vector. Note that the initial Eulerian vertex is proportional to k and does not tend to zero if $\mathbf{q}$ and $\mathbf{p}$ go to zero. The main technical difference between the quasi-Lagrangian and the conventional (in terms of Eulerian velocity) description of turbulence is that the wave vector $\mathbf{k}$ is no longer preserved in the dynamic vertex V since it is *not* proportional to $\delta(\mathbf{k}+\mathbf{q}+\mathbf{p})$. This is a consequence of the spatial inhomogeneity of the theory due to the choice of a definite reference point $\mathbf{r}_0$ in the definition of the QL velocity (12). We will use Eq. (13) in Wyld's perturbation approach [2,3].

Natural objects of Wyld's diagrammatic expansion are the bare vertex $V_{\alpha\beta\gamma}$ and dressed propagators which are the Green's function $\tilde{G}_{\alpha\beta}$ and the double correlator $\tilde{F}_{2,\alpha\beta}$. The former is defined as the susceptibility of the average QL velocity field u_α to a force ϕ_β which is added to the right-hand side of the equation of motion (13). In the QL approach propagators depend on $\mathbf{r}_0, \omega, \mathbf{k}$, and $\mathbf{k}_1$ and are *not* proportional to $\delta(\mathbf{k}-\mathbf{k}_1)$. Nevertheless, QL propagators become diagonal in $\mathbf{k}$ *by integration* with respect to ω because this results in simultaneous QL propagators $\tilde{G}_{\alpha\beta}(\mathbf{k})$ and $\tilde{F}_{\alpha\beta}(\mathbf{k})$ coinciding with the Eulerian ones $G_{\alpha\beta}(\mathbf{k})$ and $F_{\alpha\beta}(\mathbf{k})$ [2,3]

$$\int \tilde{F}_{2;\alpha\beta}(\mathbf{k},\mathbf{k}_1,\omega)d\omega/(2\pi)^3 = F_{2;\alpha\beta}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}_1)\ . \tag{14}$$

We consider first the diagrammatic expansion for the triple correlator $\tilde{F}_{3,\,\alpha\beta\gamma}$ of QL velocity. Figure 2(b) shows dashed lines trisecting diagrams by cutting through wavy lines which represent double correlators $\tilde{F}_{2,\,\alpha\beta}$. Each $\frac{1}{3}$ plane contains one entrance into the diagram. While diagrams for mass operators have vertices V as entrances, diagrams for $\tilde{F}_3$ have three external legs which are propagators (Green's function or double correlator). Sometimes it is useful to show external legs explicitly as is done with one leg (having arguments $\mathbf{k},\omega$) in Fig. 2(c). In this figure we introduce the intermediate objects A and B which have one $(\mathbf{k},\omega)$ entrance of the first type (vertex) and two other entrances of the second type (legs—propagators). In such a manner one can split the analytical expression for $\tilde{F}_3$ into two parts that are proportional, respectively, to $\tilde{F}_2 * \tilde{A}$ and to $\tilde{G} * \tilde{B}$. Here the $*$ operator designates summation over vector indices and $\mathbf{k}$ integration: $C_{\alpha\beta}(\mathbf{k},\mathbf{k}',\omega) = A * B = \int A_{\alpha\gamma}(\mathbf{k},\mathbf{k}'',\omega)B_{\gamma\beta}(\mathbf{k}'',\mathbf{k},\omega)dk''$. An important point is that *in the limit $k \ll k_1 \simeq k_2$ the main contribution to $\tilde{F}_3$ is given by the $\tilde{F}_2 * \tilde{A}$ term.* Indeed one can easily estimate the ratio $\tilde{A}/\tilde{B}$ as $\tilde{G}(\mathbf{k}_1,\mathbf{k}_1',\omega_1)/\tilde{F}_2(\mathbf{k}_1,\mathbf{k}_1',\omega_1)$—see Fig. 2(b). Taking into account that $\tilde{F}_2(\mathbf{k}_1,\mathbf{k}_1',\omega_1)/\tilde{G}(\mathbf{k}_1,\mathbf{k}_1',\omega_1)$ is close to the simultaneous double correlator $F_2(k_1)$ at $k_1 \simeq k_1'$, one has the following estimate for a ratio:

$$\mathcal{K}_1 = \frac{\tilde{G} * \tilde{B}}{\tilde{F}_2 * \tilde{A}} \simeq \frac{k_1 F_2(k_1)}{kF(k)}\ . \tag{15}$$

In the inertial interval $F_2(k) \propto k^{-y}$ (y is the static exponent, $y = 11/3$ for KO-41 model) and $\mathcal{K}_1 \simeq (k/k_1)^{(y-1)} \ll 1$. Clearly, the above considerations (classification of diagrams with respect to the type of one of the legs), show that the estimate (15) is valid for correlators of any order $n \geq 3$. That is why in the limit $k = \kappa \ll k_j$, $j = 2, \ldots, N$, the main contribution to N-order correlators comes from the $F * A$ term:

$$\tilde{F}_{N;\alpha\beta\gamma,\ldots}(\boldsymbol{\kappa}, \omega_1; \mathbf{k}_2, \omega_2; \mathbf{k}_3, \omega_3, \ldots) = \int F_{2;\alpha\alpha'}(\kappa, \boldsymbol{\kappa}', \omega_1) \times \tilde{A}_{N;\alpha'\beta\gamma,\ldots}(\boldsymbol{\kappa}', \omega_1; \mathbf{k}_2, \omega_2; \mathbf{k}_3, \omega_3, \ldots) d\boldsymbol{\kappa}'/(2\pi)^3 \ . \quad (16)$$

In order to obtain simultaneous N-order correlators of Eulerian velocities $F_{N;\alpha\beta\gamma,\ldots}(\boldsymbol{\kappa}, \mathbf{k}_2, \mathbf{k}_3, \ldots)$ one must integrate (16) with respect to all frequencies $\omega_1, \omega_2, \omega_3, \ldots$ [2]. The main contribution to $\int d\omega_1$ comes from the region $\omega_1 \simeq \kappa^z \ll \omega_j \simeq k_j^z$ because of $\kappa \ll k_j$. Here z, the dynamic exponent, equals 2/3 for the Kolmogorov model. It is important that the function $\tilde{A}$ depend only weakly on ω_1. The diagrams for $\tilde{A}$ do not contain any propagators depending explicitly on ω_1 alone. The ω_1 dependence of $\tilde{A}$ arises only via arguments like $(\omega_1 + \omega_j + \cdots)$. Therefore $\partial \tilde{A}/\partial \omega_1 \sim \tilde{A}/\omega_j$ and one may neglect the ω_1 dependence of $\tilde{A}$. This enables us to perform the ω_1 integration with the help of (14). Thus one has

$$F_{N;\alpha,\beta,\ldots,\mu}(\boldsymbol{\kappa}, \mathbf{k}_2, \ldots, \mathbf{k}_N)\delta(\boldsymbol{\kappa} + \mathbf{k}_2 + \cdots + \mathbf{k}_N) = F_{2;\alpha\alpha'}(\boldsymbol{\kappa}) A_{N;\alpha'\beta\gamma,\ldots}(\boldsymbol{\kappa}, \mathbf{k}_2, \ldots, \mathbf{k}_N) \ , \ A_{N;\alpha'\beta\gamma,\ldots}(\boldsymbol{\kappa}, \mathbf{k}_2, \ldots, \mathbf{k}_N) = \int \tilde{A}_{N;\alpha'\beta\gamma,\ldots}(\boldsymbol{\kappa}, 0; \mathbf{k}_2, \omega_2; \mathbf{k}_3, \omega_3, \ldots) \frac{d\omega_2}{(2\pi)^3} \frac{d\omega_3}{(2\pi)^3} \cdots \ . \quad (17)$$

Taking into account that function A contains the dynamical QL vertex $V(\boldsymbol{\kappa}, \mathbf{k}', \mathbf{k}'') \propto \kappa$ we come to the conclusion that for small κ this correlator is proportional to $\kappa F_2(\kappa)$.

The above result may be extended to the case in which not only one argument of F_N κ but several of these are smaller than others: $\kappa_1, \kappa_2, \ldots, \kappa_\ell \ll k_{\ell+1}, \ldots, k_N$ (because of conservation of momentum $\ell \leq N - 2$). Clearly in this case the main contribution to the N-order correlator derives from diagrams in which the external legs with small wave vectors $\boldsymbol{\kappa}_j$ are double correlators only and *are not* Green's functions. Each of these legs gives a factor $\kappa_j F_2(\kappa_j)$ in the κ dependence of F_N. As a consequence one obtains the asymptotic relation (4).

Next we will prove Eq. (5). To this end let us use the classification of diagrams for the N-order correlator shown in Fig. 2. Consider the contribution of the first QL diagram with one $\tilde{F}_2(\boldsymbol{\kappa}, \kappa', \omega')$ in the principal intersection [2,3]. In order to compute the simultaneous correlator of the Eulerian velocity one may integrate the corresponding QL correlator with respect to all frequencies. In the limit (6) it is possible to neglect the ω dependence of both ("left" and "right") functions $\tilde{A}$. The reason for this has been already discussed. Performing the ω' integration with the help of (14) one obtains a relation similar to (17)

$$F_{N;\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m}(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n, \ldots, \mathbf{k}_N)\delta(\mathbf{k}_1 + \cdots \mathbf{k}_N) = A_{n+1;\alpha_1,\alpha_2,\ldots,\alpha_n,\gamma}(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n, -\boldsymbol{\kappa}) \times F_{2;\gamma\delta}(\kappa) A_{1+m;\delta\beta_1,\beta_2,\ldots,\beta_m}(\boldsymbol{\kappa}, \mathbf{k}_{n+1}, \ldots, \mathbf{k}_N) \ . \quad (18)$$

Here the functions A are given by (17). It is clear that (5) follows from (17) as well as the relation $A \propto \kappa$.

Now we will prove relation (7). For isotropic turbulence $F_{2;\alpha,\beta}(\mathbf{k}) = P^{\perp}_{\alpha\beta}(\mathbf{k})F_2(k)$, $P^{\perp}_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta/k^2 = P^{\perp}_{\alpha\gamma}(\mathbf{k})P^{\perp}_{\gamma\beta}(\mathbf{k})$. Clearly $F_{2;\alpha,\beta}$ satisfies the relation

$$F_{2;\alpha,\beta}(\mathbf{k}) = F_{2;\alpha,\gamma}(\mathbf{k})F_{2;\gamma,\beta}(\mathbf{k})/F_2(k) \ , \quad (19)$$

which is inserted into (18). Equation (17) shows that $A_{n+1}F_2$ and F_2A_{1+n} in (18) are F_{n+1} and F_{1+n}, correspondingly. In such a manner Eqs. (17)–(19) yield (7). Note that contributions to F_N given by the next diagrams in Fig. 3 (with two or more lines of double correlators in the principal intersection) do not contain the factor $\kappa^2 F_2(\kappa)$. Therefore such contributions are smaller than the value of the first diagram (with one double correlator in the principal intersection) by a factor $\mathcal{K}_2 \simeq \kappa^2 F_2(\kappa)/k^2 F_2(k)$. For KO-41 spectra $\mathcal{K}_2 \simeq (\kappa/k)^{5/3}$.

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