Extended universality in moderate-Reynolds-number flows

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(Received 22 September 1993)

In the inertial interval of turbulence one asserts that the velocity structure functions $S_n(r)$ scale like $r^{n\zeta_n}$. Recent experiments indicate that $S_n(r)$ has a more general universal form $[rf(r/\eta)]^{n\zeta_n}$, where η is the Kolmogorov viscous scale. This form seems to be obeyed on a range of scales that is larger than power law scaling. It is shown here that this extended universality stems from the structure of the Navier-Stokes equations and from the property of the locality of interactions. The approach discussed here allows us to estimate the range of validity of the universal form. In addition, we examine the possibility that the observed deviations from the classical values of $\zeta_n=\frac{1}{3}$ are due to the finite values of the Reynolds numbers and the anisotropy of the excitation of turbulence.

PACS number(s): 47.27.−i

I. INTRODUCTION

The degree of excitation of a turbulent flow is characterized by its Reynolds number $\mathrm{Re}\equiv UL/\nu$, where L is the outer scale, U is the typical velocity difference across this scale, and ν is the kinematic viscosity of the fluid. One expects that at high values of Reynolds number turbulent flows would exhibit scaling laws; the structure functions of the longitudinal velocity fluctuations $S_q(r)$ are expected [1] to behave like

$$S_q(r)\equiv\langle \delta u_r(\mathbf{x})^q\rangle \sim U\left[\frac{r}{L}\right]^{q\zeta_q}, \tag{1.1}$$

where $\delta u_r(\mathbf{x})\equiv[\mathbf{u}(\mathbf{x}+\mathbf{r},t)-\mathbf{u}(\mathbf{x},t)]\cdot\mathbf{r}/r$, $\mathbf{u}(\mathbf{x},t)$ is the velocity field, ζ_q are scaling exponents, and $\langle\ \rangle$ denotes an average over time. It is well known, however [1], that the observation of clean scaling laws in turbulence requires very high Reynolds numbers. Equation (1.1) is expected to hold only for scales significantly larger than the Kolmogorov viscous scale η, where $\eta\sim L(\mathrm{Re}/\mathrm{Re}_c)^{-3/4}$ and Re_c is the critical Re value for the onset of turbulence (of the order of 10^2–10^3). At most laboratory flows the Reynolds number is not high enough to have a very sizable scaling range. The influence of the viscous cutoff is felt already on scales of the order of 100η.

To overcome this difficulty, it was suggested recently [2] to examine the universal scaling properties of velocity structure functions in a different way. Rather than examining the r dependence of each q-order structure functions separately, Ref. [2] studied the functional dependence of one structure function on the other. The upshot of the data analysis was that it appears that $S_q(r)$ can be written as

$$S_q(r)=C_q\left[rf\left[\frac{r}{\eta}\right]\right]^{q\zeta_q}, \tag{1.2}$$

where $f(r/\eta)$ is a scaling function which is *the same* [2] for all q. It appears that Eq. (1.2) holds for lower scales than (1.1), i.e., to scales of the order of a few η. Since the values of S_q in this range of scales are exponentially small compared to their value in the inertial range, one gets a significant increase in the range of linear behavior by plotting the logarithm of $S_q(r)$ vs the logarithm of $S_{q'}(r)$. The authors of Ref. [2] found a linear relationship that extended over 4–6 orders of magnitude in such plots. The aim of this paper is to explain the theoretical basis of (1.2), to predict the lower range of its validity, and to estimate the exponents. In the experiment the exponents turned out nonclassical. We shall examine the possibility that this result stems from corrections to scaling. We shall refer to Eq. (1.2) as the relation of "extended universality."

It will be shown below that the explanation of (1.2) is based on two equally important factors. First is the fact that the viscosity and the time derivative appear in the Navier-Stokes equations [1,3] in terms which are linear in the velocity field:

$$\partial\mathbf{u}/\partial t+\mathbf{u}\cdot\nabla\mathbf{u}-\nu\nabla^2\mathbf{u}-\nabla p=0\ ,\quad \nabla\cdot\mathbf{u}=0 \tag{1.3}$$

where p is the pressure. This fact translates to an important feature of the nth-order correlation functions of velocity fluctuations: while the two-point Green's function (response function of the velocity field to external perturbations) is an explicit function of the viscosity, the two-point correlator *and* all the higher-order correlation functions depend on the viscosity only implicitly through their dependence on the Green's function [4]. We shall demonstrate this important fact using renormalized perturbation theory in Sec. II and employ it as the first fundamental property behind Eq. (1.2).

The second ingredient of our explanation of the extended universality is the property of "locality." By locality we mean here two different things. One is locality in the sense of Kolmogorov [1,5], meaning that velocity fluctuations $\delta u_r(\mathbf{x})$ interact most effectively with velocity fluctuations $du_{r'}(\mathbf{x})$ for which $r'=O(r)$. The second meaning is locality in the sense that structure functions $S_n(r)$ are related to nth-order velocity correlation functions in $\mathbf{k}$ representation such that the region in which

$k \approx 1/r$ is most significant. Both these properties will be demonstrated in Secs. III and IV and used to reach the final result Eq. (1.2).

Of course, a proper derivation of Eq. (1.2) must include the limits of its validity. We shall see that the conditions of locality which allows us to derive Eq. (1.2) can be written as the condition

$$2\zeta_2(r) \equiv \frac{d \ln S_2(r)}{d \ln r} < 2 . \tag{1.4}$$

For sufficiently small scales where the exponential decay of $S_2(r)$ has already set in the condition (1.4) is violated. Notice that in the scaling regime the value of $2\zeta_2(r)$ is about $\frac{2}{3}$. As pure power-law behavior is lost at smaller scales, we can think of the logarithmic derivative (1.4) as an r-dependent scaling exponent $\zeta_2(r)$. The extension of the universal behavior (1.2) compared to pure power-law behavior is given by the range in which $2\zeta_2(r)$ goes between $\frac{2}{3}$ and 2. Thus the borderline of (1.4), and therefore of the applicability of (1.2), should be closer to the Kolmogorov scale η than the breakdown of the power-law form of $S_2(r)$. We shall see that the gain is in about one order of magnitude of length scales.

It will turn out that in the theory that we shall describe below the value of the scaling exponents ζ_q is $\frac{1}{3}$ for all q. The experiment of Ref. [2] found agreement with Eq. (1.2) but with nonclassical exponents, and in particular $2\zeta_2 = 0.701$. We shall argue in Sec. V that it is very likely that in these moderate-Reynolds-number experiments the difference from classical exponents stems from corrections to scaling due to the anisotropy of the flow in the large scales. This anisotropy decays only slowly with diminishing scales (as a power law), and this may result in a correction to scaling. We estimated the correction to $2\zeta_2$, and found that

$$2\zeta_2 - \tfrac{2}{3} = \left[\frac{\text{const} \times \eta}{L} \right]^{2/5} \sim \mathrm{Re}^{-3/10} . \tag{1.5}$$

If this is the right mechanism for the deviation in ζ_2 from its $\frac{1}{3}$ value, then Eq. (1.5) is a prediction that can be tested in experiments such as those of Ref. [2]. At present there is no experimental evidence for a Re dependence of ζ_2, and therefore it is possible that the observed anomalous scaling stems from a deeper cause.

II. HIGH-ORDER CORRELATION FUNCTION EXPRESSED IN TERMS OF TWO-POINT PROPAGATORS

The aim of this section is to recall the renormalized perturbation approach [4,6] to the Navier-Stokes (NS) equations which will allow us to express high-order correlation functions in terms of two-point correlation functions. We shall first introduce the necessary notation and then study the property of locality.

A. Reminder of the diagrammatic perturbation approach

The three fundamental quantities in terms of which all higher-order correlation functions can be expressed are the Green's function of $G_{\alpha,\beta}(\mathbf{k},\omega)$, the two-point correlation function $F_{\alpha,\beta}(\mathbf{k},\omega)$, and the bare vertex of the NS equations $\Gamma_{\alpha\beta\gamma}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ which are defined as

$$G_{\alpha,\beta}(\mathbf{k},\omega)\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega') = \langle \delta u_\alpha(\mathbf{k},\omega)/\delta\phi^*_\beta(\mathbf{k}',\omega') \rangle , \tag{2.1}$$

$$F_{\alpha,\beta}(\mathbf{k},\omega)\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega') = \langle u_\alpha(\mathbf{k},\omega)u_\beta(\mathbf{k}',\omega') \rangle , \tag{2.2}$$

where $\mathbf{u}(\mathbf{k},\omega)$ is the Fourier component of the velocity field $\mathbf{u}(\mathbf{x},t)$ and ϕ is an external Gaussian force that is going to be eliminated in the course of the calculation. At the present time it is used to define the averaging procedure, which is done with respect to realizations of this force. We are not going to define the vertex here, because at any rate the perturbation theory is done in quasi-Lagrangian coordinates with a different vertex that is going to be discussed later.

The well-known Dyson-Wyld equations for isotropic systems in terms of isotropic $G(\mathbf{k},\omega)$ and $F(\mathbf{k},\omega)$ are [4,6]

$$G(\mathbf{k},\omega) = \frac{1}{\omega + i\nu k^2 - \Sigma(\mathbf{k},\omega)} , \tag{2.3}$$

$$F(\mathbf{k},\omega) = |G(\mathbf{k},\omega)|^2 \Phi(\mathbf{k},\omega) . \tag{2.4}$$

The relations between the scalar propagators G and F and the tensors propagators $G_{\alpha,\beta}$ and $F_{\alpha,\beta}$ are $G_{\alpha,\beta}(\mathbf{k},\omega) = P_{\alpha,\beta}(\mathbf{k})G(\mathbf{k},\omega)$ and $F_{\alpha,\beta}(\mathbf{k},\omega) = P_{\alpha,\beta}(\mathbf{k})F(\mathbf{k},\omega)$, where $P_{\alpha,\beta}(\mathbf{k})$ is the transverse projection operator [1] $(\delta_{\alpha,\beta} - k_\alpha k_\beta / k^2)$. The self-energy term $\Sigma(\mathbf{k},\omega)$ and the nonlinear noise function $\Phi(\mathbf{k},\omega)$ are given in terms of an infinite series of integrals which involve again $G(\mathbf{k},\omega), F(\mathbf{k},\omega)$, and the bare vertex. It is important to stress that neither $\Sigma(\mathbf{k},\omega)$ nor $\Phi(\mathbf{k},\omega)$ contains the viscosity ν explicitly. For example, the lowest-order contributions to $\Sigma(\mathbf{k},\omega)$ and $\Phi(\mathbf{k},\omega)$ are [4,6]

$$\Sigma_2(\mathbf{k},\omega) = \frac{1}{8\pi^2} \int \Gamma_{\alpha\beta\gamma}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\Gamma_{\beta\gamma\alpha}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) \times G(\mathbf{k}_1,\omega_1)F(\mathbf{k}_2,\omega_2)\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2) \times \delta(\omega+\omega_1+\omega_2) d\mathbf{k}_1 d\mathbf{k}_2 d\omega_1 d\omega_2 , \tag{2.5}$$

$$\Phi_2(\mathbf{k},\omega) = \frac{1}{16\pi^2} \int \Gamma^2_{\alpha\beta\gamma}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)F(\mathbf{k}_1,\omega_1)F(\mathbf{k}_2,\omega_2) \times \delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\delta(\omega+\omega_1+\omega_2) \times d\mathbf{k}_1 d\mathbf{k}_2 d\omega_1 d\omega_2 . \tag{2.6}$$

Also all the higher-order contributions are written as functionals of the same three objects, without explicit dependence on the viscosity.

In exactly the same way the higher-order correlation functions depend on the same three objects. For example, the three-point correlation function $F^{(3)}$ is defined as

$$F^{(3)}_{\alpha\beta\gamma}(\mathbf{k}_1,\omega_1;\mathbf{k}_2,\omega_2;\mathbf{k}_3,\omega_3)\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\delta(\omega_1+\omega_2+\omega_3) = \langle u_\alpha(\mathbf{k}_1,\omega_1)u_\beta(\mathbf{k}_2,\omega_2)u_\gamma(\mathbf{k}_3,\omega_3) \rangle . \tag{2.7}$$

Within this perturbative approach one develops an infinite series representation for $F^{(n)}$, and, for example, one derives for $F^{(3)}$ and $F^{(4)}$

$$F^{(3)}_{\alpha\beta\gamma}(\mathbf{k}_1,\omega_1;\mathbf{k}_2,\omega_2;\mathbf{k}_3,\omega_3)$$
$$=\{\Gamma_{\alpha\beta\gamma}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)G_1F_2F_3$$
$$+\Gamma_{\gamma\alpha\beta}(\mathbf{k}_3,\mathbf{k}_1,\mathbf{k}_2)G_3F_1F_2$$
$$+\Gamma_{\beta\gamma\alpha}(\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_1)G_2F_3F_1\}+\cdots , \quad (2.8)$$

$$F^{(4)}_{\alpha\beta\gamma\delta}(\mathbf{k}_1,\omega_1;\mathbf{k}_2,\omega_2;\mathbf{k}_3,\omega_3;\mathbf{k}_4,\omega_4)$$
$$=\tfrac{1}{2}\{\Gamma_{\alpha\beta\sigma}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q})\Gamma_{\sigma\gamma\delta}(-\mathbf{q},\mathbf{k}_3,\mathbf{k}_4)G_1F_2F_3F_4G(\mathbf{q},\Omega)$$
$$+\text{all permutations}\}+\cdots , \quad (2.9)$$

where G_j and F_j are shorthand notation for $G(\mathbf{k}_j,\omega_j)$ and $F(\mathbf{k}_j,\omega_j)$, respectively, and $\mathbf{q}=-\mathbf{k}_1-\mathbf{k}_2$ and $\Omega=-\omega_1-\omega_2$. The permutations in (2.9) are all the $4!-1$ possible pair permutations on the indices. The ellipses stand for higher-order terms. We see again that only the two point propagators F and G and the vertex Γ appear in all these expressions.

B. Connection to the structure functions of velocity differences

The theory sketched in Sec. II A is in terms of the $\mathbf{k},\omega$ representation of correlation functions. We need now to connect to the experimentally measured structure functions of velocity differences (1.1). This is done in two steps. First, we need to integrate out the frequency dependence, to obtain simultaneous correlation functions, denoted $\tilde{F}^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n)$. For example,

$$\tilde{F}_{\alpha,\beta}(\mathbf{k})=2\pi\int\int F_{\alpha\beta}(\mathbf{k},\omega)\delta(\omega+\omega_1)\frac{d\omega}{2\pi}\frac{d\omega_1}{2\pi} , \quad (2.10)$$

$$\tilde{F}^{(3)}_{\alpha\beta\gamma}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$$
$$=2\pi\int\int\int F^{(3)}_{\alpha\beta\gamma}(\mathbf{k}_1,\omega_1;\mathbf{k}_2,\omega_2;\mathbf{k}_3,\omega_3)$$
$$\times\delta(\omega_1+\omega_2+\omega_3)\frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\frac{d\omega_3}{2\pi} , \quad (2.11)$$

etc. Second, we need to Fourier transform back from $\mathbf{k}$ representation to $\mathbf{r}$ representation. Given the simultaneous functions $\tilde{F}^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n)$ we can compute the structure functions $S_n(r)$ via the following relation:

$$S_n(r)=(2\pi)^3\int\cdots\int\tilde{F}^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n)\delta\left[\sum_{j=1}^{n}\mathbf{k}_j\right]$$
$$\times\prod_{j=1}^{n}\left\{[1-\exp(i\mathbf{k}_j\cdot\mathbf{r})]\frac{d\mathbf{k}_j}{(2\pi)^3}\right\} . \quad (2.12)$$

To implement the program of computing the structure functions $S_n(r)$ from Eq. (2.12) we need now to examine the property of the locality of interactions. This examination will lead to an understanding of the range of validity of the extended universality.

III. THE PROPERTY OF LOCALITY AND ITS IMPLICATIONS: EXTENDED SCALING RELATIONS AND EXTENDED UNIVERSALITY

The perturbative expansion done in Eulerian coordinates is plagued with divergences order by order so that a resummation is needed. It was shown that a useful resummation is obtained by employing quasi-Lagrangian velocities. Using Lagrangian velocities one eliminates the effect of sweeping of small scale structures by large-scale structures which is responsible for the divergences in the Eulerian representation. The Eulerian and Lagrangian velocities are related by [7,8,9]

$$\mathbf{u}(\mathbf{r},t)=\mathbf{v}\left[\mathbf{r}_0,\mathbf{r}-\int_{-\infty}^{t}\mathbf{v}(\mathbf{r}_0,\mathbf{r}_0,\tau)d\tau,t\right] . \quad (3.1)$$

In terms of $\mathbf{v}$ as the fundamental field one can develop a perturbation approach that was shown [7] to be free of divergences in both the ir and the uv regimes. There is a price to pay: in the quasi-Lagrangian formulation the homogeneity of space is lost, leading to some unpleasant technical difficulties in the theory. Happily, these technicalities were dealt with [7,8], and we can use for our purposes just the essential facts.

One such fact is that the Eulerian vertex $\Gamma_{\alpha\beta\gamma}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)$ is replaced by the quasi-Lagrangian vertex $V_{\alpha\beta\gamma}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)$ which has very different asymptotic properties. Namely, $\Gamma_{\alpha\beta\gamma}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\approx k$ for any value of k_1 or k_2. On the other hand, for $k\gg k_1$ or $k\gg k_2$, $V_{\alpha\beta\gamma}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)$ is of the order of the smallest wave vector:

$$V_{\alpha\beta\gamma}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\sim\min(k,k_1,k_2) . \quad (3.2)$$

This makes the Lagrangian vertex much smaller than the Eulerian one when k_1 or k_2 is much smaller than k. The reason is of course that the latter contains the sweeping interactions whereas the former does not. The Lagrangian vertex describes just the nonlinear dynamic interactions.

A. Locality of dynamical interactions

In the following we are going to seek solutions for the two-point propagators in the form

$$G(\mathbf{k},\omega)=\frac{1}{\gamma(\mathbf{k})}g\left[\frac{\omega}{\gamma(\mathbf{k})}\right] , \quad (3.3)$$

$$F(\mathbf{k},\omega)=\frac{\tilde{F}(\mathbf{k})}{\gamma(\mathbf{k})}f\left[\frac{\omega}{\gamma(\mathbf{k})}\right] . \quad (3.4)$$

In the inertial interval the solution should be scale invariant, i.e.,

$$\gamma(k)\sim k^z , \quad \tilde{F}(k)\sim k^{-y} . \quad (3.5)$$

We shall not assume such a scale invariant form, but rather study the possibility that $\gamma(k)$ and $\tilde{F}(k)$ differ from (3.5). One can actually find solutions of a form more general than power laws, and these apply on a wider range of scales than the scale invariant form (3.5). In fact, we shall see that the crucial quantity is the deriva-

tive of $\ln\tilde{F}(k)$ with respect to $\ln k$:

$$-\frac{d\ln\tilde{F}(k)}{d\ln k}\equiv y(k)\ . \tag{3.6}$$

When k increases, $y(k)$ increases due to viscous dissipation.

In studying locality one needs to examine both the uv and the ir regimes of k vectors. The issue of locality has been examined already before in the context of the scale invariant solutions (3.5) for y close to the Kolmogorov value of $\frac{11}{3}$. It was shown [7] that there are neither uv nor ir divergences. Since the solution that we seek now has faster decay for large k vectors than the scale invariant solution, we do not need to examine again the uv regime; we are guaranteed convergence there. On the other hand, for sufficiently large values of $y(k)$ the ir convergence can be lost. We thus need to study carefully the limit of convergence of our theory with the forms (3.3), (3.4).

To study the locality, examine first the typical integrals appearing in (2.5) and (2.6). After transforming to quasi-Lagrangian coordinates, we need to replace the vertices $\Gamma_{\alpha\beta\gamma}$ with the vertices $V_{\alpha\beta\gamma}$. Then we break the integration over k_2 to the two regions $[0,k^*]$ and $[k^*,\infty]$, with $k^*\ll k$. The ir regime corresponds to the first interval. Examine first (2.5). Due to the δ function, k_1 is $-(k+k_2)$, and therefore $k_1\approx -k$ in this regime. In a scaling theory the most relevant frequencies are $\omega\sim O(\gamma(k))$. Therefore in the region $k_2\ll k$, also $\omega_2\ll\omega$, and $\omega_1\approx-\omega$. Taking all this into account, and using the asymptotic properties of the quasi-Lagrangian vertices (3.2), we find the ir contribution to (2.5):

$$\Sigma_{\rm ir}(\mathbf{k},\omega)=C_1 G(\mathbf{k},\omega)I(k^*)\ , \tag{3.7}$$

$$I(k^*)=\int_0^{k^*}(k_2)^4\tilde{F}(k_2)dk_2\ , \tag{3.8}$$

with C_1 being a constant. Similarly, one can find that the ir contribution to the lowest order diagram of Φ is

$$\Phi_{\rm ir}(\mathbf{k},\omega)=C_2 F(\mathbf{k},\omega)I(k^*)\ . \tag{3.9}$$

Analysis of higher-order diagrams reveals that in each order the ir contribution is proportional to a power of $I(k^*)$, with the power being determined by the number of loops in the diagram. Therefore, the ir convergence is determined by the convergence of $I(k^*)$. The latter is safely convergent as long as $y(k)$ of Eq. (3.6) is smaller than 5 throughout the regime of small k. In other words, $y(k)$ should tend to a limit smaller than 5 as k tends to zero.

In studying the higher-order correlation functions we find that to lowest order there are no integrals appearing; cf. (2.8) and (2.9). In higher-order contributions integrals appear, and their ir locality property can be analyzed in precisely the same way, leading to the same boundary of locality.

B. Extended scaling relation

Using locality we can derive now a relation between $\gamma(k)$ and $\tilde{F}(k)$. To this aim we study first the series expansion for Φ, whose lowest-order contribution is Eq. (2.6). Using locality it is clear that the main contribution to $\Phi_2[k,\omega\approx\gamma(k)]$ comes from the region where k_1 and k_2 are of the order of k and where ω_1 and ω_2 are of $O(\gamma(k))$. We can estimate the integral (2.6) by replacing $d\mathbf{k}$ by k^3, $d\omega$ by $\gamma(k)$, and $F(k_i,\omega_i)$ by $\tilde{F}(k)/\gamma(k)$. This results in the estimate

$$\Phi_2[k,\omega\approx\gamma(k)]\approx k^5[\tilde{F}(k)]^2/\gamma(k)\ . \tag{3.10}$$

The analysis of higher-order contributions to Φ is as straightforward and the result is

$$\Phi_{2n}[k,\omega\approx\gamma(k)]=\Phi_2[k,\omega\approx\gamma(k)][\Lambda(k)]^{n-1}\ , \tag{3.11}$$

where the "expansion parameter" $\Lambda(k)$ is

$$\Lambda(k)=k^5\tilde{F}(k)/\gamma(k)^2\ . \tag{3.12}$$

This quantity can be evaluated by integrating the Wyld equation (2.4) over frequencies. Again the main contribution to the integral comes from $\omega\approx\gamma(k)$, and according to (3.3) the result of the integration is

$$\tilde{F}(k)\approx\Phi[k,\omega\approx\gamma(k)]/\gamma(k)\ . \tag{3.13}$$

Estimating $\Phi[k,\omega\approx\gamma(k)]$ as $\Phi_2[k,\omega\approx\gamma(k)]$, we get, using (3.10), that $\Lambda(k)\approx 1$ or, in other words,

$$\gamma(k)^2\approx k^5\tilde{F}(k)\ . \tag{3.14}$$

As we have found that $\Lambda(k)$ is of order 1, all the terms in the expansion of Φ are of the same order. It is customary in this type of theory to assume that this means that Φ is of the order of each of its terms. This is not a trivial assumption and it needs justification. Indeed, it has been justified under certain conditions [8] and we refer the reader to Refs. [1,6,8] for further discussions on the general validity of this assumption. The crucial point is that making this assumption justifies Eq. (3.14) as a self-consistent result with all the diagrammatic series and not only to one-loop order. In the scale invariant situation, where Eq. (3.5) is obeyed, it leads to the well-known scaling relation $2z+y=5$. Our Eq. (3.14) has a wider range of validity which extends to the borderline of locality $y=5$. We shall argue below that in fact $\gamma(k)$ has the physical meaning of an "eddy turnover time" for "eddies" of size $1/k$. We shall refer to Eq. (3.14) as the "extended scaling relation."

C. Higher-order correlation functions in k space

In this subsection we shall evaluate the higher-order simultaneous correlation functions $\tilde{F}^{(3)}_{\alpha\beta\gamma}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$, $\tilde{F}^{(4)}_{\alpha\beta\gamma\delta}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$, etc. in the regime that all the wave vectors k_i are of the same order of magnitude. We shall examine explicitly the lowest-order contributions (2.8) and (2.9) and learn from them how to reach conclusions that are valid to all orders due to the properties studied in the preceding subsection.

Begin with Eq. (2.11) and employ the same rule as before, i.e., ω is replaced by $\gamma(k)$. The result is

$$\tilde{F}^{(3)}_{\alpha\beta\gamma}(\mathbf{k},\mathbf{k},\mathbf{k}) \approx \gamma(k)^2 F^{(3)}_{\alpha\beta\gamma}(\mathbf{k},\omega\approx\gamma(k);\mathbf{k},\omega\approx\gamma(k);\mathbf{k},\omega\approx\gamma(k)) \ . \tag{3.15}$$

Similarly we can find the nth-order expression

$$\tilde{F}^{(n)}_{\alpha\beta\gamma\ldots}(\mathbf{k},\mathbf{k},\ldots,\mathbf{k}) \approx \gamma(k)^{n-1} F^{(n)}_{\alpha\beta\gamma\ldots}(\mathbf{k},\omega\approx\gamma(k);\mathbf{k},\omega\approx\gamma(k);\ldots;\mathbf{k},\omega\approx\gamma(k)) \ . \tag{3.16}$$

Next we need to evaluate the right-hand side of Eq. (3.15) using Eq. (2.8), but replacing the Eulerian vertex with the quasi-Lagrangian one. Since $V(\mathbf{k},\mathbf{k},\mathbf{k})$ is of order k we find

$$F^{(3)}_{\alpha\alpha\alpha}(\mathbf{k},\omega\approx\gamma(k);\mathbf{k},\omega\approx\gamma(k);\mathbf{k},\omega\approx\gamma(k)) \approx k[\tilde{F}(k)]^2/[\gamma(k)]^3 \ . \tag{3.17}$$

Substituting in (3.15) we find

$$\tilde{F}^{(3)}(k,k,k) \approx k[\tilde{F}(k)]^2/[\gamma(k)] \ . \tag{3.18}$$

By looking at the higher-order contributions to $\tilde{F}^{(3)}$ one discovers that as long as locality is obeyed, the difference of each term from (3.18) is again in powers of $\Lambda(k)$ which as we saw is of $O(1)$. With the same assumption that the sum of the terms is of the order of each of the terms we conclude that (3.18) is valid to all orders.

Repeating the procedure for $\tilde{F}^{(4)}(k,k,k,k)$ we find from (2.9) that

$$\tilde{F}^{(4)}(k,k,k,k) \approx \tilde{F}(k)[k\tilde{F}(k)/\gamma(k)]^2 \ . \tag{3.19}$$

The general relation can be guessed already (and checked by the diligent reader):

$$\tilde{F}^{(n)}(k,\ldots,k) \approx \tilde{F}(k)[k\tilde{F}(k)/\gamma(k)]^{n-2} \ . \tag{3.20}$$

This result, together with our extended scaling relation (3.14), leads to the final estimate for the higher-order correlations in k space:

$$\tilde{F}^{(n)}(k,\ldots,k) \approx [\tilde{F}(k)]^{n/2} k^{3(1-n/2)} \ . \tag{3.21}$$

Notice that in the inertial range this relation is a triviality that results from dimensional analysis. However, this result, as derived here in an extended range of scales down to scales which are influenced by the viscosity, is not at all trivial and it requires the property of locality. We reiterate that the viscosity appears in the higher-order functions only through its influence on the two-point function $\tilde{F}(k)$. Outside the region of locality (3.21) is not obeyed. This will give us the borderline of validity of the extended universality. What remains now is to transform this result to r space and to compare with the experimental findings.

IV. LOCALITY OF RELATIONS BETWEEN STRUCTURE FUNCTIONS AND CORRELATION FUNCTIONS

Physically it is reasonable to expect that the relation between the structure functions $S_n(r)$ and the correlation functions $\tilde{F}^{(n)}(k_1,\ldots,k_n)$ is local in the sense that the main contribution to the integral in Eq. (2.12) comes from the region of $k_1\approx k_2\approx\cdots\approx 1/r$. We need to examine, however, the borderline of this locality in order to employ the results of the previous sections, which are valid only within a region of ir locality that was found there. To our surprise we find that in fact *the two localities have exactly the same boundary* $y(k)=5$. We demonstrate this fact in this section.

A. Two-point functions

The two-point structure function $S_2(r)$ is written as

$$S_2(r) = \int \tilde{F}(\mathbf{k}_1)|1-\exp(i\mathbf{k}_1\cdot\mathbf{r})|^2 \frac{d\mathbf{k}_1}{(2\pi)^3} \ . \tag{4.1}$$

As before, we do not worry about uv divergences for obvious reasons. The ir problem is studied by again examining the integral in the interval $[0,k^*]$, with $k^*\ll 1/r$. In this region (4.1) turns into

$$S_{2,\mathrm{ir}}(r) \approx C_3 r^2 I(k^*) \ , \tag{4.2}$$

where $I(k^*)$ is the same integral as in (3.9). Obviously, the boundary of locality is the same for S_2 as before.

B. Third-order structure function

Begin with Eq. (2.12) for $n=3$. The infrared region has contributions from two regimes: (i) one of the arguments of $\tilde{F}^{(3)}$, say k_1, is small, $k_1\le k^*\ll 1/r$, but k_2 and k_3 are of $O(1/r)$; and (ii) all three arguments are small $k_1\approx k_2\approx k_3\le k^*\ll 1/r$. Due to the δ function in (2.12) we cannot have only two arguments being small. In region (i) we expand the exponential in k_1 and gain one factor of $k_1 r$. The asymptotic properties of the simultaneous third-order correlation function in the regime (i) were studied in [10], i.e., for κ small:

$$\tilde{F}^{(3)}(\kappa,k_2,k_3) \approx \kappa\tilde{F}(\kappa)h(k_2) \ , \tag{4.3}$$

with $h(k_2)$ being a function of k_2 that will not matter for our discussion. Using this in Eq. (2.12) we get

$$S^{(\mathrm{i})}_{3,\mathrm{ir}}(r) \approx rI(k^*)\int h(k_2)|1-\exp(i\mathbf{k}_2\cdot\mathbf{r})|^2 \frac{d\mathbf{k}_2}{(2\pi)^3} \ . \tag{4.4}$$

We learn that the boundary of locality is again determined by the same integral $I(k^*)$.

In the second regime we expand all the exponentials and use Eq. (3.22) for $n=3$. We find that in this regime

$$S^{(\mathrm{ii})}_{3,\mathrm{ir}}(r) \approx C_4 r^3 I_3(k^*) \ , \tag{4.5}$$

where

$$I_n(k^*) = \int_0^{k^*} [k^5\tilde{F}(k)]^{n/2} \frac{dk}{k} \ . \tag{4.6}$$

Since the boundary of locality in all the integrals I_n is the same as before we conclude that the borderline of locality for calculating S_3 is as before $y(k)=5$.

C. Fourth- and higher-order structure functions

Examining Eq. (2.12) for $n=4$ we see that the ir region has contributions from three regimes of the arguments $k_1,\ldots,k_4$, i.e., (i) one of them small, (ii) two of them small, or (iii) all of them small. The analysis of regions (i) and (iii) are entirely analogous to the analysis of regions (i) and (ii) in the preceding section. Again, the asymptotic form of the fourth- (and, in fact, the nth-) order simultaneous correlation function when one of its arguments is small is known and it is proportional to the small k vector and to the two-point correlator with the same argument [11]. To analyze regime (ii) we need to know the asymptotic form of $\tilde{F}^{(4)}$ when two of its arguments are small. The result is [12]

$$\tilde{F}^{(4)}(\kappa_1,\kappa_2,k_3,k_4)\propto(\kappa_1+\kappa_2)\tilde{F}^{(3)}[\kappa_1,\kappa_2,-(\kappa_1+\kappa_2)] . \quad (4.7)$$

Completing the analysis we find

$$S^{(\mathrm{i})}_{4,\mathrm{ir}}(r)\propto I_2(k^*) , \quad (4.8)$$

$$S^{(\mathrm{ii})}_{4,\mathrm{ir}}(r)\propto I_3(k^*) , \quad (4.9)$$

$$S^{(\mathrm{iii})}_{4,\mathrm{ir}}(r)\propto I_4(k^*) . \quad (4.10)$$

The conclusion is that the boundary of locality is again unchanged.

In a similar fashion we can analyze all the higher-order correlation functions. The general rule [12] is that

$$\tilde{F}^{(n+m)}(\kappa_1,\ldots,\kappa_n,k_{n+1},\ldots,k_m) \propto(\kappa_1+\cdots+\kappa_n)\tilde{F}^{(n+1)} \times[\kappa_1,\ldots,\kappa_n,-(\kappa_1+\cdots+\kappa_2)] . \quad (4.11)$$

Using this we find the surprising fact that the boundary of locality is the same for all the higher-order structure functions.

V. EXTENDED UNIVERSALITY AND THE SCALING EXPONENTS

A. The main result

As long as we are within the region of locality we can estimate $S_n(r)$ from (2.12) and (3.20) simply by replacing k_j by $1/r$ and $d\mathbf{k}_j$ by $1/r^3$. We find

$$S_2(r)\approx r^{-3}\tilde{F}(1/r) , \quad (5.1)$$

$$S_3(r)\approx r^{-9/2}[\tilde{F}(1/r)]^{3/2} . \quad (5.2)$$

It is worthwhile to note that (5.1) allows us to express $\gamma(k\approx 1/r)$ in terms of $S_2(r)$, i.e., $\gamma(k\approx 1/r)\approx\sqrt{S_2(r)}/r$. This determines $1/\gamma(k)$ as the characteristic turn over time of velocity fluctuations of scale r. This fact is trivial in the inertial interval. This is not at all trivial for scales in the vicinity of η since another time scale can be formed from νk^2. This is therefore another way of presenting the extended universality.

Equations (5.1) and (5.2) can be generalized for any n. We find

$$S_n(r)\approx[r^{-3}\tilde{F}(1/r)]^{n/2} . \quad (5.3)$$

This result can be represented as our final form which is the statement of extended universality

$$S_n(r)\approx C_n r^{n/3}[f(r/\eta)]^{n/3} , \quad (5.4)$$

where

$$f(r/\eta)=[r^{-11/3}\tilde{F}(1/r)]^{3/2} . \quad (5.5)$$

An equivalent form, which can be useful in experiments, is

$$S_n(r)\approx[S_3(r)]^{n/3} . \quad (5.6)$$

We stress that although (5.6), like (3.20), looks like a dimensional estimate, it is nontrivial because it extends to r values that are in the vicinity of η where dimensional estimates should fail. In fact, what remains now is to estimate how close to η can r be before Eqs. (5.5) and (5.6) lose their validity.

B. The boundary of validity of the extended universality

To estimate the range of validity of (5.5) and (5.6) we use the following form of the simultaneous correlation function $\tilde{F}(k)$:

$$\tilde{F}(k)=Ak^{-11/3}\exp(-ak\eta) . \quad (5.7)$$

This form has theoretical support [13] and had been successfully used to fit experimental data [14,15]. In (5.7) A is a dimensional constant and a is a nondimensional parameter. Using this form we can compute the logarithmic derivative (3.6) and find

$$y(k)=\tfrac{11}{3}+ak\eta . \quad (5.8)$$

The borderline of locality is obtained when $y(k_b)=5$, or when

$$ak_b\eta=\tfrac{4}{3} . \quad (5.9)$$

Deviation from pure power law is seen when $y(k)$ deviates, say, by 0.1 from its value $\frac{11}{3}$. This means that one loses the power-law behavior when $ak\eta$ is about 0.1. On the contrary, (5.9) means that we can go a full order of magnitude up in k before we lose the extended universality and the extended scaling relations. This is the main result of this paper. Note that this conclusion, which has been derived here using the fit (5.7), is really more general and it is simply a statement of the fact that $y(k)$ can change all the way from $\frac{11}{3}$ to 5 in about one decade of scales.

C. The scaling exponents

The theory presented above derives the extended universality with the standard Kolmogorov exponents. In the experimental work that challenged us to achieve this derivation the exponents that were measured differed from the Kolmogorov prediction. Instead of exponents $n/3$ in (5.6) the experiment yielded nonclassical exponents. For example, for $n=2$ the experimental fit is of an exponent 0.701.

It is possible that for the low to moderate Reynolds

numbers employed in this experiment, the deviations from the exponents $n/3$ result from boundary effects. The finite size of the system can reflect itself in "corrections to scaling" that are slow to disappear upon increasing k or decreasing r. We shall therefore analyze this possibility now to assess better the experimental evidence.

In the case of nonisotropic excitation of the flow the corrections to scaling are known [16]. If there is a preferred direction $\mathbf{n}$ on the large scale (such as mean flow direction), the correction $\delta\tilde{F}$ to the simultaneous correlation function is

$$\frac{\delta\tilde{F}(k)}{\tilde{F}(k)} \approx \frac{(\mathbf{k}\cdot\mathbf{n})^2}{k^2[kL]^{2/3}} , \tag{5.10}$$

where L here is the system size. In fact, it can be argued that the same slow decay of the effect of the boundaries is expected when the forcing is isotropic.

To estimate roughly how such a correction to scaling affects the measured exponents, we model [17] the spectrum $\tilde{F}(k)$ according to

$$\tilde{F}(k) \approx Bk^{-11/3}[1+(kL)^{-2/3}]\exp(-a\eta k) , \tag{5.11}$$

where we arbitrarily chose a positive sign to the correction to scaling term. Instead of (5.8) we shall have now

$$y(k) = \tfrac{11}{3} + a\eta k + \frac{2}{3[1+(kL)^{2/3}]} . \tag{5.12}$$

As a function of k this expression has a minimum which is rather flat and can lead to an apparent exponent y_{app}. We estimate the value of this exponent by the minimum of (5.12) which is

$$y_{\text{app}} = \frac{11}{3} + \frac{10}{9}\left[\frac{9\eta a}{4L}\right]^{2/5} . \tag{5.13}$$

Since we do not have good numerical information about a and L, we should not try to seriously attempt to draw numerical conclusions from (5.13). Notwithstanding, if we arbitrarily take the value of a to be of order 1, then for η/L of the order of 10^{-3}–10^{-4} we can reproduce the measured numbers in the experiment.

On the other hand, it is obvious from Eq. (5.13) that the correction to the value $y = \frac{11}{3}$ should decrease as a function of the Reynolds number. In a separate publication [17] we calculated that this dependence is

$$y_{\text{app}} - \tfrac{11}{3} \sim \mathrm{Re}^{-3/10} . \tag{5.14}$$

This is a sufficiently strong Re dependence to be noticed in the range of Re that is studied by the experimental group of Ciliberto. We were notified by this author [18] that his results seem to indicate no significant Re dependence. If this is verified by further experiments, we would tend to accept that the measured values of the scaling exponents stem from a different reason. Of course, this opens up again the important and unsolved problem of the origin of multiscaling in fluid mechanics, which is certainly beyond the scope of this paper, see Ref. [20].

VI. CONCLUSIONS

Let us first summarize the main points of this paper. We have shown that a consequence of the structure of the NS equations and the property of the locality of interactions is that the structure functions $S_n(r)$ have the form (1.2), which we refer to as "extended universality." The range of validity of this form is extended to lower scales as compared with Eq. (1.1), which is the is the form of scale invariance. We estimate this extension to be about one order of magnitude in length scales. Due to the action of viscous dissipation, the actual values of $S_n(r)$ can be already 2–3 orders of magnitude smaller than their values in the scale invariant regime. Thus, by plotting in log-log plots one structure function against the other, one gains 2–3 orders of magnitudes of linear relations compared to direct log-log plots of $S_n(r)$ vs r. This is the main observation of Ref. [2] and we believe that we have given here an adequate justification to this observation.

Another issue is the value of the scaling exponents that were extracted from the experiment [19]. As is well known, it is possible that at high Reynolds numbers one sees deviations from the Kolmogorov predictions, but it is also possible that all the observed deviations are actually preasymptotic [20]. We examined the possibility that the deviations seen in Ref. [2] stem from corrections to scaling, and we attempted to estimate these corrections for the conditions of the experiment. Equation (5.14) can be easily tested in the experiment. At present the experimental results indicate no significant Re dependence in the measured exponents, and this increases the credibility of the existence of a deeper reason for the deviations from the classical scaling exponents. How to derive such exponents from the theory is a different and much harder issue [20].

ACKNOWLEDGMENTS

We benefited enormously from discussions with Roberto Benzi and Sergio Ciliberto. We thank both for sharing with us their results and insights prior to publication. The idea that the scaling exponents are corrected by finite size effects was raised also by Victor Yakhot and we thank him for discussing this issue with us. This work was supported in part by the U.S.–Israel Binational Science Foundation and the Basic Research Foundation administered by the Israel Academy of Sciences and Humanities.

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