

Viscous Lengths in Hydrodynamic Turbulence are Anomalous Scaling Functions

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(Received 6 May 1996)

It is shown that the idea that scaling behavior in turbulence is limited by one outer length L and one inner length η is untenable. Every n th order correlation function of velocity differences $\mathcal{F}_n(\mathbf{R}_1, \mathbf{R}_2, \dots)$ exhibits its own crossover length η_n to dissipative behavior as a function of R_1 . This depends on n and on the remaining separations R_2, R_3, \dots . One result is that when separations are of the same order R , this scales as $\eta_n(R) \sim \eta(R/L)^{x_n}$ with $x_n = (\zeta_n - \zeta_{n+1} + \zeta_3 - \zeta_2)/(2 - \zeta_2)$, ζ_n the scaling exponent of the n th order structure function. We derive an infinite set of scaling relations that bridge the exponents of correlations of gradient fields to the exponents ζ_n , including the "bridge relation" for the scaling exponent of dissipation fluctuations $\mu = 2 - \zeta_6$. [S0031-9007(96)01435-4]

PACS numbers: 47.27.Gs, 05.40.+j, 47.27.Jv

The aim of this Letter is to expose the fact that the notion of the dissipative length in hydrodynamic turbulence is a rich and interesting concept whose complexity exceeds the expectations of established models and standard theories [1]. Indeed, during a few decades the thinking about the universal small scale structure of turbulence was dominated by Kolmogorov's picture of energy cascade through an "inertial interval" which is limited on one side by the integral scale of turbulence L and on the other side by the Kolmogorov viscous scale $\eta = (\nu^3/\bar{\epsilon})^{1/4}$, where ν and $\bar{\epsilon}$ are the fluid's kinematic viscosity and the mean energy flux in the turbulent flow, respectively. During the last decade there has been a growing concern about the inability of Kolmogorov's theory to cope with the increasing experimental evidence for multiscaling (or multifractal) behavior of higher order structure functions. Together with the concern about the statistical theory there arose a realization that the uniqueness of the viscous length is suspicious. Paladin and Vulpiani [2] and also Frisch and Vergassola [3] used the multifractal model of turbulence to assess the characteristic viscous lengths associated with the higher order structure functions of velocity differences

$$S_{2n}(\mathbf{R}_1) = \langle |\mathbf{w}(\mathbf{r}_1|\mathbf{r}'_1)|^{2n} \rangle, \quad (1)$$

$$S_{2n+1}(\mathbf{R}_1) = \hat{\mathbf{R}}_1 \cdot \langle \mathbf{w}(\mathbf{r}_1|\mathbf{r}'_1) |\mathbf{w}(\mathbf{r}_1|\mathbf{r}'_1)|^{2n} \rangle, \quad (2)$$

where $\mathbf{w}(\mathbf{r}_1|\mathbf{r}'_1, t) \equiv \mathbf{u}(\mathbf{r}'_1, t) - \mathbf{u}(\mathbf{r}_1, t)$ and $\mathbf{u}(\mathbf{r}, t)$ is the velocity field of the fluid, $\mathbf{R}_1 \equiv \mathbf{r}'_1 - \mathbf{r}_1$, and $\hat{\mathbf{R}}_1 \equiv \mathbf{R}_1/R_1$. In homogeneous and locally isotropic turbulences $S_n(R)$ is a function of the magnitude of \mathbf{R} , and the viscous length is that value of R at which the functional dependence of $S_n(R)$ changes from a nontrivial power law $S_n(R) \sim R^{\zeta_n}$ to a trivial power law that stems from a Taylor expansion of the velocity differences, $S_n(R) \sim R^n$. The multifractal model leads to a prediction that this length depends on the order n . In this Letter we argue that a proper discussion of crossovers to dissipative behavior requires the analysis of functions richer than structure functions. Firstly, we state that the fundamental object to

analyze is the n -point correlation of velocity differences

$$\mathcal{F}_n(\mathbf{r}_1, \mathbf{r}'_1 \cdots \mathbf{r}_n, \mathbf{r}'_n) = \langle \mathbf{w}(\mathbf{r}_1|\mathbf{r}'_1) \cdots \mathbf{w}(\mathbf{r}_n|\mathbf{r}'_n) \rangle, \quad (3)$$

which is an n -rank tensor. All the separations $R_i \equiv |\mathbf{r}'_i - \mathbf{r}_i|$ and $r_{ij} \equiv |\mathbf{r}_i - \mathbf{r}_j|$ are within the "inertial range." It is generally accepted that this correlation function is a homogeneous function of its arguments, i.e.,

$$\mathcal{F}_n(\lambda \mathbf{r}_1, \lambda \mathbf{r}'_1 \cdots \lambda \mathbf{r}'_n) = \lambda^{\zeta_n} \mathcal{F}_n(\mathbf{r}_1, \mathbf{r}'_1 \cdots \mathbf{r}'_n). \quad (4)$$

It should be understood that quantities like (1) are obtained from (3) by fusing some coordinates together. (In this case all $\mathbf{r}_{ij} \rightarrow 0$ and all $\mathbf{R}_i \rightarrow \mathbf{R}$.) In this process of fusion one crosses the viscous scale, and it is important to understand how to do this.

Our discussion will not call for any *ad hoc* model of turbulence. It will be based on two solid building blocks, one being the Navier-Stokes equations, and the other the fusion rules that were derived recently. The fusion rules appear naturally in the analytic theory of Navier-Stokes turbulence [4–7] and passive scalar turbulent advection [7–9], and they determine the analytic structure of the n -order correlations functions (3) when a group of coordinates tend towards each other. The fusion rules were derived in [7] for systems in which Eq. (4) holds with universal scaling exponents (i.e., the scaling exponents do not depend on the detailed form of the driving of the turbulent flows). The fusion rules address the asymptotic properties of \mathcal{F}_n when a group (or groups) of coordinates tend towards a common coordinate within each group, while all the other coordinates remain separated by a large distance R . There are two particular examples of fusion rules that we will employ in this Letter. The first pertains to the fusion of one pair of points. When the distance between one pair is small, $R_1 \sim \rho$, and the separations between all the other coordinates are much larger, $R_i \sim R$ for $i \neq 1$, then to leading order in ρ/R

$$\mathcal{F}_n \propto S_n(R)S_2(\rho)/S_2(R). \quad (5)$$

The second situation pertains to the case in which we have two groups of fusing coordinates separated by a large

distance R . When there is a group of p points separated by a typical distance ρ_1 , and a group of $n - p$ points separated by a typical distance ρ_2 with a large distance R between the groups, then

$$\mathcal{F}_n \sim S_n(R)S_p(\rho_1)S_{n-p}(\rho_2)/S_p(R)S_{n-p}(R). \quad (6)$$

These forms hold as long as ρ , ρ_1 , and ρ_2 are in the inertial range.

The Navier-Stokes equations for an incompressible velocity field $\mathbf{u}(\mathbf{r}, t)$ may be written in the form

$$\dot{\mathbf{u}}(\mathbf{r}, t) + \mathbf{\bar{P}}[\mathbf{u}(\mathbf{r}, t) \cdot \nabla]\mathbf{u}(\mathbf{r}, t) = \nu \nabla^2 \mathbf{u}(\mathbf{r}, t). \quad (7)$$

Here ν is the kinematic viscosity and $\mathbf{\bar{P}}$ is the transverse projector. Given the equation of motion we can take the time derivative of Eq. (3). We find

$$\dot{\mathcal{F}}_n = \sum_{j=1}^n \langle \dot{\mathbf{w}}(\mathbf{r}_1|\mathbf{r}'_1, t) \cdots \dot{\mathbf{w}}(\mathbf{r}_j|\mathbf{r}'_j, t) \cdots \dot{\mathbf{w}}(\mathbf{r}_n|\mathbf{r}'_n, t) \rangle.$$

Substituting Eq. (7), and considering the stationary state in which $\partial \mathcal{F}_n / \partial t = 0$, we find the balance equations

$$\mathcal{D}_n(\mathbf{r}_1, \mathbf{r}'_1; \cdots \mathbf{r}_n, \mathbf{r}'_n) = \mathcal{J}_n(\mathbf{r}_1, \mathbf{r}'_1; \cdots \mathbf{r}_n, \mathbf{r}'_n). \quad (8)$$

The term \mathcal{J}_n originates from the viscosity term in (7),

$$\mathcal{J}_n(\mathbf{r}_1, \mathbf{r}'_1; \cdots \mathbf{r}_n, \mathbf{r}'_n) = \nu \sum_{j=1}^n (\nabla_j^2 + \nabla_{j'}^2) \langle \mathbf{w}(\mathbf{r}_1|\mathbf{r}'_1) \cdots \mathbf{w}(\mathbf{r}_j|\mathbf{r}'_j) \cdots \mathbf{w}(\mathbf{r}_n|\mathbf{r}'_n) \rangle. \quad (9)$$

The term \mathcal{D}_n stems from the nonlinear term, and it needs a bit of algebra to bring to the exact form

$$\mathcal{D}_n^{\alpha_1 \alpha_2 \cdots \alpha_n}(\mathbf{r}_1, \mathbf{r}'_1; \cdots \mathbf{r}_n, \mathbf{r}'_n) = \int d\mathbf{r} \sum_{j=1}^n P_{\alpha_j \beta}(\mathbf{r}) \times \langle w_{\alpha_1}(\mathbf{r}_1|\mathbf{r}'_1) \cdots L^\beta(\mathbf{r}_j, \mathbf{r}'_j, \mathbf{r}) \cdots w_{\alpha_n}(\mathbf{r}_n|\mathbf{r}'_n) \rangle, \quad (10)$$

$$L^\beta(\mathbf{r}_j, \mathbf{r}'_j, \mathbf{r}) \equiv \frac{1}{n} \sum_{k=1}^n \{ w_\gamma(\mathbf{r}_j - \mathbf{r}|\mathbf{r}_k) \nabla_j^\gamma + w_\gamma(\mathbf{r}'_j - \mathbf{r}|\mathbf{r}'_k) \nabla_{j'}^\gamma \} \times w_\beta(\mathbf{r}_j - \mathbf{r}|\mathbf{r}'_j - \mathbf{r}). \quad (11)$$

We are going to argue now that when all the separations R_j are of the same order of magnitude R , the interaction term has a very simple evaluation, i.e.,

$$\mathcal{D}_n \sim S_{n+1}(R)/R. \quad (12)$$

To this aim we need to prove that the integral is local in the sense that it converges in the ultraviolet and in the infrared.

As the coordinate \mathbf{r} is being integrated over, the most dangerous ultraviolet contribution comes from the region of small r . In this region the projection operator can be evaluated as $1/r^3$. Other coalescence events of \mathbf{r} with other coordinates contribute less divergent integrands since the projection operator is not becoming singular. When r becomes small, there are two possibilities: (i) $\mathbf{r}_j \neq \mathbf{r}_k$ and (ii) $\mathbf{r}_j = \mathbf{r}_k$. In the first case the correlation function itself is analytic in the region $r \rightarrow 0$, and we

can expand it in a Taylor series $\text{const} + \mathbf{B} \cdot \mathbf{r} + \cdots$, where \mathbf{B} is an \mathbf{r} -independent vector. The constant term is annihilated by the projection operator. The term linear in \mathbf{r} vanishes under the $d\mathbf{r}$ integration due to $\mathbf{r} \rightarrow -\mathbf{r}$ symmetry. The next term which is proportional to r^2 is convergent in the ultraviolet. In the second case we have a velocity difference across the length r . Accordingly, we need to use the fusion rule (6), and we learn that the leading contribution is proportional to r^{ζ_2} . This is sufficient for convergence in the ultraviolet. We note that the derivative with respect to r_j cannot be evaluated as $1/r$ when $\mathbf{r}_j = \mathbf{r}_k$. Rather, it is evaluated as the inverse of the distance between \mathbf{r}_j and the nearest coordinate in the correlation function.

To understand the convergence of \mathcal{D}_n when the integration variable r becomes very large we consider the relevant geometry as shown in Fig. 1. There is one velocity difference across the coordinates $\mathbf{r}_j - \mathbf{r}$ and $\mathbf{r}'_j - \mathbf{r}$ (which is shown on the right of the figure), $n - 1$ velocity differences across coordinates that are all within a ball of radius R (at the left of the figure), and one velocity difference across the large distance r which is much larger than R . In the notation of this figure the leading order contribution for large r is obtained from the fusion rules (6) for the situation on the right and (5) for the geometry on the left. The resulting evaluation for the leading term is $r^{\zeta_{n+1}}(R_j/r)^{\zeta_2}(R/r)^{\zeta_{n-1}}$. On the face of it, this term is near dangerous. For any anomalous scaling the integral converges since $\zeta_{n+1} \leq \zeta_{n-1} + \zeta_2$ due to Hoelder inequalities. This convergence seems slow. However, the situation is, in fact, much safer. If we take into account the precise form of the second-order structure function in the fusion rules, we find that the divergence with respect to \mathbf{r}_j translates, in fact, to $\partial S_2^{\beta\gamma}(\mathbf{R}_j)/\partial R_{j\gamma}$, which is zero due to incompressibility. The next order term is convergent even for simple (K41) scaling. This completes the proof of locality of (10). The conclusion is that the main contribution to the integral in (10) comes from the region $r \sim R$. Therefore the integral can be evaluated by straightforward power counting leading to (12). It should be stressed that a more detailed analysis demonstrates that when the separations between the coordinates that do not involve velocity

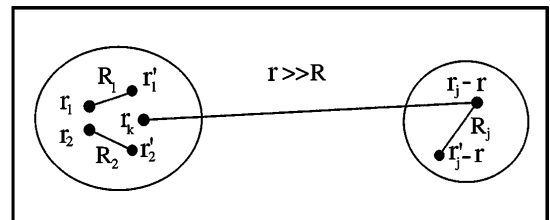


FIG. 1. Typical geometry with $n - 1$ velocity differences in a ball of radius R on the left separated by a large distance $r \gg R$ from a pair of points on the right.

differences (i.e., separations like r_{jk} but not R_j) go to zero, the evaluation does not change.

The evaluation of the quantity \mathbf{J}_n is more straightforward. When all the separations R_j and r_{ij} are of the same order R , the correlator in (9) is evaluated simply as $S_n(R)$. The Laplacian is then of the order of $1/R^2$. We note that when $\nu \rightarrow 0$ (which is the limit of infinite Reynolds number Re), this term becomes negligible compared to D_n . The ratio J_n/D_n is evaluated as $\nu S_n(R)/RS_{n+1}(R)$, which for fixed R vanishes in the limit $\nu \rightarrow 0$. Thus the ‘‘balance equation’’ becomes a *homogeneous* integro-differential equation $\mathcal{D}_n = 0$ which may have scale-invariant solutions with anomalous scaling exponents $\zeta_{n+1} \neq (n+1)/3$. It should be stressed that the evaluation (12) remains correct for every term in \mathcal{D}_n , but various terms cancel to give zero in the homogeneous equation, *provided that the scaling exponent ζ_n is chosen correctly*. To make this important point clear we exemplify it with the simple case $n = 2$ for which \mathcal{D}_n can be greatly simplified. Consider the scalar object $\mathcal{F}_2(\mathbf{r}_1, \mathbf{r}'_1, \mathbf{r}_2, \mathbf{r}'_2) = \langle \mathbf{w}(\mathbf{r}_1|\mathbf{r}'_1) \cdot \mathbf{w}(\mathbf{r}_2|\mathbf{r}'_2) \rangle$. The terms in the scalar balance equation for this case are exactly

$$\begin{aligned} \mathcal{D}_2(\mathbf{r}_1, \mathbf{r}'_1, \mathbf{r}_2, \mathbf{r}'_2) &= d[S_3(r_{12'}) - S_3(r_{12})]/2dr_1 \\ &\quad + d[S_3(r_{12}) - S_3(r_{12'})]/2dr_1', \quad (13) \\ J_2(\mathbf{r}_1, \mathbf{r}'_1, \mathbf{r}_2, \mathbf{r}'_2) &= \nu \{ \nabla_1^2 [S_2(r_{12'}) - S_2(r_{12})] \\ &\quad + \nabla_1'^2 [S_2(r_{12}) - S_2(r_{12'})] \}. \quad (14) \end{aligned}$$

When all the separations are of the order of R we can see explicitly that $J_2 \sim \nu S_2(R)/R^2$ which is much smaller than each term in D_2 . Considering the scale invariant solution $S_3(R) = AR^{\zeta_3}$, where A is a dimensional coefficient, we see that

$$\begin{aligned} \mathcal{D}_2(\mathbf{r}_1, \mathbf{r}'_1, \mathbf{r}_2, \mathbf{r}'_2) &= \frac{\zeta_3 A}{2} \\ &\quad \times \left[r_{12'}^{\zeta_3-1} - r_{12}^{\zeta_3-1} + r_{12}^{\zeta_3-1} - r_{12'}^{\zeta_3-1} \right]. \end{aligned}$$

Obviously, the solution for $\mathcal{D}_2 = 0$ requires the unique choice $\zeta_3 = 1$ which is the known exponent for S_3 [1]. The coefficient A is now determined as $\bar{\epsilon}$ which is the mean energy dissipation per unit mass and unit time.

There is a crossover from the scale invariant solution of the homogeneous equation to dissipative solutions when J_2 becomes comparable to any of the terms in \mathcal{D}_2 . This happens when at least one of the separations appearing in (14) becomes small enough. Denoting the smallest separation as r_m we evaluate $J_2 \sim \nu S_2(r_m)/r_m^2$. From this we can estimate, using the balance equation, $S_2(r_m) \sim [S_3(R)/\nu R] r_m^2 \sim \bar{\epsilon} r_m^2/\nu$. In the inertial range we have $S_2(r) \sim (\bar{\epsilon} r)^{2/3} (r/L)^{\zeta_2-2/3}$. The viscous scale η_2 for the second-order structure function is then determined from finding where these two expressions are of the same order of magnitude, i.e.,

$$\bar{\epsilon} \eta_2^2/\nu = (\bar{\epsilon} \eta_2)^{2/3} (r/L)^{\zeta_2-2/3}. \quad (15)$$

Using the outer velocity scale U_L we estimate $\bar{\epsilon} \sim U_L^3/L$ and end up with the result

$$\eta_2 \sim L \text{Re}^{-1/(2-\zeta_2)}. \quad (16)$$

We conjecture that a similar mechanism operates in the general case of \mathcal{D}_n . (In fact, we expect the mechanism to operate for any $n > 3$; $n = 3$ is a special case that will not be discussed explicitly.) As long as all the separations are in the inertial interval \mathbf{J}_n is negligible. When one separation, e.g., r_{12} diminishes towards zero, and all the other separations are of the order of R , the internal cancellations leading to the homogeneous equation $\mathcal{D}_n = 0$ disappear and \mathcal{D}_n is evaluated as in (12). The term \mathbf{J}_n is now dominated by one contribution that can be written in short-hand notation as $\nu \nabla_1^2 F_n(r_{12}, \{R\})$. We can solve for $F_n(r_{12}, \{R\})$ in this limit:

$$F_n(r_{12}, \{R\}) \approx r_{12}^2 S_{n+1}(R)/\nu R. \quad (17)$$

On the other hand, we have, from the fusion rule (6), the form of the same quantity when r_{12} is still in the inertial range, i.e., $F_n(r_{12}, \{R\}) \approx S_2(r_{12})S_n(R)/S_2(R)$. To estimate the viscous scale η_n we find when these two evaluations are of the same order. The answer for $n \neq 3$ is

$$\eta_n(R) = \eta_2 \left(\frac{R}{L} \right)^{x_n}, \quad x_n = \frac{\zeta_n + \zeta_3 - \zeta_{n+1} - \zeta_2}{2 - \zeta_2}. \quad (18)$$

We note that the Hoelder inequalities guarantee that $x_n > 0$ and increases with n . We see that the viscous ‘‘length’’ is actually an anomalous scaling function.

Next we show that in the same spirit we can derive important (and exact) scaling relations between the exponents ζ_n of the structure functions and exponents involving correlations of the dissipation field. We consider correlations of the type

$$\mathcal{K}_\epsilon^{(n)} \equiv \langle \epsilon(\mathbf{x}_1) \mathbf{w}(\mathbf{r}_1|\mathbf{r}'_1) \cdots \mathbf{w}(\mathbf{r}_n|\mathbf{r}'_n) \rangle \propto R^{-\mu_n^{(1)}}, \quad (19)$$

$$\mathcal{K}_{\epsilon\epsilon}^{(n)} \equiv \langle \epsilon(\mathbf{x}_1) \epsilon(\mathbf{x}_2) \mathbf{w}(\mathbf{r}_1|\mathbf{r}'_1) \cdots \mathbf{w}(\mathbf{r}_n|\mathbf{r}'_n) \rangle \propto R^{-\mu_n^{(2)}}, \quad (20)$$

where R is a typical separation between any pair and $\epsilon(\mathbf{r}) \equiv \nu |\nabla \mathbf{u}(\mathbf{r})|^2$, and we are interested in the scaling relations between the exponents μ_n and the exponents ζ_n . Note that $\mu_0^{(2)}$ in this notation is the well-studied [10,11] exponent of dissipation fluctuation, which is denoted as μ . This relation is almost at hand for $\mu_n^{(1)}$. We see this by writing

$$\mathcal{K}_\epsilon^{(n)} = \nu \lim_{r_{12} \rightarrow 0} \nabla_1 \nabla_2 \mathcal{F}_{n+2}(r_{12}, \{R\}). \quad (21)$$

Using the result (17) we find immediately

$$\mu_n^{(1)} = 1 - \zeta_{n+3}. \quad (22)$$

The scaling relations satisfied by $\mu_n^{(2)}$ require considerations of the second time derivative of the correlation (3):

$$\ddot{\mathcal{F}}_n = \sum_{i,j=1}^n \langle w(\mathbf{r}_1|\mathbf{r}'_1, t) \cdots \dot{w}(\mathbf{r}_i|\mathbf{r}'_i, t) \cdots \dot{w}(\mathbf{r}_j|\mathbf{r}'_j, t) \cdots w(\mathbf{r}_n|\mathbf{r}'_n, t) \rangle. \quad (23)$$

Using the Navier-Stokes equations for the time derivatives we derive a new balance equation $\mathcal{D}_n^{(2)} + \mathcal{B}_n^{(2)} = \mathcal{J}_n^{(2)}$ where, using the definition (11),

$$\begin{aligned} \mathcal{D}_n^{(2)} = & \int d\mathbf{r} d\mathbf{r}' \sum_{i,j=1}^n \mathcal{P}(\mathbf{r}) \mathcal{P}(\mathbf{r}') \\ & \times \langle w(\mathbf{r}_1|\mathbf{r}'_1) \cdots \mathcal{L}(\mathbf{r}_i, \mathbf{r}'_i, \mathbf{r}) \cdots \mathcal{L}(\mathbf{r}_j, \mathbf{r}'_j, \mathbf{r}') \cdots w(\mathbf{r}_n|\mathbf{r}'_n) \rangle. \end{aligned} \quad (24)$$

Using the fusion rules and following steps similar to those described above, we can prove that the integrals over \mathbf{r} and \mathbf{r}' converge. Accordingly, when all the separations are of the order of R , every term in $\mathcal{D}_n^{(2)}$ is evaluated as $S_{n+2}(R)/R^2$. The term $\mathcal{J}_n^{(2)}$ takes on the form

$$\begin{aligned} \mathcal{J}_n^{(2)} = & \nu^2 \sum_{i,j=1}^n (\nabla_i^2 + \nabla_j^2) (\nabla_j^2 + \nabla_i^2) \\ & \times \langle w(\mathbf{r}_1|\mathbf{r}'_1) \cdots w(\mathbf{r}_i|\mathbf{r}'_i) \cdots w(\mathbf{r}_j|\mathbf{r}'_j) \cdots w(\mathbf{r}_n|\mathbf{r}'_n) \rangle. \end{aligned} \quad (25)$$

As before, when all the separation in this quantity are of the order of R , the Laplacian operators introduce a factor of $1/R^2$ and the evaluation of this quantity is $\mathcal{J}_n^{(2)} \sim \nu^2 S_n(R)/R^4$. Clearly this is negligible compared to typical terms in $\mathcal{D}_n^{(2)}$. The quantity $\mathcal{B}_n^{(2)}$ contains a cross contribution with one Laplacian operator and one nonlinear term with a projection operator. The integral is again local, and one can show that the evaluation is $\mathcal{B}_n^{(2)} \sim \nu S_{n+1}(R)/R^3$, which is also negligible compared to typical terms in $\mathcal{D}_n^{(2)}$.

Now we consider the fusion of two pairs of coordinate, e.g., $r_{12} \rightarrow 0$ and $r_{34} \rightarrow 0$. As before, the cancellations in $\mathcal{D}_n^{(2)}$ are eliminated and the evaluation of a typical term becomes the evaluation of the quantity. The other two terms in the balance equation also become of the same order because the Laplacian operators ∇_1^2 and ∇_3^2 are evaluated as r_{12}^{-2} and r_{34}^{-2} , respectively. As before we can consider the resulting balance equation as a differential equation for $F_n(r_{12}, r_{34}, \{R\})$. The leading term in this equation is

$$\begin{aligned} 4\nu^2 \nabla_1^2 \nabla_2^2 F_n(r_{12}, r_{34}, \{R\}) \approx & \mathcal{B}_n^{(2)} + \mathcal{D}_n^{(2)} \\ \sim & S_{n+2}(R)/R^2. \end{aligned}$$

The solution is

$$\mathcal{F}_n(r_{12}, r_{34}, \{R\}) \sim r_{12}^2 r_{34}^2 S_{n+2}(R)/\nu^2 R^2. \quad (26)$$

Finally, we can write the quantities $\mathcal{K}_{\epsilon\epsilon}^{(n)}$ in terms of the correlation function as

$$\mathcal{K}_{\epsilon\epsilon}^{(n)} = \nu^2 \lim_{r_{12}, r_{34} \rightarrow 0} \nabla_1 \nabla_2 \nabla_3 \nabla_4 \mathcal{F}_{n+4}(r_{12}, r_{34}, \{R\}). \quad (27)$$

Using (26) here we end up with the evaluation

$$\mathcal{K}_{\epsilon\epsilon}^{(n)} \sim S_{n+6}/R^2 \propto R^{-\mu_n^{(2)}}, \quad \mu_n^{(2)} = 2 - \zeta_{n+6}. \quad (28)$$

For the standard exponent $\mu = \mu_0^{(2)}$ we choose $n = 0$ and obtain the phenomenologically proposed ‘‘bridge relation’’ $\mu = 2 - \zeta_6$. To our best knowledge this is the first solid derivation of this scaling relation. In general, if we have p dissipation fields correlated with n velocity differences the scaling exponent can be found by considering p time derivatives of (3), with the final result

$$\mu_n^{(p)} = p - \zeta_{n+3p}. \quad (29)$$

We see that Eqs. (22), (28), and (29) can be guessed if we assert that *for the sake of scaling purposes* the dissipation field $\epsilon(\mathbf{r})$ can be swapped in the correlation function with $w^3(\mathbf{r}_1|\mathbf{r}'_1)/R_1$, where R_1 is the characteristic scale. This reminds one of the Kolmogorov refined similarity hypothesis. We should stress that (i) our result does not depend on this uncontrolled hypothesis, and (ii) it does not imply the correctness of this hypothesis. Our result is implied by the refined similarity hypothesis, but not vice versa.

In summary, we provided an infinite set of bridge relations that show that the scaling exponents of correlations of gradient fields can be expressed in terms of the set ζ_n characterizing the structure functions. Our results follow from the Navier-Stokes equations, from the fusion rules, and from the conjecture that in fusing two coordinates in the balance equation for any n removes the cancellation as has been demonstrated explicitly for $n = 2$.

This work was supported in part by the German Israeli Foundation, the Minerva Center for Nonlinear Physics, and the Naftali and Anna Backenroth-Bronicki Fund for Research in Chaos and Complexity.

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