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Saturation of turbulent drag reduction in dilute polymer solutions

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Abstract. – Drag reduction by polymers in turbulent wall-bounded flows exhibits universal and non-universal aspects. The universal maximal mean-velocity profile was explained in a recent theory. The saturation of this profile and the crossover back to the Newtonian plug are non-universal, depending on Reynolds number $\mathcal{R}e$, concentration of polymer c_{p} and the degree of polymerization N_{p}. We explain the mechanism of saturation stemming from the finiteness of extensibility of the polymers, predict its dependence on c_{p} and N in the limit of small c_{p} and large $\mathcal{R}e$, and present the excellent comparison of our predictions to experiments on drag reduction by DNA.

The onset of turbulence in fluid flows is accompanied by a significant increase in the drag. This drag poses a real technological hindrance to the transport of fluids and to the navigation of ships. It is interesting therefore that the addition of long-chained polymers to wall-bounded turbulent flows can result in a significant reduction in the drag [1]. The basic experimental knowledge of the phenomenon had been reviewed and systematized by Virk [2]; the increase in mean velocity profile as a function of the distance from the wall depends on the characteristics of the polymer and its concentration, but cannot exceed a universal asymptote known as the "Maximum Drag Reduction" (MDR) curve which is independent of the polymer's concentration or its characteristics. When the concentration is not large enough, the mean velocity profile follows the MDR for a while and then crosses back to a Newtonian-like profile, known as the "Newtonian plug" cf. fig. 1. Recently, the nature of the MDR and the mechanism leading to its establishment were explained [3]. For Newtonian flows the momentum flux is dominated by the so-called Reynolds stress, leading to a logarithmic (von-Karman) dependence of the mean velocity on the distance from the wall [4]. With the addition of polymers, while momentum is produced at a fixed rate by the forcing, polymer stretching results in a suppression of the Reynolds stress (and thus of the momentum flux from the bulk to the wall). Accordingly, the mean velocity in the channel must increase. It was shown that when the concentration of the polymers is large enough, there exists a new logarithmic law for the mean velocity with a slope that fits existing numerical and experimental data. The law is universal, thus explaining the MDR asymptote. It turned out that the polymer stretching in channel geometry gives rise to an extremely anisotropic polymer conformation tensor $\boldsymbol{R}$ [5,6]. The trace of this tensor is dominated by R_{xx}, where x is the streamwise direction. This component decreases with the distance from the wall (the y-direction) like $1/y$. The effective viscosity, however, is proportional to R_{yy} [3], and this component *increases linearly* with the distance from the wall. In fact, it was shown that in the MDR region $R_{yy}S(y) \approx$ const, where $S(y)$ is the mean shear, $S(y) \sim 1/y$, leading to a logarithmic profile identified as the MDR by Virk [2]. With this profile the reduction in the momentum flux from the bulk to the wall overwhelms the increased dissipation that results from the increased viscosity. Thus the mean momentum increases in the bulk, and this is how drag reduction is realized. In [7] it was demonstrated by DNS that Navier-Stokes flows with viscosity profiles that vary linearly with the distance from the wall indeed show drag reduction in close correspondence with the phenomena seen in full viscoelastic simulations.

The aim of this letter is to provide a theory of the crossovers from the MDR to the Newtonian plug for small polymer concentration $c_{\rm p}$ and large $\mathcal{R}e$. We show that the mechanism for crossover in that limit is the saturation of the increase of effective viscosity: at some point there is no enough polymer to supply the necessary linear profile of effective viscosity. We will develop the theory for a channel geometry, and then apply the results to a rotating-disk experiment with DNA as the drag reducing agent. For a channel of width $2L$ we denote by x, y and z the streamwise, the wall normal and the spanwise directions, respectively. The only non-zero mean-velocity component is $V(y)$ in the streamwise direction. The traditional wall

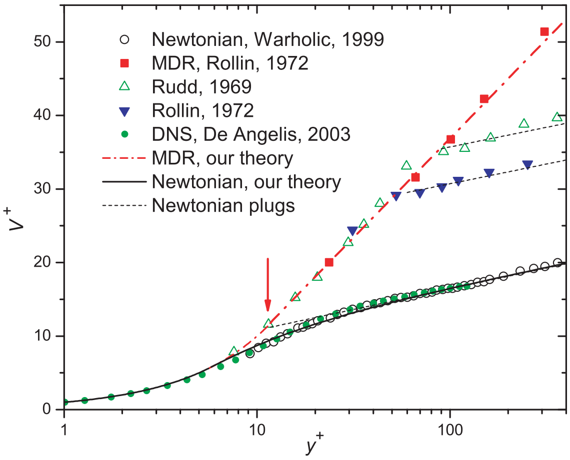

Fig. 1 – (Colour on-line) Mean normalized velocity profiles as a function of the normalized distance from the wall during drag reduction. The data points from numerical simulations (green circles) [8] and the experimental points (open circles) [9] represent the Newtonian results. The red data points (squares) [10] represent the Maximum Drag Reduction (MDR) asymptote. The dashed red curve represents the theory of [3] which agrees with the universal law (4). The arrow marks the crossover from the viscous linear law (2) to the asymptotic logarithmic law (4). The blue filled triangles [10] and green open triangles [11] represent the crossover, for intermediate concentrations of the polymer, from the MDR asymptote to the Newtonian plug. This letter presents a theory of this crossover.

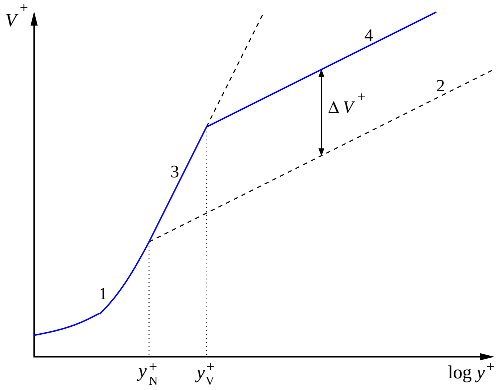

Fig. 2 – Schematic mean-velocity profiles. Region 1: $y^+ < y^+_{\rm N}$, viscous sublayer. Region 2: $y^+ > y^+_{\rm N}$, logarithmic layer for the turbulent Newtonian flow. Region 3: $y^+_{\rm N} < y^+ < y^+_{\rm V} = (1+Q)\,y^+_{\rm N}$, MDR asymptotic profile in the viscoelastic flow. Region 4: $y^+ > y^+_{\rm V}$, Newtonian plug in the viscoelastic flow.

units are

$$\mathcal{R}e_\tau \equiv L\sqrt{p'L}/\nu_0\,, \qquad y^+ \equiv y\mathcal{R}e_\tau/L, \qquad V^+ \equiv V/\sqrt{p'L}. \tag{1}$$

Here $p' \equiv -\partial p/\partial x$ is the pressure gradient in the streamwise direction x. $\mathcal{R}e_\tau$ is known as the friction Reynolds number. ν_0 is the kinematic viscosity of the neat fluid. In terms of these variables, the predictions of the standard Newtonian theory and the theory of [3] are summarized with the help of fig. 2 in which the Newtonian and viscoelastic velocity profiles are shown schematically in the reduced coordinates (1). In the Newtonian flow there are two distinct regions: a viscous sublayer (region 1) with the reduced velocity given by

$$V^+(y^+) = y^+, \tag{2}$$

and a logarithmic layer (region 2), in which velocity is given by Prandtl-Karman law:

$$V^+(y^+) = \kappa_{\rm N}^{-1}\log y^+ + B_{\rm N} \ \text{(Prandtl-Karman law)}. \tag{3}$$

Here $\kappa_{\rm N}^{-1} \simeq 2.29$ and $B_{\rm N} \simeq 6.13$ [12]. The mean-velocity profile in the viscoelastic case consists of three regions [2]: a viscous sublayer, a logarithmic polymeric sublayer (region 3 in fig. 2) with the slope greater than the Newtonian one:

$$V^+(y^+) = \kappa_{\rm V}^{-1}\log y^+ + B_{\rm V} \ \text{(MDR profile)}, \tag{4}$$

($\kappa_{\rm V}^{-1} \simeq 11.7$, $B_{\rm V} \simeq -17$ [2]), and a Newtonian plug (region 4). In the last region, the velocity follows a log law with the Newtonian slope, but with some velocity increment ΔV^+:

$$V^+ = \kappa_{\rm N}^{-1}\log y^+ + B_{\rm N} + \Delta V^+. \tag{5}$$

Note that the three profiles, eqs. (2), (3), and (4), intercept at one point $y^+ = y^+_{\rm N} \simeq \kappa_{\rm V}^{-1} \simeq 11.7$.

The increment ΔV^+ which determines the amount of drag reduction is in turn determined by the crossover from the MDR to the Newtonian plug (see fig. 2). We refer to this crossover point as $y^+_{\rm V}$. To measure the quality of drag reduction, we introduce a dimensionless drag reduction parameter,

$$Q \equiv \frac{y^+_{\rm V}}{y^+_{\rm N}} - 1. \tag{6}$$

The velocity increment ΔV^+ is related to this parameter as follows:

$$\Delta V^+ = \left(\kappa_{\rm V}^{-1} - \kappa_{\rm N}^{-1}\right) \log\left(y_{\rm V}^+/y_{\rm N}^+\right) = \alpha \log(1+Q). \tag{7}$$

Here $\alpha \equiv \kappa_{\rm V}^{-1} - \kappa_{\rm N}^{-1} \simeq 9.4$. The Newtonian flow is then a limiting case of the viscoelastic flow corresponding to $Q = 0$.

The crossover point $y_{\rm V}^+$ is non-universal, depending on $\mathcal{R}e$, the number of polymers per unit volume $c_{\rm p}$, the chemical nature of the polymer, etc. According to the theory of drag reduction [3], the total viscosity of the fluid $\nu_{\rm tot}(y^+) = \nu_0 + \nu_{\rm p}(y^+)$ (where $\nu_{\rm p}(y^+)$ is the polymeric contribution to the viscosity which is proportional to Ryy) is linear in y^+ in the MDR region:

$$\nu_{\rm tot}(y^+) = \nu_0 y^+/y_{\rm N}^+\,, \qquad y_{\rm N}^+ < y^+ < y_{\rm V}^+\,. \tag{8}$$

When the concentration of polymers is small and $\mathcal{R}e$ is large enough, *the crossover to the Newtonian plug at* $y_{\rm V}^+$ *occurs when the polymer stretching can no longer provide the necessary increase of the total fluid viscosity.* In other words, in that limit the crossover is due to the finite extensibility of the polymer molecules. Obviously, the polymeric viscosity cannot be greater than $\nu_{\rm p\,max}$, which is the viscosity of the fully stretched polymers. Thus, the total viscosity is limited by $\nu_0 + \nu_{\rm p\,max}$. Equating $\nu_0 + \nu_{\rm p\,max}$ and $\nu_{\rm tot}(y_{\rm V}^+)$ gives us the crossover position

$$y_{\rm V}^+ = y_{\rm N}^+(\nu_0 + \nu_{\rm p\,max})/\nu_0\,. \tag{9}$$

It follows from eq. (6) that the drag reduction parameter is determined very simply by

$$Q = \nu_{\rm p\,max}/\nu_0\,, \qquad c_{\rm p}\ \text{small}, \qquad \mathcal{R}e\ \text{large}. \tag{10}$$

At this point we need to relate the maximum polymeric viscosity $\nu_{\rm p\,max}$ to the polymer properties. To this aim, we estimate the energy dissipation due to a single, fully stretched, polymer molecule. In a reference frame co-moving with the polymer's center of mass, the fluid velocity can be estimated as $u \simeq r\nabla u$ (the polymer's center of the mass moves with the fluid velocity due to negligible inertia of the molecule). The friction force exerted on the i-th monomer is estimated using Stokes law,

$$F_i \simeq \rho_0 \nu_0\, a\, \delta u_i = \rho_0 \nu_0\, a\, r_i \nabla u\,, \tag{11}$$

where a is the effective hydrodynamic radius of one monomer (depending on the chemical composition), and r_i is the distance of the i-th monomer from the center of the mass. In a fully stretched state $r_i \simeq a\,i$ (the monomers are aligned along a line). The energy dissipation rate (per unit volume) is equal to the work performed by the external flow

$$\begin{aligned} -\frac{{\rm d}E}{{\rm d}t} &\simeq c_{\rm p} \sum_{i=1}^{N_{\rm p}} F_i \delta u_i \simeq \rho_0 \nu_0 a^3 c_{\rm p} N_{\rm p}^3 (\nabla u)^2 \\ &\equiv \rho_0 \nu_{\rm p\,max} (\nabla u)^2. \end{aligned} \tag{12}$$

We thus can estimate $\nu_{\rm p\,max}$:

$$\nu_{\rm p\,max} = \nu_0 a^3 c_{\rm p} N_{\rm p}^3\,. \tag{13}$$

Finally, the drag reduction parameter Q is given by

$$Q = a^3 c_{\rm p} N_{\rm p}^3\,, \qquad c_{\rm p}\ \text{small}, \qquad \mathcal{R}e\ \text{large}. \tag{14}$$

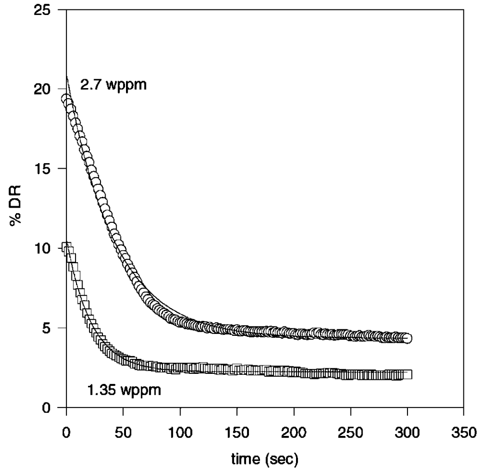

Fig. 3 – %DR in a rotating-disk experiment with λ-DNA as the drag reducing polymer. Note that the %DR is proportional to c_{p}. When the length N_{p} reduces by a factor of 2 and, simultaneously, c_{p} increases by factor of 2, the %DR reduces by a factor of 4.

This is the central theoretical results of this letter, relating the concentration c_{p} and degree of polymerization N_{p} to the increment in mean velocity ΔV^{+} via eq. (7).

A particularly interesting experiment suitable for testing our prediction was described in [13]. Here turbulence was produced in a rotating-disk apparatus, with λ-DNA molecules used to reduce the drag. The Reynolds number was relatively high (the results below pertain to $\mathcal{R}e \approx 1.2 \times 10^{6}$) and the initial concentrations of DNA relatively low (results employed below pertain to 2.70 and 1.35 wppm). During the experiment DNA degrades; fortunately, the degradation is very predictable: double-stranded molecules with 48502 bp in size degrade to double-stranded molecules with 23100 bp. Thus invariably the length N_{p} reduces by a factor of approximately 2, and the concentration c_{p} increases by a factor of 2. The experiment followed the drag reduction efficacy measured in terms of the percentage drag reduction defined by

$$\%\mathrm{DR} = \frac{T_{\mathrm{N}} - T_{\mathrm{V}}}{T_{\mathrm{N}}} \times 100, \tag{15}$$

where T_{N} and T_{V} are the torques needed to maintain the disk to rotate at a particular Reynolds number $\mathcal{R}e$ without and with polymers, respectively. The main experimental results which are of interest to us are summarized in fig. 3. We see from the experiment that both initially (with un-dergraded DNA) and finally (with degraded DNA) the %DR is proportional to c_{p}. Upon degrading, which amounts to decreasing the length N_{p} by a factor of approximately 2 and, simultaneously increasing c_{p} by factor of 2, %DR decreases by a factor 4.

The flow geometry is rather complicated: with a rotating disk the *linear* velocity depends on the radius, and the local $\mathcal{R}e$ is a function of the radius. The drag reduction occurs however in a relatively small near-wall region, where the flow can be considered as a flow near the flat plate. Thus, we consider an equivalent channel flow —with the same $\mathcal{R}e$ and a half width L of the order of height/radius of the cylinder. In this plane geometry the torques in (15) should be replaced by the pressure gradients $p'_{\mathrm{N,V}}$:

$$\%\mathrm{DR} = \frac{p'_{\mathrm{N}} - p'_{\mathrm{V}}}{p'_{\mathrm{N}}} \times 100. \tag{16}$$

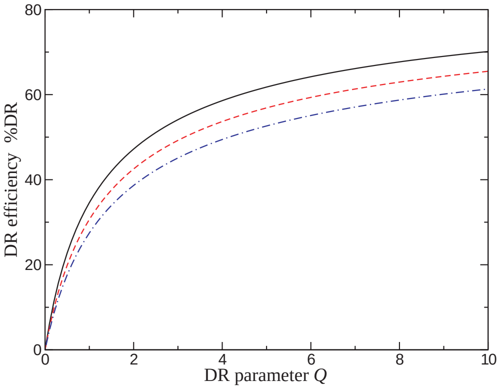

Fig. 4 – Drag reduction efficiency %DR as a function of the drag reducing parameter Q for different Reynolds numbers $\mathcal{R}e$: 1.2×10^5, 1.2×10^6, and 1.2×10^7 (from top to bottom).

In order to relate %DR with the drag reduction parameter Q, we re-write eq. (5) in natural units:

$$V(y) = \sqrt{p'L}\left[\kappa_{\mathrm{N}}^{-1} \log\left(y\sqrt{p'L}/\nu_0\right) + B_{\mathrm{N}} + \Delta V^+\right]. \tag{17}$$

With constant $\mathcal{R}e$ the centerline velocity $V_0 = V(L)$ is kept fixed:

$$\mathcal{R}e \equiv \frac{V_0 L}{\nu_0} = \mathcal{R}e_\tau\left[\kappa_{\mathrm{N}}^{-1} \log \mathcal{R}e_\tau + B_{\mathrm{N}} + \Delta V^+\right]. \tag{18}$$

This equation implicitly determines the pressure gradient and therefore the %DR as a function of Q and $\mathcal{R}e$. The set of eqs. (7) and (18) is readily solved numerically, and the solution for three different values of $\mathcal{R}e$ is shown in fig. 4. The middle curve corresponds to $\mathcal{R}e = 1.2 \times 10^6$, which coincides with the experimental conditions [13]. One sees, however, that the dependence of %DR on the $\mathcal{R}e$ is rather weak. One important consequence of the solutions shown in fig. 4 is that for small Q (actually for $Q \le 0.5$ or $\%\mathrm{DR} \le 20$), %DR is approximately a linear function of Q. The experiments [13] lie entirely within this linear regime, in which we can linearize eq. (18) and find an approximate solution for the %DR:

$$\%\mathrm{DR} = \frac{2\alpha Q}{\kappa_{\mathrm{N}}^{-1}\log(e\mathcal{R}e_\tau^0) + B_{\mathrm{N}}} \times 100. \tag{19}$$

Here $\mathcal{R}e_\tau^0$ is the friction Reynolds number for the Newtonian flow, *i.e.* the solution of eq. (18) for $\Delta V^+ = 0$.

It is interesting to note that while the %DR depends on the Reynolds number, the ratio of different %DRs does not (to O(Q)):

$$\frac{\%\mathrm{DR}^{(1)}}{\%\mathrm{DR}^{(2)}} = \frac{Q^{(1)}}{Q^{(2)}} = \frac{\nu_{\mathrm{p\,max}}^{(1)}}{\nu_{\mathrm{p\,max}}^{(2)}}. \tag{20}$$

This result, together with eq. (13), rationalizes completely the experimental finding of [13] summarized in fig. 3. During the DNA degradation, the concentration of polymers increases by a factor of 2, while the number of monomers N_{p} decreases by the same factor. This means that %DR should decrease by a factor of 4, as is indeed the case.

We have presented a theory of the crossover from the universal MDR mean-velocity profiles to the Newtonian plug. The experiment discussed does not display a direct measurement of the velocity profiles and the crossovers. Nevertheless, we have connected the theory to measured percentages of drag reduction, and tested the theory against experiments in which λ-DNA is used as the drag-reducing agent. The experimental results pertain to high $\mathcal{R}e$ and small c_{p}, where we can assert that *the crossover results from exhausting the stretching of the polymers such that the maximal available viscosity is achieved.* In the linear regime that pertains to this experiment, the degradation has a maximal effect on the quality of drag reduction Q, leading to the precise factor of 4 in the results shown in fig. 3. Larger values of the concentration of DNA will exceed the linear regime as is predicted by fig. 4; then the degradation is expected to have a smaller influence on the drag reduction efficacy. It is worthwhile to test the predictions of this theory also in the nonlinear regime.

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