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Cluster dynamics of planetary waves

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Abstract – The dynamics of nonlinear atmospheric planetary waves is determined by a small number of independent wave clusters consisting of a few connected resonant triads. We classified the different types of connections between neighboring triads that determine the general dynamics of a cluster. Each connection type corresponds to substantially different scenarios of energy flux among the modes. The general approach can be applied directly to various mesoscopic systems with 3-mode interactions, encountered in hydrodynamics, astronomy, plasma physics, chemistry, medicine, etc.

Introduction. – Planetary-scale motions in the ocean and atmosphere are due to the shape and rotation of the Earth, and play a crucial role in the problems of weather and climate prediction [1]. Oceanic planetary waves affect the general large-scale ocean circulation, can intensify the ocean currents such as the Gulf Stream, as well as push them off their usual course. For example, a planetary wave can push the Kuroshio Current northwards and affect the weather in North America [2]. Atmospheric planetary waves detach the masses of cold, or warm, air that become cyclones and anticyclones and are responsible for day-to-day weather patterns at mid-latitudes [3]. Recently a novel model of intra-seasonal oscillations in the Earth atmosphere has been developed [4] in terms of isolated resonant triads of planetary waves (for wave numbers $m, \ell \leqslant 21$). The complete cluster structure depends on the spectral-domain size, both for atmospheric [5] and oceanic [6] planetary waves. In particular, an enlargement of at least some of the clusters is possible, with growing of the spectral domain. In some cases this yields the energy flux between previously isolated clusters. To justify the basic model [4] one has to understand whether or not existing clusters are capable to adopt external energy via this mechanism. In [7] the isomorphism (one-to-one correspondence) between clusters of similar structure and corresponding dynamical systems has been established. This allows for the study of the dynamical behavior of similar clusters. It is shown both in numerical simulations [8,9] and in laboratory experiments [10] that the dynamics of mesoscopic wave systems does not obey the statistical description (wave kinetic equations). It rather needs a special investigation.

In this letter we show that the general dynamics of big clusters in mesoscopic systems with 3-mode interactions is determined by the connection types between neighbor triads. In particular, we analyzed clusters of atmospheric planetary waves in the spectral domain $m, \ell \leqslant 1000$ which allowed us to justify the model suggested in [4].

Triad dynamics. – Consider three planetary (Rossby) waves with frequencies ω_1, ω_2 and ω_3, which satisfy the conditions of time and space synchronism:

$$\begin{cases} \omega_1 + \omega_2 = \omega_3, \\ m_1 + m_2 = m_3, \\ |\ell_1 - \ell_2| \leqslant \ell_3 \leqslant \ell_1 + \ell_2, \\ m_j \leqslant \ell_j, \ j = 1, 2, 3, \\ \ell_i \neq \ell_j, \ i \neq j, \ i, j = 1, 2, 3, \\ \ell_1 + \ell_2 + \ell_3 \ \text{ is odd}, \end{cases} \tag{1}$$

and $\omega \propto m/[\ell(\ell+1)]$. The first three equations correspond to a three-wave resonance on a sphere while the two last equations provide a nonzero coupling coefficient in the corresponding dynamical system (see [4] for more details).

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This is the simplest possible cluster that is described by the dynamical system

$$\dot{B}_1 = ZB_2^* B_3, \quad \dot{B}_2 = ZB_1^* B_3, \quad \dot{B}_3 = -ZB_1 B_2. \qquad (2)$$

Here $\dot{B}_j = \mathrm{d}B_j/\mathrm{d}t$, $B_i = \alpha_i A_i$ is a time derivative with α_i being explicit functions of the longitudinal wave numbers ℓ_j, A_j are modes amplitudes, and Z is the interaction coefficient which is also some function of the wave numbers. Equations (2) are symmetric with respect to the exchange of two low-frequency modes $1 \Leftrightarrow 2$. The mode with the highest frequency (which in this paper will be denoted by the subscript "3") is a special mode. The system eq. (2) has two independent conservation laws

$$\begin{cases} I_{23} = |B_2|^2 + |B_3|^2 = (E\,N_1 - H)N_{23}/N_1N_2N_3\,, \\ I_{13} = |B_1|^2 + |B_3|^2 = (E\,N_2 - H)N_{13}/N_1N_2N_3\,, \\ I_{12} = I_{13} - I_{23} = |B_1|^2 - |B_2|^2\,, \end{cases} \qquad (3)$$

which are linear combinations of the energy E and enstrophy H, defined as

$$E = E_1 + E_2 + E_3\,, \quad H = N_1E_1 + N_2E_2 + N_3E_3. \qquad (4)$$

Here E_j is the energy of the j-mode and $N_j = \ell_j(\ell_j + 1)$. The solutions of eqs. (2) are Jacobian elliptic functions, and whether or not its dynamics is periodic is determined by the energy in the ω_3-mode (for details see [4]).

To understand the dynamics of the energy flow within a cluster, the first step would be to initiate a small amount of chosen modes and to study afterwards the energy exchange within a cluster. Thus, we begin with discussing the evolution of the triad amplitudes with special initial conditions, when only one mode is substantially excited. If $B_1(t=0) \gg B_2(t=0)$ and $B_1(t=0) \gg B_3(t=0)$, then $I_{23}(t=0) \gg I_{13}(t=0)$. The integrals of motion are independent of time, therefore $I_{13} \gg I_{23}$ at each instant of time and hence $|B_1(t)|^2 \gg |B_2(t)|^2$. Moreover, $|B_1(t)|^2 \gg |B_3(t)|^2$ at every instant. Indeed, the assumption $|B_1(t)|^2 \lesssim |B_3(t)|^2$ yields $I_{13} \simeq I_{23}$, which is not the case. This means that the ω_1-mode, being the only substantially exited at $t=0$ cannot share its energy with the two other modes in a triad. The same is true for the ω_2-mode. In this context we call the modes with frequencies $\omega_1 < \omega_3$ and $\omega_2 < \omega_3$ passive modes, or *P-modes*.

The conservation laws (3) cannot restrict the growth of the P-modes from initial conditions when only the ω_3-mode is exited. In this case the P-mode amplitudes will grow exponentially: $|B_1(t)|\,, |B_2(t)| \propto \exp[\,|ZB_3(t=0)|t]$ until all the modes will have comparable magnitudes of the amplitudes. Therefore we call the ω_3-mode an active mode, or *A-mode*. The A-mode, being initially excited, is capable to share its energy with two P-modes within a triad.

Connection types within a cluster. – An arbitrary cluster in our wave system is a set of connected triads. A cluster consisting of two triads that are connected via one common mode is called a *butterfly*; a cluster of three triads with one common mode is called a *triple star*. The general dynamics of a cluster depends on the type of the connecting mode, which is common for the neighbor triads. Correspondingly, we can distinguish three types of butterflies (with PP-, AP- and AA-connections), four triple stars (with PPP-, PPA-, PAA- and AAA-connections), etc. We begin with considering the butterfly and the triple star dynamics; we then discuss an actual dynamics of the more involved but concrete topology of connected triads in atmospheric planetary waves.

A *PP-butterfly* consists of two triads a and b, with wave amplitudes B_{ja}, B_{jb}, $j = 1, 2, 3$, that are connected *via* one common mode, $B_{1a} = B_{1b}$ which is passive in both triads. The equations of motion for this system read

$$\begin{cases} \dot{B}_{1a} = Z_aB_{2a}^*B_{3a} + Z_bB_{2b}^*B_{3b},\ B_{1a} = B_{1b}, \\ \dot{B}_{2a} = Z_aB_{1a}^*B_{3a}\,, \quad \dot{B}_{2b} = Z_bB_{1a}^*B_{3b}\,, \\ \dot{B}_{3a} = -Z_aB_{1a}B_{2a}\,,\ \dot{B}_{3b} = -Z_bB_{1a}B_{2b}. \end{cases} \qquad (5)$$

An examination of eqs. (5) shows that they have three integrals of motion:

$$\begin{cases} I_{23a} = |B_{2a}|^2 + |B_{3a}|^2\,, \quad I_{23b} = |B_{2b}|^2 + |B_{3b}|^2\,, \\ I_{ab} \ = |B_{1a}|^2 + |B_{3a}|^2 + |B_{3b}|^2. \end{cases} \qquad (6)$$

The first two, I_{23a} and I_{23b}, do not involve the common mode $B_{1a} = B_{1b}$, and are similar to the integral I_{23}, eq. (3), for an isolated triad. Similarly to the case of the evolution of a triad from the initial conditions with an excited passive mode, the following conclusion can be made. If at $t=0$ the amplitudes of one triad substantially exceed two remaining amplitudes of the butterfly, that is, if $|B_{1a}|, |B_{2a}|, |B_{3a}| \gg |B_{2a}|, |B_{3a}|$, then this relation persists. In other words, in a PP-butterfly any of two triads, a or b, having initially very small amplitudes, will be unable to adopt energy from the second triad during its nonlinear evolution.

An *AP-butterfly* consists of two triads a and b, with wave amplitudes that are connected via the common mode $B_{3a} = B_{1b}$, which is active in one triad (a for concreteness) and is passive in the second triad (b). In this case equations and integrals of motion are:

$$\begin{cases} \dot{B}_{1a} = Z_aB_{2a}^*B_{3a}\,, \quad \dot{B}_{3b} = -Z_bB_{3a}B_{2b}\,, \\ \dot{B}_{2a} = Z_aB_{1a}^*B_{3a}\,, \quad \dot{B}_{2b} = Z_bB_{3a}^*B_{3b}\,, \\ \dot{B}_{3a} = -Z_aB_{1a}B_{2a} + Z_bB_{2b}^*B_{3b}\,, \end{cases} \qquad (7)$$

$$\begin{cases} I_{12a} = |B_{1a}|^2 - |B_{2a}|^2\,, \quad I_{23b} = |B_{2b}|^2 + |B_{3b}|^2\,, \\ I_{ab} \ = |B_{1a}|^2 + |B_{3a}|^2 + |B_{3b}|^2. \end{cases} \qquad (8)$$

The integrals I_{12a} and I_{23b}, do not involve a common mode $B_{3a} = B_{1b}$; they are similar to the corresponding integrals I_{12} and I_{23}, for the isolated triad. In the case, when the triad a is excited at $t=0$ much stronger then the b-triad (in which case $I_{12a} \gg I_{23b}$) then the smallness of the positively definite integral of motion I_{23b} prevents the triad b from adopting energy from the triad a during all the evolution. The situation is different, when triad b is excited initially and $I_{23b} \gg I_{12a}$. In this case the initial energy of the triad b can be shared with the triad a. The smallness of I_{12a} only required that during the evolution $|B_{1a}| \approx |B_{2a}|$.

For an *AA-butterfly* with a common active mode in both triads ($B_{3a} = B_{3b}$) we have:

$$\begin{cases} \dot{B}_{1a} = Z_a B_{2a}^* B_{3a}\,, & \dot{B}_{1b} = -Z_b B_{2b}^* B_{3a}\,, \\ \dot{B}_{2a} = Z_a B_{1a}^* B_{3a}\,, & \dot{B}_{2b} = Z_b B_{1b}^* B_{3a}\,, \\ \dot{B}_{3a} = -Z_a B_{1a} B_{2a} - Z_b B_{1b} B_{2b}. & \end{cases} \tag{9}$$

$$\begin{cases} I_{12a} \;=\; |B_{1a}|^2 - |B_{2a}|^2\,, \quad I_{12b} = |B_{1b}|^2 - |B_{2b}|^2\,, \\ I_{ab} \;\;\, = \; |B_{1a}|^2 + |B_{3a}|^2 + |B_{3b}|^2. \end{cases} \tag{10}$$

Again, the integrals I_{12a} and I_{12b} do not involve a common mode $B_{3a} = B_{1b}$ and are similar to I_{12}, eqs. (3), for an isolated triad. A simple analysis of these integrals of motion shows that the energy that is initially held in one of the triads will be dynamically shared between both triads.

Finally, we consider one of the triple-star clusters, *e.g.* the APP-star, in which a common mode is active in the a-triad and passive in the b- and c-triad: $B_{3a} = B_{1b} = B_{1c}$. This system has four integrals of motion:

$$\begin{cases} I_{12a} = |B_{1a}|^2 - |B_{2a}|^2\,, \\ I_{23b} = |B_{2b}|^2 + |B_{3b}|^2\,, \quad I_{23c} = |B_{2c}|^2 + |B_{3c}|^2\,, \\ I_{abc} = |B_{1a}|^2 + |B_{3a}|^2 + |B_{3b}|^2 + |B_{3c}|^2. \end{cases} \tag{11}$$

Similarly to butterflies, there exists one integral of motion for each connected triad, that does not involve the common mode: these are the integrals I_{12a}, I_{23b}, and I_{23c}, which are the same as the corresponding integrals in the isolated triad. The integrals I_{23b} and I_{23c} prevent the b- and c-triads (which are connected via a P-mode) from adopting energy from the initially excited a-triad. In those cases when the b- and/or c-triads are initially exited, the a-triad can freely adopt their energy via the connecting A-mode.

Any triad that is connected to a cluster (no matter how big the cluster would be) via its passive mode cannot adopt energy from the cluster, if the triad is not excited initially. On the other hand, a triad that is connected to any cluster via an active mode can adopt energy from the cluster during its nonlinear evolution.

Topology and cluster dynamics for atmospheric planetary waves. – The frequencies of atmospheric planetary waves are

$$\omega_j = -2\,\Omega\, m_j/\ell_j(\ell_j + 1). \tag{12}$$

The negative sign indicates wave propagation opposite to the rotation of the Earth (east to west with frequency Ω). The integers ℓ_j with and $m_j \leqslant \ell_j$ describe the eigen-mode structure, which is j-spherical harmonics with ℓ_j and $\ell_j - m_j$ zeros in the longitudinal and latitudinal directions, correspondingly. The Diophantine eqs. (12) and $\omega_1 + \omega_2 = \omega_3$, have many solutions, each of them describing an exact resonance of *ideal* planetary waves (ignoring the real Earth topography, etc.). In this approximation we can describe all important resonant triads in the Earth atmosphere, see table 1. First we restricted ourself to the so-called meteorologically significant spectral domain with wave numbers $\ell, m \leqslant 21$ [4]. In this domain we have found in [4] four isolated triads, denoted as $\Delta_1, \ldots, \Delta_4$, three PP-butterflies $\bowtie_{5,6}$, $\bowtie_{7,8}$, and $\bowtie_{9,10}$, involving six triads $\Delta_5, \ldots, \Delta_{10}$ and one, further called *caterpillar*, $\boxtimes_{11-16}$, consisting of six triads $\Delta_{11}, \ldots, \Delta_{16}$ with three PP-, one AP- and one AA-connection.

Table 1: In the first 3 columns the following data in the domain $m, \ell \leqslant 21$ are given: cluster's form, triad numbers and modes within a cluster; in the last two —numbers of connecting triads and their modes are given that enlarge the corresponding cluster when the spectral domain $m, \ell \leqslant 1000$ is regarded.

Clust.	$\mathcal{N}_1$	Modes $[m, \ell]$	$\mathcal{N}_2$	Connecting triads
Δ_1	1	[4,12] [5,14] [9,13]	–	–
Δ_2	2	[3,14] [1,20] [4,15]	2.1	[4,15] [10,24] [14,20]
			2.2	[1,20] [14,29] [15,28]
			2.3	[1,20] [15,75] [16,56]
Δ_3	3	[6,18] [7, 20] [13,19]	3.1	[2,15] [5,24] [7,20]
Δ_4	4	[1,14] [11,21] [12,20]	4.1	[1,14] [9,27] [10,24]
$\bowtie_{5,6}$	5	[2,6] [3,8] [5,7]	5.1	[4,14] [9,27] [13,20]
	6	[2,6] [4,14] [6,9]		
$\bowtie_{7,8}$	7	[6,14] [2,20] [8,15]	7.1	[2,20] [11,44] [13,35]
	8	[3,6] [6,14] [9,9]	7.2	[2,20] [30,75] [32,56]
			7.3	[32,56] [26,114] [58,69]
$\bowtie_{9,10}$	9	[3,10] [5,21] [8,14]	–	–
	10	[8,11] [5,21] [13,13]		
$\boxtimes_{11-16}$	11	[2,14] [17,20] [19,19]	11.1	[2,14] [18,27] [20,24]
	12	[1,6] [2,14] [3,9]	11.2	[6,44] [14,21] [20,24]
	13	[3,9] [8,20] [11,14]	11.3	[9,35] [11,20] [20,24]
	14	[1,6] [11,20] [12,15]	11.4	[3,20] [45,75] [48,56]
	15	[9,14] [3,20] [12,15]		
	16	[2,7] [11,20] [13,14]		

In this letter we show that the topological structure of clusters in the extended spectral domain $m, \ell \leqslant 1000$ is richer, in particularly, some clusters are enlarged by new resonances formed by modes with $m, \ell > 21$. To illustrate this mechanism, we introduced the following rows in table 1: 1) $\mathcal{N}_1$ —the number of clusters in the spectral domain $m, \ell \leqslant 21$, with the same numeration as given in [4]; 2) Modes —resonance clusters belonging in the same spectral domain as in 1); 3) $\mathcal{N}_2$ —the number of the additional clusters in the spectral domain $m, \ell \leqslant 75$; 4) Connecting triads —additional resonance clusters which appeared in the spectral domain as in 3). The topological structure of resonance clusters in the spectral domain $m, \ell \leqslant 75$ is shown in fig. 1, where numbers inside of triangles correspond to the numeration $\mathcal{N}_1$, $\mathcal{N}_2$ in table 1.

There exist altogether 1965 isolated triads and 424 clusters consisting of 2 to 3691 connected triads, among them 235 butterflies, 95 triple-triad clusters, etc. —see the histogram in fig. 2. The three largest clusters consist of 14, 16 and 3691 connected triads. For the clarity of presentation we did not display on the histogram the

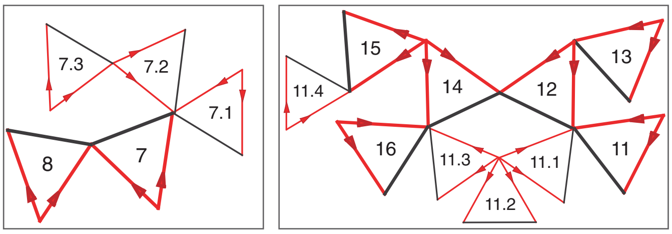

Fig. 1: (Color online) Triads belonging to butterfly $\bowtie_2$ (left) and to caterpillar (right) are drawn by bold (red) lines while new, connected to them, triads appearing in the spectral domain $m, \ell \leqslant 1000$ are drawn by thin lines. (Red) arrows are coming from an active mode and show directions of the energy flux. The numbers inside each triangle correspond to the numeration in table 1.

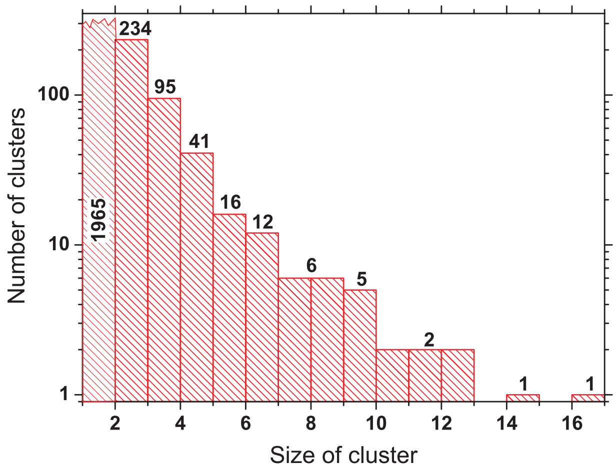

Fig. 2: (Color online) Horizontal axes denotes the number of triads in the cluster while vertical axes shows the number of corresponding clusters.

largest 3691-cluster, which we will call further the *monster*.

It can be seen that 82.2% of all clusters are isolated triads. Their dynamics has been investigated in [4] in all details. The energy oscillates between three modes in the triads, the period of this oscillation being much larger than the wave period. It was found to be inversely proportional to the root mean square of the wave amplitude.

235 clusters ($\simeq$10.5%) are butterflies. Among them there are 131 PP-, 69 AP- and 35 AA-butterflies. The butterfly dynamics is restricted by three integrals of motion, and it can be shown that the phase space of butterflies is four-dimensional. In [4] only PP-butterflies have been considered. For the initial conditions studied in this letter their dynamics is similar to that of two isolated triads.

Preliminary numerical simulations show that only in the case when the initial levels of excitation in both triads are compatible, a periodic energy exchange is observed *not only* within a triad but also between two connected triads.

The 95 triple-triad clusters include 66 linear clusters with two pair connections, 25 "3-stars" with triple connections and 4 triangles (with three pair connections). A similar classifications can be performed for all the remaining clusters. For example, the monster includes one mode (218,545), participating in 10 triads, three modes, participating in 9 triads, 5 modes —in 8 triads, 23 —in 7, 50 —in 6, 90 —in 5, 236 —in 4, 550 —in 3, and 1428 modes —in 2 triads (butterflies). The analysis of their dynamical behavior depends critically on the type of connection, as was shown above.

For example, the 16-triad cluster can be divided into "almost separated" parts (connected through PP-connections) parts: 5 triads, one AA- and one AP-butterfly, one AAP-star and one AAA-star with an AP-connected triad. The overall qualitative conclusion is that even big clusters are dynamically not very different from separated small clusters with active connections, AA-butterflies, AAA-stars, etc.

To clarify the dynamics of the first sixteen triads $\Delta_1, \dots, \Delta_{16}$, it is important to establish their (possible) connection to the clusters in a bigger spectral domain. We have found that the triad Δ_1 remains isolated, Δ_3 turned into an isolated PA-butterfly. It can be proven [11] that these objects remain "forever" isolated, even if the size of the bigger domains goes to infinity. The triads Δ_2 and Δ_4 became parts of the monster but are connected with it via P-modes; in this sense they are practically separated. Three initial butterflies, $\bowtie_{5,6}$, $\bowtie_{7,8}$, $\bowtie_{9,10}$ have PP-connections and thus their dynamics does not differ much from the dynamics of an isolated triad. Moreover, an increase of the spectral domain to $m, \ell \leqslant 1000$ does not change the situation substantially: the butterfly $\bowtie_{9,10}$ remains isolated in an arbitrarily large (even infinite) domain, $\bowtie_{7,8}$ became part of a 5-triad cluster, also with PP- and PPP-type of connections (see fig. 1), $\bowtie_{5,6}$ is now part of the monster but only via PP-connections. The caterpillar gains one P-connected triad (11.4 in table 1) and one AAA-star (11.1, 11.2 and 11.3 triads) with two P-connections, see fig. 1. Therefore almost all 16 triads, (except of the AA-butterfly $\bowtie_{15,16}$) can be considered as completely or as almost separated from the rest of the system.

Conclusions. – In the physically relevant domain of atmospheric planetary waves ($m, \ell \leqslant 1000$, when the mode scale is larger than the height of the Earth atmosphere) we have determined and described the topology of all clusters that are formed by resonantly interacting planetary modes. The cluster set contains isolated triads and sets of 2-, 3-, ..., 16 and 3691 connected triads, with 2- 3-, ..., 9-mode and (maximum) 10-mode connections.

Analyzing the integrals of motion we suggested a classification i) of triad modes into two types —active (A) and passive (P), and ii) of connection types between triads — AA, AP and PP. We have shown that in AA-butterflies the energy can flow in both directions, in AP-butterflies only from one triad to the second one (and not vice versa), while in PP-butterflies the triads are "almost isolated". We have also shown that the dynamical behavior of bigger clusters can be similarly characterized by connection types like AAA, AAP, ..., etc.

As a first approximation, almost all triads in the meteorologically significant domain of table 1, can be considered as completely or almost separated from the rest of atmospheric planetary waves, and therefore energy oscillations within them can lead to intra-seasonal oscillations in the Earth's atmosphere, as suggested in [4].

Our analysis is based on the structure of the dynamical system (2) and has two advantages: i) it does not need to exploit the explicit form of the interaction coefficients Z; ii) it is completely analytical without using numerics (which will be required for more general initial conditions). Thus our results can be used directly for arbitrary resonant 3-wave systems governed by these equations, *e.g.* drift waves, gravity-capillary waves, etc.

A touchstone of any theory is, of course, an experiment. Numerical experiments with resonance clusters described by barotropic vorticity equation are now on the way, preliminary results confirm our theoretical conclusions. The next step of utmost importance would be to study some physical mechanisms that might destroy clusters. Mathematically speaking, one can always introduce a big enough resonance width

$$\Omega = \omega_1 + \omega_2 - \omega_3 > 0$$

such that a substantial part of the cluster's energy will be re-distributed among other waves *via* nonresonant interactions (see fig. 2 [12]). From the physical point of view, the source of the resonance broadening might have substantially different reasons —from baroclinic instabilities due to the effects of the free surface at large scales, to the effects due to the Earth topography at smaller scales, to the increasing the level of turbulence, say, in summer due to increasing sun activity, and thus going into the regime of fully developed wave turbulence, to the inclusion of dissipation and forcing. These effects can also be combined, of course. The problem of the utmost importance is therefore to study the resonance clusters behavior in the situation when at least some of these effects are included. The analysis applied to climatic variations on geological scales "typically give indications of low dimensionality and (...) the hope of justifying the modelling of weather or climate in terms of a small set of ordinary differential equations" [13]. But it does not mean, of course, that the overall energy flux will stay nicely regular. Indeed, as was shown in [13], a special choice of instabilities included into (2) will cause appearance of strange attractors. The study of this transition from regular to chaotic regimes in resonant clusters and mutual interrelations of relevant physical parameters is the subject of our further research.

Last but not least. Whereas it is quite difficult to check experimentally the theory for planetary waves, it can be done much easier within a framework of laboratory experiments with some water waves, *e.g.* gravity-capillary waves. Some preliminary program of laboratory experiments with this type of waves has been worked out in [14] but only for simplest triad's clusters for at that time the algorithms of cluster computing [15,16] were not developed yet. Presently this program can easily be elaborated for clusters of some more complicated structure. Considerably more sophisticated preliminary work is needed for planning laboratory experiments with gravity water waves. As was shown in [17], a 3-wave resonance system differs principally from wave systems in which 4- and more wave resonances are allowed. Indeed, in a 3-wave resonances system, each nonlinear resonance generates a new scale, while already in a 4-wave system this is not necessarily true. Indeed, beginning with 4-wave resonances, different types of energy fluxes have been pointed out: scale resonances (as in a 3-wave system), angle resonances (formed by two couples of wave vectors with pairwise equal lengths) and mixed cascades. In this case, clusters have more complicated structures presented explicitly in [18] for 4-wave interactions of gravity water waves. The qualitative dynamics of small clusters formed by resonant quartets is briefly as follows: "One wave mode may typically participate in many angle resonances and only one scale resonance. Thus, one can split large clusters into "reservoirs", each formed by a large number of angle quartets in quasi-thermal equilibrium, and which are connected with each other by sparse links formed by scale quartets."(see [18]). This means, in particular, that scale resonances cause spectrum anisotropy. On the other hand, one can easily compute what ratio aspect of the sides of a laboratory tank should be chosen in order to suppress scale resonances. In this case, regular patterns on the water surface are to be expected, similar to what was observed in [19–21]. It would be interesting to study these experimental data in order to establish these nearly permanent patterns observed, which can be attributed to some specific resonance clusters. The existence of independent resonance clusters can shed some light on the origin of the Benjamin-Feir instability [22] or the McLean instability [23] (see also Discussion in [18]).

Another interesting and even more complicated area of further investigations would be the study of the dynamical behavior of resonant quintets, sextets and so on. The methods developed in [15,16] allow to compute the clusters and corresponding wave frequencies. This information can be afterwards used, for instance, for investigating some special types of shallow water instabilities [24]. In systems containing simultaneously quartets and quintets [25], the explicit construction of clusters can possibly yield the explanation of the competing regimes between these two types of instabilities. We point out here three possible scenarios due to the change of 1) boundary conditions, 2) the frequency range under the study, and 3) the initial distribution of energies among the modes of a cluster. In the first two cases, some quartets can be suppressed, thus turning the quintets into the principal clusters (and *vice versa*, of course). In the third case, a lot of different sub-scenarios are possible: quartet and quintet connected *via* one common mode can become "independent" if the energy in the common mode is too small; a quartet can "die out" if its initial energy is not enough for nonlinear interactions; the same for the quintet; the energy can flux from the quartet to the quintet *via* the common mode, in dependence on the energy distribution among the other modes in the quartet thus yielding the dying out of the quartet; the same for the quintet; etc. Combining the methods [15,16] and results of [25] will help to single out some of the possibilities immediately. For instance, one can compute wave numbers corresponding to a resonant cluster, put them into the eq. (2.11) from [25] and check whether coupling coefficient(s) in this cluster is (are) nonzero. Another advantage would be the construction of the topological structure of the complete cluster set as is done in [18] instead of resonance curves: it gives a general overview of the nonlinear dynamics rather than a (possible) subset of resonances formed with a fixed wave vector (in particular cases this subset can be empty, of course). One more advantage would be the following. As was shown in [25], instability analysis is based on the properties of the roots of the polynomial given by eq. (3.10) (see also fig. 3a–d therein). In some cases a qualitative instability analysis can help but "to get qualitative results in general case one should solve (3.10)" (ref. [25], p. 313). This computations are very involved while the polynomial has degree 4 and its coefficients depend on the dynamical coupling coefficients. On the other hand, the knowledge of explicit wave numbers for modes forming a resonance cluster will turn the coefficients of (3.10) into known constants and the problem can be easily solved.

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