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Equations of motion and conservation laws in a theory of stably stratified turbulence

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Abstract

This paper is part of an invited talk given at the international conference 'Turbulent Mixing and Beyond'. We consider non-isothermal fluid flows and revise simplifications of basic hydrodynamic equations for such flows, arriving eventually at a generalization of the Oberbeck–Boussinesq approximation valid for arbitrary equation of state including both non-ideal gases as well as liquids. The proposed approach is based on a suggested general definition of potential temperature. Special attention is paid to the energy conservation principle: the proposed approximation exactly preserves the total mechanical energy by approximate equations of motion. It is emphasized explicitly the importance for any turbulent boundary layer model to respect the conservation laws.

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1. Introduction

In this paper, which has a partially pedagogical character, we reconsider simplifications of the basic hydrodynamic equations of motion for non-isothermal fluid flows, which can serve as a basis for a consistent theory of atmospheric turbulent boundary layers with stable temperature stratification. The reason for doing so stems from the fact that textbook derivations of the celebrated Oberbeck–Boussinesq approximation are too specialized either to fluids whose density variations are very small or to ideal gases and cannot be applied, for example, to humid air. Our generalization of this approximation is based on a generalized definition of the potential temperature suggested below, which is valid for arbitrary equation of state, while its standard definition, widely used in meteorology, is applicable only to ideal gases.

We discuss also a form of mechanical energy (consisting of the kinetic and potential energies) that is exactly preserved by the resulting approximate equations of motion. This integral of motion plays a crucial role in modeling stably stratified atmospheric turbulent boundary layers (see our other paper in this issue) allowing us to control further approximations with the goal of a simple, but still adequate description of turbulent fluxes that determine basic physics of the turbulent boundary layers.

This paper is part of an invited talk given at the international conference 'Turbulent Mixing and Beyond' devoted to long-standing tasks and challenging problems in fluid dynamics, turbulence, geophysics and statistics.

2. Generalization of the Oberbeck–Boussinesq approximation for non-ideal gases and liquids

2.1. Basic hydrodynamic equations

The system of hydrodynamic equations describing a fluid in which the temperature is not uniform consists of the Navier–Stokes equation for the fluid velocity, $\boldsymbol{\mathcal{U}}(\boldsymbol{r},t)$, a continuity equation for the space- and time-dependent (total) density of the fluid, $\rho(\boldsymbol{r},t)$, and of the heat balance equation for the (total) ntropy per unit mass, $\mathcal{S}(\boldsymbol{r},t)$, [1]:

$$\rho\left(\frac{\partial}{\partial t}+\boldsymbol{\mathcal{U}}\cdot\nabla\right)\boldsymbol{\mathcal{U}}=-\nabla p+\boldsymbol{g}\,\rho+\nabla\cdot\mu\,\nabla\boldsymbol{\mathcal{U}}\,, \tag{1a}$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{\mathcal{U}})=0\,, \tag{1b}$$

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$$\rho\Big(\frac{\partial}{\partial t}+\boldsymbol{\mathcal{U}}\cdot\nabla\Big)\mathcal{S}=\nabla\cdot\kappa\,\nabla\boldsymbol{S}. \tag{1c}$$

Here p is the pressure, $\boldsymbol{g}=-\hat{\boldsymbol{z}}g$ is the vertical acceleration due to gravity, and μ and κ are the (molecular) dynamical viscosity and heat conductivity.

These equations are considered with boundary conditions that maintain the solution far from the equilibrium state, at which $\boldsymbol{\mathcal{U}}=\mathcal{S}=0$. These boundary conditions are $\boldsymbol{\mathcal{U}}=0$ at zero elevation, $\boldsymbol{\mathcal{U}}=\text{const}$ at a high elevation of a few kilometers. This reflects the existence of a wind at high elevation, but we do not attempt to model the physical origin of this wind in any detail. The only important condition with regard to this wind is that it maintains a momentum flux toward the ground that is prescribed as a function of the elevation. Similarly, we assume that a stable temperature stratification is maintained such that the heat flux towards the ground is prescribed as well. In the entropy balance equation (1*c*), we have already neglected the viscous entropy production term, $\propto\mu|\nabla\boldsymbol{\mathcal{U}}|^2$, assuming that the temperature gradients are large enough such that the thermal entropy production term on the rhs of equation (1*c*) dominates. Actually, this assumption is very realistic in meteorological and oceanographic applications. For simplicity of the presentation, we restrict ourselves to relatively small elevations and disregard in equation (1*a*) the Coriolis force (for more details, see [2]).

On the other hand, we assume that the temperature and density gradients in the entire turbulent boundary layer are sufficiently small to allow employment of local thermodynamic equilibrium. In other words, we assume the validity of the equation of state, and that the entropy $\mathcal{S}$ is a state function of the local values of the density and pressure:

$$\rho=\rho(T,p)\,,\qquad \mathcal{S}=\mathcal{S}(\rho,p). \tag{2}$$

In the same manner we will neglect the temperature dependence of the dissipation parameters ν and κ.

Pressure fluctuations caused by turbulent velocity fluctuations $\boldsymbol{u}$ propagate in a compressible medium with the sound velocity c_{s}, causing time-dependent density fluctuations of the order of $(u/c_{\text{s}})^2\rho_0$, where ρ_0 is the mean density. Assuming that the square of the turbulent Mach number $M_{\text{T}}^2\equiv(u/c_{\text{s}})^2$ is small compared with unity, we can neglect in equation (1*b*) the partial time derivative, hence $\nabla\cdot(\rho\boldsymbol{\mathcal{U}})=0$ (see e.g. [1]). Even tropical hurricanes of category 5 have mean wind velocity U below $300\,\text{km}\,\text{h}^{-1}$. Usually, the turbulent velocity fluctuations u are less than $U/10$, i.e. even in these extreme conditions $u<30\,\text{km}\,\text{h}^{-1}$ and $M_{\text{T}}^2<10^{-3}$ (with $c_{\text{s}}\simeq1200\,\text{km}^{-1}$). Therefore the incompressibility approximation $\nabla\cdot(\rho\boldsymbol{\mathcal{U}})=0$ is well justified in atmospheric physics. In the ocean, where the sound velocity is even larger and water velocities even smaller, this approximation is quite excellent.

2.2. Isentropic basic reference state

In quiet air, without turbulence, the pressure and the density depend on the elevation z simply due to gravity. For example, in full thermodynamic equilibrium the temperature is uniform and z-independent, and the density decreases exponentially with the elevation. However, this equilibrium model of the atmosphere is not realistic, and cannot be used as a reference state about which the actual dynamics is considered. A much better reference suggestion is a state in which the entropy is space homogeneous. In this model, the thermal conductivity (leading to the temperature homogeneity) is neglected with respect to heat transfer due to the vertical adiabatic mixing of air, leading to a z-independent entropy. We refer to the isentropic model as a 'basic reference state' and denote this state of the system with a subscript '$_{\text{b}}$':

$$\mathcal{S}_{\text{b}}=\mathcal{S}(\rho_{\text{b}},p_{\text{b}})=\text{const}\,,\qquad \rho_{\text{b}}=\rho(T_{\text{b}},p_{\text{b}}). \tag{3}$$

The first of equations (3) relates the gradients of the pressure and density in this state:

$$0=\nabla\mathcal{S}_{\text{b}}=\left(\frac{\partial\mathcal{S}_{\text{b}}}{\partial\rho_{\text{b}}}\right)_{\text{p}}\nabla\rho_{\text{b}}+\left(\frac{\partial\mathcal{S}_{\text{b}}}{\partial p_{\text{b}}}\right)_{\rho}\nabla p_{\text{b}}. \tag{4a}$$

Another relation between ρ_{b} and p_{b} follows from the condition of hydrostatic equilibrium:

$$\nabla p_{\text{b}}=\boldsymbol{g}\rho_{\text{b}}. \tag{4b}$$

Equations (4) together with the first of equations (3) determine the density, pressure and temperature profiles in the isentropic basic reference state.

2.3. Hydrodynamic equations in generalized Oberbeck–Boussinesq approximation

2.3.1. Equations of motion. Let us denote the deviations of the total density, pressure, temperature and entropy from the basic reference state as follows:

$$\begin{aligned}\hat{\rho}&\equiv\rho-\rho_{\text{b}}, & \hat{p}&\equiv p-p_{\text{b}}\,,\\ \hat{T}&\equiv T-T_{\text{b}}\,, & \hat{\mathcal{S}}&\equiv\mathcal{S}-\mathcal{S}_{\text{b}}.\end{aligned} \tag{5}$$

Following Oberbeck [3] and Boussinesq [4], assume that these deviations are small: $\hat{\rho}\ll\rho_{\text{b}}$, $\hat{p}\ll p_{\text{b}}$. Then one simplifies the full system of hydrodynamic equations (1) and rewrites them in terms of the fluid velocity $\boldsymbol{\mathcal{U}}$ and the small deviations $\hat{p}$ and $\hat{\mathcal{S}}$ (instead of $\hat{\rho}$). The first step is very simple: because of equation (4*b*)

$$\nabla p-\boldsymbol{g}\rho=\nabla\hat{p}-\boldsymbol{g}\,\hat{\rho}. \tag{6a}$$

Next we should relate the deviations $\hat{p}$, $\hat{\rho}$ and $\hat{\mathcal{S}}$. In the linear approximation, equation (2) yields

$$\hat{\mathcal{S}}=\left(\partial\mathcal{S}_{\text{b}}/\partial p_{\text{b}}\right)_{\rho}\hat{p}+\left(\partial\mathcal{S}_{\text{b}}/\partial\rho_{\text{b}}\right)_{\text{p}}\hat{\rho}. \tag{6b}$$

With the help of equations (4) this gives

$$\boldsymbol{g}\,\hat{\rho}=\frac{\nabla\rho_{\text{b}}}{\rho_{\text{b}}}\,\hat{p}-\boldsymbol{\beta}\,\frac{\rho_{\text{b}}}{c_{\text{p}}}\,T_{\text{b}}\,\hat{\mathcal{S}}. \tag{6c}$$

Here, $\boldsymbol{\beta}\equiv\boldsymbol{g}\,\tilde{\beta}$ is the buoyancy parameter, $\tilde{\beta}$ is the thermal expansion coefficient and c_{p} is the isobaric specific heat in the basic reference state (for thermodynamic relations see, for example, [5]):

$$\tilde{\beta}\equiv-\frac{1}{\rho_{\text{b}}}\left(\frac{\partial\rho_{\text{b}}}{\partial T_{\text{b}}}\right)_{\text{p}}=-\frac{c_{\text{p}}}{\rho_{\text{b}}\,T_{\text{b}}}\left(\frac{\partial\rho_{\text{b}}}{\partial\mathcal{S}_{\text{b}}}\right)_{\text{p}}. \tag{6d}$$

Now equation (6*a*) in the linear approximation yields

$$\nabla p - \boldsymbol{g}\rho = \rho_{\rm b}\left[\nabla\left(\frac{p_{\rm b}}{\rho_{\rm b}}\right) + \boldsymbol{\beta}\,\frac{T_{\rm b}}{c_p}\,\hat{\mathcal{S}}\right]. \tag{6e}$$

Then equation (1*a*) can be approximated as

$$\left(\frac{\partial}{\partial t} + \boldsymbol{\mathcal{U}}\cdot\nabla\right)\boldsymbol{\mathcal{U}} = -\nabla\left(\frac{\hat{p}}{\rho_{\rm b}}\right) - \boldsymbol{\beta}\frac{T_{\rm b}}{c_{\rm p}}\,\hat{\mathcal{S}} + \frac{1}{\rho_{\rm b}}\nabla\cdot\mu\nabla\boldsymbol{\mathcal{U}}. \tag{7}$$

2.3.2. Generalized potential temperature. To proceed, we generalize the notion of potential temperature Θ (see e.g [6]), which is traditionally defined as the temperature that a volume of *dry air* at a pressure $p(z)$ and temperature $T(z)$ would attain when adiabatically compressed to the pressure p_* that exists at zero elevation $z = 0$. This potential temperature can be explicitly computed for an ideal gas with the result

$$\bar{\Theta}(z) \equiv T_*\big(p_*/p(z)\big)^{(\gamma-1)/\gamma}, \tag{8}$$

where γ is the ratio of isobaric to isochoric specific heats, $\gamma \equiv c_{\rm p}/c_{\rm v}$, and T_* is the temperature at zero elevation.

We want to generalize the notion of the potential temperature for an arbitrary stratified fluid requiring that in the isentropic basic reference state it would be constant $\Theta_* = T_*$. A second requirement is that the definition will agree with equation (8) for an ideal gas. Accordingly we define

$$\bar{\Theta}(z) = T_*\exp\left[\left(\mathcal{S}(z) - \mathcal{S}_{\rm b}\right)/c_{\rm p}\right]. \tag{9}$$

For more details, see also [7]. Indeed, if we employ the equation of state and the equation for the entropy of an ideal gas, i.e.

$$p = \rho\,T\,, \qquad \mathcal{S} = \ln\left(p^{c_{\rm v}}/\rho^{c_{\rm p}}\right) + \text{const}, \tag{10}$$

one can easily check that equation (8) is recaptured.

2.3.3. Resulting equations. For small deviations of Θ from the basic reference state value T_*, i.e. up to the linear order, equation (9) gives:

$$\Theta_{\rm d} \equiv \bar{\Theta} - T_* = T_*\hat{\mathcal{S}}/c_{\rm p}. \tag{11}$$

Now we can present equations (1*a*), (1*b*) (with $\partial\rho_{\rm t}/\partial t = 0$, as explained) and (1*c*) as follows:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{\mathcal{U}}\cdot\nabla\right)\boldsymbol{\mathcal{U}} = -\nabla\left(\frac{p}{\rho_{\rm b}}\right) - \boldsymbol{\beta}\,\Theta_{\rm d} + \nu\,\Delta\boldsymbol{\mathcal{U}}\,, \tag{12a}$$

$$\nabla\cdot\left(\rho_{\rm b}\boldsymbol{\mathcal{U}}\right) = 0\,, \tag{12b}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{\mathcal{U}}\cdot\nabla\right)\Theta_{\rm d} = \chi\,\Delta\,\Theta_{\rm d}. \tag{12c}$$

The dissipative terms are important only in the narrow region of the viscous sublayer, where we can safely neglect the z-dependence of $\rho_{\rm b}$, μ and κ, and consider the dynamical viscosity $\nu = \mu/\rho_{\rm b}$ and dynamical thermal conductivity $\chi = \kappa/\rho_{\rm b}$ as some z-independent constants.

Note that for turbulence in liquids (water, etc) one can simplify these equations further. There one can neglect the effect of adiabatic cooling (together with the compressibility), and simply use another reference state with constant temperature and density:

$$T = T_*, \qquad \rho_{\rm b} = \rho_*, \qquad p_{\rm b} = p_* + g\rho_* z. \tag{13}$$

For this reference state, the standard reasoning (see e.g. [1]) yields the same equations as equations (12) in which again $\tilde{\beta}$ is given by equation (6*d*) and it is a parameter characterizing a particular fluid. In this case $\rho_{\rm b} = \rho_*$, independent of z, and the potential temperature $\Theta = T$, such that $\Theta_{\rm d} = T_{\rm d}$ is a deviation of the total temperature T from its ground (bottom, or whatever) level T_*.

In the suggested equation (12), the situation is more general since we do not assume that the reference state has the simple form (13) (with $\rho_{\rm b} = \text{const}$). Importantly, on the rhs of equation (12*a*) the density $\rho_{\rm b}(z)$ is operated on by the gradient, and the buoyancy term $-\boldsymbol{\beta}\Theta_{\rm d}$ involves $\Theta_{\rm d} \neq T_{\rm d}$, the deviation of the *potential temperature* defined by equation (11). This definition for liquids has nothing in common with the standard meteorological definition (8). Notice also that for an ideal gas $\tilde{\beta} = 1/T$, and equation (12*a*) coincides with that suggested in the book [8].

3. Conservation of total mechanical energy

3.1. Dynamical integral of motion

It is important to realize that the *approximate* equations (12) *exactly* conserve an *approximate* expression for the total mechanical energy of the system in the dissipationless limit. This total energy is the sum of the kinetic, $\mathcal{E}_{\rm K}$, and the potential energy $\mathcal{E}_{\rm P}$ (calculated in the basic reference state):

$$\mathcal{E}_{\rm K} \equiv \int {\rm d}\boldsymbol{r}\,\rho_{\rm b}\,\frac{|\boldsymbol{\mathcal{U}}|^2}{2}\,, \qquad \mathcal{E}_{\rm P} \equiv \int {\rm d}\boldsymbol{r}\,\rho_{\rm b}\boldsymbol{\beta}\cdot\boldsymbol{r}\,\Theta_{\rm d}. \tag{14}$$

One can check by direct substitution that this sum of energies is conserved by equations of motion (equations (12)) when $\nu = \chi = 0$, i.e. the sum $\mathcal{E}_{\rm K} + \mathcal{E}_{\rm P}$ is a dynamical integral of motion.

3.2. Statistical integral of motion

We show now that the potential energy of a stratified turbulent flow $\mathcal{E}_{\rm P}$, equation (14), can be presented in a turbulent regime as a sum of (time-independent) potential energy of the basic reference state, $\bar{\mathcal{E}}_{\rm P}$ and a 'turbulent' potential energy, associated with temperature fluctuations, $\tilde{\mathcal{E}}_{\rm P}$, as was first mentioned by Richardson [9]. Actually, it is more instructive to discuss this issue in a more general case, when the stratification is caused by some 'internal' parameter of the fluid, ξ, not necessarily the potential temperature. It can be salinity of water in a sea, humidity of air, the concentration of particles co-moving with the fluid as Lagrangian tracers, etc.

In the general case then the equation for the potential energy of a stratified fluid has the form:

$$\mathcal{E}_{\rm P} = g\int \rho(\boldsymbol{r})z\;{\rm d}x\,{\rm d}y\,{\rm d}z. \tag{15a}$$

In the basic reference state, the potential energy reaches its minimum value, referred to as the basic potential energy, $\bar{\mathcal{E}}_{\rm P}$:

$$\bar{\mathcal{E}}_{\rm P} = g \int \rho_{\rm b}(z) z \, {\rm d}x \, {\rm d}y \, {\rm d}z. \qquad (15b)$$

Clearly, in the equilibrium, the fluid density $\rho_{\rm b}(z)$ decreases with the elevation: ${\rm d}\rho_{\rm b}/{\rm d}z < 0$. In a turbulent state the density deviates from its reference value: $\rho(\boldsymbol{r}, t) = \rho_{\rm b}(z) + \tilde{\rho}(\boldsymbol{r}, t)$, and the mean potential energy $\langle \mathcal{E}_{\rm P} \rangle$ exceeds $\bar{\mathcal{E}}_{\rm P}$:

$$\langle \mathcal{E}_{\rm P} \rangle = \bar{\mathcal{E}}_{\rm P} + \tilde{\mathcal{E}}_{\rm P}. \qquad (15c)$$

We compute the turbulent potential energy, $\tilde{\mathcal{E}}_{\rm P}$, for the case when the internal parameter ξ (temperature, etc) is co-moving with the fluid element as a Lagrangian marker. In the Lagrangian approach we can consider ρ as the Lagrangian marker and introduce a variable $z(\rho, t)$, which is understood as an elevation of the fluid element with the density ρ. Noticing that $z{\rm d}z = \frac{1}{2}{\rm d}z^2$, and integrating equation (15*a*) by parts with respect to z^2, we can present $\langle \mathcal{E}_{\rm P} \rangle$ in the Lagrangian approach as

$$\langle \mathcal{E}_{\rm P} \rangle = -\frac{g}{2} \int \langle [z(\rho, t)]^2 \rangle \, {\rm d}x \, {\rm d}y \, {\rm d}\rho. \qquad (16)$$

As a result of turbulent motion, the elevation $z(\rho, t)$ at given ρ fluctuates and can be decomposed into the mean and fluctuating parts:

$$z(\rho, t) = z_{\rm b}(\rho) + \tilde{z}(\rho, t), \qquad \langle \tilde{z}(\rho, t) \rangle = 0. \qquad (17)$$

The substitution of $z^2(\rho, t) = z_{\rm b}^2 + 2 z_{\rm b} \tilde{z} + \tilde{z}^{\,2}$ in equation (16) leads to three contributions to the potential energy. The first one (originating from $z_{\rm b}^2$) describes the basic potential energy, equation (15*b*). The second contribution, which is linear in $\tilde{z}$, disappears because $\langle \tilde{z} \rangle = 0$. The last one describes the turbulent potential energy:

$$\tilde{\mathcal{E}}_{\rm P} = -\frac{g}{2} \int \langle [\tilde{z}(\rho, t)]^2 \rangle \, {\rm d}x \, {\rm d}y \, {\rm d}\rho. \qquad (18)$$

Relating the density fluctuations, $\tilde{\rho}$, around the basic reference state density profile $\rho_{\rm b}(z)$ to $\tilde{z}$

$$\delta\rho \equiv \frac{{\rm d}\rho_{\rm b}(z)}{{\rm d}z} \, \delta z, \qquad (19)$$

and returning to the Eulerian description in equation (18), one has

$$\tilde{\mathcal{E}}_{\rm P} = -\frac{g}{2} \int \left\langle \tilde{\rho}^{\,2} \right\rangle \left[\frac{{\rm d}\,\rho_{\rm b}(z)}{{\rm d}z} \right]^{-1} {\rm d}x \, {\rm d}y \, {\rm d}z. \qquad (20)$$

Here, we used the transformation formula, similar to equation (19): ${\rm d}\rho = [{\rm d}\rho_{\rm d}(z)/{\rm d}\,z]\,{\rm d}z$. Equation (20) allows one to introduce a local density of turbulent potential energy per unit mass,

$$E_{\rm P} = -\frac{g}{2} \left[\frac{{\rm d}\,\rho_{\rm b}(z)}{{\rm d}z} \right]^{-1} \left\langle \tilde{\rho}^{\,2} \right\rangle \Big/ \rho_{\rm b}, \qquad (21a)$$

such that

$$\tilde{\mathcal{E}}_{\rm P} = \int \rho_{\rm b} E_{\rm P} \, {\rm d}x \, {\rm d}y \, {\rm d}z. \qquad (21b)$$

For the particular case of temperature stratification

$$\frac{{\rm d}\,\rho_{\rm b}(z)}{{\rm d}z} = \rho_{\rm b} \tilde{\beta} \, \frac{{\rm d}\Theta}{{\rm d}z}, \qquad \tilde{\rho} = \rho_{\rm b} \tilde{\beta} \theta, \qquad \beta = g\tilde{\beta}\,, \qquad (22)$$

where $\Theta = \langle \Theta_{\rm d} \rangle$ is the mean potential temperature, $\theta = \Theta_{\rm d} - \Theta$ is the fluctuation of potential temperature, and the turbulent potential energy per unit mass for the stratified turbulent boundary layers is

$$E_{\rm P} = \beta E_\theta / S_\Theta, \qquad (23)$$

where $E_\theta = \left\langle \theta^2 \right\rangle / 2$ and $S_\Theta = {\rm d}\Theta/{\rm d}z$.

Clearly, a consistent statistical description of turbulent flows must conserve (in the dissipationless limit) the total mechanical energy, which for stratified flows consists of kinetic energy of the mean flow, kinetic energy of turbulent velocity fluctuations, $\boldsymbol{u}$, and turbulent potential energy, $E_{\rm P}$. To respect this conservation law, one has to discuss explicitly the balance equation for the turbulent kinetic energy that includes components of the Reynolds stress tensor $\langle u_i u_j \rangle$, and the balance equation for the potential energy, which is proportional to $\langle \theta^2 \rangle$. Unavoidably, the turbulent heat flux vector $\langle \boldsymbol{u}\theta \rangle$ is involved in the game and requires a separate balance equation for itself. Therefore, a consistent statistical description of the stratified turbulent flows demands an explicit consideration of the whole set of second-order one-point, simultaneous (cross)-correlation functions: $\langle u_i u_j \rangle$, $\langle \boldsymbol{u}\theta \rangle$ and $\langle \theta^2 \rangle$. Our version of such a consistent treatment is presented elsewhere in this issue.

4. Summary

We have presented here a generalization of the Oberbeck–Boussinesq approximation for temperature stratified flows. We have shown via a detailed discussion that this approach is acceptable for fluids, where the local thermodynamic equilibrium is established, i.e. the equation of state is valid. Thus the applicability of the approximation is much wider than for just ideal gases or liquids. For example, it is well established for humid air accounting for which is important for realistic meteorological prognoses, for salt water, hence oceanography, or even for fluids near the critical point met in engineering applications. The paper also shows that the proposed generalization respects the conservation of total mechanical energy. Obeying the conservation laws is of principal importance in a construction of consistent models, and a violation of this rule may lead to unphysical predictions. For stratified turbulent boundary layers, this requirement lays in an explicit consideration of not only the mean profiles, but also of all relevant second-order, one-point, simultaneous correlation functions of all fluctuating fields.

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