Reply to "Comment on 'Dynamics of the density of quantized vortex lines in superfluid turbulence' "

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This is a Reply to Nemirovskii's Comment [Phys. Rev. B **94**, 146501 (2016)] on Khomenko *et al.* [Phys. Rev. B **91**, 180504 (2015)] in which a new form of the production term in Vinen's equation for the evolution of the vortex-line density $\mathcal{L}$ in the thermal counterflow of superfluid ^{4}He in a channel was suggested. To further substantiate the suggested form which was questioned in the Comment, we present a physical explanation for the improvement of the closure suggested in Khomenko *et al.* [Phys. Rev. B **91**, 180504 (2015)] in comparison to the form proposed by Vinen. We also discuss the closure for the flux term, which agrees well with the numerical results without any fitting parameters.

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A complete dynamical theory of quantized vortex tangles continues to be a challenge, requiring careful attention to numerous details [1–5]. The Comment by S. Nemirovskii [hereafter referred to as (SN)] requires an additional clarification of the approximations and assumptions made in our approach [2] to the dynamics of the vortex-line density $\mathcal{L}(\boldsymbol{r},t)$ in superfluid turbulence.

The vorticity in superfluid ^{4}He is quantized: It is constrained to vortex-line singularities of fixed circulation $\kappa = h/M$, where h is Planck's constant and M is the mass of the ^{4}He atom. The smallness of the vortex-line core radius $a_0 \simeq 10^{-8}$ cm allows us to consider them as directional geometrical lines $\boldsymbol{s}(\xi,t)$, traditionally parametrized [3] by an arclength ξ.

A good *phenomenological* starting point in the analysis of the vortex-line dynamics is the Vinen equation [6] for the evolution of $\mathcal{L}(t)$ in the space homogeneous (thermally driven) superfluid ^{4}He flow with a counterflow velocity $V_{\rm ns}$,

$$\frac{d\mathcal{L}(t)}{dt} = \mathcal{P}(t) - \mathcal{D}(t). \tag{1}$$

Here the production and the decay terms $\mathcal{P}(t)$ and $\mathcal{D}(t)$ are expressed in terms of $\mathcal{L}(t)$ and $V_{\rm ns}$ only.

In Ref. [2] we reconsidered the evolution equation for $\partial\mathcal{L}(\boldsymbol{r},t)/\partial t$ starting from the *microscopic* Schwarz equation [3,4] for the length of the vortex-line segment $\delta\xi$. This equation contains two temperature-dependent dimensionless mutual friction parameters α and α',

$$\frac{1}{\delta\xi}\frac{d\,\delta\xi}{dt} = \alpha V_{\rm ns}(\boldsymbol{s},t)\cdot(\boldsymbol{s}'\times\boldsymbol{s}'') + \boldsymbol{s}'\cdot\boldsymbol{V}^{s\,\prime}_{\rm nl} - \alpha'\boldsymbol{s}''\cdot\boldsymbol{V}_{\rm ns}. \tag{2}$$

We agree with SN that this equation includes more terms than Eq. (4a) of Ref. [2]. For that reason we used in Eq. (4a) an "≈" sign instead of an "=" to indicate that only the most dominant contributions were retained for the case of counterflow turbulence in a channel. The relative importance of the different terms of (2) is shown in Fig. 1 using the following normalization:

$$y^\dagger = y/H, \quad \mathcal{P}^\dagger = \kappa^3\mathcal{P}/\langle V_{\rm ns}^2\rangle^2, \quad \mathcal{D}^\dagger = \kappa^2\mathcal{J}/\langle V_{\rm ns}^2\rangle^{3/2}. \tag{3}$$

We agree with SN that any decomposition of superfluid velocities may be considered arbitrary. We decided to follow Schwarz [4] and to assign the contribution proportional to $|s''|^2$ to the decay term. There are two reasons for that. First, this term should remain active after switching off the counterflow and thus should not vanish when $V_{\rm ns} = 0$. Second, it should be positive definite. All the other contributions were included in the production term. Having clarified the form of Eq. (2) in Ref. [2], we get straightforwardly to Eq. (6a) in Ref. [2] for the production term,

$$\mathcal{P}(y,t) = \alpha\mathcal{L}(y)\langle\boldsymbol{V}_{\rm ns}\cdot(\boldsymbol{s}'\times\boldsymbol{s}'')\rangle_{x,z}. \tag{4a}$$

Here $\langle\cdots\rangle_{x,z}$ denotes a y-dependent average over the vortex tangle residing in thin slices of width δy parallel to the channel wall.

Equation (4a) involves a smooth macroscopic field $\boldsymbol{V}_{\rm ns}$ that changes slowly along the vortex line and a factor $(\boldsymbol{s}'\times\boldsymbol{s}'')$ that is defined by a local structure of the tangle and changes fast along the vortex line. That allows us to take the slow $[(y,t)$-dependent$]$ factor $\boldsymbol{V}_{\rm ns}$ out of the average and account for the scalar product by retaining only a streamwise projection of $(\boldsymbol{s}'\times\boldsymbol{s}'')$,

$$\begin{aligned}\mathcal{P}(y,t) &\approx \alpha\mathcal{L}(y)\boldsymbol{V}_{\rm ns}(y)\cdot\langle\boldsymbol{s}'\times\boldsymbol{s}''\rangle_{x,z}\\ &= \alpha\mathcal{L}(y)V_{\rm ns}(y)\langle(\boldsymbol{s}'\times\boldsymbol{s}'')_x\rangle_{x,z}.\end{aligned} \tag{4b}$$

We see that rigorous analysis of the production term (4b) in Eq. (1) for $d\mathcal{L}(t)/dt$ requires an equation for the streamwise projection $(\boldsymbol{s}'\times\boldsymbol{s}'')_x$. One can find it analyzing an equation for $d\langle(\boldsymbol{s}'\times\boldsymbol{s}'')_x\rangle_{x,z}/dt$, which can be derived from Eq. (2). Unfortunately, the equation for $d\langle(\boldsymbol{s}'\times\boldsymbol{s}'')_x\rangle_{x,z}/dt$ involves even more complicated statistical characteristics of the random vortex tangle. A way to close such an infinite *unclosed* chain of coupled equations is to find a proper *closure* approximation (hereafter referred to for shortness as *closure*) which expresses $(\boldsymbol{s}'\times\boldsymbol{s}'')_x$ in terms of macroscopic objects $V_{\rm ns}$ and $\mathcal{L}$.

A possible closure for the streamwise projection $(\boldsymbol{s}'\times\boldsymbol{s}'')_x$ follows from the observation that its value is equal to that of the local curvature: $|\boldsymbol{s}'\times\boldsymbol{s}''| = |\boldsymbol{s}''|$. Assuming for a moment that on average,

$$\langle(\boldsymbol{s}'\times\boldsymbol{s}'')_x\rangle_{x,z} \simeq \langle|\boldsymbol{s}''|\rangle_{x,z}, \tag{5}$$

and accepting the common and reasonably well-justified approximation that $\langle|s''|\rangle_{x,z} \propto \sqrt{\mathcal{L}}$ (see the solid and dashed red lines in Fig. 2), we end up with Vinen's form of the

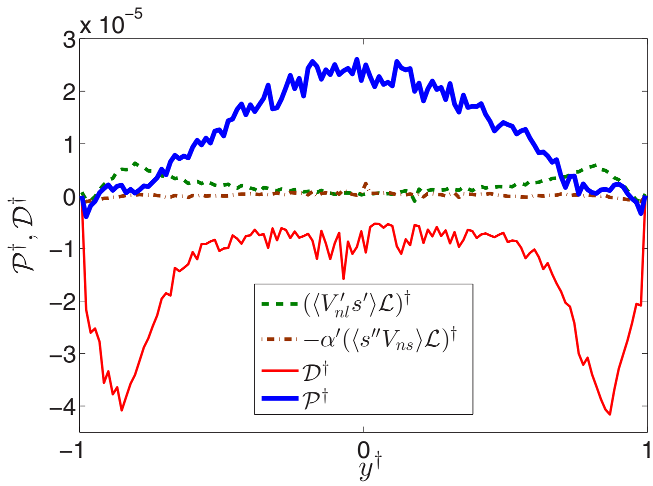

FIG. 1. Relative contributions of the neglected terms in the microscopic vortex dynamics (2) to $\partial\mathcal{L}(\boldsymbol{r},t)/\partial t$ are shown by dashed and dot-dashed lines. The accounted for contributions to $\mathcal{P}$ and $\mathcal{D}$ are shown by the thick and thin solid lines, respectively. All quantities are normalized according to Eq. (3). (Parabolic profile $T = 1.6$ K.)

production term,

$$\mathcal{P}_1 \sim \alpha V_{\rm ns}\mathcal{L}^{3/2}. \tag{6a}$$

Note however that the assumption (5) is not fulfilled even on a qualitative level. On one hand, from the theoretical viewpoint, Eq. (5) is in contradiction with simple symmetry considerations: In the absence of the counterflow there is no preferred direction in the problem (far away from the wall), and one expects $\langle(\boldsymbol{s}'\times\boldsymbol{s}'')_x\rangle_{x,z} = 0$, whereas $\langle|s''|\rangle_{x,z}$ has a well-defined value. On the other hand, comparing in Fig. 2 the calculated wall-normal profiles of $\langle(\boldsymbol{s}'\times\boldsymbol{s}'')_x\rangle_{x,z}$ (blue line) and $\langle|\boldsymbol{s}'\times\boldsymbol{s}''|\rangle_{x,z}$ (red line), we find that these profiles demonstrate completely different behaviors.

We therefore reach an important conclusion, formulated in Ref. [2], that the traditional form $\mathcal{P}_1$ of the production term, given by Eq. (6a), contradicts both the results of the numerical simulations in the channel and the symmetry arguments.

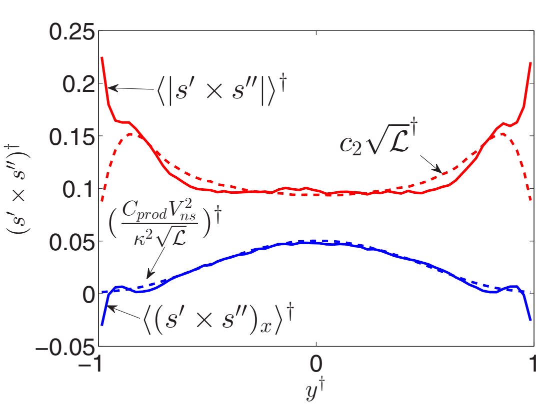

FIG. 2. Comparison of the streamline projection of $\langle\boldsymbol{s}'\times\boldsymbol{s}''\rangle$ (blue line) with its modulus (red line) for the parabolic profile. The blue and red dashed lines denote properly normalized profiles of $V_{\rm ns}^2/\sqrt{\mathcal{L}}$ and $\sqrt{\mathcal{L}}$. All quantities are normalized by $\sqrt{\langle V_{\rm ns}^2\rangle}/\kappa$.

To find a better closure approximation for the production term, let us replace $\boldsymbol{V}_{\rm ns}\to-\boldsymbol{V}_{\rm ns}$. It is natural to expect that $\langle(\boldsymbol{s}'\times\boldsymbol{s}'')_x\rangle_{x,z}\to\langle(\boldsymbol{s}'\times\boldsymbol{s}'')_x\rangle_{x,z}$, i.e., $\langle(\boldsymbol{s}'\times\boldsymbol{s}'')_x\rangle_{x,z}$ is an even function of $\boldsymbol{V}_{\rm ns}$ (equal to zero for $\boldsymbol{V}_{\rm ns}=0$). Assuming analyticity, we conclude that for small $V_{\rm ns}$ the function $\langle(\boldsymbol{s}'\times\boldsymbol{s}'')_x\rangle_{x,z}\propto V_{\rm ns}^2$. If so, simple dimensional reasoning gives

$$\langle(\boldsymbol{s}'\times\boldsymbol{s}'')_x\rangle_{x,z}\sim V_{\rm ns}^2/(\kappa^2\sqrt{\mathcal{L}}). \tag{6b}$$

We admit that these arguments are not rigorous. Nevertheless their conclusion (6b) is well supported by the numerical simulation: See the solid and dashed blue lines in Fig. 2. With the closure (6b), Eq. (4b) results in the form

$$\mathcal{P}\quad\Rightarrow\quad\mathcal{P}_3=\alpha C_{\rm prod}\sqrt{\mathcal{L}}V_{\rm ns}^3/\kappa^2, \tag{6c}$$

which was suggested in Ref. [2].

Another comment of SN pertains to the decay term, which in his words "can be extracted from dimensional analysis, which gives

$$\mathcal{D}=C\kappa\mathcal{L}^2." \tag{7a}$$

In fact this is *not* the case. As was explained after Eq. (1) of Ref. [2], this form was obtained by Vinen under the assumption that $\mathcal{D}$ is independent of $V_{\rm ns}$. The validity of this assumption is not obvious; the fact that this form is applicable even for $V_{\rm ns}\neq0$ (as demonstrated, e.g., in Fig. 2(b) of Ref. [2]) is a nontrivial statement. To clarify this point further, consider an analytical expression for $\mathcal{D}$ that follows from Eq. (2) in the local induction approximation [2,4]:

$$\mathcal{D}(y,t)=\frac{\alpha\kappa\,\ln(R/a_0)}{4\pi}\langle|s''|^2\rangle_{x,z}, \tag{7b}$$

where R is the mean radius of curvature.

This object is again an even function of $V_{\rm ns}$, but, in contrast to $\langle(\boldsymbol{s}'\times\boldsymbol{s}'')_x\rangle_{x,z}$ it has a nonzero value for $V_{\rm ns}=0$. Our support for the form (7a) of the decay term simply means that for the parameters of the numerical simulation in Ref. [2] the contribution proportional to $V_{\rm ns}^2$ is small with respect to the zero-order term (7a).

In his Comment SN also expressed doubts about the flux term. In particular, citing Ref. [1], "One more serious objection concerns the choice of flux in the form Eq. (10) of the discussed paper. It is not motivated and looks strange." To clarify the issue let us take one step backward and recall that we possess a microscopic equation that describes the dynamics of the vortex tangle which as stated is not very useful for the coarse-grained macroscopical description. The goal is therefore to obtain a description that employs only a few macroscopic fields: the vortex-line density, the superfluid, and normal fluid velocity. Obviously, this goal embodies a strong assumption; there is no guarantee that this is possible in every particular situation.

Consider a form for the flux term of the form of Eq. (9) below. First, we stress that this form agrees perfectly with the numerical results without any fitting parameters, see Fig. 3. Second, we can explain the origin of Eq. (9) by considering

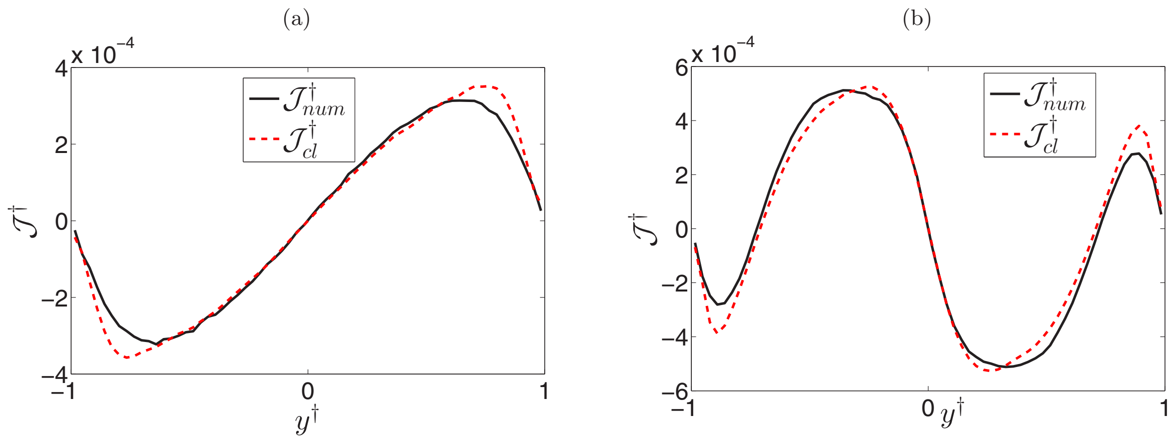

FIG. 3. Comparison of numerically measured flux and its modeling. Panel a: parabolic profile. Panel b: nonparabolic profile. The solid lines denote numerical results, and the dashed lines correspond to the closure (9). All quantities are normalized according to Eq. (3).

Eq. (6c) in our Ref. [2] for the flux (in the wall-normal direction $\hat{\boldsymbol{y}}$),

$$\mathcal{J}_y(y) = \mathcal{L}(y)\langle V_y^{\text{drift}}\rangle_{x,z}. \tag{8a}$$

Here the drift velocity is given by

$$\boldsymbol{V}^{\text{drift}} = \boldsymbol{V}^s + (\alpha - \alpha'\boldsymbol{s}') \times \boldsymbol{s}' \times \boldsymbol{V}_{\text{ns}}. \tag{8b}$$

The main contribution to the y component of the drift velocity is the second term, caused by mutual friction, as is clearly seen in Fig. 4. In its turn, the main contribution to the friction term comes from the part proportional to α. All this allows us to approximate the flux term as written in Eq. (6c) of Ref. [2],

$$\mathcal{J}_y(y) \approx \alpha\mathcal{L}(y)\langle V_{\text{ns},x}s'_z\rangle_{x,z} \approx \alpha V_{\text{ns}}\mathcal{L}(y)\langle s'_z\rangle_{x,z}. \tag{8c}$$

In the last equation we took the slowly varying function V_{ns} out of the average.

Next, we note that the only source of vorticity in superfluids is the quantized vortex lines. Therefore, $\kappa\mathcal{L}(y)\langle s'_z\rangle_{x,z}$ is a $\hat{z}$ component of vorticity, calculated microscopically. On the other hand, on the macroscopic level, the vorticity is defined by $\nabla \times \boldsymbol{V}_s$. By equating the z components of these expressions, i.e.,

$$\kappa\mathcal{L}(y)\langle s'_z\rangle_{x,z} = \frac{dV_s}{dy}, \tag{8d}$$

we end up with an expression for the flux without any fitting parameters,

$$\mathcal{J}_y(y,t) \approx \frac{\alpha}{\kappa}V_{\text{ns}}\frac{dV_s}{dy}. \tag{9}$$

Finally, in the Introduction of Ref. [1] SN listed some steps in the analysis that he deems "questionable." These require further clarification.

(1) SN noticed that "The authors of Ref. [2] have stated in the abstract of their paper as "To overcome this difficulty we announce here an approach that employs an inhomogeneous channel flow which is excellently suitable to distinguish the implications of the various possible forms of the desired equation." SN refers to the difficulty of inferring dynamical equations from stationary flows. Of course, in general, if one knows nothing about the system, it is difficult to say much about the temporal dependence studying only stationary states. But in our case we do know the microscopic equations that

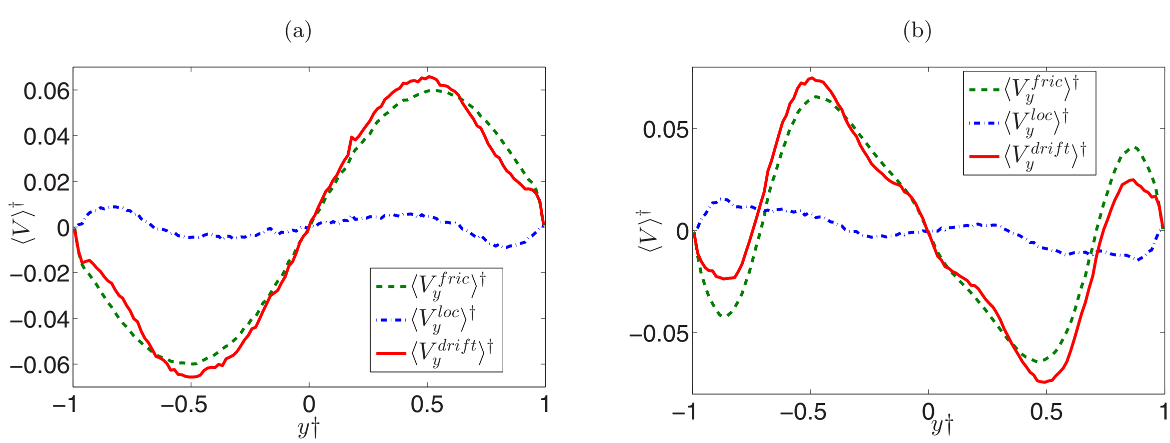

FIG. 4. Comparison of the components of the drift velocity. Panel a: parabolic profile. Panel b: nonparabolic profile. The velocities are normalized by $\sqrt{\langle V_{\text{ns}}^2\rangle}$.

describe the dynamics of the vortex tangle. The whole problem discussed in Ref. [2] is how to estimate these expressions in terms of macroscopic variables.

(2) In his Comment [1] SN discussed the limitations required to present an equation for $\partial\mathcal{L}(\boldsymbol{r},t)$ in the closed form

$$\frac{\partial\mathcal{L}(\boldsymbol{r},t)}{\partial t}=\mathcal{F}(\mathcal{L},V_{\rm ns}). \tag{10}$$

Clearly, in order to describe some averaged quantities defined by the configurations of the vortex tangle in terms of $V_{\rm ns}$ and $\mathcal{L}$, the statistics of the vortex tangle should be in a quasiequilibrium. Whether the same equation also describes the equilibration process is an open question that requires further study. In particular, in Ref. [2] we demonstrated that under the stated conditions the suggested closure form (10) exists and agrees very well with the numerics.

(3) SN makes the statement that the relaxation time of $\mathcal{L}(t)$ is much larger than that of $\boldsymbol{s}''$; the latter has enough time to adjust to the change in $\mathcal{L}(t)$, i.e., $|s''|\propto\sqrt{\mathcal{L}}$. Then, the self-preservation assumption is valid, and therefore the production term has a classical form $\mathcal{P}\propto|V_{\rm ns}|\,\mathcal{L}^{3/2}$.

We definitely disagree with this statement. The fact that s'' changes faster than $\mathcal{L}$ does not imply that the production term has the classical form chosen by Vinen. This was explicitly shown in our Ref. [2]. The deep reason is that $\mathcal{P}$ is not proportional to $\langle|s''|\rangle$. In reality $\mathcal{P}\propto\langle s'\times s''\rangle$. This term has completely different properties from $\langle|s''|\rangle$ as we explained above.

To summarize, we have shown that reasonable closure estimates of the microscopic terms appearing in the equations of motion of the density of vortex lines can be achieved. The resulting macroscopic equations were shown to be in satisfactory agreement with detailed numerical simulations. We do expect that additional order parameters may be required to describe more complex superfluid flows, but this is a challenge for the future.

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