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Kolmogorov Spectra of Turbulence

Wave Turbulence

Chapter 2

STATISTICAL DESCRIPTION OF WEAK WAVE TURBULENCE

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Statistical description of weak wave turbulence

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Main Symbols

$a(\boldsymbol{k},t)$, a_k, $b(\boldsymbol{k},t)$, b_k, $c(\boldsymbol{k},t)$, c_k	wave amplitudes
d	dimensionality of the $\boldsymbol{k}$-space
E	energy
$\varepsilon(\boldsymbol{k})$	energy density in the $\boldsymbol{k}$-space
$E(k)$	energy density in the k-space
g	gravity acceleration
g_m	magnetic-to-mechanical ratio
$\boldsymbol{H}$, H	magnetic field
$\mathcal{H}$	Hamiltonian
$\hbar$	Planck constant
$I_k\{n(\boldsymbol{k}',t)\}$, $I(\boldsymbol{k})$, I_k	collision integral
$\boldsymbol{k}$	wave vector
k	wave number
m	scaling index of interaction coefficient
$n(\boldsymbol{k},t)$, n_k	wave density in the $\boldsymbol{k}$-space
N	total number of waves
$\boldsymbol{p}$	energy flux in the $\boldsymbol{k}$-space
$P(k)$, P_k, P	energy flux in the k-space
$\mathit{\Pi}$	total momentum of waves
$\boldsymbol{R}$	momentum flux in the $\boldsymbol{k}$-space
S	entropy
$T(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)=T_{1234}$	coefficient of four-wave interaction
$V(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)=V_{123}$	coefficient of three-wave interaction
α	index of wave frequency
$\mathit{\Gamma}(k)$	external growth-rate (or decrement)
$\delta(x)$	Dirac's delta-function
$\omega(\boldsymbol{k})$, ω_k	wave frequency
ξ	dimensionless variable
$\propto$	proportional
$\approx$	approximately equal
$\simeq$	of the same order

2 Statistical Description of Weak Wave Turbulence

> It is a thing which you can easily explain twice before anybody knows what you are talking about
>
> *A.MILNE The House at Pooh Corner*

In this chapter we shall go over from the dynamic description of a wave systems to the statistical one. This will be done in terms of pair correlators of a wave field. They represent the occupation numbers (density) of waves in k-space. In Sect. 2.1 we shall obtain the kinetic equation for the occupation numbers of waves as the main mathematical objects dealt with in the first volume of this book. In Sect. 2.2 we shall study the general properties of these equations and derive the stationary equilibrium solutions. Section 2.2.3 deals with general conditions which are necessary for the existence Kolmogorov spectra.

2.1 Kinetic Wave Equation

2.1.1 Equations of Motion

In the preceding chapter we have applied the Hamiltonian method to the motion in continuous media like fluids, plasmas, solids and the planetary atmosphere, have considered the structure of the Hamiltonian for weak nonlinearities and given its coefficients for some physical examples. The canonical equations of motion in terms of complex wave amplitudes $c(k, t)$, i.e.:

$$i\frac{\partial c(\boldsymbol{k},t)}{\partial t} - \omega_k c(\boldsymbol{k},t) = \frac{\delta H_{int}}{\delta c^*(\boldsymbol{k},t)} \tag{2.1.1}$$

is the main result. After application of the procedure outlined in Sect. 1.1.3, the interaction Hamiltonian $\mathcal{H}_{int}$ includes only the terms describing only real processes for which the energy and momentum conservation laws are valid. In a medium supports propagating waves for which the three-wave decay processes are allowed, we have:

$$\mathcal{H}_{\text{int}} = \mathcal{H}_3 = \frac{1}{2}\int \big[V_{123}c_1^*c_2c_3 + \text{c.c.}\big]\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \ . \tag{2.1.2}$$

In this case

$$\begin{aligned} i\frac{\partial c(\boldsymbol{k},t)}{\partial t} - \omega_k c(\boldsymbol{k},t) = \int \Big[\frac{1}{2}V_{k12}c_1c_2\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \\ + V_{1k2}^*c_1c_2^*\delta(\boldsymbol{k}_1 - \boldsymbol{k} - \boldsymbol{k}_2)\Big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 \ . \end{aligned} \tag{2.1.3}$$

The first term of the right-hand side of this equation describes the $k \rightarrow 1+2$ decay processes with conservation laws

$$\omega_k = \omega_1 + \omega_2\ , \quad \boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2\ , \tag{2.1.4a}$$

$[\omega_j \equiv \omega(\boldsymbol{k}_j)]$ and the reverse $1+2 \rightarrow k$ processes. The second term in (2.1.3) describes the confluence $k+2 \rightarrow 1$ processes with

$$\omega_k + \omega_2 = \omega_1\ , \quad k + k_2 = k_1\ . \tag{2.1.4b}$$

If the three-wave processes (2.1.4) are forbidden, the Hamiltonian (2.1.2) vanishes under the nonlinear canonical transformation (1.1.28). In that case, $\mathcal{H}_{\text{int}}$ includes only the four-wave scattering processes of type $2 \rightarrow 2$:

$$\mathcal{H}_{\text{int}} = \mathcal{H}_4 = \frac{1}{4}\int T_{12,34}c_1^*c_2^*c_3c_4\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 . \tag{2.1.5}$$

and the canonical equations of motion have the form:

$$\begin{aligned} i\frac{\partial c(\boldsymbol{k},t)}{\partial t} &- \omega_k c(\boldsymbol{k},t) \\ &= \frac{1}{2}\int T_{k123}c_1^*c_2c_3\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3))\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \ . \end{aligned} \tag{2.1.6}$$

The right-hand side of this equation describes the dynamics of scattering processes

$$\omega_k + \omega_1 = \omega_2 + \omega_3, \quad \boldsymbol{k} + \boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3 \ .$$

2.1.2 Transition to the Statistical Description

The dynamic equations (2.1.3, 6) describe the time evolution of wave amplitudes $|c(\boldsymbol{k},t)|$ and of their phases $\varphi(\boldsymbol{k},t)$:

$$c(\boldsymbol{k},t) = |c(\boldsymbol{k},t)|\exp[i\varphi(\boldsymbol{k},t)] \ . \tag{$*$}$$

At weak nonlinearity and with a large number of excited waves, such description is generally highly redundant: it includes the slow evolution of amplitudes (constant in the linear approximation) and the fast but uninteresting phase dynamics

$$\phi(\boldsymbol{k},t) \approx \omega(\boldsymbol{k})t\,,$$

which leave the amplitude evolution virtually unaffected. This redundancy is eliminated by the transition from the dynamic description of a wave system in terms of $|c(\boldsymbol{k},t)|$ and $\varphi(\boldsymbol{k},t)$ to the statistical one, in terms of the correlation functions of the field $c(\boldsymbol{k},t)$. Consistent statistical averaging consist of two steps, the construction of an ensemble of dynamic equation solutions and specification of the rule for using them to compute the averages. Then one should consider only those averaging conditions (i.e., correlation properties of the field) which are compatible with the dynamic equations and are independent of time. In other words, it is necessary to introduce an invariant measure on the manifold of wave field configurations. Using the smallness of the nonlinearity we can go over to a statistical description in terms of perturbation theory in $\mathcal{H}_{\text{int}}$.

In the zeroth approximation in $\mathcal{H}_{\text{int}}$ we have a free field with the trivial evolution that the amplitudes $c(\boldsymbol{k},t) = c(\boldsymbol{k})$ are constant and the phases are given by Eq. (*). For the "slow" wave phase

$$\psi(\boldsymbol{k},t) = \varphi(\boldsymbol{k},t) - \omega(\boldsymbol{k})t$$

by virtue of (2.1.1) we get

$$\frac{\partial\psi(\boldsymbol{k},t)}{\partial t} = 0\,, \quad \text{at} \quad \mathcal{H}_{\text{int}} = 0\,,$$

i.e., the $\psi(\boldsymbol{k})$ phases are in a state of indifferent equilibrium. Consequently, any small random perturbation like medium inhomogeneities and the interaction with a thermostat drives them into a chaotic regime. Apart from external reasons there is also an "internal" reason of phase chaotization: the dispersion (i.e., the $\boldsymbol{k}$-dependence) of wave frequencies. Because of dispersion, even initially correlated harmonics with different $\boldsymbol{k}$'s undergo phase randomization as time progresses. Thus in describing a free wave field it would be natural to average by the ensemble of chaotic (random) phases, i.e., to use random-phase approximation. After such an averaging, only correlators that are independent of the wave phase will be nonzero. For example,

$$\begin{aligned}
&\langle c_{\boldsymbol{k}}\rangle = \langle |c_{\boldsymbol{k}}| \exp(i\varphi_{\boldsymbol{k}})\rangle = 0\,,\\
&\langle c_{\boldsymbol{k}}\, c_{\boldsymbol{k}'}\rangle = \langle |c_{\boldsymbol{k}}||c_{\boldsymbol{k}'}| \exp(i\varphi_{\boldsymbol{k}} + i\varphi_{\boldsymbol{k}'})\rangle = 0\,, \qquad (2.1.7a)\\
&\langle c_{\boldsymbol{k}} c^*_{\boldsymbol{k}'}\rangle = \langle |c_{\boldsymbol{k}}||c_{\boldsymbol{k}'}| \exp(i\varphi_{\boldsymbol{k}} - i\varphi_{\boldsymbol{k}'})\rangle = n(\boldsymbol{k})\delta(\boldsymbol{k} - \boldsymbol{k}')\ .
\end{aligned}$$

Evidently, all odd-order correlators vanish. Only such correlators be nonzero if the number of factors $c(\boldsymbol{k},t)$ in them coincides with the number of $c^*(\boldsymbol{k},t)$, and if the wave vector at one of the factors $c(\boldsymbol{k},t)$ coincides with that for (any) other $c^*(\boldsymbol{k},t)$. Only then will the averaged expression be phase-independent. Later on we shall need the fourth- and six-order correlators

$$\begin{aligned}&\langle c_1^* c_2^* c_3 c_4\rangle\\&= n(\boldsymbol{k}_1)n(\boldsymbol{k}_2)[\delta(\boldsymbol{k}_1-\boldsymbol{k}_3)\delta(\boldsymbol{k}_2-\boldsymbol{k}_4)+\delta(\boldsymbol{k}_1-\boldsymbol{k}_4)\delta(\boldsymbol{k}_2-\boldsymbol{k}_3)]\,,\end{aligned} \tag{2.1.7b}$$

$$\begin{aligned}&\langle c_1^* c_2^* c_3^* c_4 c_5 c_6\rangle\\&= n(\boldsymbol{k}_1)n(\boldsymbol{k}_2)n(\boldsymbol{k}_3)\big[\delta(\boldsymbol{k}_1-\boldsymbol{k}_4)[\delta(\boldsymbol{k}_2-\boldsymbol{k}_5)\delta(\boldsymbol{k}_3-\boldsymbol{k}_6)\\&+\delta(\boldsymbol{k}_2-\boldsymbol{k}_6)\delta(\boldsymbol{k}_3-\boldsymbol{k}_5)]+\delta(\boldsymbol{k}_1-\boldsymbol{k}_5)[\delta(\boldsymbol{k}_2-\boldsymbol{k}_4)\delta(\boldsymbol{k}_3-\boldsymbol{k}_6)\\&+\delta(\boldsymbol{k}_2-\boldsymbol{k}_6)\delta(\boldsymbol{k}_3-\boldsymbol{k}_4)]+\delta(\boldsymbol{k}_2-\boldsymbol{k}_4)[\delta(\boldsymbol{k}_3-\boldsymbol{k}_5)\delta(\boldsymbol{k}_2-\boldsymbol{k}_4)\\&+\delta(\boldsymbol{k}_2-\boldsymbol{k}_5)\delta(\boldsymbol{k}_3-\boldsymbol{k}_4)]\big]\ .\end{aligned} \tag{2.1.7c}$$

It is seen from these examples that the random fourth-order correlator decomposes to various products of pair correlators. This implies Gaussian statistics the free wave field. Of course, the above considerations are not a proof of this fact. With nmore rigor the problem of statistical properties of the free and weakly interacting fields will be discussed in the second volume of this book.

We consider only ergodic systems for which $\langle\,\rangle$ equally well to averaging over the ensemble or over the times that are much longer than the time of fast phase evolution and much smaller than the time of slow amplitude evolution. In (2.1.7), the wave field was suggested to be statistically homogeneous in space, therefore the correlator

$$\langle c(\boldsymbol{k},t)c^*(\boldsymbol{k}',t)\rangle \propto \delta(\boldsymbol{k}-\boldsymbol{k}')\ .$$

The classical pair correlator $n(\boldsymbol{k},t)$ is proportional to the quantum-mechanical occupation numbers $N(\boldsymbol{k},t)$:

$$n(\boldsymbol{k},t)=\hbar N(\boldsymbol{k},t), \tag{2.1.8}$$

i.e. to the "number of the quanta" of the respective Bose field. Therefore we shall call $n(\boldsymbol{k},t)$ the number of waves, in spite of the fact that $n(\boldsymbol{k},t)$ is a dimensional value. For weak nonlinearity we may use perturbation theory in $\mathcal{H}_{\text{int}}$ to obtain an equation closed with respect to $n(\boldsymbol{k},t)$.

2.1.3 The Three-Wave Kinetic Equation

To calculate $\partial n(\boldsymbol{k},t)/\partial t$, we shall multiply the equation (2.1.3) by $c^*(\boldsymbol{k},t)$, the complex conjugated equation by $c(\boldsymbol{k}',t)$, subtract the latter from the former and average. Setting $\boldsymbol{k}=\boldsymbol{k}'$, we obtain:

$$\begin{aligned}\frac{\partial n(\boldsymbol{k},t)}{\partial t}=\operatorname{Im}\int\big[&V_{k12}J_{k12}\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\\&-2V_{1k2}J_{1k2}\delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2)]\,d\boldsymbol{k}_1d\boldsymbol{k}_2\ .\end{aligned} \tag{2.1.9a}$$

Here

$$J_{123}\delta(\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)=\langle c_1^*c_2c_3\rangle \tag{2.1.9b}$$

is the triple correlation function. For a free field, $J_{123} = J^0_{1,23} = 0$. This implies that in first order perturbation theory in $\mathcal{H}_{\text{int}}$ we have

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = 0 \ .$$

In order to calculate $\partial n(\boldsymbol{k},t)/\partial t$ in second order perturbation theory in $\mathcal{H}_{\text{int}}$, one should know J_{123} in the first order. Using the definition (2.1.19b) and the equations of motion (2.1.3), we calculate $\partial J/\partial t$:

$$\begin{aligned}
&[i\frac{\partial}{\partial t} + (\omega_1 - \omega_2 - \omega_3)]J(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3; t) \\
&\qquad\qquad = \int \big[-\frac{1}{2}V^*_{145}J_{4523}\delta(\boldsymbol{k}_1 - \boldsymbol{k}_4 - \boldsymbol{k}_5) \\
&\qquad\qquad\quad +V^*_{425}J_{1534}\delta(\boldsymbol{k}_4 - \boldsymbol{k}_2 - \boldsymbol{k}_5) \\
&\qquad\qquad\quad +V_{435}J_{1524}\delta(\boldsymbol{k}_4 - \boldsymbol{k}_3 - \boldsymbol{k}_5)\big]\, d\boldsymbol{k}_4 d\boldsymbol{k}_5 \ .
\end{aligned} \tag{2.1.10a}$$

Here

$$J_{1234}\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4) = \langle c^*_1 c_2 c_3 c_4 \rangle \tag{2.1.10b}$$

is a quadruple correlation function. As we are interested in $J^{(1)}_{123}$ (in the first order), one should substitute into the right-hand side of (2.1.10a) J_{1234} in the zeroth order in $\mathcal{H}_{\text{int}}$, i.e., J^0_{1234} for the free field. In accordance with (2.1.8), this correlator is expressed via the pair correlators $n(\boldsymbol{k},t)$. As a result we have

$$\begin{aligned}
&\left[i\frac{\partial}{\partial t} + (\omega_1 - \omega_2 - \omega_3)\right]J_1(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3; t) \\
&\quad = V^*_{1,23}[n_1 n_3 + n_1 n_2 - n_2 n_3] = A_{123} \ .
\end{aligned} \tag{2.1.10c}$$

Remaining constantly within the second order of the perturbation theory, one should neglect the time dependence of $n(\boldsymbol{k},t)$ on the right-hand side of (2.1.10c), i.e., set A =const. Then this equation may be solved

$$\begin{aligned}
&J(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_1; t) = B\exp(-i\varDelta\omega t) + A_{123}/\varDelta\omega, \\
&\varDelta\omega = \omega_1 - \omega_2 - \omega_3 \ .
\end{aligned} \tag{2.1.10d}$$

By substituting the first term in (2.1.9a), we get at $t \neq 0$ an integral of a fast oscillating function, its contribution decreases with increase of t, and becomes nonessential at the times larger than $1/\omega(\boldsymbol{k})$. The second term from (2.1.10d) gives for J_1 an expression depending via $n(\boldsymbol{k}_j, t)$ parametrically on the slow time:

$$J_1(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3; t) = \frac{V^*_{123}(n_1 n_2 + n_1 n_3 - n_2 n_3)}{\omega_1 - \omega_2 - \omega_3 + i\delta} \ . \tag{2.1.11}$$

To the denominator we have added the term $i\delta$ to circumvent the pole. It may be obtained in a consistent procedure by considering the free wave field at $t \to \infty$, and adiabatically slowly including the interaction [for $t \gg 1/\omega(\boldsymbol{k})$]. The sign of δ can also be determined by accounting for the presence of small wave damping. Substituting (2.1.11) into (2.1.9a) and using $\mathrm{Im}\{\omega + i\delta) = -\pi\delta(\omega)$, we obtain the three-wave kinetic equation:

$$\begin{aligned} \frac{\partial n(\boldsymbol{k},t)}{\partial t} &= \pi \int \Big[|V_{k12}|^2 f_{k12} \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\delta(\omega_{\boldsymbol{k}}-\omega_1-\omega_2) \\ &+ 2|V_{1k2}|^2 f_{1k2}\delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2)\delta(\omega_1-\omega_{\boldsymbol{k}}-\omega_2)\Big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 \ , \end{aligned} \tag{2.1.12a}$$

$$f_{k12} = n_1 n_2 - n_{\boldsymbol{k}}(n_1+n_2), \quad n_j = n(\boldsymbol{k}_j, t). \tag{2.1.12b}$$

Most of the first volume is devoted to studying the solutions of this equation. Now we shall discuss at a qualitative level the conditions for application of it.

2.1.4 Applicability Criterion of the Three-wave Kinetic Equation (KE)

Let us consider a distribution $n(\boldsymbol{k},t)$ in the form of a broad packet with $\Delta k \simeq k$. The characteristic time $1/\gamma(\boldsymbol{k})$ of variations of the occupation numbers $n(\boldsymbol{k},t)$ may be estimated from the kinetic equation (2.1.12):

$$\begin{aligned} &\gamma(\boldsymbol{k}) \simeq \left|V_{k,kk}\right|^2 \frac{n_{\boldsymbol{k}} k^d}{\omega_{\boldsymbol{k}}} \simeq \left|V_{kkk}\right|^2 \frac{N}{\omega_{\boldsymbol{k}}} \ , \\ &N = \int n(\boldsymbol{k})\, d\boldsymbol{k} \ . \end{aligned} \tag{2.1.13}$$

Perturbation theory and random-phase approximation are applicable if this characteristic time is significantly larger than time required for phase chaotization estimated to equal roughly $1/\Delta\omega_{\boldsymbol{k}} \simeq 1/\omega_{\boldsymbol{k}}$. Here $\Delta\omega_{\boldsymbol{k}}$ is the difference between the packet wave frequencies. Comparison of these times yeilds the following applicability criterion for the kinetic equation:

$$\xi_1(\boldsymbol{k}) = \left|V_{kkk}\right|^2 N\omega_{\boldsymbol{k}} \ll 1 \ . \tag{2.1.14}$$

Let us derive the same criterion by calculating directly the ratio

$$\xi_2(\boldsymbol{k}) = \frac{\langle \mathcal{H}_{\mathrm{int}} \rangle}{\langle \mathcal{H}_2 \rangle} = \frac{\mathrm{Re} \int V_{123} J_{123} \delta(\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3}{\int \omega(\boldsymbol{k}) n(\boldsymbol{k})\, d\boldsymbol{k}} \ . \tag{2.1.15}$$

Substituting here (2.1.11) for J_1, we get

$$\xi_2(\boldsymbol{k}) = \frac{\int \left|V_{123}\right|^2 f_{123} (\omega_1-\omega_2-\omega_3)^{-1} \delta(\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3}{\int \omega(\boldsymbol{k}) n(\boldsymbol{k})\, d\boldsymbol{k}} \ . \tag{2.1.16}$$

Rough estimates of this equation for a wave packet with $\varDelta k \simeq k$ yeild $\xi_1 \simeq \xi_2$. Thus the criterion (2.1.14) ensures the smallness of the interaction Hamiltonian $\langle \mathcal{H}_{\mathrm{int}} \rangle$ as compared to the Hamiltonian of the free wave field $\langle \mathcal{H}_2 \rangle$. This is exactly same as the applicability condition of perturbation theory. For narrow wave distributions the applicability of the kinetic equation should be treated more accurately. For example, the interaction of three narrow packets should obey the equation (1.5.9), so the nonlinear interaction time is by vitue of (1.5.10):

$$t_{\mathrm{int}} \simeq \frac{1}{2\pi} |V_{123} c_{\max}| \ . \tag{2.1.17}$$

Let all the three packets in the initial state have approximately equal amplitudes $c_1 \simeq c_2 \simeq c_3 \simeq c_{\max}$ and widths $l_1 \simeq l_2 \simeq l_3 \simeq l$, and let them evolve in such a way that they collide at a certain moment, i.e., they arrive into the same spatial region. All three packets overlap each other during the time of collision

$$t_{\mathrm{col}} \simeq \frac{1}{v_{123}}, \quad v_{123} = \max\bigl[|\boldsymbol{v}_1 - \boldsymbol{v}_2|, |\boldsymbol{v}_1 - \boldsymbol{v}_3|, |\boldsymbol{v}_2 - \boldsymbol{v}_3|\bigr] \ . \tag{2.1.18}$$

If $t_{\mathrm{int}} \gg t_{\mathrm{col}}$, the amplitude and phase of each packet will change only slightly during one collision. It is clear that the interaction of an ensemble of such packets may be described statistically assuming their phases to be almost random, i.e., using the kinetic equation. The applicability criterion for the kinetic equation thus derived has a form

$$\xi_{123} = \frac{t_{\mathrm{col}}}{t_{\mathrm{int}}} \simeq (2\pi)^{3/2} \bigl| v_{123} c_{\max} \bigr| \frac{l}{V_{1,23}} \ll 1 \ . \tag{2.1.19}$$

Using (1.5.8), we calculate

$$\langle |c_j(\boldsymbol{r}, t)|^2 \rangle = \frac{1}{(2\pi)^3} \int n_j(\boldsymbol{\kappa}_j + \boldsymbol{k}_j) \, d\boldsymbol{\kappa}_j = \frac{N_j}{(2\pi)^3} \ . \tag{2.1.20}$$

Taking into account that the width of the packet in the coordinate representation l is related to $\varDelta k$ as follows: $l \simeq 1/\varDelta k$, we obtain

$$\xi_{123} \simeq \bigl| V_{123} \bigr| \frac{\sqrt{N}}{\varDelta k v_{123}} \ll 1 \ . \tag{2.1.21}$$

If we set here $\varDelta k \simeq k,\ v_{123} \simeq v_j \simeq \partial\omega(k_j)/\partial k_j,\ k v_j \simeq \omega_j$, the expression for ξ_{123} will coincide with (2.1.14) and (2.1.16). This is, of course, not accidental, but a consequence of the fact that all the three inferences are equivalent. Our reasoning in the coordinate representation was the most detailed one, therefore the applicability criterion (2.1.21) allows us to clarify more subtle things. In particular, it yeilds that for waves with the linear dispersion law $\omega(k) = ck$, the classical kinetic equation is inapplicable at any excitation level of the system. Indeed, at $\omega(k) \propto k$, it follows from the conservation laws (2.1.4) that $\boldsymbol{k}_1 \parallel \boldsymbol{k}_2 \parallel \boldsymbol{k}_3$. Then $\boldsymbol{v}_1 = \boldsymbol{v}_2 = \boldsymbol{v}_3 = c\boldsymbol{k}_j/k_j$ and, by (2.1.18), $v_{123} = 0$. In the coordinate language, the inapplicability of the KE in this case may be easily understood: all interacting packets travel in the same direction with identical velocities and, therefore, they can not diverge. As a result an arbitrary weak interaction will lead to substantial correlation phases of the wave packet.

As to be demonstrated later, there exists a difference between two- and three-dimensional cases in the applicability of the kinetic equation to acoustic turbulence (see Sect. 3.2.1). In the three-dimensional case KE formally undergoes a transition to the non-dispersive limit and it may be shown that at low excitation level KE is applicable [see also remark after (2.1.25) below].

Now let us consider a wave system with the power dispersion law

$$\omega(k) = ak^{\alpha}, \quad \alpha > 1\,,$$

and let $\boldsymbol{k}_1 = 2\boldsymbol{k},\ \boldsymbol{k}_2 = \boldsymbol{k} + \boldsymbol{\kappa},\ \boldsymbol{k}_3 = \boldsymbol{k} - \boldsymbol{\kappa}$. It follows from the conservation law (2.1.4) that

$$\begin{aligned} &\omega(2k) = 2\omega(k) + \hat{L}\kappa^2, \\ &\hat{L}\kappa^2 = \sum_{i,j} \kappa_i \kappa_j \frac{\partial^2 \omega}{\partial k_i \partial k_j} = \frac{v\kappa^2}{k} + \frac{\omega''(\kappa k)^2}{k^3}\,, \\ &v = \frac{\partial \omega(k)}{\partial k}, \quad \omega'' = \frac{\partial^2 \omega(k)}{\partial k^2} \end{aligned} \tag{2.1.22}$$

and whence

$$(\kappa/k)^2 \simeq \alpha - 1\ . \tag{2.1.23a}$$

It is easy to see that all differences in group velocities in (2.1.20b) are of the same order and

$$v_{123} \simeq v(\kappa/k). \tag{2.1.23b}$$

Thus the criterion for the applicability of kinetic equation has the form

$$\xi_{\boldsymbol{k}} \simeq \frac{|V_{123}|\sqrt{N}k}{v\kappa \varDelta k} \ll 1\ , \tag{2.1.24a}$$

or, after substitution of (2.1.23a) for κ,

$$\xi_{\boldsymbol{k}} \simeq \frac{|V_{123}|}{v\varDelta k}\sqrt{\frac{N}{\alpha - 1}} \ll 1\ . \tag{2.1.24b}$$

But for the dispersion law is $\omega(k) = c_s k(1+\mu k^2)$, criterion (2.1.24a) remains valid; but for κ we have, instead of (2.1.23a),

$$(\kappa/k)^2 \simeq \mu k^2\ .$$

The applicability criterion for the kinetic equation in this case is

$$\xi_{\boldsymbol{k}} \simeq \frac{|v_{123}\sqrt{N}}{c_s k \sqrt{\mu} \Delta k} \ll 1 \ . \tag{2.1.25}$$

As we shall show elsewhere, the strict criterion for weakly dispersive waves depends on dimensionality d of the media(see also Sect. 3.2 bellow). The criterion (2.1.25) holds only for $d = 2$. For the case $d = 3$, (2.1.25) should be replaced by the criterion $|V_{123}|\sqrt{N}/c_s k \ll ln^{-1}(\mu k^2)^{-1}$.

In addition to the motion of packets limiting the time of their interaction, one should also take into account the dispersion diffusion of packets. The time of such a diffusion process is

$$t_{\text{dif}} \simeq 1/\hat{L}(\Delta k)^2 \ , \tag{2.1.26}$$

where the $\hat{L}$ operator is given by the equation (2.1.22). When the diffusion is dominant (at $t_{\text{dif}} < t_{\text{col}}$), we obtain instead of the criterion (2.1.21) [and the equation (2.1.24-25) following from it], the inequality

$$\xi_{\boldsymbol{k}} \simeq t_{\text{int}}/t_{\text{dif}} \simeq \frac{|V_{1,23}|\sqrt{N}}{\hat{L}(\Delta k)^2} \ll 1 \ . \tag{2.1.27}$$

It should bre borne in mind that all our deductions about the applicability criterion of the kinetic equation are of qualitative character. More strictly this question can be discussed in terms of the Wyld's diagram technique.

2.1.5 The four-wave Kinetic Equation

In the non-decay case, resonant conditions (2.1.4) can not be satisfied so that the right-hand side of the kinetic equation (2.1.12) vanishes identically. Consequently, the four-wave processes should be taken into account. The respective kinetic equation is derived in the same way as the one for three-wave processes. Namely, using (2.1.6), we calculate $\partial n(\boldsymbol{k}, t)/\partial t$:

$$\frac{\partial n(\boldsymbol{k}, t)}{\partial t} = \text{Im} \int T_{k123} J_{k123} \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) \, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \ , \tag{2.1.28a}$$

where J_{k123} is the fourth-order correlation function (2.1.10b). In the zeroth order in interaction it is determined by the equation (2.1.8b). This gives

$$\frac{\partial n(\boldsymbol{k}, t)}{\partial t} = n(\boldsymbol{k}, t) \text{Im} \int T_{kk'kk'} n(\boldsymbol{k}') \, dk' \ . \tag{2.1.28b}$$

As $T_{1234} = T^*_{3412}$, the right-hand side becomes zero and $\partial n/\partial t = 0$ to first order in the interaction. Using the scheme employed in Sect. 2.1.3 fotr the derivation of $J(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$, we obtain a first-order addition to $J(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)$:

$$J_1(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4; t) = T^*_{1234} f_{k123} \big(\omega_1 + \omega_2 - \omega_3 - \omega_4 + i\delta\big)^{-1}, \tag{2.1.28c}$$

$$f_{k123} = n_2 n_3 (n_1 + n_{\boldsymbol{k}}) - n_1 n_{\boldsymbol{k}} (n_2 + n_3) \ . \tag{2.1.29a}$$

In deriving that expression, we have represented the sixth-order correlator for the noninteracting field via the product of pair correlators by (2.1.8c) and neglected the time derivative $\partial J_1(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4; t)/\partial t$ like in the three-wave case. Note that we neglected the nonlinear renormalization of the frequency $\mathrm{Re} \int T_{kk'kk'} n(\boldsymbol{k}')\, dk'$ since its contribution to $\partial n(\boldsymbol{k}, t)/\partial t$ is only to the next order. By substituting (2.1.28c) into (2.1.28a), we get the four-wave kinetic equation

$$\begin{aligned} \frac{\partial n(\boldsymbol{k}, t)}{\partial t} = \frac{\pi}{2} \int & \left| T_{k123} \right|^2 f_{k123} \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) \\ & \times \delta(\omega_{\boldsymbol{k}} + \omega_1 - \omega_2 - \omega_3)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \ . \end{aligned} \tag{2.1.29b}$$

Its applicability criterion may also be obtained in terms of interacting packets. In contrast to (2.1.20a), t_{int} for the four-wave processes has the form

$$\frac{1}{t_{\text{int}}} = (2\pi)^3 |T c_{\max}|^2 \simeq TN \ . \tag{2.1.30a}$$

The most rigirous applicability criterion of the four-wave KE will be obtained in the case of self-interaction within a single wave packet which is narrow in the k-space. We denote its center by k_0 and its width by Δk. In the r-representation this will be a smooth packet with an envelope of a width $l \simeq 1/\Delta k$. The interaction process must be restricted by the diffusion of the packet during the time t_{dif} (2.1.26). Thus, a necessary condition for the applicability of the kinetic equation (2.1.29) is

$$\xi_{\boldsymbol{k}} \simeq t_{\text{int}}/t_{\text{dif}} \simeq TN/L(\Delta k)^2 \ll 1 \, , \qquad L \simeq \omega'' \ . \tag{2.1.30b}$$

2.1.6 The quantum Kinetic Equation

The above derivation of the kinetic equation is valid in the classical limit, when $n(\boldsymbol{k}) \gg \hbar$, i.e., the quantum-mechanical occupation numbers (2.1.8) $N(k) \gg 1$. In deriving the quantum KE, it is convenient to start from the Hamiltonian operator for a wave system in the second quantization representation. One can go over from the Hamilton function $\mathcal{H}$ expressed in terms of complex canonical variables to the operator $\hat{\mathcal{H}}$ by substituting for $c_{\boldsymbol{k}}$, $c^*_{\boldsymbol{k}}$ the Bose operators of creation $\hat{c}^+$ and annihilation $\hat{c}$ of wave-field quanta, respectively (see, e.g.[2.2]):

$$c^*_{\boldsymbol{k}} \to \sqrt{n} \hat{c}^+_{\boldsymbol{k}}, \quad c_{\boldsymbol{k}} \to \sqrt{n} \hat{c}_{\boldsymbol{k}}. \tag{2.1.31}$$

The anticommutator of $\hat{c}, \hat{c}^+$ is known to be nonzero:

$$\left\{ \hat{c}(\boldsymbol{k}), \hat{c}^+(\boldsymbol{k}') \right\} = \hat{c}(\boldsymbol{k}) \hat{c}^+(\boldsymbol{k}') - \hat{c}^+(\boldsymbol{k}) \hat{c}(\boldsymbol{k}') = \delta(\boldsymbol{k} - \boldsymbol{k}') \ , \tag{2.1.32}$$

therefore we encounter in $\mathcal{H}_2$ a problem related to the order of the operators, which amounts to choosing a reference point for the energy. If we assume

$$\mathcal{H}_2 = \frac{1}{2}\int \hbar\omega(\boldsymbol{k})\big[\hat{c}^+(\boldsymbol{k})\hat{c}(\boldsymbol{k}) + \hat{c}(\boldsymbol{k})\hat{c}^+(\boldsymbol{k})\big]\, d\boldsymbol{k}\ , \tag{2.1.33}$$

then the $\hat{\mathcal{H}}$-eigenvalues will coincide with the energy of a system of noninteracting oscillators

$$E = \int \hbar\omega_{\boldsymbol{k}}(N_{\boldsymbol{k}} + \frac{1}{2})\, d\boldsymbol{k}\ . \tag{2.1.34}$$

The presence of $\hbar\omega_{\boldsymbol{k}}/2$ terms in this integral indicates that even in the ground state (vacuum), when all the occupation numbers $N_{\boldsymbol{k}}$ are zero, there is quantum uncertainty in the coordinates and velocities of atoms comprising our "continuous" medium. This quantum-mechanical effect is called zero oscillations, and the terms $\hbar\omega_{\boldsymbol{k}}/2$, accordingly, the energy of zero oscillations. It should be noted that the integral in (2.1.34) defining the full energy of zero oscillations of a continuous medium diverges in the region of large k's. But as any medium (a solid, fluid, gas or plasma) consists of separate particles, integration in (2.1.34) is cut at some $k_{\max} \simeq 1/l$, where l is an average distance between the particles. It is customary to write all annihilation operators on right-hand side of the interaction operator $\mathcal{H}_{\mathrm{int}}$, then its average value in the vacuum state will be zero. Thus, we obtain from (2.1.2,5,31)

$$\mathcal{H}_3 = \frac{\hbar^{3/2}}{2}\int \Big[V_{123}\hat{c}_1^+\hat{c}_2\hat{c}_3 + \mathrm{H.c.}\Big]\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3\ , \tag{2.1.35}$$

$$\mathcal{H}_4 = \frac{\hbar^2}{4}\int T_{1234}\hat{c}_1^+\hat{c}_2^+\hat{c}_3\hat{c}_4\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4. \tag{2.1.36}$$

The quantum-mechanical states of a medium may be classified according to the states of a noninteracting wave field whose Hamiltonian $\mathcal{H}_2$ (2.1.33) is the sum of the Hamiltonians of noninteracting harmonic oscillators. The energy spectrum of an oscillator with frequency ω is known to be $\hbar\omega(N+1/2)$, where $N = 0, 1, 2, \ldots$, with all states N being nondegenerate. Therefore for an oscillator there exists a complete set of fuctions ψ_0, ψ_1, …, ψ_N, …. They may be denoted by $\psi_N = |N\rangle$. Such a choice of functions is called the second quantization representation. The matrix elements of operators $\hat{c}$ and $\hat{c}^+$ are [2.2]:

$$\langle N-1|\hat{c}|N\rangle = \langle N|\hat{c}^+|N-1\rangle = \sqrt{N}, \tag{2.1.37}$$

being zero between other states. It is seen that the creation operator $\hat{c}$ increases the number of quanta N by a unity while $\hat{c}^+$ reduces it by a unity. One can introduce an operator describing the number of quanta $\hat{N} = \hat{c}^+\hat{c}$, which is diagonal in the $|N\rangle$ representation:

$$\langle N|\hat{N}|N\rangle = N. \tag{2.1.38}$$

the question is whether the oscillator description using the average value of the occupation number N is complete or not. The answer may be found using an analogy with classical mechanics where an oscillator is characterized by the energy $\propto N$ and the phase ϕ. In quantum mechanics, the energy of an oscillator and its phase may not be measured simultaneously. In a state of a definite energy the phase is absolutely random. Thus the description of a quantum system using the distribution function $N_{\boldsymbol{k}}$ is completely equivalent to the random-phase approximation (2.1.7) used by us in the classical approach. A more complete description of a quantum system allowing for its phase correlations is possible in terms of a density matrix (where the numbers $N_{\boldsymbol{k}}$ are its diagonal elements). Thus, in the random-phase approximation we obtain an equation describing the variation rate of the number of quanta of a field $N_{\boldsymbol{k}}$ with a given wave vector $\boldsymbol{k}$. The $N_{\boldsymbol{k}}$ value decreases in the decay processes

$$|N_{\boldsymbol{k}},\, N_1,\, N_2\rangle \longrightarrow |N_{\boldsymbol{k}}-1,\, N_1+1,\, N_2+1\rangle, \quad \boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2 \tag{2.1.39a}$$

and the confluence processes

$$|N_{\boldsymbol{k}},\, N_1,\, N_2\rangle \longrightarrow |N_{\boldsymbol{k}}-1,\, N_1-1,\, N_2+1\rangle, \quad \boldsymbol{k}+\boldsymbol{k}_1=\boldsymbol{k}_2, \tag{2.1.39b}$$

and increases in the reverse processes. According to quantum-mechanical perturbation theory, the probability of these processes is equal to the product of $2\pi/\hbar$ with the squared modulus of a matrix element of the interaction Hamiltonian and by the δ-function of the energy difference between the initial and final states. For example, for the sum of the decay processes (2.1.39a) with different $\boldsymbol{k}_1$ and with $\boldsymbol{k}_2=\boldsymbol{k}-\boldsymbol{k}_1$, the probability is

$$\begin{aligned} W_{\mathrm{a}} = 2\pi \int \big|\langle N_{\boldsymbol{k}},\, N_1,\, N_2|\hat{c}_{\boldsymbol{k}}\hat{c}_1^{+}\hat{c}_2^{+}|N_{\boldsymbol{k}}-1,\, N_1+1,\, N_2+1\rangle_{\mathrm{a}}\big|^2 \\ \big|V_{k12}\big|^2 \times \hbar^2\delta(\omega_{\boldsymbol{k}}-\omega_1-\omega_2)\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,d\boldsymbol{k}_1 d\boldsymbol{k}_2\ . \end{aligned} \tag{2.1.40}$$

Here the matrix element of the operator $\hat{c}_{\boldsymbol{k}}\hat{c}_1^{+}\hat{c}_2^{+}$ should be calculated here for the free field. It decomposes to the product of matrix elements from each degree of freedom of the field

$$\langle\,\rangle_{\mathrm{a}} = \langle N_{\boldsymbol{k}}|\hat{c}_{\boldsymbol{k}}|N_{\boldsymbol{k}}-1\rangle\,\langle N_1|\hat{c}_1^{+}|N_1+1\rangle\,\langle N_2|\hat{c}_2^{+}|N_2+1\rangle\ . \tag{2.1.41a}$$

This procedure is, in effect, equivalent to the uncoupling of the classical correlators (2.1.7) used in deducing the expression (2.1.11) for the classical triple correlator. Using (2.1.37), we obtain from (2.1.41a)

$$|\langle\,\rangle_{\mathrm{a}}|^2 = N_{\boldsymbol{k}}(N_1+1)(N_2+1)\ , \tag{2.1.41b}$$

which, together with (2.1.40), gives the final expression for the probability W_{a}. Clearly,

$$\frac{\partial N(\boldsymbol{k},t)}{\partial t} = -(W_{\mathrm{a}}+W_{\mathrm{b}}) + (\tilde{W}_{\mathrm{a}}+\tilde{W}_{\mathrm{b}})\ .$$

W_a and W_b are the probabilities of the processes (2.1.39a,b), which reduce $N_{\boldsymbol{k}}$ by a unity, and $\tilde{W}_\mathrm{a}$ and $\tilde{W}_\mathrm{b}$ are the probabilities of the reverse processes increasing $N_{\boldsymbol{k}}$ by a unity. Computing in a similar way W_b, $\tilde{W}_\mathrm{a}$ and $\tilde{W}_\mathrm{b}$, we get finally

$$\frac{\partial N(\boldsymbol{k},t)}{\partial t} = \pi\hbar \int \Big[|V_{k12}|^2 F_{k12} \delta(\omega_{\boldsymbol{k}} - \omega_1 - \omega_2)\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \\ -2|V_{1k2}|^2 F_{1k2}\delta(\omega_1 - \omega_{\boldsymbol{k}} - \omega_2)\delta(\boldsymbol{k}_1 - \boldsymbol{k} - \boldsymbol{k}_2)\Big] \, d\boldsymbol{k}_1 d\boldsymbol{k}_2 \tag{2.1.42a}$$

with

$$F_{k12} = (N_1 + 1)N_2N_3 - N_1(N_2 + 1)(N_3 + 1) \; . \tag{2.1.42b}$$

The factor 1/2 before V_{k12} eliminates double counting of individual processes. The expression (2.1.42b) may be rewritten in the form

$$F_{123} = N_2N_3 - N_1(N_2 + N_3 + 1) \; . \tag{2.1.42c}$$

Neglecting unity in this expression and replacing $N(k_j)$ by $\hbar n(k_j)$, the quantum KE (2.1.42) go over into the classical one (2.1.12). In a similar way we obtain the quantum kinetic equation for scattering processes:

$$\begin{aligned} \frac{\partial N(\boldsymbol{k},t)}{\partial t} =& \frac{\pi}{2} \int |T_{k123}|^2 F_{k123} \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) \\ & \times \delta(\omega_{\boldsymbol{k}} + \omega_1 - \omega_2 - \omega_3)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \;, \\ F_{k123} =& (N_{\boldsymbol{k}} + 1)(N_1 + 1)N_2N_3 - N_{\boldsymbol{k}}N_1(N_2 + 1)(N_3 + 1) \\ =& N_2N_3(N_1 + N_{\boldsymbol{k}} + 1) - N_1N_{\boldsymbol{k}}(N_2 + N_3 + 1) \; . \end{aligned} \tag{2.1.43}$$

For $N_{\boldsymbol{k}} \gg 1$ it goes over to the classical kinetic equation (2.1.29).

For those familiar with solid state theory, it is customary to use the kinetic equation to describe weak nonequilibrium kinetics (see, for example, *Ziman* [2.3]). Here we shall use kinetic equation to describe turbulence, i.e., the states which are rather far from the equilibrium. Aas we have seen,this use of the kinetic equation is possible since its applicability is ensured by the weakness of the nonlinearity (rather than the proximity to the equilibrium), and averaging is possible due to the statistics of a free wave field (but not the thermodynamic equilibrium).

In the opposite (quantum) limit $N_{\boldsymbol{k}} \ll 1$, waves behaves like a particles and equation (2.1.43) goes over into the well-known Bolzmann kinetic equation for classical particles with energy $\varepsilon = \hbar\omega_{\boldsymbol{k}}$ and momentum $p = \hbar k$

$$\frac{\partial N_p}{\partial t} = \frac{\pi}{2}\hbar^{1-2d} \int |T_{p123}|^2 \left(N_2N_3 - N_1N_p\right) \delta(\boldsymbol{p} + \boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{p}_3) \\ \delta(\varepsilon_p + \varepsilon_1 - \varepsilon_2 - \varepsilon_3)\, d\boldsymbol{p}_1 d\boldsymbol{p}_2 d\boldsymbol{p}_3 \tag{2.1.44}$$

2.2 General Properties of Wave Kinetic Equations

2.2.1 Conservation Laws

The existenceof conservation laws for kinetic equations is an important problem which turns out to be more involved than expected. Let us first attempt to obtain the energy integral. As seen above, the initial dynamic equations for the wave amplitudes, conserve the full Hamiltonian $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_{\mathrm{int}}$. The kinetic equation is valid in the limit $\mathcal{H}_{\mathrm{int}} \ll \mathcal{H}_2$; the interaction is in this case described by the collision integral whose form is entirely determined by resonance processes. In this case, every elementary interaction preserves the sum of wave frequencies. Thus, in the weakly nonlinear limit the role of energy should be played by the quadratic term of the Hamiltonian $\mathcal{H}_2$. The kinetic equation should retain the averaged value, $\langle \mathcal{H}_2 \rangle$. However, for a statistically homogeneous field we have

$$\langle c(\boldsymbol{k}) c^*(\boldsymbol{k}_1) \rangle = n_{\boldsymbol{k}} \delta(\boldsymbol{k} - \boldsymbol{k}_1)$$

and the Hamiltonian $\mathcal{H}_2$ is infinite. The finite value is its density per unit volume. Considering a wave system in a box with sides of length L then we have for $\delta(\boldsymbol{k})$ at $k = 0 \quad (2\pi/L)^d$, and see that the value

$$E = \int \omega(k) n(\boldsymbol{k}, t)\, d\boldsymbol{k} \tag{2.2.1}$$

is the spatial density of the average wave energy. Indeed, it is easy to see that (2.2.1) may be an integral of motion in the equations (2.1.12.29) if some additional conditions are satisfied. The formal reason for the vanishing dE/dt is the presence of the frequency δ-functions ensuring energy conservation in every elementary act of wave interaction. For example, in equation (2.1.12):

$$\begin{aligned} \frac{dE}{dt} = \int & (\omega_{\boldsymbol{k}} - \omega_1 - \omega_2) \big| V_{k12} \big|^2 \delta(\omega_{\boldsymbol{k}} - \omega_1 - \omega_2) \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \\ & \times (n_1 n_2 - n_{\boldsymbol{k}} n_1 - n_{\boldsymbol{k}} n_2)\, d\boldsymbol{k} d\boldsymbol{k}_1 d\boldsymbol{k}_2 = 0 \ . \end{aligned} \tag{2.2.2}$$

This equation is valid if its integrals converge. Assuming that there are no singularities of $n(\boldsymbol{k}, t)$ at $k = 0$ and the power law of the functions $\omega(k)$ and V_{k12} at $k \to \infty$, we see that the integrals in (2.2.2) converge if in the region of large k the occupation numbers $n(\boldsymbol{k}, t)$ desceases more rapidly than k^{-m-d}. Here m is the homogeneity index of V_{k12} at $k \to \infty$: and d the dimension of the $\boldsymbol{k}$-space. In order to explain the physical meaning of this condition, we should introduce the idea of energy flux in the $\boldsymbol{k}$-space. The equation (2.2.2) allows us to write the kinetic equation (2.1.12) in the form of a differential equation, i.e., as a continuity equation for energy density $\varepsilon(\boldsymbol{k}, t) = \omega(k) n(\boldsymbol{k}, t)$

$$\frac{\partial \varepsilon(\boldsymbol{k}, t)}{\partial t} + \operatorname{div} \boldsymbol{p}(\boldsymbol{k}, t) = 0 \ , \tag{2.2.3}$$

where $\boldsymbol{p}(\boldsymbol{k},t)$ is the energy flux in the $\boldsymbol{k}$-space given by

$$\operatorname{div}\boldsymbol{p}(\boldsymbol{k},t) = -\omega(k)I(\boldsymbol{k},t)\ . \tag{2.2.4}$$

The symbols $I(\boldsymbol{k},t) = I_{\boldsymbol{k}}\{n(\boldsymbol{k},t)\} = I_{\boldsymbol{k}}$ denote here and below the *collision integral*, i.e. the right-hand side of the kinetic equation (2.1.12) or (2.1.29). The variation rate of the full energy of the wave system dE/dt, according to (2.2.3) must be equal to the integral over a spheric surface of infinite radius from the normal component of the vector $\boldsymbol{p}(\boldsymbol{k},t)$. The condition of a faster fall-off of $n(\boldsymbol{k},t)$ as compared to k^{-m-d} is necessary to ensure that this integral goes to zero at $k \to \infty$, see (2.2.4). It may be well that this is satisfied. Moreover, even if the initial distribution is rather localized, the power distributions corresponding to a constant flux are formed in a natural way in the evolution process at $k \to 0$ or $k \to \infty$. Such asymptotics may set in at an infinite interval during the finite period of time (as it will be described in Chap. 4). Due to this, the problems related to the existence of integrals of motion in the kinetic equation are not quite trivial. Thus, some value may be conserved only for a limited period of time (see Sect. 4.3 below). One of the main goals of this book is to demonstrate that given the infinite k-space the "naively" determined integrals of motion may prove to be fictitious and are not really conserved.

Violation of the conservation law of the kinetic equation surely implies also its violation for the initial dynamic equation which is, however, more difficult to show.

The four-wave kinetic equation (2.1.29) may also be represented in the form of a continuity equation like (2.2.3). In this case, full energy of a wave system is conserved if at $k \to \infty$, the value $n(\boldsymbol{k},t)$ decreases with k more rapidly than $k^{-d-2m/3}$. As mentioned in the previous chapter, the four-wave scattering processes do not change the number of waves. Therefore, the kinetic equation (2.1.29) as well as the dynamic one (2.1.5b), have an additional integral of motion

$$N = \int n(\boldsymbol{k},t)\,d\boldsymbol{k}\ . \tag{2.2.5}$$

Integrating (2.1.29) over $d\boldsymbol{k}$ this is readily seen to hold. The integrand changes after the interchange $k \leftrightarrow k_1$ or $k_1 \leftrightarrow k_2$ of dummy integration variables. Owing to conservation of the N value, the four-wave kinetic equation may be written in the divergent form

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} + \operatorname{div}\boldsymbol{q}(\boldsymbol{k},t) = 0\,, \quad \operatorname{div}\boldsymbol{q}(\boldsymbol{k},t) = -I(\boldsymbol{k},t)\ . \tag{2.2.6}$$

The quantity $\boldsymbol{q}(\boldsymbol{k},t)$ is called the flux of the number of waves or the wave action flux. The latter name is due to the fact that the value $n(\boldsymbol{k},t) = \varepsilon(\boldsymbol{k},t)/\omega(k)$ has the meaning of density of adiabatic wave invariant in phase space.

Besides thefrequencies δ-functions, the kinetic equations also contain the δ-functions of wave vectors, leading to conservation of the full momentum of the wave system

$$\boldsymbol{\Pi} = \int \boldsymbol{k} n(\boldsymbol{k}, t)\, d\boldsymbol{k} = \int \boldsymbol{\pi}(\boldsymbol{k}, t)\, d\boldsymbol{k} \ , \qquad \frac{\partial \boldsymbol{\pi}(\boldsymbol{k}, t)}{\partial t} + \operatorname{div} \boldsymbol{R}(\boldsymbol{k}, t) = 0 \ . \tag{2.2.7}$$

Here the momentum flux $\boldsymbol{R}$ is a second order tensor.

It should be pointed out that, contrary to E, the N and $\boldsymbol{\Pi}$ values are exact integrals of motion, i.e., they are preserved within the framework of the starting dynamic equations (for distributions that go fast to zero).

In the general (nondegenerate) case the quantities E, N and $\boldsymbol{\Pi}$ form a complete set of integrals of motion for the kinetic equations . Additional integrals may only appear in degenerate cases. Let us consider, for example, the three-wave kinetic equation (2.1.12). The resonance surface for interaction in the space of vectors $\boldsymbol{k}_1, \boldsymbol{k}_2$ is given by the condition

$$\omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) = \omega(\boldsymbol{k}_1 + \boldsymbol{k}_2) \ . \tag{2.2.8a}$$

If the function $\omega(\boldsymbol{k})$ is not uniquely retrieved from the form of this surface, i.e., if there is another function $f(\boldsymbol{k})$ $[f(\boldsymbol{k}) \neq A\omega(k) + (\boldsymbol{B} \cdot \boldsymbol{k})]$ satisfying on the surface (2.2.8a) the same condition

$$f(\boldsymbol{k}_1) + f(\boldsymbol{k}_2) = f(\boldsymbol{k}_1 + \boldsymbol{k}_2) \ , \tag{2.2.8b}$$

then the dispersion law $\omega(\boldsymbol{k})$ is called degenerative. Here the quantity

$$F = \int f(\boldsymbol{k}) n(\boldsymbol{k}, t)\, d\boldsymbol{k}$$

will be the integral of motion of kinetic equation (2.1.12). The nondegeneracy criterion for $\omega(\boldsymbol{k})$ and examples of the degenerate dispersion laws are given in [2.4].

For example, let us consider shallow-water gravitational-capillary waves with the dispersion law (1.2.39)

$$\omega(k) = ck(1 + k^2/2k_*^2) \ .$$

If the motion is almost one-dimensional $k_x \gg k_y$ the dispersion law has the form

$$\omega(k_x, k_y) \approx ck_x(1 + k_x^2/2k_*^2 + k_y^2/2k_x^2) = Ap + p^3/16 + 3q^2/p \ .$$

That dispersion law corresponds to Kadomtsev-Petviashvili equation (1.5.4) and coincide with (1.5.5a) with $\beta > 0$ and $p = 2k_x(c/k_*^2)^{1/3}$ and $q = k_y(c^2/k_*)^{1/3}/\sqrt{3}$. In the four-dimensional space of vectors k_1, k_2 the resonance condition (2.2.8a) defines the three-dimensional surface. That surface can be parametrized by [2.4]:

$$p_1 = 2(\xi_1 - \xi_2), \quad q_1 = \xi_1^2 - \xi_2^2, \quad p_2 = 2(\xi_2 - \xi_3), \quad q_2 = \xi_2^2 - \xi_3^2 \ ,$$
$$\omega(p_1, q_1) = \frac{A}{2}(\xi_1 - \xi_2) + 2(\xi_1^3 - \xi_2^3) \ ,$$
$$\omega(p_2, q_2) = \frac{A}{2}(\xi_2 - \xi_3) + 2(\xi_2^3 - \xi_3^3) \ .$$

Thus, if we introduce

$$f(p,q) = \varphi(\xi_1) - \varphi(\xi_2) = \varphi\left(\frac{q}{p} + \frac{p}{4}\right) - \varphi\left(\frac{q}{p} - \frac{p}{4}\right)$$

with the arbitrary even function φ, then f satisfies condition (2.2.8b). Therefore we have an infinite set of integrals of motion for this case, which corresponds to an integrability of Kadomtsev-Petviashvili equation.

2.2.2 Boltzmann's H-theorem and Thermodynamic Equilibrium

The dynamic Hamilton equations are invariant with respect to time reversal, i.e., the transformations $t \to -t$, $\mathcal{H} \to -\mathcal{H}$. The kinetic equations obtained after the averaging procedure, however, describe an irreversible evolution towards thermodynamic equilibrium. The mathematical statment of irreversibility is the theorem of entropy growth which is similar to the Boltzmann's H-theorem for gas kinetics [2.5]. Indeed, let us consider, for example, the time dependence of the entropy

$$S(t) = \int \ln[n(\boldsymbol{k}, t)]\, d\boldsymbol{k} \ , \tag{2.2.9}$$

of a wave system that obeys the three-wave kinetic equation (2.1.12). We obtain

$$\begin{aligned} \frac{dS}{dt} &= \int \frac{\partial n(\boldsymbol{k},t)}{\partial t} \frac{d\boldsymbol{k}}{n(\boldsymbol{k},t)} = \int \frac{I(\boldsymbol{k},t)}{n(\boldsymbol{k},t)}\, d\boldsymbol{k} \\ &= \int \left|V_{k12}\right|^2 \delta(\omega_{\boldsymbol{k}} - \omega_1 - \omega_2)\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \\ &\times \frac{\left(n_1 n_2 - n_{\boldsymbol{k}} n_1 - n_{\boldsymbol{k}} n_2\right)^2}{n_{\boldsymbol{k}} n_2 n_3}\, d\boldsymbol{k} d\boldsymbol{k}_1 d\boldsymbol{k}_2 > 0 \ . \end{aligned} \tag{2.2.10}$$

Similar inequalities may be derived for the four-wave and quantum kinetic equations. In the latter case, one should proceed from the exact expression for entropy of a Bose gas

$$S_B(t) = \int \left\{[1 + N(\boldsymbol{k},t)] \ln[1 + N(\boldsymbol{k},t)] - N(\boldsymbol{k},t) \ln N(\boldsymbol{k},t)\right\} d\boldsymbol{k} \ , \tag{2.2.11}$$

whose classical limit [at $N(\boldsymbol{k}) = n(\boldsymbol{k})/\hbar \gg 1$] is (2.2.9). Differentiating (2.2.11) and substituting $N(\boldsymbol{k})$ from (2.1.42) we get

$$\begin{aligned}\frac{dS_B}{dt} &= \int \left|V_{k12}\right|^2 \delta(\omega_{\boldsymbol{k}} - \omega_1 - \omega_2)\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \\ &\times \left\{N(\boldsymbol{k},t)[N(\boldsymbol{k}_1,t)+1][N(\boldsymbol{k}_2,t)+1] - N(\boldsymbol{k}_1,t)N(\boldsymbol{k}_2,t)[N(\boldsymbol{k},t)+1]\right\} \\ &\times \ln \frac{[N(\boldsymbol{k}_1,t)+1][N(\boldsymbol{k}_2,t)+1]N(\boldsymbol{k},t)}{N(\boldsymbol{k}_1,t)N(\boldsymbol{k}_2,t)[N(\boldsymbol{k},t)+1]}\, d\boldsymbol{k}d\boldsymbol{k}_1d\boldsymbol{k}_2 > 0 \ .\end{aligned}$$

The inequality follows since $(x-y)\ln(x/y) > 0$ holds at any x, y.

Thus, the entropy of a closed wave system can only increase. The thermodynamic equilibrium corresponds to the maximum of entropy given a constant total energy. Making use of the method of Lagrange multiplier, we obtain from (2.2.11)

$$\frac{\delta}{\delta N(k)}(S - \lambda E) = \ln \frac{N(k)+1}{N(k)} = 0 \ .$$

The λ parameter is given by the temperature of the system $\lambda = T/\hbar$, hence we have the Planck distribution

$$N(k) = \{\exp[\hbar\omega(k)/T] - 1\}^{-1} \ . \tag{2.2.12a}$$

In the quantum limit [when $\hbar\omega = \varepsilon \gg T$ and $N(k) \ll 1$] (2.2.12a) gives the *Maxwell distribution* $N(\varepsilon) = \exp(-\varepsilon/T)$, which is the stationary solution of the Boltzmann kinetic equation (2.1.44). In a classical limit $\hbar\omega(k) \ll T$ and (2.2.12a) goes over to the *Rayleigh-Jeans distribution*

$$n(k) = \hbar N(k) = \frac{T}{\omega(k)} \ , \tag{2.2.12b}$$

which may be obtained directly, by varying (2.2.9). The equilibrium solution (2.2.12b) corresponds to the equipartition of energy in the degrees of freedom $\varepsilon(k) = T$.

If we demand that, apart from energy, the total momentum of the system be also nonzero and constant, we obtain the *drift equilibrium distribution*

$$n(k) = \frac{T}{\omega(k) - (\boldsymbol{k}\cdot\boldsymbol{u})} \ . \tag{2.2.13}$$

Here $\boldsymbol{u}$ is the drift velocity of a wave system relative to the chosen reference system.

Upon direct substitution it is seen that (2.2.12a) is a stationary solution of (2.1.42,43), and (2.2.12b,13) are the stationary solutions of (2.1.12,29). In this case every term in the collision integral vanishes separately. The energy flux on equilibrium distributions is identically zero $\boldsymbol{p}(\boldsymbol{k}) = 0$.

The four-wave kinetic equation has three integrals of motion, therefore the general equilibrium distribution depends on three constants:

$$n(k) = \frac{T}{\omega(k) - (\boldsymbol{k}\cdot\boldsymbol{u}) - \mu} \ . \tag{2.2.14}$$

Here μ is the chemical potential. As (2.1.29) is invariant relative to substitution $\omega(k) \to \omega(k) - \mu$, one can set $\mu = 0$, which corresponds to the definite choice of zero energy.

2.2.3 Stationary Nonequilibrium Distributions

Under influance of external effects the wave system deviates from the thermodynamic equilibrium. Under sufficiently powerful effects all wave excitations in different media lead to instability and exponential growth of wave occupation numbers at the linear stage. Damping of waves is possible due to interactions with inhomogeneities of the medium, quasi-particles of other types etc. We shall describe all these effects, both of wave generation and relaxation, by the same function $\mathit{\Gamma}(\boldsymbol{k})$:

$$\frac{\partial n(\boldsymbol{k}, t)}{\partial t} = I(\boldsymbol{k}, t) + \mathit{\Gamma}(\boldsymbol{k}) n(\boldsymbol{k}, t) \ .$$

In the regions of the $\boldsymbol{k}$-space where $\mathit{\Gamma}(\boldsymbol{k})$ is positive it defines the growth-rate of wave instability, and where $\mathit{\Gamma}(k) < 0$, the damping decrement.

We shall call turbulence a highly nonequilibrium state of the wave system when the occupation numbers deviate from equilibrium in some regions of the $\boldsymbol{k}$-space. The kinetic equation, which is valid at weak nonlinearity, describes the so-called weak turbulence.

Let us discuss the necessary requirements to be satisfied by the $\mathit{\Gamma}(\boldsymbol{k})$ to ensure existence of the stationary distribution $n(\boldsymbol{k})$

$$\mathit{\Gamma}(\boldsymbol{k}) n(\boldsymbol{k}) + I_{\boldsymbol{k}}\{n(\boldsymbol{k}')\} = 0 \tag{2.2.15}$$

[2.6]. These requirements are based on the general properties of the collision integral $I(\boldsymbol{k})$ which is a functional of $n(\boldsymbol{k}_1)$. First, it follows from the H-theorem that $\int I(\boldsymbol{k}) n^{-1}(\boldsymbol{k})\, d\boldsymbol{k} > 0$ [see (2.2.10)]. Thus, it is necessary

$$\int \mathit{\Gamma}(\boldsymbol{k})\, d\boldsymbol{k} < 0 \ . \tag{2.2.16}$$

to satisfy (2.2.15). The physical meaning of this condition is obvious: for the existence of a nonequilibrium steady state the environment should provide a constant output of entropy from the system.

Since energy and momentum conserve as waves interact with each other, in the stationary distribution we have

$$\int \mathit{\Gamma}(\boldsymbol{k}) \omega(\boldsymbol{k}) n(\boldsymbol{k})\, d\boldsymbol{k} = 0 \ , \tag{2.2.17}$$

$$\int \mathit{\Gamma}(\boldsymbol{k}) \boldsymbol{k}\, n(\boldsymbol{k})\, d\boldsymbol{k} = 0 \ . \tag{2.2.18}$$

These conditions should be satisfied at convergence of their integrals, i.e. with the fluxes becoming zero at $k \to \infty$.

It is seen from (2.2.16) that the function $\Gamma(\boldsymbol{k})$ ensuring a steady state should be sign-alternating, i.e., describe both sources and sinks of wave energy. The mutual disposition of a sources and sinks in k-space should not at all be arbitrary. Let us consider, for example, an isotropic arrangement. Let us introduce the energy density in the space of wave numbers

$$E(k) = (2k)^{d-1}\pi\varepsilon(k) = (2k)^{d-1}\pi\omega(k)n(k)$$

and the respective flux $P(k)$ (also in spherical normalization):

$$\frac{dP(k)}{dk} = -(2k)^{d-1}\pi\omega(k)n(k)\ .$$

Then the stationary kinetic equation (2.2.15) may be written as

$$\frac{dP(k)}{dk} = \Gamma(k)I(k)\ . \tag{2.2.19}$$

Let us integrate it from some k_m to infinity:

$$P(\infty) - P(k_\mathrm{m}) = \int_{k_\mathrm{m}}^{\infty} \Gamma(k)E(k)\,dk\ . \tag{2.2.20}$$

Assuming the occupation numbers to decrease rather rapidly [faster than k^{-m-d} for three-waves KE (2.1.12) and $k^{-d-2m/3}$ for four-waves KE (2.1.29)], we have $P(\infty) = 0$. Nearly in all cases of wave turbulence (see Chap. 1), $m + d > \alpha$ and $2m/3 + d > \alpha$, where m, α are the indices of interaction coefficient and the dispersion law at $k \to \infty$. These inequalities imply that with the growth of k in the region of large k the nonequilibrium stationary distributions should decrease more rapidly than the equilibrium ones. This means that at sufficiently large k, the energy flux is positive $P(k) > 0$. Indeed, for the equilibrium solution we have $\varepsilon(k)$ =const and $P(k) = 0$ while more localized distributions have a flux directed towards the regions with smaler the energy density $\varepsilon(k)$, i.e. towards to large k [positiveness of the flux may also be shown strictly, see below Sect. 3.1 (3.1.18)]. Returning to (2.2.20), we see that for the stationary distribution $\exists k_\mathrm{m}\, \forall k > k_\mathrm{m}$:

$$\int_{\boldsymbol{k}}^{\infty} \Gamma(k')E(k')\,dk' = \int_{\boldsymbol{k}}^{\infty} (2k')^{d-1}\pi\Gamma(k')\omega(k')n(k')\,dk' < 0\ . \tag{2.2.21}$$

Thus, a necessary condition for the existence of a nonequilibrium steady state is energy damping in the region of large k. For the four-wave kinetic equation the condition

$$\int \Gamma(\boldsymbol{k})n(\boldsymbol{k})\,d\boldsymbol{k} = 0 \tag{2.2.22}$$

following from conservation of the total number of waves should also be satisfied in addition to (2.2.17–18). From this one can easily obtain (in isotropic situation too) the necessity of at least one more sink, i.e. a region with negative $\Gamma(k)$. Indeed, if the function Γ changes its sign only once, it is easy to prove that it is impossible to satisfy the equations (2.2.16) and (2.2.22) simultaneously, since $\omega(k)$ is assumed to be a monotonic function.

A very common situation is the case when the pumping and damping regions in the k-space well separated (by a large region called the inertial interval where $\Gamma(\boldsymbol{k}) \approx 0$). In this region the stationary turbulent distribution should satisfy

$$I_{\boldsymbol{k}}\{n(\boldsymbol{k}_1)\} = 0 \ . \tag{2.2.23}$$

At the ends of the inertial interval the solution $n(\boldsymbol{k})$ should match source and sink. Clearly, the equilibrium distributions (2.2.12–14), though reducing the collision integral $I(\boldsymbol{k})$ to zero, cannot match external sources and sinks.

Determination of nonequilibrium stationary distributions is based on the universality hypothesis, according to which the form of solution in the inertial interval should be independent of the structure of $\Gamma(\boldsymbol{k})$ in the pumpimg and damping regions. The expression for $n(\boldsymbol{k})$ may contain only the integral characteristics of the source that determine fluxes of the integrals of motion. Such a universality implies a step-by-step (cascade) transfer of energy (or another integral) over different scales from the source to the sink. The necessary condition is locality of interaction in the k-space; motions of strongly differing scales should produce a weak effect on each other.

Thus, the program for determinaning stationary weak turbulent distributions is:

1) find the universal solutions of equation (2.2.23) transmitting the fluxes of conserved quantities;
2) verify the locality of the resulting distributions, i.e. convergence of collision integrals;
3) match the universal solutions with the source and the sink.

In order to make sure that the solutions obtained are physically realizable, one should see if they are stable.

The first three items are covered in Chapt. 3; the third item, in Sect. 3.4 and the stability problem is solved in Chap. 4.

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