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Kolmogorov Spectra of Turbulence

Wave Turbulence

Chapter 3

STATIONARY SPECTRA OF WEAK WAVE TURBULENCE

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Stationary Spectra of Weak Wave Turbulence

Contents

Main Symbols

So the Bellman would cry: and the crew would reply,
"They are merely conventional signs!"

L.CAROLL The Hunting of the Shark

$a(\boldsymbol{k},t)$, a_k, $b(\boldsymbol{k},t)$, b_k, $c(\boldsymbol{k},t)$, c_k	wave amplitudes
d	dimensionality of the $\boldsymbol{k}$-space
E	energy
$\varepsilon(\boldsymbol{k})$	energy density in the $\boldsymbol{k}$-space
$E(k)$	energy density in the k-space
g	gravity acceleration
g_m	magnetic-to-mechanical ratio
$\boldsymbol{H}$, H	magnetic field
$\mathcal{H}$	Hamiltonian
$\hbar$	Planck constant
$I_k\{n(\boldsymbol{k}',t)\}$, $I(\boldsymbol{k})$, I_k	collision integral
$\boldsymbol{k}$	wave vector
k	wave number
m	scaling index of interaction coefficient
$n(\boldsymbol{k},t)$, n_k	wave density in the $\boldsymbol{k}$-space
N	total number of waves
$\boldsymbol{p}$	energy flux in the $\boldsymbol{k}$-space
$P(k)$, P_k, P	energy flux in the k-space
Π	total momentum of waves
$\boldsymbol{R}$	momentum flux in the $\boldsymbol{k}$-space
S	entropy
$T(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) = T_{1234}$	coefficient of four-wave interaction
$V(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = V_{123}$	coefficient of three-wave interaction
α	index of wave frequency
$\Gamma(k)$	external growth-rate (or decrement)
$\delta(x)$	Dirac's delta-function
$\omega(\boldsymbol{k})$, ω_k	wave frequency
ξ	dimensionless variable
$\propto$	proportional
$\approx$	approximately equal
$\simeq$	of the same order

3 Stationary Spectra of Weak Wave Turbulence

This is key chapter of the first volume. Here we deal with stationary distributions of weak turbulence. In Sections 3.1–4 we describe the universal Kolmogorov-like spectra in the inertial interval. We obtain the Kolmogorov spectra as exact solutions of kinetic equations for scale-invariant media, both isotropic and non-isotropic and for nearly scale-invariant ones. Section 3.1.5 deals with the structure of a stationary spectrum in the pumping and damping regions.

3.1 Kolmogorov Spectra of Weak Turbulence in Scale-Invariant Isotropic Media

What I tell you three times is true.

L.CAROLL The Hunting of the Shark

This part of the book is the highlight of the first volume. We shall transform the qualitative arguments about turbulence spectra into exact formulas, and the locality hypothesis into a strict theorem. This will be done in three steps. First, we shall start from dimensional analysis and, using a specific form of the kinetic equation, find the form of the stationary turbulence spectrum in the case of complete self-similarity. If this is not the case despite the scale invariance of Hamiltonian coefficients, still the form of the spectrum can be obtained quite easily. For that purpose it is sufficient to demand the flux to be a constant. That means that the flux expressed in terms of the collision integral must be proportional to the zeroth power of the wave vector. Such procedures can not always be relied upon or taken for granted; so they will not guarantee the existence of the stationary spectrum, which can be ensured only by proving the locality of interaction. Thus, as a second step, we shall discuss the structure of the asymptotic form of the kinetic equation and work out the locality criterion. Finally we shall acquaint the reader with a little miracle of the theory of wave turbulence:

so-called Zakharov-Kraichnan transformations. They factorize the collision integral. As a result one can (i) prove directly that Kolmogorov spectra reduce the collision integral to zero and (ii) find that the Rayleigh-Jeans and Kolmogorov distributions are the only universal stationary power solutions of kinetic equation.

3.1.1 Dimensional Estimations and Self-similarity Analysis

This section deals with universal flux distributions corresponding to constant fluxes of integrals of motion in the k-space. In this subsection we shall show that for scale-invariant media, these solutions may be obtained from dimensional analysis (see also [3.1,2]).

For complete self-similarity we shall first discuss the possible form of universal flux distributions $n(k)$ and the corresponding energy spectra $E(k) = (2k)^{d-1}\omega(k)n(k)$. We shall recall how to find the form of the spectrum $E(k)$ for the turbulence of an incompressible fluid: in this case here is only one relevant parameter, the density ρ; and $E(k)$ may be expressed via ρ, k and the energy flux P. Comparing the dimensions, we obtain

$$E(k) \simeq P^{2/3}k^{-5/3}\rho^{1/3} \tag{3.1.1}$$

which is the famous Kolmogorov-Obukhov "5/3 law" [3.3,4].

As we have seen in Sect. 1.1, in the case of wave turbulence there are always two relevant parameters. We can choose the medium density to be the first one. In contrast to eddies, waves have frequencies, which may be chosen as the second parameter. The frequency enables us to arrange dimensionless parameter

$$\xi = \frac{Pk^{5-d}}{\rho\omega^3(k)} \ ,$$

so $E(k)$ may be determined from dimensional analysis up to an approximation of the unknown dimensionless function $f(\xi)$:

$$E(k) = \rho\omega_k^2 k^{d-6} f\left(Pk^{5-d}/\rho\omega_k^3\right) \ . \tag{3.1.2}$$

In particular, if we demand that $\omega(k)$ be eliminated from (3.1.2), we obtain $f(\xi) \propto \xi^{2/3}$, and (3.1.2) coincides with (3.1.1). In the case of weak wave turbulence the connection between $P(k)$ and $n(k)$ follows from the stationary kinetic equation:

$$dP(k)/dk = -(2k)^{d-1}\pi\omega(k)I(k) \tag{3.1.3}$$

which holds in the limit $\xi \ll 1$.

For the three-wave kinetic equation (2.1.12) $I(k) \propto n^2(k)$ and $n(k) \propto P^{1/2}$, and for the four-wave one, $n(k) \propto P^{1/3}$. These expressions may be unified into one:

$$n(k) \propto P^{1/(j-1)} ,$$

here j is the number of waves participating in an elementary interaction act ($j = 3$ for decay case or $j = 4$ for non-decay one). From this one can easily find the form of the function of the dimensionless argument at $\xi \to 0$

$$f(\xi) \propto \xi^{1/(j-1)}$$

and the Kolmogorov spectrum for weak-turbulence

$$\begin{aligned} E(k) &\simeq \rho^{(j-2)/(j-1)} \omega_k^{(2j-5)(j-1)} P^{1/(j-1)} k^{(dj-2d-6j+1)/(j-1)} , \\ n(k) &\simeq \rho^{(j-2)/(j-1)} \omega_k^{(j-4)/(j-1)} P^{1/(j-1)} k^{(10-d-5j)/(j-1)} . \end{aligned} \tag{3.1.4}$$

Separately for the three-wave processes the non-equilibrium distribution $n(k)$ reads

$$n(k) \simeq (P\rho)^{1/2} \omega_k^{-1/2} k^{-(5+d)/2} . \tag{3.1.5}$$

In the case of the four-wave interaction, apart from the distribution with energy flux derived from (3.1.4),

$$n(k) \simeq (P\rho^2)^{1/3} k^{-(10+d)/3} , \tag{3.1.6a}$$

a solution describing the constant flux of wave action Q may exist [in spherical normalization $Q(k) = (2k)^{d-1}\pi q(k)$]. It is evident from dimensional analysis that such a distribution may be obtained by substituting in (3.1.6a) for P the quantities $\omega(k)Q$:

$$n(k) \simeq (Q\rho^2)^{1/3} \omega_k^{1/3} k^{-(10+d)/3} . \tag{3.1.6b}$$

Let us discuss the somewhat more general situation when the dispersion law and the interaction coefficients are scale-invariant:

$$\begin{aligned} &\omega(\lambda k) = \lambda^\alpha \omega(k), \quad V(\lambda \boldsymbol{k}, \lambda \boldsymbol{k}_1, \lambda \boldsymbol{k}_2) = \lambda^m V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) , \\ &T(\lambda \boldsymbol{k}, \lambda \boldsymbol{k}_1, \lambda \boldsymbol{k}_2, \lambda \boldsymbol{k}_3) = \lambda^m T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) . \end{aligned} \tag{3.1.7a}$$

As shown in the previous chapter, this is possible not only in the case of complete self-similarity (without a length parameter), but when a parameter with the dimension of length is present, i.e., in the case of second-order self-similarity.

It follows from (3.1.7) that $\omega(k)$ is a power function

$$\omega(k) = \beta k^\alpha \tag{3.1.7b}$$

and the interaction coefficients may be written as

$$\begin{aligned} |V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)|^2 &= V_0^2 k^{2m} f_1(\boldsymbol{k}_1/k, \boldsymbol{k}_2/k) , \\ |T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)|^2 &= T_0^2 k^{2m} f_2(\boldsymbol{k}_1/k, \boldsymbol{k}_2/k, \boldsymbol{k}_3/k) . \end{aligned} \tag{3.1.7c}$$

Here V_0, T_0 are the dimensional constants; f_1, f_2 are dimensionless functions.

The exponent s_0 of the Kolmogorov stationary solution

$$n(k) = Ak^{-s_0}$$

with a constant energy flux is found directly from (3.1.3). For example, for the three-wave case

$$\begin{aligned} P(k) =& \pi \int_0^k \omega(k) I(k)(2k)^{d-1}\, dk \\ =& k^{2(m+d-s_0)} V_0^2 A^2 a(m, d, \alpha, s_0) \ . \end{aligned} \qquad (3.1.8a)$$

Here $a(m, d, \alpha, s_0)$ is the dimensionless integral defined below, see (3.1.13). Demanding that the flux be constant $P(k) = P$, we obtain the stationary distribution

$$n(k) = (P/aV_0^2)^{1/2} k^{-s_0}, \quad s_0 = m + d \ . \qquad (3.1.9)$$

Likewise for the four-wave kinetic equation

$$n(k) = (P/aT_0^2)^{1/3} k^{-s_0}, \quad s_0 = 2m/3 + d \ . \qquad (3.1.10a)$$

It should be pointed out that the indices of the Kolmogorov distribution that transports the energy flux are independent of the index of the dispersion law. On the contrary, the equilibrium distributions, as shown in Sect. 2.2, depend neither on the interaction coefficients nor on space dimension, and are entirely determined by the wave dispersion law.

We have already encountered the $(m + d)$ and $(2m/3 + d)$ power indices in the previous section when we discussed the conditions for energy conservation of wave systems. As we have seen, if $n(k) \to k^{-s}$ at $k \to \infty$ and $s > m + d$ (or $2m/3 + d$ in the four-wave case), then the flux $P \to 0$ at $k \to \infty$. The distributions (3.1.9–10a) with $s = s_0$ correspond to a constant energy flux in the k-space. The direction of the flux is determined by the sign of the integral $a(m, d, \alpha)$ – see below (3.1.13).

Similarly to (3.1.10a), from the stationary kinetic equation $\partial Q(k)/\partial k = (2k)^{d-1}\pi I(k)$ one can obtain the stationary distribution with the flux of wave action:

$$n(k) = (\beta Q/aT_0^2)^{1/3} k^{-x_0}, \quad x_0 = \frac{2m}{3} + d - \frac{\alpha}{3} \ . \qquad (3.1.10b)$$

In the case of complete self-similarity, the interaction coefficients may be expressed via $\rho, \omega(k), k$:

$$V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) = \rho^{-1/2} \omega_k^{1/2} k^{(5-d)/2} f_1(\boldsymbol{k}_1/k, \boldsymbol{k}_2/k),$$

$$T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \rho^{-1} k^{5-d} f_2(\boldsymbol{k}_1/k, \boldsymbol{k}_2/k, \boldsymbol{k}_3/k) \ .$$

The distributions (3.1.9,10) will go over respectively to (3.1.5,6).

3.1.2 Exact Stationary Solutions of Three-Wave Kinetic Equations

Let us verify directly that (3.1.9,10) are the solutions of the stationary kinetic equation (2.2.23) and determine the constant $a(m,d,\alpha)$. Integrating (2.2.23) from zero to k and taking into account the boundary condition (the presence of the source $\Gamma(k)$ at $k_0 \ll k$), we obtain (3.1.8a), which we here represent as:

$$\pi \int_0^k (2k)^{d-1} \omega(k) \Gamma(k) n(k)\, dk = P = -\pi \int_0^k \omega(k) I(k) (2k)^{d-1}\, dk \;. \quad (3.1.8b)$$

In this section we shall consider the case of the three-wave collision integral $I(k)$ (2.1.12). If a medium is isotropic, the interaction coefficient V_{k12} is invariant with regard to rotations in $\boldsymbol{k}$-space and, consequently, like the frequency $\omega(k)$, may be considered to be a function of only the absolute values of wave vectors $\boldsymbol{k}, \boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Since the Kolmogorov solution (3.1.9) is also isotropic, only the δ-function of wave vectors is to be averaged over angles in the $\boldsymbol{k}_1, \boldsymbol{k}_2$ spaces. This may be conveniently done by a direct angular integration. For $d = 2$

$$\int \delta(k - k_1 \cos\theta_1 - k_2 \cos\theta_2)\delta(k_1 \sin\theta_1 + k_2 \sin\theta_2)\, d\theta_1 d\theta_2$$
$$= (k k_1 \sin\theta_1^0)^{-1} = \Delta_2^{-1} \,,$$

where Δ_2, the area of a triangle formed by the vectors k, k_1, k_2;

$$\Delta_2 = (1/2)\Big[2\big(k^2 k_1^2 + k^2 k_2^2 + k_1^2 k_2^2\big) - k^4 - k_1^4 - k_2^4\Big]^{1/2} \;.$$

In the same way, for $d = 3$ introducing the spherical coordinate system (k, θ, φ), we obtain

$$\int \delta(k - k_1 \cos\theta_1 - k_2 \cos\theta_2)\delta(k_1 \sin\theta_1 \sin\varphi_1 + k_2 \sin\theta_2 \sin\varphi_2)$$
$$\times \delta(k_1 \sin\theta_1 \cos\varphi_1 + k_2 \sin\theta_2 \cos\varphi_2)\, d\cos\theta_1 d\varphi_1 d\cos\theta_2 d\varphi_2$$
$$= \int \delta(k - k_1 \cos\theta_1 - k_2 \cos\theta_2)\delta(k_1 \sin\theta_1 + k_2 \sin\theta_2) \frac{\sin\theta_1\, d\theta_1 d\theta_2}{2k_2}$$
$$= \frac{1}{2kk_1k_2} = \frac{\Delta_3^{-1}}{2} \;.$$

The functions $\Delta_d(k, k_1, k_2)$ are independent of the signs of terms in the arguments of the δ-function, are invariant relative to rearrangements of its arguments and nonzero if the vectors with lengths k, k_1, k_2 can be used to form a triangle. Let us now go over from integration over the absolute values of k_1, k_2 to integration over frequencies $\omega(k_1) = \omega_1$, $\omega(k_2) = \omega_2$

$$\pi(2k)^{d-1}\frac{I(k)}{v(k)} = I(\omega) = \int_0^\infty\int_0^\infty d\omega_1 d\omega_2 \Big[(R(\omega,\omega_1,\omega_2) - R(\omega_1,\omega,\omega_2) - R(\omega_2,\omega,\omega_1)\Big] . \tag{3.1.11a}$$

Here

$$R(\omega,\omega_1,\omega_2) = \beta^{-2}|V(k,1,2)|^2 2^{d-1}\Delta_d^{-1}(\omega\omega_1\omega_2)^{-1+d/\alpha}$$
$$\times\, \delta(\omega-\omega_1-\omega_2)\Theta(\omega-\omega_2)\Theta(\omega-\omega_1)[n_1 n_2 - n(\omega)(n_1+n_2)] \tag{3.1.11b}$$
$$v(k) = d\omega(k)/dk, \quad n_j = n(\omega_j), \quad \omega_j = \omega(k_j) .$$

The expression for the flux (3.1.8b) may also be rewritten as an integral over frequencies

$$P(\omega) = -\int_0^\omega \omega' I(\omega')\, d\omega'$$
$$= \int_0^\omega \omega'\, d\omega' \int_0^\infty\int_0^\infty d\omega_1 d\omega_2 [R(\omega',\omega_1,\omega_2) - R(\omega_1,\omega',\omega_2) - R(\omega_2,\omega',\omega_1)] .$$

After rearrangements in the second and third terms $\omega_1 \leftrightarrow \omega'$ and $\omega_2 \leftrightarrow \omega'$, we obtain

$$P(\omega) = \int_0^\omega \omega'\, d\omega' \int_0^\infty W(\omega',\omega_1)\big[n(\omega')n(\omega_1) - n(\omega')n(\omega'-\omega_1) - n(\omega')n(\omega'+\omega_1)\big]\, d\omega_1 . \tag{3.1.11c}$$

The form of homogeneous function W is specified by comparison with (3.1.11a)

$$W(\omega_1,\omega_2) = \omega_2^\gamma f(\omega_1/\omega_2)\,, \quad \gamma = \frac{2m+2d}{\alpha} - 3 .$$

Indeed, regarding f_1 (3.1.7c) as being a function of ω_1/ω we obtain, e.g. for $d=3$

$$f(x) = [x(1-x)]^{(d-1-\alpha)/\alpha} f_1(x, 1-x) .$$

So all the information about interactions contained in fact in one number m and in one dimensionless function $f(x)$ of single variable, where $f(x)$ is some structural function expressed via f_1. To avoid possible misunderstanding we should point out that $n(\omega)$ in (3.1.11) and below denotes wave density in the space of wave numbers (but not frequencies) taken as a function of ω : $n(\omega) = n[\omega(k)] = n(k)$. Thus, we have to solve a nonlinear integral equation

$$P(\omega) = \text{const} = P \ .$$

At $P = 0$ the equation has an equilibrium solution $n(\omega) = T/\omega$. A general stationary solution may depend both on P and T (see Sect. 4.1.2 below). Let us derive a turbulent solution for $P \neq 0$ and $T = 0$. Indeed, the existence of a damping region is necessary for turbulence to be steady. According to the Kolmogorov concept, the expression for the spectrum does not depend on damping features, while in the presence of damping it is necessary to set $T = 0$, since no thermal reservoir can exist in this case. Of course, the temperature of the turbulent fluid is not zero, otherwise the fluid would be frozen. One should neglect T in some frequency range (where the turbulent distribution exists). Indeed, according to the very definition of turbulence, the level of excitation is supposed to be much higher than the equilibrium level, so that the latter is negligible. In Sect. 3.1.2 we shall elaborate on the relation between P and T in turbulence

Locality of Interaction. Let us first discuss the convergence of the collision integral (3.1.11a) for power distributions $n(\omega_k) = k^{-s} = \omega^{-s/\alpha}$. We have a sum of integrals of power functions, obviously each of them diverges either at $\omega_1 \to 0$ or at $\omega_1 \to \infty$. However, the structure of the kinetic equation provides the reduction of divergences by one power of ω_1 at $\omega_1 \to \infty$

$$n(\omega_1 - \omega) - n(\omega_1) \approx \omega \frac{\partial n(\omega_1)}{\partial \omega_1} \propto n(\omega_1)\frac{\omega}{\omega_1}$$

and by two powers (i.e. by ω_1^2) at $\omega_1 \to 0$.

Indeed, let at $x \to 0$, the structural function (3.1.7c) $f_((x, 1 - x) \equiv f_1(x) \to x^{m_1}$. This implies that at $k_1 \ll k$ the interaction coefficient behaves like:

$$|V(k, k_1, k_2)|^2 \propto k_1^{m_1} k^{2m-m_1} \ .$$

Then the second and third terms in (3.1.11a) converge at $\omega_1 \to \infty$ if the condition

$$s > s_2 = 2m - m_1 + d + 1 - 2\alpha \tag{3.1.12a}$$

is satisfied. Such a reduction takes place at $\omega_1 \to 0$

$$n(\omega - \omega_1) - n(\omega) \approx \omega_1 \frac{\partial n(\omega)}{\partial \omega} \propto n(\omega)\frac{\omega_1}{\omega} \ .$$

The divergences at $\omega_1 \to 0$ are also mutually cancelled with divergences at $\omega_1 \to \omega$ [here one should take into account that, owing to the symmetry of matrix elements $V_{k12} = V_{k21}$, the structural function $f_1(k_1/k, k_2/k) = f_1(x, y)$ satisfies the condition $f_1\big(x^\alpha, 1 - x^\alpha\big) = f_1\big(1 - x^\alpha, x^\alpha\big)$ on the resonance surface $\omega(k_1) + \omega(k_2) = \omega(k)$ (that is $x^\alpha + y^\alpha = 1$)]. As a result, an additional factor ω_1/ω arises and the convergence criterion of the collision integral at small frequencies is:

$$s < s_1 = m_1 + d - 1 + 2\alpha \ . \tag{3.1.12b}$$

Thus, if $s_1 > s_2$, i.e.,

$$2m_1 > 2m + 2 - 4\alpha \ , \tag{3.1.12c}$$

then there exists an interval of s exponents ensuring locality of interaction. It is important to note that the Kolmogorov exponent $s_0 = m+d$ always lies exactly in the middle of the "locality interval": $s_0 = (s_1 + s_2)/2$ [3.5]. This is due to the fact that on the Kolmogorov distribution, the contributions to interactions of all scales, from small to large ones, level out implying the "balance of interactions". The main contribution to the collision integral $I(\omega)$ comes from the frequency range $\omega_1 \simeq \omega$, indicating locality of interaction. Thus, the collision integral over the Kolmogorov spectrum either converges at $\omega_1 \to 0$ and $\omega_1 \to \infty$ or diverges at both limits.

It should be pointed out that we demanded convergence of the collision integral only for isotropic distributions, in particular, on the Kolmogorov solution itself. That property will be referred to as stationary locality. For stationary locality the kinetic equation has a stationary Kolmogorov solution. One can, however, raise the question about arbitrary (especially anisotropic) perturbations of the Kolmogorov spectrum. In Sects. 4.1,2 we shall see that there are cases when the collision integral describing the behavior of weak anisotropic perturbations against the background of the stationary solution has no locality interval at all. It is also possible that the evolution of perturbations of Kolmogorov distribution is determined by interaction with the ends of the inertial interval (for details see Sect. 4.2). It would be natural to call that property "evolution non-locality". Unless stated otherwise,interaction locality will be used below in the sense of stationary locality. It should be noted, that realization of the equilibrium spectra does not presuppose interaction locality.

Signs of Fluxes. Let us now substitute into (3.1.11) $n(\omega) = A\omega^{-s/\alpha}$. Under the locality condition the collision integral over the power distribution is also a power function

$$I(\omega) = \omega^{\sigma-2}\big(V_0 A\big)^2 I(s), \quad \sigma = \frac{2(m+d-s)}{\alpha} \ ,$$

where $I(s)$ is a dimensionless integral. Hence, the expression for the flux is:

$$P = \omega^{\sigma}\big(V_0 A\big)^2 \frac{I(s)}{\sigma} \ . \tag{3.1.13a}$$

As we see, at $s = s_0 = m + d$ the expression (3.1.13a) contains an indeterminacy of the form 0/0, as the collision integral over the Kolmogorov solution should vanish, $I(m + d) = 0$, see below (3.1.14c). Evaluating the indeterminacy using the L'Hospital's rule, we obtain an expression where the energy flux is proportional to the derivative of the collision integral with respect to the index of solution [3.6,7]. From (3.1.11a,b) we get

$$\begin{aligned} P =& \big(V_0 A\big)^2 a(m,d,\alpha) = \big(V_0 A\big)^2 \left(\frac{dI(s)}{ds}\right)_{s_0} \\ =& \big(V_0 A\big)^2 \int_0^\infty \ln(1 = x)[x(1+x)]^{-s_0/\alpha} f_1(x) \\ & \times \Big[(1+x)^{s_0/\alpha} - x^{s_0/\alpha} - 1\Big]\, dx \; . \end{aligned} \tag{3.1.13b}$$

Such an expression can be obtained from (3.1.11c) by changing the order of integration. The sign of P is determined by the last square bracket which is positive if $s_0/\alpha > 1$; as mentioned earlier, the energy flux is positive, i.e., directed to large k if the Kolmogorov distribution decays for growing k more rapidly than the equilibrium one. In the other case with $s_0 < \alpha$, the Kolmogorov solution does not exist, since it would correspond to imaginary value of the constant A.

The interaction locality provides convergence of integral (3.1.13b), i.e., finiteness of the flux. Thus, the power solution (3.1.9) corresponds to a constant energy flux and owing to (2.3.1) should be a solution of the kinetic equation in the inertial interval.

Zakharov-Kraichnan's Transformations. Using the conformal transformations suggested independently by *Zakharov* and *Kraichnan* [3.8] it can be directly verified that the collision integral becomes zero on the Kolmogorov distribution. The method of conformal transformations allows to obtain all power solutions of the stationary kinetic equation. Let us substitute into (3.1.11a) the distribution in the form $n(\omega) = \omega^{-\nu}$. Then we rearrange the second term

$$\omega_1 = \omega \frac{\omega}{\omega_1'}, \qquad \omega_2 = \omega_2' \frac{\omega}{\omega_1'} \tag{3.1.14a}$$

using the extension factor ω_1/ω, and perform a similar manipulation with the third term (with substitution $\omega_1 \leftrightarrow \omega_2$):

$$\omega_1 = \omega_1' \frac{\omega}{\omega_2'}, \qquad \omega_2 = \omega \frac{\omega}{\omega_2'} \; . \tag{3.1.14b}$$

In (3.1.14), the integration limits 0 and ∞ exchange places, therefore these relations are only valid for converging integrals, i.e., in the case of locality. After those rearrangements the collision integral is factorized and acquires the simple form

$$I(\omega) = \int\int_0^\infty d\omega_1 d\omega_2 \Big[1 - (\omega/\omega_1)^x - (\omega/\omega_2)^x\Big] R(\omega, \omega_1, \omega_2)$$

$$= \int_0^\infty \left[1 - \left(\frac{\omega}{\omega_1}\right)^x - \left(\frac{\omega}{\omega - \omega_1}\right)^x\right] \left[1 - \left(\frac{\omega}{\omega_1}\right)^\nu - \left(\frac{\omega}{\omega - \omega_1}\right)^\nu\right] \quad (3.1.14c)$$

$$\times\, \omega_1^\nu (\omega - \omega_1)^\nu W(\omega_1, \omega - \omega_1)\, d\omega_1, \quad x = 2\frac{m+d}{\alpha} - 2\nu - 1\ ,$$

where W is a positive function. As we see, the integrals vanish at $\nu = 1$ and $\nu = (m + d)/\alpha$, for all other ν they retain their signs. Thus the Rayleigh-Jeans and Kolmogorov distributions are the only universal stationary power solutions of the kinetic equation (2.1.12). Each of them is a one-parameter solution. The question with regard general to solutions of the stationary kinetic equation which depend on several parameters is still open. Partly, this problem will be discussed in Sect. 4.1 and, for the particular case of acoustic turbulence, in Sect. 5.1.

The expression (3.1.13b) for the flux can be obtained as a derivative of collision integral (3.1.14c).

Examples. Let us now consider particular examples of wave systems with the decay (power) dispersion laws in isotropic media. There are two such examples among the wave systems mentioned in Chap. 1: the capillary waves on shallow (1.1.39) and deep (1.1.40) water. In both examples $d = 2$.

Let us start with the shallow-water case. The dispersion law (1.1.39a) is quadratic: $\alpha = 2$. The interaction coefficient has an extremely simple form (1.1.39b), $m = 2$, $f_1(x) = 1$, $m_1 = 0$. The Kolmogorov index $s_0 = m + d = 4$, $n(k) \propto k^{-4}$, the locality conditions (3.1.12) are satisfied. The dimensionless constant a is equal to the integral from (3.1.13b) which here has a form

$$a = \int_0^\infty x^{-3/2} \ln(1 + x)\, dx = 2\pi\ .$$

Thus, the Kolmogorov solution (3.1.9) for shallow-water waves is:

$$n(k) = P^{1/2} 8\sqrt{\pi} \left(\frac{\rho h}{\sigma}\right)^{1/4} k^{-4}\ . \quad (3.1.15a)$$

It was first obtained by *Kats and Kontorovich* [3.9].

For waves on the surface of deep water $\alpha = 3/2$, $m = 9/4$. Taking into account

$$\lim_{k_1 \to 0} |V(k, k_1, k_2)|^2 \propto k_1^{m_1} k^{2m - m_1}\ ,$$

(3.1.13b) yields $m_1 = 7/2$, and the locality conditions (3.1.12) are also satisfied. The index of this solution is $s_0 = 17/4$. In this case one should insert the function

$$f_1(x) = [x(1-x)]^{2/3}\{(1-x^{2/3})^2(1-x)^{-1/3} + [1-(1-x)^{2/3}]^2 x^{-1/3} - [x^{2/3}-(1-x)^{2/3}]^2\}^2\{4x^{4/3}(1-x)^{4/3} - [1-x^{4/3}-(1-x)^{4/3}]^2\}^{-1/2} \ .$$

into the integral (3.1.13b) specifying a. The respective Kolmogorov solution was obtained by *the Zakharov and Filonenko* [3.10]:

$$n(k) = \left(\frac{P}{a}\right)^{1/2} 8\pi \left(\frac{4\rho^3}{\sigma}\right)^{1/4} k^{-17/4} \ . \qquad (3.1.15b)$$

From dimensional analysis it follows that the Kolmogorov distribution for capillary waves on the surface of an arbitrary-depth fluid may be written as

$$n(k) = P^{1/2}\left[\frac{k}{\omega(k)}\right]^{1/2} k^{-4}\phi(kh) \qquad (3.1.15c)$$

where $\phi(x)$ is a dimensionless function. At $x \to 0$, $\phi \to x^{1/2}$ and (3.1.15c) goes over to (3.1.15a); at $x \to \infty$, $\phi \to$const and (3.1.15c) goes over to (3.1.15b).

3.1.3 Exact Stationary Solutions for Four-wave Kinetic Equations

Let us now consider the kinetic equation (2.1.29) and, assuming the distribution to be isotropic $n(\boldsymbol{k}) = n[\omega(k)] = n(\omega)$, average in it over angles to get

$$\begin{aligned}\frac{\partial N(\omega,t)}{\partial t} = \int\int_0^\infty\int U(\omega,\omega_1,\omega_2,\omega_3)[n(\omega_1)n(\omega_2)n(\omega_3)\\ +n(\omega)n(\omega_2)n(\omega_3) - n(\omega)n(\omega_1)n(\omega_2) - n(\omega)n(\omega_1)n(\omega_3)]\\ \times\delta(\omega+\omega_1-\omega_2-\omega_3)\, d\omega d\omega_1 d\omega_2 d\omega_3 \;=\; I(\omega) \ .\end{aligned} \qquad (3.1.16)$$

Here $N(\omega)$ is the wave density in the frequency space

$$N(\omega) = (2k)^{d-1}\pi\left(\frac{dk}{d\omega}\right) n(\omega) \ ,$$

and $U(\omega,\omega_1,\omega_2,\omega_3)$ is the result of angle averaging of the function

$$2^d\pi^2|T_{k123}|^2\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)(kk_1k_2k_3)^{d-1}v_k v_1 v_2 v_3, \quad v_i = \frac{d\omega(k_i)}{dk_i} \ .$$

In spite of the isotropy of the medium, the coefficient of the four-wave interaction $T_{k123} = T(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$ cannot be reduced to a function of wave numbers only (as in the three-wave case). This is due to the fact that the triangle is completely specified by side lengths, whereas the quadrangle is not.

As the constant in the dispersion law may be eliminated by a simple rearrangement (1.4.22), we shall set further $\omega(0) = 0$.

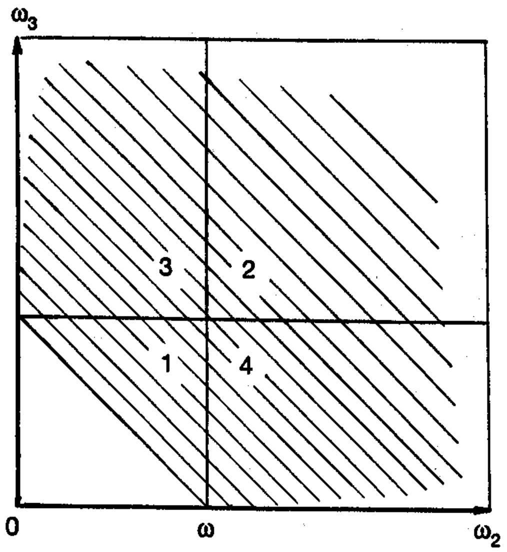

Fig. 3.1. The integration region for the integral given by (3.1.17)

Exact Solutions. Let us find the stationary power solution $n(\omega)$ of the kinetic equation (3.1.16). Integration over $d\omega_1$ gives

$$I(\omega) = \int_{\Omega} d\omega_2 d\omega_3 U(\omega, \omega_2+\omega_3-\omega, \omega_2, \omega_3) n(\omega_2) n(\omega_2+\omega_3-\omega) \\ \times n(\omega) n(\omega_3) \left[n^{-1}(\omega) + n^{-1}(\omega_2+\omega_2-\omega) - n^{-1}(\omega_2) - n^{-1}(\omega_3) \right] . \tag{3.1.17}$$

In (3.1.17) we integrate over the (shaded) region Ω in Fig. 3.1. It has a form of a square $\omega_2 > 0$, $\omega_3 > 0$ without the lower left-side corner. This region is divided into four subregions: in the one designated by 2 in Fig. 3.1 ($\omega_2, \omega_3 > \omega$) we substitute the variables as follows

$$\omega_2 = \frac{\omega\omega_2'}{\omega_2'+\omega_3'-\omega}, \quad \omega_3 = \frac{\omega\omega_3'}{\omega_2'+\omega_3'-\omega} , \tag{3.1.18a}$$

and in subregions 3,4

$$\omega_2 = \frac{\omega^2}{\omega_2'}, \quad \omega_3 = \frac{\omega\omega_3'}{\omega_2'} , \\ \omega_3 = \frac{\omega^2}{\omega_3'}, \quad \omega_2 = \frac{\omega\omega_2'}{\omega_3'} . \tag{3.1.18b}$$

In these transformations the subregions 2,3,4 will go over to subregion 1.

The function $U(\omega, \omega_1, \omega_2, \omega_3)$ obeys the same symmetry conditions as T_{k123}

$$U(\omega, \omega_1, \omega_2, \omega_3) = U(\omega_1, \omega, \omega_2, \omega_3) = U(\omega, \omega_1, \omega_3, \omega_2) \\ = U(\omega_2, \omega_3, \omega, \omega_1) .$$

Besides, U is a homogeneous function

$$U(\lambda\omega, \lambda\omega_1, \lambda\omega_2, \lambda\omega_3) = \lambda^\gamma U(\omega, \omega_1, \omega_2, \omega_3) \ ,$$
$$\gamma = \frac{2m+3d}{\alpha} - 4 \ .$$

that we can write in the form

$$U(\omega, \omega_1, \omega_2, \omega_3) = \omega^\gamma f_3(\omega_1/\omega, \omega_2/\omega, \omega_3/\omega)$$

where f_3 is a dimensionless function of three variables. Therefore the Zakharov's transformations (3.1.18) will convert the equation (3.1.17) into:

$$\begin{aligned} I(\omega) = \int_1 = \int_0^\omega d\omega_3 \int_{\omega-\omega_3}^{\omega} d\omega_2 U(\omega, \omega_2+\omega_3-\omega, \omega_2, \omega_3) \\ \times\left[\omega^x + (\omega_2+\omega_3-\omega)^x - \omega_2^x - \omega_3^x)\right] [\omega(\omega_2+\omega_3-\omega)\omega_2\omega_3]^{-x} \quad (3.1.19) \\ \times\left[1 + \left(\frac{\omega_2+\omega_3-\omega}{\omega}\right)^y - \left(\frac{\omega_2}{\omega}\right)^y - \left(\frac{\omega_3}{\omega}\right)^y\right] = 0 \end{aligned}$$

with $y = 3x - 3 - \gamma$. The integrand in (3.1.19) becomes zero in four points

$$x = 0, \quad x = 1, \quad 3x - 3 - \gamma = 0, \quad 3x - 3 - \gamma = 1$$

and due to positiveness of the U function it retains it sing at all other x. Thus (3.1.17) has four universal power-solutions

$$n(\omega) = C_1/\omega, \qquad n(\omega) = C_2 \ ; \qquad (3.1.20a, b)$$

$$n(\omega) = C_3\omega^{-(3+\gamma)/3}, \qquad n(\omega) = C_4\omega^{-(4+\gamma)/3} \ . \qquad (3.1.21a, b)$$

The solutions (3.1.20) correspond to thermodynamic equilibrium. They are the limiting cases of solutions (2.2.14) with $u = 0$ at $\mu \to 0$ and $\mu \to \infty$.The solutions (3.1.21) are the Kolmogorov solutions; rewriting the exponents, they coincide with (3.1.10): function (3.1.21a)

$$n(\omega) \propto \omega^{-x_Q}, \qquad x_Q = \frac{2m+3d}{3\alpha} - \frac{1}{3}$$

corresponds to the flux of the action spectrum, and (3.1.21b)

$$n(\omega) \propto \omega^{-x_P}, \qquad x_P = x_Q + \frac{1}{3} = \frac{2m+3d}{3\alpha} \ .$$

to the energy flux spectrum.

The Kolmogorov solution holds for local turbulence, i.e., the integrals in (3.1.19) should converge. This is easy to check directly in every particular case.

Signs of Fluxes. Let us now discuss the directions of transmitted fluxes in the ω-space. It is clear that [similarly to the three-wave case (3.1.8,13)], the fluxes will be proportional to the derivatives of the collision integral with respect to the index of solutions x taken at the respective $x = x_Q$ or $x = x_P$

$$\begin{aligned} Q = &\int_0^1 dy_2 \int_{1-y_2}^1 dy_3 f_3(y_2, y_3)(y_2 + y_3 - 1)^{-x_Q}(y_2 y_3)^{-x_Q} \\ &\times \ln\left(\frac{y_2 + y_3 - 1}{y_2 y_3}\right)\left[1 + (y_2 + y_3 - 1)^{x_Q} - u_2^{x_Q} - y_3^{x_Q}\right] , \end{aligned} \tag{3.1.22a}$$

$$\begin{aligned} P = &\int_0^1 dy_2 \int_{1-y_2}^1 dy_3 f_3(y_2, y_3)(y_2 + y_3 - 1)^{-x_P}(y_2 y_3)^{-x_P} \\ &\times [(y_2 + y_3 - 1)\ln(y_2 + y_3 - 1) - y_2 \ln y_2 - y_3 \ln y_3] \\ &\times \left[1 + (y_2 + y_3 - 1)^{x_Q} - u_2^{x_Q} - y_3^{x_Q}\right] , \end{aligned} \tag{3.1.22b}$$

Thus, one index m and one dimensionless function $f(y_2, y_3)$ defined on the triangle $y_2, y_3 < 1$, $y_2 + y_3 > 1$ exhaust in this case all the information about interaction. Here $y_2 = \omega_2/\omega$, $y_3 = \omega_3/\omega$. Since U and f_3 do not change sign, it is sufficient to analyze the behavior of the last square bracket in that expression because

$$\ln\left(\frac{y_2 + y_3 - 1}{y_2 y_3}\right) < 0 \qquad \text{and}$$
$$[(y_2 + y_3 - 1)\ln(y_2 + y_3 - 1) - y_2 \ln y_2 - y_3 \ln y_3] > 0$$

at $y_2, y_3 < 1$, $y_2 + y_3 > 1$. Therefore, it is necessary to consider

$$\phi(x, y_2, y_3) = 1 + (y_2 + y_3 - 1)^x - y_2^x - y_3^x \ .$$

It is easy to establish that $\operatorname{sign}\phi = \operatorname{sign} x(x + 1)$. Thus (see also [3.7]),

$$\operatorname{sign} Q = \operatorname{sign}\left(\frac{\partial\phi}{\partial x}\right)_{x=x_Q} = -\operatorname{sign}(x_Q - 1)x_Q \ , \tag{3.1.22c}$$

$$\operatorname{sign} P = \operatorname{sign}\left(\frac{\partial\phi}{\partial x}\right)_{x=x_P} = \operatorname{sign}(x_P - 1)x_P \ . \tag{3.1.22d}$$

Most of the physical examples known to us, including those given in Chap. 1, correspond to the Kolmogorov spectra decaying with the growth of ω more rapidly than the equilibrium spectrum: $x_Q > 1$. In terms of the initial indices the latter condition is written as $2m + 3d > 4\alpha$. The only exception is given by two-dimensional Schrödinger equation which corresponds to $m = 0$, $d = 2$, $\alpha = 2$ (see Sect. 5.3.2 below).

Equation(3.1.16) may be cast in the divergent form as a continuity equation for $n(\omega)$:

$$\frac{\partial N(\omega,t)}{\partial t} = \frac{\partial^2 K(\omega,t)}{\partial \omega^2} \ , \tag{3.1.23}$$

where

$$\begin{aligned} K(\omega,t) = &\int (\omega+\omega_1-\omega_2-\omega_3) U(\omega_2+\omega_3-\omega_1,\omega_1,\omega_2,\omega_3) \\ &\times \big\{ \big[n(\omega_2+\omega_3-\omega_1) + n(\omega_1) \big] n(\omega_3) n(\omega_2) - n(\omega_1) \\ &\times n(\omega_2+\omega_3-\omega_1) \big[n(\omega_2) + n(\omega_3) \big] \big\} \, d\omega_1 d\omega_2 d\omega_3 \ . \end{aligned} \tag{3.1.24}$$

The integration in (3.1.24) is performed over the range limited by the inequalities

$$\omega_1 > 0, \quad \omega_2 > 0, \quad \omega_3 > 0, \qquad \omega_1 < \omega_2 + \omega_3 < \omega + \omega_1 \ .$$

It follows from (3.1.23) that the wave action flux $Q(\omega,t) = -\partial K(\omega,t)/\partial\omega$. Because of energy conservation, the equation (3.1.16) may also be written in a form

$$\frac{\partial E(\omega,t)}{\partial t} = \omega \frac{\partial N(\omega.t)}{\partial t} = -\frac{\partial}{\partial \omega} \left[K(\omega,t) - \omega \frac{\partial K(\omega,t)}{\partial \omega} \right] ,$$

[differing from (3.1.23)] which is a continuity equation for $E(\omega)$. For the energy flux $P(\omega,t)$ we have then

$$P(\omega,t) = K(\omega,t) - \omega \frac{\partial K(\omega,t)}{\partial \omega} = K(\omega,t) + \omega Q(\omega,t) \ .$$

A general isotropic stationary solution of (2.3.23) may depend on four parameters T, μ, P, Q (see Sect. 4.1.2 below). For an equilibrium system, one should set $P = Q = 0$, which corresponds $K \equiv 0$. The general turbulent solution of (3.1.23) corresponds to $T = \mu = 0$ and to the constant flux:

$$K(\omega) = P - \omega Q \ . \tag{3.1.25}$$

The physical meaning of this solution is that at $\omega = 0$ there is an energy source of intensity P, and at $\omega = \infty$, the wave action source of intensity $-Q$. The fluxes of these quantities flow into opposite directions: $P > 0$, $Q < 0$. If the intensity of one of the sources is zero, the solution (3.1.25) is the power type in the whole space and coincides with one of the solutions of (3.1.21).

In a real situation, the existence of a stationary distribution requires the presence of damping regions both at large and small ω (see Sect. 2.2), even if there is only one source. Let us consider, for example, the situation schematically indicated in Fig. 3.2: at $\omega = \omega_2$ there is a wave source generating N_2 waves per unit time; at $\omega = \omega_1$ and $\omega = \omega_3$ there are two sinks absorbing accordingly N_1 and N_3 waves.

By virtue of conservation of energy and of the total number of waves, the equations

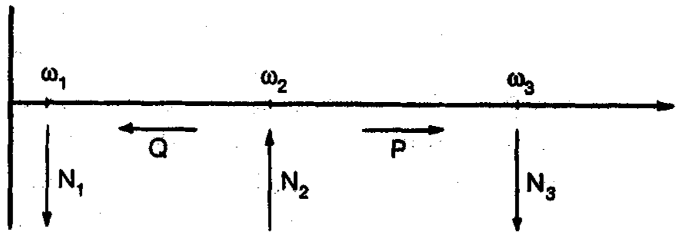

Fig. 3.2. The directions of fluxes in the situation with one source and two sinks

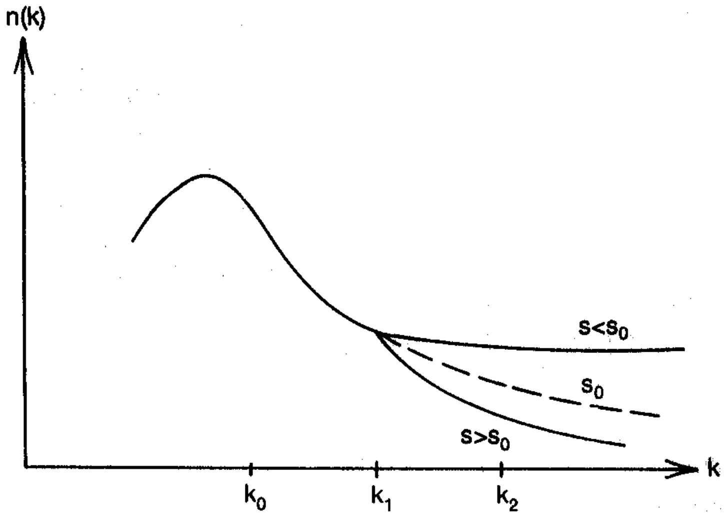

Fig. 3.3. The function $n(k)$ is plotted for different parameters S

$$N_2 = N_1 + N_3, \qquad \omega_2 N_2 = \omega_1 N_1 + \omega_3 N_3 \tag{3.1.26}$$

should be satisfied in the steady state. From this we easily obtain

$$N_1 = N_2 \frac{\omega_3 - \omega_2}{\omega_3 - \omega_1}, \qquad N_3 = N_2 \frac{\omega_2 - \omega_1}{\omega_3 - \omega_1} \ .$$

It is seen that at a sufficiently large left inertial interval (i.e., at $\omega_1 \ll \omega_2 < \omega_3$), the whole energy is absorbed by the right sink: $\omega_2 N_2 \approx \omega_3 N_3$. Similarly, at $\omega_3 \gg \omega_2 > \omega_1$, we have $N_1 \approx N_2$, i.e., the wave action is absorbed at small ω. At $\omega_1 \ll \omega_2 \ll \omega_3$, the solution (3.1.10a) should serve as an intermediate asymptotics at $\omega_2 \ll \omega \ll \omega_3$, and the solution (3.1.10b), at $\omega_1 \ll \omega \ll \omega_2$. This simple deduction shows the energy and action fluxes move into opposite directions: $P > 0$ and $Q < 0$. The distributions with $Q > 0$ ($x_Q < 1$) or with $P < 0$ ($x_P < 1$), which are the formal solutions of (2.1.29), seem to exclude matching with boundary conditions (i.e., sources and sinks). This may be proved with the help of the following simple argument suggested by *Fournier and Frisch* [3.11]. Let us consider, for example, formation of the distribution on the right-hand side of the source located at $k = k_0$ (see Fig. 3.3 adopted from [3.12]). We assume that in an interval from the source to some k_1 a stationary spectrum has been established. Then, the following condition has be satisfied to allow the stationary spectrum $n_k = Ak^{-s_0}$ to extend still further into the region of large k:

If the distribution decays with growing of k at $k > k_1$ faster than the stationary distribution, then the occupation numbers $n(k,t)$ should increase with time. Conversely, for the less steep distributions the derivative of the occupation numbers with respect to time should be negative. Thus we come to a condition that $\forall k_2$ $(k_2 > k_1)$

$$\mathrm{sign}\left[\frac{\partial n(k_2,t)}{\partial t}\right] = \mathrm{sign}\,(s - s_0)\ .$$

Since k_2 is located in the inertial interval, the evolution of the occupation numbers is determined only by interaction of waves with each other, i.e., by the collision integral

$$\frac{\partial n(k_2,t)}{\partial t} = I(k_2,t)\ .$$

It is readily seen from the last two equations that formation of the stationary spectrum $n_k = Ak^{-s_0}$ in the region of large k requires for power distributions a positive derivative of the collision integral with respect to index of the solution

$$\left[\frac{\partial I(s)}{\partial s}\right]_{s_0} > 0\ .$$

But according to (3.1.13), the sign of this derivative gives the sign of a flux of the corresponding integral of motion (in this case, of energy).

It is proved in a similar way that on the left-hand side of the source, thereby can only be a spectrum with a negative flux of the integral of motion.

Thus, only those Kolmogorov solutions may be realized which transfer the flux of the integral of motion from the source to the sink but not into the opposite direction. This is obvious. We note that the above reasoning is true irrespective of the type of collision integral.

Examples. Out of all examples of wave turbulence with the non-decay dispersion law, gravitational waves on the surface of deep fluid (1.1.42), (2.2.42) are of greatest physical interest. This is the kind of waves that may be excited by the wind on the surface of seas and oceans (see, e.g., [3.13] and references therein). For these waves $d = 2$, $\alpha = 1/2$ [see (1.1.43) and (1.2.42)]. There are two Kolmogorov solutions obtained by *Zakharov and Filonenko* [3.14]. The first one transmits the wave action flux to a low-frequency region

$$n(k) = g^{1/6}\rho^{2/3}(Q/a_1)^{1/3}k^{-23/6} \tag{3.1.27}$$

and another one, the energy flux to high frequencies

$$n(k) = \rho^{2/3}(P/a_2)^{1/3}k^{-4} \; . \qquad (3.1.28)$$

Here g is acceleration due to gravity and ρ the fluid density. It is seen from the analysis of (rather complicated) integrals for the dimensionless constants a_1, a_2 that both distributions are local.

Other examples of Kolmogorov solutions (3.1.10) are Langmuir and spin waves (see Sects. 1.3.4). The dispersion laws (1.3.3) and (1.4.9a,21) have the same form, quadratic with a gap, so that $\alpha = 2$. Let us set $d = 3$. With regard to interaction coefficients, there are two possibilities:
i) $m = 2$, which corresponds to the direct interaction of spin waves (1.4.9b);
ii) $m = 0$, which corresponds to the interaction of plasmons via virtual ion-sound wave (1.3.14) and the interaction of magnons in antiferromagnets (1.4.20). Case ii) also corresponds to the nonlinear Schrödinger equation (1.4.24, 1.5.3) which defines the "turbulence of envelopes". In particular, the equation describes the "turbulence of light" in the nonlinear dielectrics.

In the case i), a solution with energy flux is

$$n(k) = Ak^{-13/3} \; , \qquad (3.1.29a)$$

where we have for the Langmuir waves [see (1.3.5)]

$$A = (P/a_1)^{1/3} n_0^{2/3} \qquad (3.1.29b)$$

and for the spin waves [see (1.4.9b)]

$$A = (P/a_2)^{1/3}(\beta g_m)^{-2/3} \; . \qquad (3.1.29c)$$

The Kolmogorov solution possessing an action flux is

$$n(k) = Bk^{-11/3} \; , \qquad (3.1.30a)$$

where B reads for plasmons

$$B = (Q/b_1)^{1/3}\omega_p^{1/3}(r_D n_0)^{2/3} \qquad (3.1.30b)$$

and for magnons

$$B = (Q/b_2)^{1/3}(M/\beta g_m)^{1/3} \; . \qquad (3.1.30c)$$

The analysis shows that the dimensionless constants a_1, a_2, b_1, b_2 are given by the convergent integrals, i.e., the solutions (3.1.29–30) are local.

In the case ii), only distributions with wave action flux are local:

$$n(k) = Ck^{-7/3} \; . \qquad (3.1.31a)$$

The quantity C reads for the Langmuir waves [see (1.3.14)]

$$C = (Q/c_1)^{1/3}(r_D n_0 T)^{2/3}\omega_p^{-1/3} \; , \qquad (3.1.31b)$$

and for the spin waves

$$C = (Q/c_2)^{1/3} g_m^{-4/3} \omega_{ex}^{-1/2} \omega_a^{1/6} . \qquad (3.1.31c)$$

Solutions (3.1.29–31) were obtained by *Zakharov* [3.8]. As far as distribution (3.1.10a) is concerned, it is in this case nonlocal. the solution carrying a constant energy flux has the form

$$n(k) = DP^{1/3} k^{-3} ln^{-2/3}[k/k_0]$$

with k_0 being the pumping frequency [3.37].

Coming to the end, let us discuss the applicability range of weak-turbulence approximation for the Kolmogorov solutions (3.1.9–10). On the power distributions $n(k) \propto k^{-s}$, the characteristic time of nonlinear interaction depends on k in the following way

$$t_{NL} \propto k^z, \qquad z = \alpha + (s_0 - d)(j - 2) - 2m . \qquad (3.1.32)$$

This formula is valid for both (2.1.12) and (2.1.29), j is the number of participants in the elementary interaction act. The applicability parameter of weak-turbulence approximation (2.1.14) is:

$$\xi^{-1}(k) = \omega(k) t_{NL} \propto k^{z+\alpha} . \qquad (3.1.33a)$$

For the distribution possessing an energy flux we have $s_0 = d + 2m/(j-1)$ and

$$\xi^{-1}(k) \propto k^y, \qquad y = 2\alpha - 2m/(j-1) . \qquad (3.1.33b)$$

At $\alpha(j-1) > m$, the applicability criterion of the theory of weak turbulence is violated at small k and in the case of $\alpha(j-1) < m$, at large ones. For the distributions possessing a wave action flux we have $j = 4$, $s_0 = d + 2m/3 - \alpha/3$, and the ξ parameter behaves as follows

$$\xi^{-1}(k) \propto k^x, \qquad x = 2(2\alpha - m)/3 . \qquad (3.1.33c)$$

At $2\alpha > m$, the turbulence becomes strong in the region of small k and at $2\alpha < m$, of large k.

3.1.4 Exact Power Solutions of the Boltzmann equation

The Maxwell equilibrium distribution $N(\varepsilon) = \exp(-\varepsilon/T)$ decreases exponentially with energy. It is interesting that non-equilibrium power distributions can be obtained as exact stationary solutions of Boltzmann kinetic equation [3.16]. Moreover, the power-law spectra of particles have been observed in various experiments: for cosmic rays [3.17], for emission current from metal irradiated by strong laser impulse [3.18], etc.

Let us consider the Boltzmann kinetic equation (2.1.44)

$$\frac{\partial N(\boldsymbol{p},t)}{\partial t} = I(\boldsymbol{p}) = \frac{\pi}{2}\hbar^{1-2d} \times \int \left|T(\boldsymbol{p},\boldsymbol{p}_1;\boldsymbol{p}_2,\boldsymbol{p}_3)\right|^2 \left[N(\boldsymbol{p},t)N(\boldsymbol{p}_1,t) - N(\boldsymbol{p}_2,t)N(\boldsymbol{p}_3,t)\right] \times \delta(\boldsymbol{p}+\boldsymbol{p}_1-\boldsymbol{p}_2-\boldsymbol{p}_3)\delta(\varepsilon_p+\varepsilon_1-\varepsilon_2-\varepsilon_3)\, d\boldsymbol{p}_1 d\boldsymbol{p}_2 d\boldsymbol{p}_3 \ . \tag{3.1.34}$$

We also suppose the scattering probability $\left|T(\boldsymbol{p},\boldsymbol{p}_1;\boldsymbol{p}_2,\boldsymbol{p}_3)\right|^2$ and energy $\varepsilon(p)$ to be a scale-invariant functions of momenta with $2m$ and α indices respectively. For isotropic distributions we can integrate over angles. Going over from $\boldsymbol{p}$ to ε-variables, we obtain

$$\frac{\partial N(\varepsilon,t)}{\partial t} = I(\varepsilon) = \int U(\varepsilon,\varepsilon_1;\varepsilon_2,\varepsilon_3)\left[N(\varepsilon,t)N(\varepsilon_1,t) - N(\varepsilon_2,t)N(\varepsilon_3,t)\right] \times \delta(\varepsilon+\varepsilon_1-\varepsilon_2-\varepsilon_3)\, d\varepsilon_1 d\varepsilon_2 d\varepsilon_3 \ . \tag{3.1.35}$$

Here the particle density in the ε-space $N(\varepsilon) = N(\boldsymbol{p})\pi(2p)^{d-1}(dp/d\varepsilon)$ and U is a homogeneous function with scaling index

$$\gamma = \frac{2m+3d}{\alpha} - 4$$

as for four-wave kinetic equation (3.1.16). Following (3.1.16-19) we integrate over $d\omega_1$ and make Zakharov transformations. Thus we find for power distribution $N(\varepsilon) = A\varepsilon^{-x}$

$$I(\varepsilon) = A^2 \int_0^{\varepsilon} d\varepsilon_3 \int_{\varepsilon-\varepsilon_3}^{\varepsilon} d\varepsilon_2 U(\varepsilon,\varepsilon_2+\varepsilon_3-\varepsilon;\varepsilon_2,\varepsilon_3)\left[\varepsilon^{-x}(\varepsilon_2+\varepsilon_3 - \varepsilon)^{-x} - \varepsilon_2^{-x}\varepsilon_3^{-x}\right]\left[1+\left(\frac{\varepsilon_2+\varepsilon_3-\varepsilon}{\varepsilon}\right)^y - \left(\frac{\varepsilon_2}{\varepsilon}\right)^y - \left(\frac{\varepsilon_3}{\varepsilon}\right)^y\right] \tag{3.1.36}$$

with $y = 2x - 3 - \gamma$.

Considering the last square bracket in (3.1.36) one can see that both $y = 0$ and $y = 1$ imply that the collision integral $I(\varepsilon)$ vanishes. So the stationary power solutions have the form [compare with (3.1.10,20)]

$$N_1(\varepsilon) = A_1\varepsilon^{-x_1}, \quad x_1 = \frac{2m+3d-\alpha}{2\alpha} \ , \tag{3.1.37a}$$

$$N_2(\varepsilon) = A_2\varepsilon^{-x_2}, \quad x_2 = \frac{2m+3d}{2\alpha} \ . \tag{3.1.37b}$$

The first solution evidently corresponds to the constant flux of particles Q, while the second to a constant energy flux P. Following the same procedure, the fluxes can be expressed in terms of $\partial I/\partial x$, so that we have $A_1 \propto \sqrt{Q}$, $A_2 \propto \sqrt{P}$. For an analysis of locality of the different cases see [3.19].

Thus, quantum kinetic equation has power non-equilibrium solutions in both the limits: $N_k \gg 1$ and $N_k \ll 1$. The respective solutions have different indices. It is naturally connected with the absence of scaling invariance in quantum kinetic equation which contains the terms of different order in N_k.

3.2 Kolmogorov Spectra of Weak Turbulence in Nearly Scale-Invariant Media

As mentioned above, the behavior of weakly nonlinear waves is completely specified by two functions, the dispersion law $\omega(k)$ and the interaction coefficients $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$ or $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ characterizing the given medium. Under the condition of scale invariance of these functions and in the absence of preferred directions, the stationary flux solutions of the kinetic equations are the power functions of the modulus of the wave vector (as shown in the preceding section). Clearly, in the case of arbitrary functions $\omega(\boldsymbol{k})$, $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$, $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ it is impossible to find the form of the flux spectra. In this section we shall describe the Kolmogorov solutions which may be obtained for the nearly scale-invariant situations.

3.2.1 Weak Acoustic Turbulence

Separate consideration is necessary for the case of weakly dispersive waves whose dispersion law

$$\omega(k) = ck + \Omega(k) \tag{3.2.1}$$

deviate only a little bit from the linear one: $\Omega(k) \ll ck$. As a rule, $\Omega(k)$ corresponds to the next term of expansion of $\omega(k)$ in powers of a small parameter. If the dispersion law is close to a linear one in the short-wave region, then the small parameter is $(ak)^{-1}$, where a is a characteristic scale. The expansion contains usually even powers and $\Omega(k) \propto ck(ak)^{-2}$. Thus, for example, the dispersion law of spin waves in antiferromagnets (1.4.18) and of ultra-relativistic particles, $\omega^2(k) = \omega_0^2 + (vk)^2$, reduces at $k \gg \omega/v$ to the form $\omega(k) \approx vk + \omega_0^2/(2vk)$. But if the long-wave region is considered (acoustic type systems), the small parameter is (ak), and $\Omega(k) \propto ca^2k^3$ [see, e.g., (1.2.22), (1.2.41), (1.3.10)].

Let us consider with wave turbulence with nearly acoustic decay dispersion law

$$\omega(k) = ck(1 + a^2k^2)$$

which, in the region of small k [the region where (3.2.1) is valid], is close to a linear one, i.e. to the power law with the exponent $\alpha = 1$. At first sight it seems as if one should expect in this region a flux spectrum close to the power Kolmogorov one (3.1.9): $n(k) \propto k^{-m-d}$ where the index of interaction coefficient for sound $m = 3/2$ [see (1.1.38, 1.2.20)]. However, we have to bare in mind that the simple criteria for weak turbulence obtained in Sect. 2.1.4 are not applicable to non-dispersive sound waves. The kinetic equation [whose solution is (3.1.9)] is not suitable for a description of that case. Thus, one can expect the realization of a weakly turbulent Kolmogorov spectrum in an intermediate region of scales where the dispersion term ca^2k^3 is much smaller than the frequency $\omega(k)$ but much greater than the characteristic inverse time of interaction [see (2.1.25)].

Under these conditions, weak acoustic turbulence is described by the kinetic equation (2.1.12) in which one should insert $\omega(k)$ and $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$ given by (1.2.22) and (1.2.20) respectively. It is readily seen that in the small dispersion region, $ak \ll 1$, the space-time synchronization condition $\omega(\boldsymbol{k}) = \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k} - \boldsymbol{k}_1)$ allows interaction only between waves propagating at close angles:

$$\theta(k, k_1) \approx \sqrt{6}|k - k_1|a \ . \tag{3.2.2}$$

Thus, to first order in ak, ak_1 [in which the dispersion parameter should be retained only in the argument of the δ-function] one can set $n(|\boldsymbol{k} - \boldsymbol{k}_1|) = n(k - k_1)$ and substitute scalar products in interaction coefficient by modulus products. Assuming an isotropic distribution $n(\boldsymbol{k}) = n(k)$, we can perform all integrations except one to obtain

$$\begin{aligned} \frac{\partial n(k,t)}{\partial t} &= 2^{d-1} b\pi (\sqrt{6}a)^{d-3} \Bigg\{ \int\limits_0^k dk_1 \big[k_1(k-k_1)\big]^{d-1} \\ &\times \big[n(k_1)n(k-k_1) - n(k)n(k_1) - n(k)n(k-k_1)\big] \\ &- 2\int_k^\infty dk_1 \big[k_1(k_1-k)\big]^{d-1} \big[n(k)n(k_1-k) - n(k_1)n(k) \\ &\quad - n(k_1)n(k_1-k)\big] \Bigg\} = I_d(k) \ . \end{aligned} \tag{3.2.3}$$

Here b is the dimensional constant in the interaction coefficient

$$|V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)|^2 = bkk_1k_2$$

and $d = 2, 3$, the dimension of the $\boldsymbol{k}$-space. In addition to the three-dimensional case, we have here also the two-dimensional one, corresponding to gravitational-capillary waves on the surface of a shallow fluid [see (1.3.19)].

The presence of the dispersion parameter a with a length dimension [see from (3.2.3)] changes structure and index of the angle-averaged interaction coefficient as compared to the scale-invariant expression (3.1.11a). That is why the Kolmogorov solution differs in general from (3.1.9). Indeed, the index of the flux spectrum s_0 is easy to find by computing the energy flux index and equating it to zero:

$$P = P(k) = -2^{d-1}\pi \int_0^k \omega(k)k^{d-1}I_d(k)\,dk \propto k^{3d-2s_0} , \qquad (3.2.4a)$$

$$n(k) = \lambda P^{1/2}k^{-s_0}, \qquad s_0 = 3d/2 . \qquad (3.2.4b)$$

We see that the resulting expression $s_0 = d + d/2$ coincides with (3.1.9) $s_0 = d + m = d + 3/2$ only in the three-dimensional case. Really, at $d = 3$ the dispersion length a disappears from the kinetic equation (3.2.3).

Thus, the kinetic equation (3.2.3) formally experiences the transition to the non-dispersive limit $a \to 0$ for $d = 3$. The three-dimensional case is intermediate. For $d > 3$, the mean time of nonlinear interaction increases when $a \to 0$, so turbulence is effectively weak even in that limit. On the other hand for $d < 3$, the interaction time decreases and turbulence becomes strong when $a \to 0$. As we shall show in the second volume of this book, the case with $d = 3$ corresponds to the first one (as for $d > 3$), since the interaction time decreases for $a \to 0$ slower then that of dispersion.

The fact that the solution (3.2.5) yields a vanishing the collision integral (3.2.3) may be verified by using Zakharov transformations (3.1.14). In this case the second term in (3.2.3) should be spit up into two identical terms: in the first one we make the substitution $k_1 \to k^2/k_1$, in the second one $k_1 \to kk_1/(k - k_1)$, and at $n_k \propto k^{-3d/2}$ we get $I_d(k) \equiv 0$. Substituting (3.2.5) into the equation $P_k = P$ (3.2.4), one can find the λ constant. By changing the order of integration as done for to (3.1.13b), we obtain

$$\lambda^{-2} = b2^{2d-2}\pi^2(\sqrt{6}a)^{d-3}I_d,$$
$$I_d = \int_0^\infty \ln(1+x)[x(1+x)]^{-1-d/2}\left[(1+x)^{3d/2} - x^{3d/2} - 1\right] dx .$$

The dimensionless integral I_d for $d = 3$ was calculated approximately in [3.20]: $I_3 \approx 0.2$, and for $d = 2$, exactly in [3.5]: $I_2 = 2$. In the two-dimensional case, $\lambda \propto \sqrt{a}$, i.e., the amplitude of stationary flux solution vanishes as the dispersion parameter goes to zero.

Let us consider now the case with a non-decay dispersion law. If the dispersion law of waves is nearly acoustic

$$\omega(k) = ck - \Omega(k), \quad 0 < \Omega(k) \ll ck, \quad \frac{\partial^2\Omega}{dk^2} > 0$$

with the dispersion addition being negative and convex function, then the three-wave processes are prohibited. The dominant role in the interaction is played by the four-wave processes with the interaction coefficient (1.1.29) arising in the second order of the perturbation theory at the expense of triple interactions via a virtual but almost real (due to smallness of Ω) intermediate wave.

$$
\begin{aligned}
T_p = &- \frac{U_{-1-212}U_{-3-434}}{\omega_3 + \omega_4 + \omega_{3+4}} + \frac{V^*_{1+212}V_{3+434}}{\omega_1 + \omega_2 - \omega_{1+2}} \\
&- \frac{V^*_{131-3}V_{424-2}}{\omega_{4-2} + \omega_2 - \omega_4} - \frac{V^*_{242-4}V_{313-1}}{\omega_{3-1} + \omega_1 - \omega_3} \\
&- \frac{V^*_{232-3}V_{414-1}}{\omega_{4-1} + \omega_1 - \omega_4} - \frac{V^*_{141-4}V_{323-2}}{\omega_{3-2} + \omega_2 - \omega_3} \ .
\end{aligned}
$$

Here $(j \pm i) = k_j \pm k_i$.

However such a coefficient $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ is not scale-invariant because of the presence of two terms with different k-dependencies in the functions $\Omega(k_j)$ in the denominator. How can we find the Kolmogorov solution in this case? An answer to this question was given in [3.21] where it was shown that in this case the four-wave kinetic equation also has isotropic Kolmogorov solutions $n(k) \propto k^{-s}$ in the weak dispersion region. Really, at $\Omega(k) \ll ck$ the main contribution to the interaction should stem from small-angle scattering when all the four vectors $\boldsymbol{k}, \boldsymbol{k}_1\boldsymbol{k}_2$ and $\boldsymbol{k}_3$ are almost parallel and the denominators in $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ are small. In this case one can set, for example:

$$
\omega(k) \pm \omega(k_j) - \omega(k \pm k_j) \approx ck \frac{k_j \theta_j^2}{2|k \pm k_j|} + \Omega(k) \pm \Omega(k_j) - \Omega(k \pm k_j) \ ,
$$

where θ_j is the angle between wave vectors $\boldsymbol{k}_j$ and $\boldsymbol{k}$. If the $\Omega(k)$ is scale-invariant $\Omega(\lambda k) = \lambda^\beta \Omega(k)$, one can perform transformations that leave the collision integral

$$
\begin{aligned}
I(\boldsymbol{k}) = \int \left| T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) \right|^2 \delta(\omega_k + \omega_1 - \omega_2 - \omega_3) \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) \\
\times n_k n_1 n_2 n_3 \left(n_k^{-1} + n_1^{-1} - n_2^{-1} - n_3^{-1} \right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3
\end{aligned}
$$

unchanged. In addition to a dilatation of the wave vector by a factor of $\lambda_j = k/k_j$ [similar to (3.1.18)] these transformations also involve multiplications of the angle by $\lambda_j^{(\beta-1)/2}$: $\theta_j' = \lambda_j^{(\beta-1)/2}\theta_j$, necessary to ensure the invariance of the denominators in T and of the arguments of the frequency δ-function. In addition, the transformations should contain a rotation which matches the vector $\boldsymbol{k}_j$ with vector $\boldsymbol{k}$. Such transformations consisting of dilatation and rotations in the $\boldsymbol{k}$-space were suggested in 1971 by *Katz and Kontorovich* [3.22]. We shall consider them in detail below, in Sect. 4.1, where we will be use them to obtain the stationary anisotropic additions to Kolmogorov distributions. For example, in the two-dimensional case the transformation converting $\boldsymbol{k}_2$ to $\boldsymbol{k}$ $(j = 2)$ but leaving $I(k)$ invariant [except for an arbitrary factor] has the form:

$$
\begin{aligned}
&k_2' = \lambda - 2k = \lambda_2^2 k_2, \quad k_1' = \lambda_2 k_3, \quad k_3' = \lambda_2 k_1 \ , \\
&\theta_2' = -\lambda_2^{(\beta-1)/2}\theta_2, \quad \theta_1' = \lambda_2^{(\beta-1)/2}(\theta_3 - \theta_2), \quad \theta_3' = \lambda_2^{(\beta-1)/2}(\theta_1 - \theta_2) \ .
\end{aligned}
$$

Splitting the collision integral up into four identical terms and subjecting three of them to transformations with $j = 1, 2, 3$, we obtain in the weak-dispersion region

$$I(\boldsymbol{k}) = \frac{k^\nu}{2} \int |T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)|^2 \delta\Big[\Omega(k) + \Omega(k_1) - \Omega(k_2) - \Omega(k_3)$$
$$+ \frac{c}{2}\big(k_1\theta_1^2 - k_2\theta_2^2 - k_3\theta_3^2\big)\Big]\delta(k + k_1 - k_2 - k_3)$$
$$\times \delta(k_1\theta_1\boldsymbol{\kappa}_1 - k_2\theta_2\boldsymbol{\kappa}_2 - k_3\theta_3\boldsymbol{\kappa}_3) n_k n_1 n_2 n_3$$
$$\times \big(n_k^{-1} + n_1^{-1} - n_2^{-1} - n_3^{-1}\big)\big(k^{-\nu} + k_1^{-\nu} - k_2^{-\nu} - k_3^{-\nu}\big)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \ .$$

Here $\boldsymbol{\kappa}_j = \boldsymbol{k}_{j\perp}/k_{j\perp}$, where $\boldsymbol{k}_{j\perp}$ is the component of the $\boldsymbol{k}_j$ vector orthogonal to the $\boldsymbol{k}$ vector (in the two-dimensional case $\kappa_j = 1$). The index ν equals to:

$$\nu = -3s + 4m - 3\beta - 1 - \frac{(\beta+1)(d-1)}{2} + 3\frac{(\beta-1)(d-1)}{2} + 4d \ .$$

In this expression m is the scaling index of the three-wave interaction coefficient V_{k12}; according to (1.2.20), $m = 3/2$. The stationary Kolmogorov solution with the constant energy flux will be obtained for $\nu = -1$, the distribution index will then be equal to:

$$s_P = (8 + 2d + \beta d - 4\beta)/3 \ . \tag{3.2.5a}$$

The choice of $\nu = 0$ reduces the collision integral to zero and yields the index

$$s_Q = (7 + 2d + \beta d - 4\beta)/3 \ . \tag{3.2.5b}$$

of the Kolmogorov solution with the constant wave energy flux. Further consideration requires concretization of the form of $\Omega(k)$, in particular, specification of the value of β. We shall restrict ourselves to systems of acoustic type for which we have

$$\Omega(k) = ca^2k^3, \quad \beta = 3 \ .$$

Below we shall be concerned with this case (see Sects. 3.4, 4.3 and 5.1). Thus, we see that for a three-dimensional medium and in the non-decay case, the Kolmogorov distributions of weak acoustic turbulence are equal to

$$n_k \propto k^{-s_Q} = k^{-10/3} \ , \tag{3.2.6a}$$

$$n_k \propto k^{-s_P} = k^{-11/3} \ . \tag{3.2.6b}$$

Those solutions can be obtained using the general formulas (3.1.10) $s_P = d + 2m/3$, $s_Q = s_P - \alpha/3$ with $\alpha = 1$, $m = 1$. Thus the index of effective interaction coefficient is equal to unity.

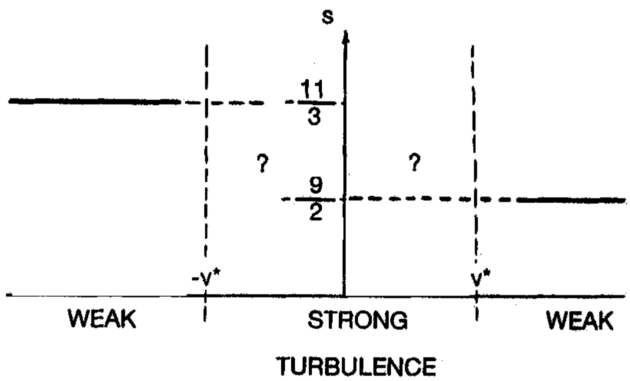

Fig. 3.4. The index of Kolmogorov spectrum for acoustic turbulence

In agreement with the criterion (3.1.22), the solution (3.2.6a) transfers the wave action flux to the long-wave region, and the solution (3.2.6b) transfers the energy flux to the short-wave region. It is interesting to compare (3.2.6b) with the distribution (3.2.5) transmitting the energy flux in the decay case: at $d = 3$ (3.2.5) gives $n_k \propto k^-9/2$. If we write the almost acoustic dispersion law in the form of (1.2.22): $\omega^2(k) = c^2k^2 + vk^4$, then, according to (3.2.5,6b), the s index of the stationary turbulent distribution depends in the short-wave region on v, as shown in Fig. 3.4.

The structure of the stationary spectrum at $|v| \lesssim |v^*|$ (in particular, in the non-dispersion case $v = 0$), when the weak turbulence criterion (2.1.25) is violated, is currently unknown. We shall discuss the available theoretical approaches to the problem of acoustic turbulence in the second volume.

The two-dimensional case (the turbulence of shallow-water gravitational waves) requires special consideration. This is due to the fact that in this region and for waves propagating in a narrow angular cone, the dynamic equations (1.3.11–33) may naturally be reduced to the known Kadomtsev-Petviashvili (KP) equation (1.5.4) (see also [3.23]):

$$\frac{\partial}{\partial x}\left[\frac{\partial \eta}{\partial t} + \sqrt{gh_0}\left(\frac{\partial \eta}{\partial t} + \frac{h_0^2}{6}\frac{\partial^3 \eta}{\partial x^3} + \eta\frac{\partial \eta}{\partial x}\right)\right] = -\frac{1}{2}\sqrt{gh_0}\frac{\partial^2 \eta}{\partial y^2}$$

Here h_0 is the depth, the dynamic variable $\eta(x, y, t)$ describes deviation of fluid surface from the unperturbed state (see Sect. 1.2.5); the motions are assumed to be weakly non-one-dimensional: $|\partial\eta/\partial x| \gg |\partial\eta/\partial y|$. The Kadomtsev-Petviashvili equation possesses a number of remarkable properties owing to its integrability, in particular, an infinite set of conservation laws (see for details [3.24]). The latter generate a series of integrals of motion of the corresponding kinetic equation. However, as shown by *Zakharov and Shulman* [3.25], the dispersion law of the weakly non-one-dimensional sound with negative dispersion (corresponding to the KP equation)

$$\omega(k_x, k_y) = ck_x\left(1 - \frac{k^2h_0^2}{6}\right) + \frac{ck_y^2}{2k_x}$$

is non-degenerate (see Sect. 2.2.1). This implies that it is impossible to find such a function $f(k)$ that is on the resonance surface given by

$$\omega(\boldsymbol{k}) + \omega(\boldsymbol{k}_1) - \omega(\boldsymbol{k}_2) - \omega(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2) = 0 \ ;$$

the same equation should be obeyed by $f(k)$:

$$f(\boldsymbol{k}) + f(\boldsymbol{k}_1) - f(\boldsymbol{k}_2) - f(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2) = 0 \ .$$

For a non-degenerate dispersion law (in contrast to the positive dispersion case), the existence of a sequence of integrals of motion of the kinetic equations leads to a degenerate interaction coefficient, viz., to its vanishing on the resonance surface, see [3,.25]. A curious reader may use a direct (and rather cumbersome) computation to verify that the interaction coefficient (1.1.29b) with $V(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \theta(k_{1x})\theta(k_{2x})\theta(k_{3x})\sqrt{k_{1x}k_{2x}k_{3x}}$, vanishes on the resonance surface. Thus, in the two-dimensional case, one should take into account the higher order terms of the interaction Hamiltonian expansion in terms of the small parameter kh_0. As a result, the coefficient of four-wave interaction will gain the h_0 factor, increasing its scaling index by unity. The Kolmogorov indices therefore are: $s_P = 10/3,\ s_Q = 3$ [3.26].

3.2.2 Media with Two Types of Interacting Waves

This a rather widespread case. It is observed in the interaction of electromagnetic and Langmuir waves in plasma, in the Mandelstam-Brillouin forced scattering upon interaction of Langmuir and ion-sound waves [3.27–29], upon interaction of spin and sound waves in magnets [3.30] and in some other physical systems.

Kinetic Equations and their Solutions. Following *Zakharov and Kuznetsov* [3.28], we shall consider the interaction of high-frequency (HF) waves with the dispersion law $\omega(k)$ and the low frequency (LF) waves with the dispersion law $\Omega(k)$ in the three-wave decay processes (1.3.11–12). The Hamiltonian of such a system

$$\mathcal{H} = \int \omega_k a_k a_k^* d\boldsymbol{k} + \int \Omega_k b_k b_k^* d\boldsymbol{k} + \int \left(V_{k12} b_k a_1 a_2^* + \text{c.c.}\right)\delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2)\, d\boldsymbol{k} d\boldsymbol{k}_1 d\boldsymbol{k}_2$$

leads to the equations of motion linear in the amplitudes of LF waves b_k. Therefore, for applicability of the weak-turbulence approximation (2.1.14c), it is sufficient to require the smallness of HF wave amplitudes. Going over to the statistical description and introducing $\langle a_k a_{k'}^* \rangle = N_k \delta(\boldsymbol{k} - \boldsymbol{k}')$, $\langle b_k b_{k'}^* \rangle = n_k \delta(\boldsymbol{k} - \boldsymbol{k}')$, one can obtain the kinetic equations:

$$\frac{\partial N(\boldsymbol{k}, t)}{\partial t} = \int \left[T(\boldsymbol{k}_2; \boldsymbol{k}, \boldsymbol{k}_1) - T(\boldsymbol{k}_2; \boldsymbol{k}_1, \boldsymbol{k})\right] d\boldsymbol{k}_1 d\boldsymbol{k}_2 \ , \tag{3.2.7a}$$

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = -\int T(\boldsymbol{k};\boldsymbol{k}_1,\boldsymbol{k}_2)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 \ , \tag{3.2.7b}$$

where

$$\begin{aligned} T(\boldsymbol{k}_2;\boldsymbol{k},\boldsymbol{k}_1) =& 2\pi\big|V(k_2,k,k_1)\big|^2\big[N(k_1)n(k_2) - N(k)n(k_1) - N(k)n(k_2)\big] \\ & \times \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\delta(\omega_k - \omega_1 - \Omega_2) \ . \end{aligned}$$

Let us assume the interaction coefficient to be scale-invariant

$$V(\lambda k,\lambda k_1,\lambda k_2) = \lambda^m V(k,k_1,k_2) \ ,$$

and the dispersion laws to be of the power-type

$$\omega(k) = \omega_0 + c_1 k^\alpha, \qquad \Omega(k) = c_2 k^\beta \ . \tag{3.2.8}$$

Obviously, with such choice of $\omega(k)$ the frequency ω_0 does not appear in (3.2.7) and, so that we may without loss of generality use $\omega_0 = 0$.

Equations (3.2.7) conserve the complete number of HF waves and the total wave energy $\varepsilon_k = \omega_k N_k + \Omega_k n_k$, and, consequently, support the thermodynamically equilibrium stationary solutions $N_k = T/(\omega_k+\mu)$, $n_k = T/\Omega_k$. We shall deal with the non-equilibrium (flux) solutions of these equations.

Let us first examine the case of complete scale invariance, when $\alpha = \beta$. Such coincidence occurs, for example, upon the interaction of electromagnetic and sound waves. We shall search for the stationary solutions of (3.2.7) in the power form

$$N_k = A k^x, \qquad n_k = B k^x \ . \tag{3.2.9}$$

In (3.2.7a), we shall map the integration domain of the second integral as determined by the decay conditions into the integration domain of the first integral. For that purpose it is convenient to introduce the coordinates k_x, k_y and the complex variable $w = k_x + ik_y$. The mapping is [3.28]:

$$w = w'\frac{w}{w'}, \quad w_1 = w\frac{w}{w'}, \quad w_2 = w''\frac{w}{w'} \ . \tag{3.2.10}$$

As a result, the integrand is factorized and in a stationary case we have

$$\int T(\boldsymbol{k}_2;\boldsymbol{k},\boldsymbol{k}_1)\Big[1-(k/k_1)^\gamma\Big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 = 0 \ ,$$

where $\gamma = 2m + 2d - \alpha - 2x$.

Equation (3.2.7b) is transformed in a similar way. Multiplication by Ω_k and addition of (3.2.7a) multiplied by ω_k, yields a second equation for the stationary case:

$$\int T(\boldsymbol{k}_2;\boldsymbol{k},\boldsymbol{k}_1)\Big[\omega_k - \omega_1(k/k_1)^{\gamma+\alpha} - \Omega_2(k/k_2)^{\gamma+\alpha}\Big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 = 0 \ .$$

From these equations, two non-equilibrium solutions may be obtained, for which $\gamma = 0$ and $\gamma = -\alpha$ or $x_1 = -m - d + \alpha/2$ and $x_2 = -m - d$. One of the equations coincides with the equation (3.2.7b) in the stationary case

$$\int T(\boldsymbol{k}; \boldsymbol{k}_1, \boldsymbol{k}_2)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 = 0 \ . \tag{3.2.11}$$

That equation defines the relationship between A and B which are constants in the distributions (3.2.9).

Analysis of Solutions of the Kinetic Equations. Let us demonstrate that the first solution ($\gamma = 0$) corresponds to the constant flux of the number of HF waves, and the second one ($\gamma = -\alpha$) corresponds to the constant energy flux.

By integrating (3.2.7a) we find the waves flux

$$P_N = -2^{d-1} \frac{\pi k^{\gamma+d}}{\gamma} \int T(\boldsymbol{k}_2; \boldsymbol{k}, \boldsymbol{k}_1) \left(k^{-\gamma} - k_1^{-\gamma} \right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 \tag{3.2.12}$$

in the case of power distributions (3.2.9).

For the energy $\varepsilon(k,t)$ one can obtain from (3.2.7)

$$\begin{aligned} \frac{\partial \varepsilon(k,t)}{\partial t} = \int \big[& \omega_k T(\boldsymbol{k}_2; \boldsymbol{k}, \boldsymbol{k}_1) - \omega_k T(\boldsymbol{k}_2; \boldsymbol{k}_1, \boldsymbol{k}) \\ & - \Omega_k T(\boldsymbol{k}; \boldsymbol{k}_1, \boldsymbol{k}_2) \big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 \ . \end{aligned} \tag{3.2.13}$$

By integrating (3.2.13) we find the energy flux

$$\begin{aligned} P_\varepsilon = & -2^{d-1} \frac{\pi k^{\gamma+d+\alpha}}{\gamma+\alpha} \int T(\boldsymbol{k}_2; \boldsymbol{k}, \boldsymbol{k}_1) \\ & \times \left(\omega_k k^{-\gamma-\alpha} - \omega_1 k_1^{-\gamma-\alpha} - \omega_2 k_2^{-\gamma-\alpha} \right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 \ . \end{aligned} \tag{3.2.14}$$

Going over to the limits ($\gamma \to 0$ and $\gamma + \alpha \to 0$) in (3.2.12,14) and taking into account the conditions (3.2.11) one can see that the first solution corresponds to a Kolmogorov spectrum with a constant flux of the number of HF waves with $P_\varepsilon = 0$. The second solution corresponds to the Kolmogorov spectrum with the energy flux $P_\varepsilon \neq 0$ and $P_N = 0$. Hence it appears that $A, B \propto P^{1/2}$, which is in full agreement with dimensional analysis.

Finally, let us determine the direction of the fluxes. For this purpose, it should be remarked that one can explicitly find n_k from the equation (3.2.11). Substitution of this distribution into (3.2.12,14) and symmetrization of the integrand give rise to:

$$\begin{aligned} P_N &= \int R_{k123} N_k N_1 N_2 N_3 \left(N_k^{-1} + N_1^{-1} - N_2^{-1} - N_3^{-1} \right) \ln \frac{k k_1}{k_2 k_3} d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3, \\ P_\varepsilon &= \int R_{k123} N_k N_1 N_2 N_3 \left(N_k^{-1} + N_1^{-1} - N_2^{-1} - N_3^{-1} \right) \\ & \qquad \times \left(\omega_k \ln k + \omega_1 \ln k_1 - \omega_2 \ln k_2 - \omega_3 \ln k_3 \right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \ , \end{aligned}$$

where

$$R_{1234} = T_{1324} + T_{3142} + T_{1423} + T_{4132} \;,$$
$$T_{1234} = T(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) = \int d\boldsymbol{k} \frac{U(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) U(\boldsymbol{k}, \boldsymbol{k}_3, \boldsymbol{k}_4)}{\int U(\boldsymbol{k}, \boldsymbol{k}_5, \boldsymbol{k}_6)[N(k_6) - N(k_5)]\, d\boldsymbol{k}_5 d\boldsymbol{k}_6} \;,$$
$$U_{123} = (2k)^d \frac{\pi^2}{8} \left| V_{123} \right|^2 \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 + \boldsymbol{k}_3) \delta(\Omega_1 - \omega_2 + \omega_3) \;,$$
$$\omega_i = \omega(k_i), \qquad \Omega_j = \Omega(k_j), \qquad N_i = N(k_i) \;.$$

The resulting expressions formally coincide with the fluxes in the four-wave interaction; the sign of the interaction coefficient R_{1234} is opposite to sign $(x + \alpha)$. Therefore, using a similar procedure as in Sect. 3.1, we obtain:

$$\operatorname{sign} P_N = \operatorname{sign}(x_1 + \alpha) \;. \tag{3.2.15}$$

For the interaction of electromagnetic and sound waves this procedure leads to ($\alpha = 1$, $m = -1/2$, $d = 3$) [3.27,28]:

$$x_1 = -2, \quad P_N < 0, \qquad x_2 = -5/2, \quad P_\varepsilon > 0 \;.$$

Thus, we have in this case one Kolmogorov solution (with the constant flux of the number of HF waves) may be realized in the long-wave region where P_N is directed inwards , and another one in the short-wave region where the energy flows outwards.

Diffusion Approximation. The homogeneity of the kernels of the kinetic equations (3.2.8) and, as a consequence, availability of exact solutions in the form of Kolmogorov spectra are first of all due to coincidence of indices α and β. This property is not observed in most examples important for applications. Among these applications is first of all the interaction of HF Langmuir waves with low frequency ion-sound waves. A typical situation for such turbulence corresponds to the frequency $\omega_k \gg \Omega_k$, i.e., the characteristic scale of the variation of the HF wave frequency is small. In this case the integrands in (3.2.7) are nearly scale-invariant. In such a situation (the so-called diffusion approximation) the equations may be simplified.

By expanding in (3.2.7) the frequency δ-function in a series of Ω and assuming isotropy of wave distributions N_k and n_k, we obtain

$$\frac{\partial n_k}{\partial t} = \int \tilde{T}(\boldsymbol{k}; \boldsymbol{k}_1, \boldsymbol{k}_2)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 \;, \tag{3.2.16}$$

$$\frac{\partial N_k}{\partial t} + \operatorname{div} \boldsymbol{p}_N = 0 \;, \tag{3.2.17}$$

$$\frac{\partial \varepsilon_k}{\partial t} = -\operatorname{div}(\omega_k \boldsymbol{p}_N) + \int \left(\Omega_k \tilde{T}_{k12} - \Omega_1 \tilde{T}_{1k2} \right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 \;, \tag{3.2.18}$$

where

$$\tilde{T}_{k12} = 2\pi \left|V_{k12}\right|^2 \left(N_1^2 + n_1 \Omega_k \frac{\partial N_1}{\partial \omega_1} \right) \delta(\boldsymbol{k} - \boldsymbol{k}_1 + \boldsymbol{k}_2)\delta(\omega_1 - \omega_2) \ ,$$
$$\boldsymbol{p}_N = \frac{\boldsymbol{k}}{k} \int \frac{\Omega_2}{\omega_k} \tilde{T}_{2k1} \, d\boldsymbol{k}_1 d\boldsymbol{k}_2$$

is the density of the flux of HF quasi-particles. These equations represent two independent systems (3.2.16,17) and (3.2.17,18) whose identity is readily verified by direct calculation. Like the starting system (3.2.7,13), equations (3.2.16–18) have solutions in the form of the Rayleigh-Jeans distributions, with vanishing fluxes P_N and P_ε $[\tilde{T}(\boldsymbol{k}; \boldsymbol{k}_1, \boldsymbol{k}_2) = 0]$.

Let us consider the solutions with constant fluxes P_N and P_ε. In the case of power solutions:

$$N_k = Ak^x, \qquad n_k = Bk^x \omega_k / \Omega_k \ . \tag{3.2.19}$$

these quantities are constant. For the spectra corresponding to P_N =const, the x index is directly determined by calculating the powers in (3.2.17):

$$x_1 = -m - d + \alpha - \beta/2 \ . \tag{3.2.20}$$

Here $A, B \propto P^{1/2}$

To find a second solution corresponding to P_ε =const, we shall subject (3.2.18) to a transformation similar to (3.2.10), which give rise to

$$x_2 = -m - d - (\beta - \alpha)/2, \quad A, B \propto P^{1/2} \ . \tag{3.2.21}$$

For both solutions the relationship between constants A and B is determined from the stationary equation (3.2.17). It yields a vanishing flux: $P_\varepsilon = 0$ at P_N =const and vice versa. The expressions for the signs which do not include the index β, remain the same. This is due to the fact that from the equations (3.2.17,18) the $\Omega_k n_k$ value is eliminated instead of n_k at $\alpha = \beta$.

Turbulence Locality. Our solutions of both diffusion and exact equations are valid, (i) only in the inertial interval, which is an intermediate range of the k-space where there is neither damping nor pumping and, (ii), when the wave interaction in this region is local. Therefore, the local Kolmogorov spectra are independent of the detailed structure of instability growth-rate and damping decrement but are determined only by the flux magnitudes.

For the spectrum to be local, it is sufficient to prove the convergence of integrals in the diffusion equations (3.2.16–18). This in turn requires the necessity to know the asymptotics of interaction coefficients $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$. We shall assume

$$V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) \to Dk^{m-m_1} k_1^{m_1} \quad \text{at} \quad k_1 \gg k$$

to hold. Then the upper convergence (at $k \to \infty$) of integrals is achieved at

$$x + 2m_1 + d - 2\alpha < -1 \ ,$$

while the lower convergence at

$$x + \alpha + \beta + d + 2(m - m_1) > 1 .$$

It should be noted that the index (3.2.21) lies exactly in the middle of the locality strip.

Some Particular Examples. We shall consider the interaction of HF Langmuir and spin waves with the sound waves:

$$\omega_k = \omega_0 + \mu k^2, \quad \Omega_q = cq . \tag{3.2.22}$$

In the diffusion approximation, $\mu k \gg c$ and in each elementary act $\omega(\boldsymbol{k}) = \omega(\boldsymbol{k} - \boldsymbol{q}) + \Omega(\boldsymbol{q})$ the HF wave with the wave vector $\boldsymbol{k}$ interacts with sound with the wave vector $q \approx 2k$, directed backwards.

For spin waves in ferromagnets, the coefficient of interaction with sound is equal to [3.30]:

$$|V_{2k1}|^2 = \frac{V_1^2}{k} \left\{ \lambda_1 \left[(\boldsymbol{p}\boldsymbol{k}_1)(\boldsymbol{k}\boldsymbol{k}_2) + (\boldsymbol{p}\boldsymbol{k}_2)(\boldsymbol{k}\boldsymbol{k}_1) \right] + \lambda_2 (\boldsymbol{k}_1\boldsymbol{k}_2)(\boldsymbol{k}\boldsymbol{p}) \right\}^2 .$$

Here V_1 is a dimensional constant: $\boldsymbol{p}$ is the vector of sound polarization; λ_1, λ_2, the dimensionless phenomenological constants of magnetoelastic interaction, they are of the order of unity. In this case we have $\alpha = 2,\ \beta = 1,\ m = 5,\ m_1 = 1$. The distributions with a constant spin wave flux into the region of large scales are

$$N_k \simeq P_N^{1/2} V_1^{-1} k^{-13/2}, \quad n_k \simeq \frac{\mu}{c} P_N^{1/2} V_1^{-1} k^{-11/2} ,$$

and with the dispersive part of energy flux $\varepsilon_k = \mu k^2 + ck$:

$$N_k \simeq P_\varepsilon^{1/2} V_1^{-1} k^{-15/2}, \quad n_k \simeq \frac{\mu}{c} P_\varepsilon^{1/2} V_1^{-1} k^{-13/2} .$$

All these distributions are local.

In antiferromagnets, the coefficient of the magnon-phonon interaction is

$$|V_{2k1}|^2 = V_2^2 k \varphi(\boldsymbol{p}, \boldsymbol{k}/k) , \tag{3.2.23}$$

where φ is a dimensionless function depending on mutual orientation of the external magnetic field, wave vector of the sound and its polarization [3.30]. Here $m = 1/2,\ m_1 = 0$. The dispersion law for spin wave (3.2.18)

$$\omega_k = \sqrt{\omega_0^2 + v^2 k^2}$$

is only in the region $k \to 0$ and $k \to \infty$ reduced to the cases considered in this subsection. At $k \gg \omega_0/v$, we have the scale-invariant case $\alpha = \beta$ with Kolmogorov solutions having the form (3.2.9), where $x = -7/2$ holds for the distribution transferring the "energy" flux to the short-wave region. Another solution (with magnon flux) corresponds to $x = -3$. Both distributions are local. At $\omega_0 \gg kv \gg c$, we have the case (3.2.2). The distributions with magnon flux to the long-wave region have the form (3.2.20):

$$N_k \simeq P_N^{1/2} V_2^{-1} k^{-2}, \quad n_k \simeq \frac{v^2}{\omega_0 c} P_N^{1/2} V_2^{-1} k^{-1} , \tag{3.2.24}$$

and those possessing an "energy" flux:

$$N_k \simeq P_\varepsilon^{1/2} V_2^{-1} k^{-3}, \quad n_k \simeq \frac{v^2}{\omega_0 c} P_\varepsilon^{1/2} V_2^{-1} k^{-2} . \tag{3.2.25}$$

The power distributions for the interacting Langmuir and ion-sound waves have exactly the same form. Really, the coefficient of interaction (1.3.13) has the same form as (3.2.23) and the dispersion law coincides with (3.2.22). It is easy to see that the solutions (3.2.24,25) are local [3.28].

The Kolmogorov Spectrum Close to the Thermodynamically Equilibrium. One interesting fact should be noted: the solutions (3.2.24) $N_k \propto k^{-2}$ and $n_k \propto k^{-1}$ coincide with ones for the thermodynamic equilibrium and can consequently transmit only zero fluxes. What is the structure of the flux spectra in this case? An answer to this question will be given following *Kanashov and Rubenchik* [3.31]. Let us substitute into (3.2.16,17) the interaction coefficient (1.3.13), introduce a new variable $\omega = k^2$, and use $z(\omega) = n(2k)2kc/\mu$. Then renormalization of time for the dimensional constant in the interaction coefficient gives:

$$\frac{\partial N}{\partial t} = \frac{2}{\sqrt{\omega}} \frac{d}{d\omega} \left(2\frac{\partial N}{\partial \omega} \int_0^\omega z(\omega')\omega' \, d\omega' + \omega^2 N^2 \right) , \tag{3.2.26a}$$

$$\frac{\partial z}{\partial t} = V = \frac{\sqrt{\omega}}{2} \left(-zN + \int_\omega^\infty N^2 \, d\omega' \right) . \tag{3.2.26b}$$

To obtain (3.2.26), we have used unity for the angular dependence of the interaction coefficient $\cos^2\theta_{12}$. However, since we shall deal only with the behavior of the distributions in the region $\omega \to 0$, the resulting asymptotics will also satisfy the exact equations (3.2.16,17). Let us consider the stationary solutions (3.2.26); due to conservation of the number of HF waves, the equation (3.2.26a) is integrated once to yield

$$2\frac{\partial N}{\partial \omega} \int_0^\omega z(\omega')\omega' \, d\omega' + \omega^2 N^2 = P_N , \tag{3.2.27a}$$

$$zN = \int_\omega^\infty N^2 \, d\omega' . \tag{3.2.27b}$$

Here the constant P_N is the flux of the number of HF waves. It is seen that at $P_N = 0$, the system (3.2.27) has a simple solution $z = 1$, $N = 1/\omega$, corresponding to (3.2.24). Equations (3.2.27) may be reduced to a system of ordinary differential equations. Let us differentiate (3.2.27b) with respect to ω, then substitute N^2 thus found into (3.2.27a) and integrate it from 0 to ω:

$$\omega P_N = 2N \int_0^\omega z(\omega')\omega' \, d\omega' - N\omega^2 z \\ -N\left(2\int_0^\omega z(\omega')\omega' \, d\omega' - \omega^2 z\right)_{\omega=0} . \tag{3.2.28}$$

Expecting that the resulting spectra will not grow too quickly at $\omega \to 0$ [see below (3.2.31)], we shall assume that the last term to become zero. Then, introducing the dimensionless variables $z \to z\sqrt{P_N}$, $N = y\sqrt{P_N}/\omega$, we obtain

$$\frac{d}{d\omega}\frac{\omega^2}{y} = -\omega^2\frac{dz}{d\omega} , \quad \frac{d}{d\omega}\frac{zy}{\omega} = -\frac{y^2}{\omega^2} . \tag{3.2.29}$$

The system (3.2.29) is homogeneous with regard to the variable ω and, therefore, introducing the variable $t = \ln\omega$, we shall reduce it to the form :

$$\frac{dz}{dy} = \frac{z+y}{y(y^2 - zy - 2)} , \tag{3.2.30a}$$

$$\frac{dz}{dy} = -\frac{z+y}{zy+1} . \tag{3.2.30b}$$

admitting a simple investigation. If the wave vector goes to zero then $t \to -\infty$. It is seen from the equation (3.2.30b) that in this case z increases monotonically. The behavior of the solutions of (3.2.30a) is shown in Fig. 3.5.

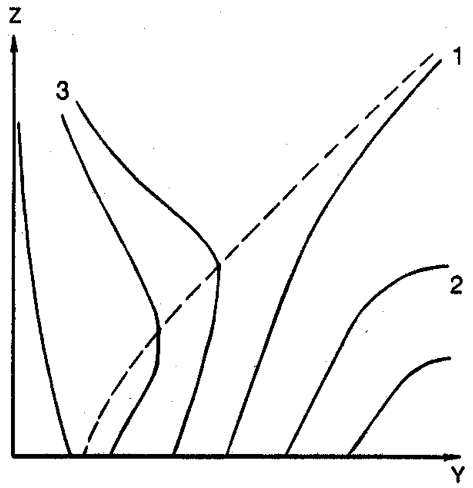

Fig. 3.5. For different integration constants z is plotted as a function of y

The dotted line corresponds to $y^2 - zy - 2 = 0$. Above it the slope of curves is negative, and below they are positive. It is easy to see that the curves 2 having the asymptotics $y \gg z$, describe the spectra of Langmuir oscillations with singularities at finite k. On the curves of type 3 the inequality $y \ll z$ is satisfied. Their spectrum of Langmuir oscillations tends to zero, $N \propto \omega$ at $\omega \to 0$. Thus the considered flux solution corresponds to a separatrix (line 1). Asymptotically, $y \simeq 2\sqrt{\ln k^2}$, $z \simeq y + 4/y$, $y \gg 1$. This asymptotic solution also satisfies the initial equations (3.2.16,17) with exact interaction coefficients, which may be verified by direct substitution.

We have shown that there are stationary non-power solutions (3.2.26) corresponding to the spectra of high-frequency and sound waves growing while k decreases. In terms of initial variables we have

$$N_k \propto \frac{y}{\omega}\ , \qquad n_k \propto \frac{z}{k}\ .$$

Since z and y vary logarithmically , the forms of the spectra differ only slightly from (3.2.24), though, as opposed to them, they provide in contrast to them the energy flux to small k.

Let us now discuss how the solutions (3.2.31) and (3.2.20)may be matched. The latter should represent asymptotics at $\omega \to \infty$:

$$N \to 2A\omega^{-3/2}\ , \qquad z \to A\omega^{-1/2}\ .$$

We shall consider the excitation of Langmuir oscillations by a small increment $\gamma_k = \gamma\delta(k - k_0)$. The solutions obtained by us contain four arbitrary constants: P_N, A and the coordinates of the solution on the integral curves at $k = k_0$. Since the spectra are described by the second-order differential equation (3.2.26a), at $k = k_0$, n_k and N_k should be continuous. Integrating the kinetic equation for the Langmuir oscillations, we obtain the condition $P_N = \gamma k_0^2 N(k_0)$. The solutions (3.2.25) correspond to the vanishing flux of the wave number P_N. Similarly to (3.2.28), one can obtain for them

$$N\left(-2\int_0^{\omega} z(\omega')\omega'\, d\omega' + \omega^2 z\right) + \frac{2}{3}A^2 = 0\ .$$

Comparing this equation with (3.2.28), we find:

$$A = \sqrt{3P_N\omega(k_0)/2}\ .$$

The four conditions obtained allow us to match the solutions. Of course, since the integral curves may be found only numerically, it is impossible to give explicit formulas to match the solutions, whereas the last formula provides an unambiguous relationship between the asymptotic solutions in the region of $k \gg k_0$ and $k \ll k_0$.

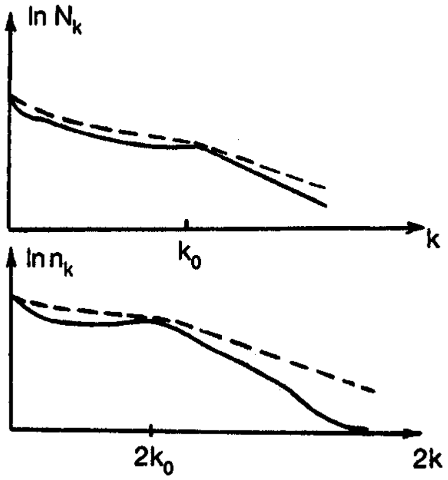

Fig. 3.6. The steady state distributions of the Langmuir waves N_k and ion sound n_k obtained by numeric simulation [3.29]

Fig. 3.6 represents the results of numerical solution of (3.2.7) with the interaction coefficients (3.1.13), (3.2.23) for isotropic excitation of HF waves with $k = k_0$ and dissipation at $k > k_0$ (simulating the Landau damping of waves on particles) [3.29]. Despite the narrowness of the source, the distributions turn out to be rather smooth and correspond qualitatively to the solutions (3.2.25) and (3.2.31) depicted by the dotted line in Fig. 3.6. The substantial deviation from the Kolmogorov solutions at $k > k_0$ is due to the presence of dissipation.

3.3 Kolmogorov Spectra of Weak Turbulence in Anisotropic Media

Another Woozle joined them!

A. MILNE The World of Winnie-the-Pooh

3.3.1 Stationary Power Solutions

In this section we shall discuss situations where the Kolmogorov solutions in anisotropic media may be calculated explicitly. In the absence of isotropy, the dispersion law and the interaction coefficient depend differently on different wave vector components. The Kolmogorov solutions can be found if in each component of k these dependences are separately scale-invariant. Such solutions were first obtained by *Kuznetsov* [3.32]. Let us consider, for example, an axially symmetric medium, like a plasma in a magnetic field. We shall direct the z axis of the choosen cylindric coordinate system along the specified direction into the $\boldsymbol{k}$-space. Let p denote the longitudinal component of the wave vector k_z and $\boldsymbol{q}$ the transverse component $\boldsymbol{k}_\perp$. We assume the dispersion law and the interaction coefficient of the three-wave interaction to be *bihomogeneous functions* of p and q (see Sects. 1.3,4):

$$\omega(p,q) = p^a q^b \ . \tag{3.3.1a}$$

$$V(\lambda p, \lambda p_1, \lambda p_2, \mu \boldsymbol{q}, \mu \boldsymbol{q}_1, \mu \boldsymbol{q}_2) = \lambda^u \mu^v V(p, p_1, p_2, \boldsymbol{q}, \boldsymbol{q}_1, \boldsymbol{q}_2) \ . \tag{3.3.1b}$$

This is just the case when the stationary Kolmogorov distributions are power functions of the components of the wave vector and may be found as exact solutions of the kinetic equation.

Let us average over the angle in our cylindric coordinate system. The kinetic equation acquires then the form

$$\begin{aligned} 2\pi q \frac{\partial n(p,q,t)}{\partial t} = & \int [\mathcal{R}(p,p_1,p_2,q,q_1,q_2) - \mathcal{R}(p_1,p,p_2,q_1,q,q_2) \\ & - \mathcal{R}(p_2,p_1,p,q_2,q_1,q)]\, dp_1 dp_2\, dq_1 dq_2 \\ \equiv & 2\pi q I(p,q) \ , \end{aligned} \tag{3.3.2a}$$

where

$$\begin{aligned} \mathcal{R}(p,p_1,p_2,q,q_1,q_2) = & U(p,p_1,p_2,q,q_1,q_2)[n(p_1,q_1)n(p_2,q_2) \\ & - n(p,q)n(p_1,q_1) - n(p,q)n(p_2,q_2)] \\ & \times \delta(p-p_1-p_2)\delta(p^a q^b - p_1^a q_1^b - p_2^a q_2^b) \ . \end{aligned} \tag{3.3.2b}$$

Without loss in generality one can consider U to depend only on p_i and the modules of $\boldsymbol{q}_i$. Then

$$U = (2\pi)^3 |V|^2 \frac{q q_1 q_2}{\Delta_2} ,$$

where the quantity $2\pi/\Delta_2$ is the result of averaging over angles of the two-dimensional δ-function of wave vectors which we encountered already in Sect. 3.1.2. In the given case

$$2\Delta_2 = \sqrt{2(q_1^2 q_2^2 + q^2 q_1^2 + q^2 q_2^2) - q^4 - q_1^4 - q_2^4} . \tag{3.3.3}$$

Thus, U is a function homogeneous in p_i of $2u$ power and in q_i of $2v$ power. Besides, $U = 0$ if segments of lengths q, q_1 and q_2 cannot be used to construct a triangle with sides of these length. Due to this, integration in the plane q_1, q_2 is performed over the shaded region shown in Fig. 3.7.

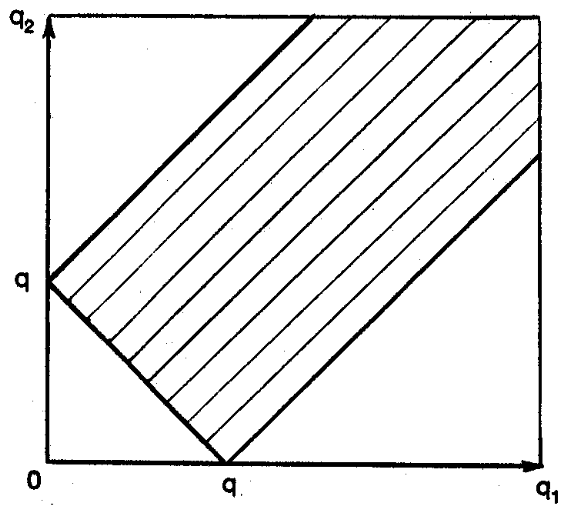

Fig. 3.7. The integration region for the collision integral (3.3.2)

Due to the presence of two δ-functions (3.3.2) has two integrals of motion: the energy $\int p^a q^b n(p,q)\, dp d\boldsymbol{q}$ and the z-component of momentum $\int pn(p,q)\, dp d\boldsymbol{q}$). Consequently, in addition to the Kolmogorov solutions with a constant energy flux

$$P \propto |p|^{2+2u} q^{4+2v} n^2(p,q), \quad n(p,q) \propto P^{1/2} |p|^{-1-u} q^{-2-v} \tag{3.3.4}$$

one can expect here stationary distributions carrying the constant momentum flux

$$\begin{aligned} R &\propto |p|^{3+2u-a} q^{4+2v-b} n^2(p,q), \\ n(p,q) &\propto R^{1/2} |p|^{(a-2u-3)/2} q^{(b-2v-4)/2} . \end{aligned} \tag{3.3.5}$$

We shall seek the stationary solutions of the equation (3.3.2) in the power form:

$$n(p,q) \propto |p|^{-x} q^{-y} \ . \tag{3.3.6}$$

Throughout this section, we shall take the z-components of wave vectorsto be positive ($p, p_1, p_2 > 0$), i.e., we consider only waves travelling in one direction along the specified axis. In this case, the distributions of the form (3.3.6) have a nonzero momentum

$$\int d\boldsymbol{q} \int_0^\infty pn(p,q)\, dp \neq 0 \ ,$$

and both stationary solutions (3.3.4–5) must exist.

Indeed, the bi-homogeneity of the collision integral (3.3.2) in p and q, like in the isotropic case, allows the transformation of the second and third terms to the first one (with accuracy of a factor). These transformations found by *Kuznetsov* [3.32] represent an analogue of separate Zakharov transformations for the longitudinal component of wave vector and the modulus of the transverse components [cf. (3.1.14)]:

$$p_1 = p^2/p_1', \quad p_2 = (p/p_1')p_2' \ , \tag{3.3.7a}$$

$$q_1 = q^2/q_1', \quad q_2 = (q/q_1')q_2' \ . \tag{3.3.7b}$$

Besides, it is clear that

$$p = (p/p_1')p_1', \quad q = (q/q_1')q_1' \ .$$

One can verify that (3.3.7b) transforms the integration domain over q_1, q_2 into itself. In this case the second term in (3.3.2) is transformed to the first one with an accuracy of a factor. A similar transformation with the third term gives, in the stationary case, an equation

$$\begin{aligned} &\int U(p,p_1,p_2,q,q_1,q_2)\delta(p-p_1-p_2)\delta(p^a q^b - p_1^a q_1^b - p_2^a q_2^b) \\ &\quad \times \left[(p_1p_2)^{-x}(q_1q_2)^{-y} - (pp_1)^{-x}(qq_1)^{-y} - (pp_2)^{-x}(qq_2)^{-y}\right] \\ &\quad \times \left[1 - \left(\frac{p}{p_1}\right)^r \left(\frac{q}{q_1}\right)^s - \left(\frac{p}{p_2}\right)^r \left(\frac{q}{q_2}\right)^s\right] dp_1 dp_2 dq_1 dq_2 = 0 \ , \end{aligned} \tag{3.3.8}$$

where $r = 2(1+u-x) - a,\ s = 2(2+v-y) - b$.

Equation (3.3.8) has four solutions. Indeed, the first square bracket vanishes either due to of the frequency δ-function by choosing

$$x_1 = a, \quad y_1 = b, \qquad n(p,q) \propto \omega^{-1}(p,q) \ , \tag{3.3.9a}$$

or due to the δ-function in the z-components of wave vector:

$$x_2 = 1, \quad y_2 = 0, \qquad n(p,q) \propto p^{-1} \ . \tag{3.3.9b}$$

The solutions (3.3.9) are the limiting cases of the general thermodynamically equilibrium solution $n(p,q,T) = T/[\omega(p,q) + cp]$, see (2.2.13).

Using the frequency δ-function to evaluate the second square bracket to zero, we set set $r = -a,\ s = -b$, which gives

$$x_3 = 1 + u, \quad y_3 = 2 + v. \tag{3.3.10a}$$

In fact we have obtained (3.1.9) for the relationship between the solution indices and the interaction coefficient separately for different wave vector components. Indeed, for the longitudinal component p the dimension of the k-space is equal to a unity, and for q, it is equal to two.

Finally, using the δ-function in p, we obtain a fourth solution corresponding to $r = -1,\ s = 0$:

$$x_4 = (3 + 2u - a)/2, \quad y_4 = (4 + 2v - b)/2\ . \tag{3.3.10b}$$

The solutions corresponding to (3.3.9) are the Kolmogorov solutions. Comparing them with (3.3.4-5), we see that (3.3.10a) corresponds to the constant energy flux

$$n(p, q) = \lambda_1 P^{1/2} |p|^{-1-u} q^{-2-v} \tag{3.3.10c}$$

and the solution (3.3.10b), to the constant flux of the z-component of momentum

$$n(p, q) = \lambda_2 R^{1/2} p^{(a-2u-3)/2} q^{(2v+4-b)/2}\ . \tag{3.3.10d}$$

It should be noted that the solution of (3.3.10c) also exists in the case when p varies from $-\infty$ to ∞.

The distributions (3.3.10) are only local systems solutions of (3.3.2), i.e., when integral converges for all singularities of a sub-integrand. This has to be verified directly for every particular case.

3.3.2 Fluxes of Integrals of Motion and Families of Anisotropic Power Solutions

Let us now discuss the directions and values of the transferred fluxes. Let us start with the distributions transferring an energy flux [3.33]. Symmetry considerations show the energy flux vector to two components: P_z and $P_\perp$ ($P_\phi = 0$). The stationary kinetic equation for power solutions (3.3.6) may be written in the form

$$\begin{aligned} \operatorname{div} \boldsymbol{P}_k &= \frac{\partial P_z}{\partial p} + \frac{1}{q}\frac{\partial}{\partial q}(q P_\perp) = -\omega(p, q) I(p, q) \\ &= -\lim_{(x,y)\to(x_3,y_3)} \left[q^{2(y_3-y)-2} |p|^{2(x_3-x)-1} I(x, y) \right] , \end{aligned} \tag{3.3.11a}$$

where $I(x, y)$ is a p- and q-independent integral derived from $I(p, q)$ (3.3.2,8) similarly to $I(s)$ in (3.1.13):

$$I(x, y) = \omega(p, q) I(p, q) |p|^{2(x-x_3)+1} q^{2(y-y_3)+2}\ .$$

Using the δ-function representation mentioned in Sect. 3.1 $\lim_{\epsilon\to 0}|x|^{\epsilon-1} = 2\delta(x)$, we shall find the explicit form of the singularities in (3.3.11a);

$$\frac{\partial P_z}{\partial p} + \frac{1}{q}\frac{\partial}{\partial q}(qP_\perp) = \frac{2}{q^2}\frac{\partial I}{\partial x}\delta(p) + \frac{2\pi}{p}\frac{\partial I}{\partial y}\delta(\boldsymbol{q}) \ . \qquad (3.3.11b)$$

The derivatives $\partial I/\partial x$ and $\partial I \partial y$ in (3.3.11b) are calculated at $x = x_3 = 1 = u$, $y = y_3 = 2 + v$.

We see that the external sources here must be nonzero in the plane $p = 0$ and on the line with $q = 0$, where the distribution (3.3.10a,c) has singularities. In the rest $\boldsymbol{k}$-space, $\operatorname{div}\boldsymbol{P} = 0$.

Due to the homogeneous dependence of all quantities on p and q, the components of the energy flux should also be the power functions p and q. The equation (3.3.11b) has a single solution of such a form:

$$P_\perp = \frac{A}{q|p|}, \quad P_z = \frac{Bp}{q^2|p|}, \quad A = \left(\frac{\partial I}{\partial y}\right)_{x_3,y_3}, \ B = \left(\frac{\partial I}{\partial x}\right)_{x_3,y_3} . \qquad (3.3.12)$$

Thus, the dependence of components of the energy flux vector on the ones of the wave vector proved to be universal, in every particular case only the A and B constants are different. The complete vector of energy flux in the $\boldsymbol{k}$-space is directed at some angle to the wave vector:

$$\frac{P_z}{P_\perp} = \frac{p}{q}\frac{A}{B} = \frac{k_z}{k_\perp}\frac{\partial I/\partial y}{\partial I/\partial x} \ . \qquad (3.3.13)$$

The case is similar to the case of the momentum flux components. Indeed, in this second-rank tensor, only two components are nonzero: R_{zz} and $R_{z\perp}$, the fluxes of the z-component of momentum, along and across the z-axis, respectively. Taking the z-component of the equation for momentum conservation $(\operatorname{div}\boldsymbol{R})_z = -pI_k$ one can construct an equation for them, which coincides with (3.3.11):

$$\frac{\partial R_{zz}}{\partial p} + \frac{1}{q}\frac{\partial}{\partial q}\left(qR_{z\perp}\right) = -\lim_{(x,y)\to(x_4,y_4)}\left[q^{2(y_4-y)-2}p^{2(x_4-x)-1}I(x,y)\right]$$
$$= \frac{2}{q^2}\frac{\partial I}{\partial x}\delta(p) + \frac{2\pi}{p}\frac{\partial I}{\partial y}\delta(\boldsymbol{q}) \ .$$

Thus, the components of momentum flux tensor also show a universal dependence (3.3.12) on the components of the wave vector :

$$R_{z\perp} = \frac{A}{q|p|}, \ R_{zz} = \frac{Bp}{q^2|p|}, \quad A = \left(\frac{\partial I}{\partial y}\right)_{x_4,y_4}, \ B = \left(\frac{\partial I}{\partial x}\right)_{x_4,y_4} . \qquad (3.3.14)$$

As we see, in anisotropic case the fluxes of conserving quantities are also specified by the derivatives of collision integral with regard to the solution index.

The presence of two indices (x and y) in power distributions (3.3.6) implies that the family of stationary solutions of the equation (3.3.8) proves to be richer than the set (3.3.9–10), as it was pointed out by *Kanashov* [3.33]. Let us consider the plot of the function $I = I(x, y)$ representing a surface in the space $I(x, y)$. The points in which the surface touches or intersects the plane $I = 0$ will be the power indices of stationary solutions of the kinetic equation. It is easily and directly verified that in the points (x_i, y_i) $i = 1, \dots, 4$ (see 3.3.9–10), the derivatives $\partial I/\partial x$ and $\partial I/\partial y$ in the general are nonzero. This means that in these points the surface $I = I(x, y)$ does not touch the plane $I = 0$ but intersects it at certain lines. In the small neighborhood of the point (x_i, y_i), the intersection line may be assumed to be a straight line given by

$$(x - x_i)\left(\frac{\partial I}{\partial x}\right)_{x_i, y_i} + (y - y_i)\left(\frac{\partial I}{\partial y}\right)_{x_i, y_i} = 0 \; . \tag{3.3.15}$$

The set of indices of stationary solutions situated on the intersection curve is really limited by the locality condition specifying a certain region in the plane (x, y). However if a solution with indices (x_i, y_i) is local, the solutions with indices close to it will also be local.

These families of power solutions $n \propto |p|^{-x} q^{-y}$ also correspond to the family of solutions of the equation $\operatorname{div} \boldsymbol{P} = 0$ at $p \neq 0$, $q \neq 0$ generalizing (3.3.12,14):

$$\begin{aligned} P_\perp &= A|p|^{2(x_3 - x) - 1} q^{2(y_3 - y) - 1} , \\ P_z &= Bp|p|^{2(x_3 - x) - 1} q^{2(y_3 - y) - 2} \; . \end{aligned} \tag{3.3.16}$$

It is easy to see that (3.3.16) satisfies the equation $\operatorname{div} \boldsymbol{P} = 0$ at $p \neq 0$, $q \neq 0$ if

$$A(y - y_3) + B(x - x_3) = 0 \; .$$

This condition coincides with (3.3.15).

It should be pointed out that locality region in the (x, y)-plane may be either two- or one-dimensional. In the latter case (as for Rossby waves — see below Sect. 5.3.5), it might be impossible to define the stationary solution with indices (x, y) different from (x_i, y_i). The definition of fluxes might be impossible as well because the derivatives ($\partial I/\partial x$ or $\partial I/\partial y$) of collision integral may not exist. Nevertheless, the collision integral converges on the Kolmogorov solution, so that we have stationary locality.

To conclude this subsection we list some naturally arising questions, the answers to which are currently not available. The set of stationary indices (x, y) has been computed only in a particular case. In the work by *Balk and Nazarenko* [3.34], the set of stationary indices was found numerically for the drift-type waves described in Sect. 1.3.2. Two curves sections were obtained on the plane (x, y). One of the curves passes through the two points corresponding to the equilibrium solutions (3.3.9) and another one through the points corresponding to the Kolmogorov solutions (3.3.10). (The authors of [3.34] also obtained the stationary index curves outside the locality region. In doing so they substituted the integral by a sum of integrals which they used in computer calculation).

Which of the stationary solutions is realized in the presence of a particular source? As seen from (3.3.12-14), everything depends on the relation between powers of sources located at $q = 0$ and at $p = 0$. If the ratio of these powers is A/B the solution (3.3.10c) is realized; if C/D, it is given by (3.3.10d). When the power ratio is equal to a number c out of the interval $(A/B, C/D)$, then one of solutions of the type (3.3.16) with x and y indices located on the curve passing through (x_3, y_3) and (x_4, y_4) should be realized. The relation $(\partial I/\partial x)/(\partial I/\partial y) = c$ must be fulfilled. But if the power ratio can not be represented in the form of a ratio of derivatives in some point (x, y) on the stationary curve in the locality region, the stationary solution must be, generally speaking, the non-power type. We note that the physical meaning and occurrence of the $n \propto p^{-x}q^{-y}$ type distribution are not quite clear, since a source capable of exciting waves simultaneously on a plane and a line in the $\boldsymbol{k}$-space [see (3.3.11)] is still considered to be rather exotic. However, it is possible that on wave excitation by a quite an arbitrary source localized in the region $p \simeq p_0$, $q \simeq q_0$, the asymptotics of a stationary distribution at $p \gg p_0$, $q \gg q_0$ will be described by some of solutions (3.3.10). For a source generating fluxes of both conserved quantities, i.e. energy and momentum, in follows from a dimensional analysis that the two-flux solution may be written in the form

$$n(p, q) = P^{1/2}|p|^{-1-u}q^{-2-v}f(\xi) \,, \tag{3.3.17}$$

where $\xi = R|p|^{a-1}q^b/P$ is a dimensionless parameter specifying the relation between the fluxes. At $\xi \to 0$, (3.3.17) should coincide with (3.3.10c), therefore $f(\xi) \to$const. In the opposite limit, $n(p, q)$ may depend only on the momentum flux : $f(\xi) \to \sqrt{\xi}$ at $\xi \to \infty$. Thus, the two-flux solution may have power asymptotic only in the regions of $\boldsymbol{k}$-space that are sufficiently remote both from the source and the surface given by the equality $\xi = 1$. The function $f(\xi)$ has not been yet calculated for any particular case either.

In Sects. 1.3,4 we gave examples of physical systems in which the dispersion law and the interaction coefficient are bihomogeneous functions of the wave vector components, i.e., they satisfy the equations (3.3.1). The examples of the Kolmogorov spectra in anisotropic media will be described in Chapt. 5.

The attentive reader has probably noted that in all the examples listed in Sects. 1.3,4 the bihomogeneity (3.3.1) was observed only for the waves whose wave vectors are located within a narrow angle range in the $\boldsymbol{k}$-space. For Kolmogorov solutions to be valid, one should prove in every case that the waves inside the angle range interact with each other by far stronger than with waves from the rest $\boldsymbol{k}$-space.

3.4 Matching Kolmogorov Distributions with Pump and Damping Regions

In the previous sections (3.1–3), we have obtained the universal nonequilibrium distributions (3.1.5–6), (3.3.3–6,9,19–21), (3.3.10), which reduce the corresponding collision integrals to zero. For each of these solutions we have found a relationship between the amplitude of distribution and the amount of flux it transmits[see (3.1.13.22), (3.2.12,14), and (3.3.13)]. In this section we shall go further and discuss, first of all, the question of how to find the flux carried away by the Kolmogorov solution fir given amplitude and spectral characteristics of a wave source. The role of the source will be played, as mentioned in Sect. 2.2.3, by the increment of some instability, i.e., the positive part of the function $\Gamma(k)$ [see below (3.4.1)]. To answer this question, it is necessary to find the form of the distribution n_k satisfying the stationary kinetic equation (2.2.15)

$$\Gamma_k n_k + I_k\{n_{k'}\} = 0 \tag{3.4.1}$$

in the region of the k-space where $\Gamma_k > 0$. In other words, one should match the Kolmogorov solution (realized far from the regions where $\Gamma_k \neq 0$) with the one for pumping region. As we shall see in this section, a spectrally narrow pump gives rise to a very interesting intermediate solution, the *pre-Kolmogorov asymptotics*, in a wide range of scales up to those strongly differing from the pumping region scale. Secondly, we shall discuss the effect of wave damping on the structure of stationary distribution, i.e., we shall take into consideration that, starting from some k_d, the function $\Gamma_k < 0$ and the occupation numbers n_k should fall off faster than the Kolmogorov law.

Consideration of at least one of the scales, the pumping region k_0 or the damping region k_d, is physically absolutely indispensable. This follows from the fact that the Kolmogorov solutions obtained in Sections 3.1–3 in an infinite interval $k \in (0,\infty)$ are the power solutions $n_k \propto k^{-s_0}$ with the wave energy density per unit volume of a medium in such distributions being infinite. Indeed, the integral

$$E = \int \omega_k n_k \, d\boldsymbol{k} \propto \int_0^\infty k^{\alpha+d-s_0-1}\, dk \tag{3.4.2}$$

diverges either at the upper or lower limit, depending on the sign of the quantity

$$h = \alpha + d - s_0 \; . \tag{3.4.3}$$

As seen, at $h < 0$, the energy-containing region is the pumping region $k \simeq k_0$, while at $h > 0$, this is the damping region for $k \lesssim k_\mathrm{d}$. The h index determines, in particular, the type of the dependence of energy flux on the characteristic scale of pumping region (Sect. 3.4.1), and the type of nonstationary behavior of weakly turbulent distributions (Chapt. 4).

Thus, in this section we shall study the stationary kinetic equation (3.4.1). This is a nonlinear integral equation, and it is only in the degenerate cases that its complete analytical solution may be obtained. Therefore in this section, along with the analytical methods, we shall use order-of-magnitude estimates and numeric simulations.

3.4.1 Matching with the Wave Source

Let us consider the isotropic turbulence in the decay case, in which the only Kolmogorov solution has a form

$$n_k^0 = \lambda P^{1/2} k^{-m-d} \tag{3.4.4}$$

and the h index is equal to

$$h = \alpha + d - s_0 = \alpha - m \; . \tag{3.4.5}$$

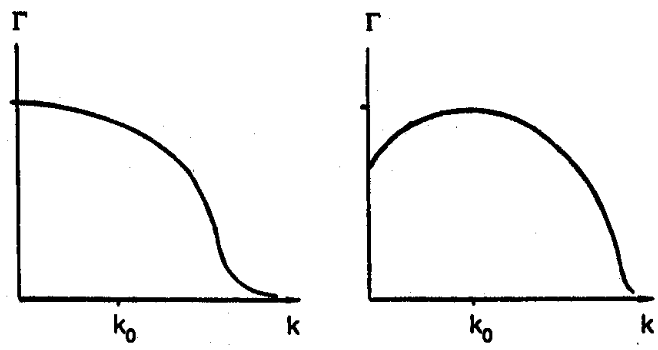

Fig. 3.8. The typical behavior of Γ_k for sources with broad spectrum around the single characteristic wave number k_0

The power absorbed by a wave system (that is the flux transmitted) P is most simply estimated for sources with a broad spectrum as depicted in Fig. 3.8. This is accomplished as follows: the energy flux directed into a medium by pumping

$$P = \int \Gamma(k)\omega(k)n(k)\, d\boldsymbol{k} \simeq \Gamma_0 \omega(k_0) n(k_0) k_0^d \tag{3.4.6}$$

is expressed via the values of occupation numbers in the pumping region $n(k_0)$. Assuming the distribution to be moderately distorted by a smooth source (not more than by a magnitude of the same order), one can estimate $n(k_0)$, extending the Kolmogorov solution (3.4.4) to small k's

$$n(k_0) \simeq \lambda P^{1/2} k_0^{-m-d} \; . \tag{3.4.7}$$

Substituting (3.4.7) into (3.4.6), we find the flux

$$P \simeq \Gamma_0^2 \omega^2(k_0) k_0^{-2m} \lambda^2 \propto \gamma_0^2 k_0^{2h} \ . \qquad (3.4.8)$$

The assumption about small distortions of the Kolmogorov distribution by a broad source is supported by Fig. 3.9 taken from the work by *Zakharov and Musher* [3.35] where the Kolmogorov spectrum of wave turbulence has first been observed in the numeric simulation. Equation (3.2.3) for three-dimensional acoustic turbulence was modelled, and the source was chosen to be $\Gamma_k = \exp[-(k-10)^2/4]$. The resulting distribution shown in Fig. 3.9 is close to the Kolmogorov solution $n_k^0 \propto k^{-9/2}$ in the $5 < k < 50$ range (Fig. 3.9 represents the $k^2 n_k$ value diminishing as $k^{-5/2}$).

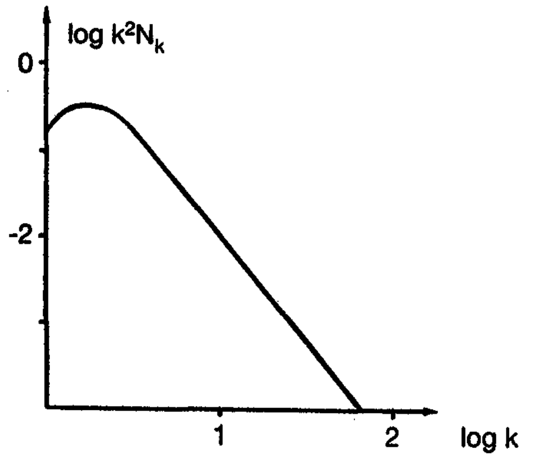

Fig. 3.9. The steady state distribution of the acoustic waves excited by broad sources

Expression (3.4.8) has a clear physical meaning: the energy flux increases when a source is shifted into the energy-containing region. Indeed, at $h > 0$, P is an increasing function of k_0, at $h < 0$, a decreasing one. This formula was obtained by *Falkovich and Shafarenko* [3.36]. At the same way one can obtain for the four-wave case

$$P \propto \Gamma_0^{3/2} k_0^{3h_1/2} \quad \text{and} \quad Q \propto \Gamma_0^{3/2} k_0^{3h_2/2} \ .$$

The quantity $h_2 = d - s_0$ indicates the position at which the action integral $N = \int n_k \, d\boldsymbol{k}$ diverge; the value h_1 corresponds to (3.4.3) with $s_0 = 2m/3 + d$.

Let us now consider the excitation of turbulence by a source which is nonzero in the small neighborhood $\triangle k$ in the vicinity of $k = k_0$, i.e., is narrow in the modulus of k ($\triangle k \ll k_0$) and, accordingly, in frequencies: $\triangle \omega_k \ll \omega(k_0) \equiv \omega_0$.

The question with regard to the structure of a stationary distribution excited by a spectrally narrow source is not quite trivial. The stationary kinetic equation (3.4.1) may be written in the form:

$$\Gamma_k n_k + I_k\{n_{k'}\} = \Gamma_k n_k - \gamma_k\{n_{k'}\} n_k + S_k\{n_{k'}\} = 0.$$

Here the collision integral is split up into two parts. The first one, $\gamma_k n_k$, contains n_k outside the integral. The quantity $\gamma_k\{n_{k'}\}$ has a meaning of a nonlinear damping decrement of a wave with wave number k in the presence of the distribution $n_{k'}$. This quantity is a functional of $n_{k'}$ and is in the three-wave case given by the single integral over $d\boldsymbol{k}$

$$\gamma_k = 2\int |V(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}-\boldsymbol{k}')|^2\delta\big(\omega(k)-\omega(k')-\omega(|\boldsymbol{k}-\boldsymbol{k}'|)\big)n(k')\,d\boldsymbol{k}'$$
$$+2\int |V(\boldsymbol{k}',\boldsymbol{k},\boldsymbol{k}'-\boldsymbol{k})|^2\delta\big(\omega(k')-\omega(k)-\omega(|\boldsymbol{k}'-\boldsymbol{k}|)\big)[n(\boldsymbol{k}'-\boldsymbol{k})-n(k')]\,d\boldsymbol{k}'$$

and in the four-wave case by the double integral

$$\gamma_k = \int |T(\boldsymbol{k},\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{k},\boldsymbol{k}_2,\boldsymbol{k}_3)|^2\delta\big(\omega(k)+\omega(|\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{k}|)-\omega_2-\omega_3\big)$$
$$\times\,[(n_2+n_3)n(\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{k})-n_2n_3]\,d\boldsymbol{k}_2 d\boldsymbol{k}_3\ .$$

As usual, $\omega_i=\omega(k_i)$, $n_i=n(k_i)$.

In the three-wave case, the remaining part of the collision integral equals to

$$S_k = \int |V(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}-\boldsymbol{k}_1)|^2\delta\big(\omega(k)-\omega(k_1)-\omega(\boldsymbol{k}-\boldsymbol{k}_1)\big)n_1 n(\boldsymbol{k}-\boldsymbol{k}_1)\,d\boldsymbol{k}_1$$
$$+2\int |V(\boldsymbol{k}_1,\boldsymbol{k},\boldsymbol{k}_1-\boldsymbol{k})|^2\delta\big(\omega(k_1)-\omega(k)-\omega(\boldsymbol{k}_1-\boldsymbol{k})\big)n_1 n(\boldsymbol{k}_1-\boldsymbol{k})\,d\boldsymbol{k}_1$$

and in the four-wave case

$$S_k = \int |T(\boldsymbol{k},\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{k},\boldsymbol{k}_2,\boldsymbol{k}_3)|^2\delta\big(\omega(k)+\omega(\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{k})$$
$$-\,\omega_2-\omega_3\big)n(\boldsymbol{k}_2+\boldsymbol{k}_2-\boldsymbol{k})n_2n_3\,d\boldsymbol{k}_2 d\boldsymbol{k}_3\ .$$

It is important to emphasize that, being the distribution integrals, the quantities γ_k and S_k should be smooth functions of the k-vector for sufficiently arbitrary $n_{k'}$. Let us write the stationary kinetic equation in the form

$$n_k = \frac{S_k\{n_{k'}\}}{\gamma_k\{n_{k'}\}-\Gamma_k} \equiv \frac{S_k}{\gamma_k-\Gamma_k}\ .$$

As a consequence of smoothness of γ_k, S_k it follows clearly that if in any region of the k-space the external pumping Γ_k has a sharp anomaly, the occupation numbers n_k should also have anomaly in this region. In other words, since $\partial\gamma_k/\partial k$ and $\partial S_k/\partial k$ are small, the derivative $\partial n_k/\partial k$ is large where $\partial\Gamma_k/\partial k$ is large. This conclusion is correct both for the decay and nondecay cases. But there arises the question whether the anomaly is essential, i.e., whether there is a sharp peak of occupation numbers in the pumping region. The answer may vary with types of interaction (three- or four-wave ones).

Let the source excite waves in a narrow interval $\triangle\omega$ around ω_0. In the case of three-wave interaction, by $2 \to 1$ confluence processes these waves will generate a peak of occupation numbers at $\omega = 2\omega_0$. By the induced $1 \to 2$ decay processes waves from a second peak can generate waves inside the first (pump) interval rather than outside it. Indeed, in the second integral of the kinetic equation (2.1.12) for $\partial n(\omega, t)/\partial t$, the terms other than zero will be these containing $n(2\omega_0)n(2\omega_0 - \omega)$. Besides, waves from the first peak will merge with waves from the second peak to generate a third peak, etc. However, due to locality of interaction, influence of remote peaks on the first one (and the resulting peak broadening) should be small. Thus, in the three-wave case it would be natural to expect the sharp peak of occupation numbers in the pump region and a chain of peaks (at multiple frequencies) generated by it. This subsection mainly deals with the properties of such solution.

In the non-decay case, wave excitations by a narrow source should occur quite differently. Indeed, now every $2 \to 2$ interaction involves four waves. If two out of four interacting waves have frequencies close to the pumping frequency, the frequencies of the other two waves can vary over a wide range. For example, scattering of two waves with frequency ω_0 can give waves with frequencies from zero to $2\omega_0$. As a result, the energy of the source might be distributed over wide regions of the k-space instead of being concentrated in narrow intervals. As indicated by computer simulation of weak Langmuir turbulence, spectrally narrow pumping does not generate a sharp distribution. Figures 3.10 and 3.11 obtained by *Hansen and Nicholson* [3.12] shows formation of Langmuir turbulence spectrum by pumping at $0.7 < \omega < 0.8$. At first pumping excites a sharply peaked distribution which is then smeared (Fig. 3.10) and in the quasi-steady state (Fig. 3.11 corresponds to $t = 100$) it is no longer pronounced.

A distribution of the same kind had been obtained by Falkovich and Ryzhenkova for optical turbulence. Figures 3.12a,b obtained by numeric simulation of the kinetic equation corresponding to nonlinear Schrödinger equation (see Sect. 1.4.3) show the steady behavior of the distribution and its index respectively. A weak anomaly of the distribution is and a strong one of the derivative $[s(\omega) = d \ln n(\omega)/d\omega]$ were observed.

The question with regard to presence or absence of a sharp peak generated by pumping in a stationary distribution is also related to whether the energy flux depends on the integral power of the source $\Gamma = \int \Gamma_k \, d\boldsymbol{k}$ or only on its maximal value $\Gamma(k_0) = \Gamma_0$. It should be noted that the absence of a sharp peak in the narrow pumping region could also be connected with the large value of γ_k (formally speaking, with the divergency of the integral for γ_k).

Let us nowreturn to the decay case.If one assumes also a smooth distribution for narrow pumping, then instead of (3.4.6), one obtains:

$$P = \int \Gamma_k \omega_k n_k \, dk \simeq k_0^{d-1} \omega(k_0) n(k_0) \int \Gamma_k \, dk \equiv \Gamma \omega_0 n(k_0) k_0^{d-1} \ .$$

Usinf in here the $n(k_0)$ from (3.4.7), yeilds then

$$P \simeq \Gamma^2 \omega_0^2 k_0^{-2m-2} \lambda^2 \propto \Gamma^2 k_0^{2(h-1)} , \tag{3.4.9}$$

which differs from (3.4.8) by the factor $(\triangle k/k_0)^2$ and contains Γ instead of Γ_0. This estimate (3.4.9) which has also been given in the literature [3.8,42], is incorrect.

Following [3.36], we shall show the dependence $P \propto k_0^{2h}$ to hold also for spectrally narrow pumping. An error made in deriving (3.4.9) is the unjustified assumption that the occupation numbers at $k \approx k_0$ exhibit a smooth behavior. As a matter of fact, the sharp function Γ_k may generate a sharp peak n_k at $k \approx k_0$, $\omega_k \approx \omega_0$. As described above, the properties of the three-wave interaction ensure that the peak will give rise to a chain of peaks at multiple frequencies $\omega_j = j\omega_0$, j is a natural number. To illustrate this, we shall give an example where the limiting case of such a solution may be found analytically. Consider the equation (3.2.3) for the two-dimensional acoustic turbulence. Instead of $\Gamma_k n_k$ we shall take an external force F_k independent of n_k:

$$\begin{aligned} I(k) = &\int_0^k k_1(k-k_1)(n_1 n_2 - n_k n_1 - n_k n_2)\, dk_1 \\ &-2\int_k^\infty k_1(k_1-k)(n_k n_2 - n_1 n_k - n_1 n_2)\, dk_1 = -F_k \ . \end{aligned} \tag{3.4.10a}$$

Here $n_1 = n(k_1)$, $n_2 = n(|k-k_1|)$. Using the substitution $f(k) = kn(k)$, this equation may be represented as:

$$-4f(k)\int_0^\infty f(k_1)\, dk_1 + \int_{-\infty}^\infty f(k_1) f(|k-k_1|)\, dk_1 = -F(k) \ . \tag{3.4.10b}$$

If we determine $f(k)$ and $F(k)$ at $k<0$ with the help of $F(-k) = f(k)$ and $F(-k) = F(k)$, then the second term on the left-hand side of (3.4.10b) is a convolution integral. Consequently, the equation (3.4.10b) may be solved using the Fourier transformation. Indeed, the Fourier transformations

$$f(\sigma) = \int_{-\infty}^\infty f(k) e^{ik\sigma}\, dk, \quad F(\sigma) = \int_{-\infty}^\infty F(k) e^{ik\sigma}\ , dk$$

rediice (3.4.10) to the algebraic equation:

$$-2f(\sigma)f(0) + f^2(\sigma) + F(\sigma) = 0 \ .$$

From this we find $f(\sigma)$ and, using the inverse Fourier transform, we obtain the stationary distribution in the form

$$\begin{aligned} n_k =& \frac{1}{k}\int_{-\infty}^\infty e^{-ik\sigma}\left[\sqrt{F(0)} - \sqrt{F(0) - F(\sigma)}\right] d\sigma \\ = \frac{\sqrt{2}}{k} & \int_{-\infty}^{\infty} e^{-ik\sigma}\left[\sqrt{\int_0^\infty F(k)\, dk} - \sqrt{\int_0^\infty \left(1 - e^{ik\sigma}\right) F(k)\, dk}\right] d\sigma \ . \end{aligned} \tag{3.4.11}$$

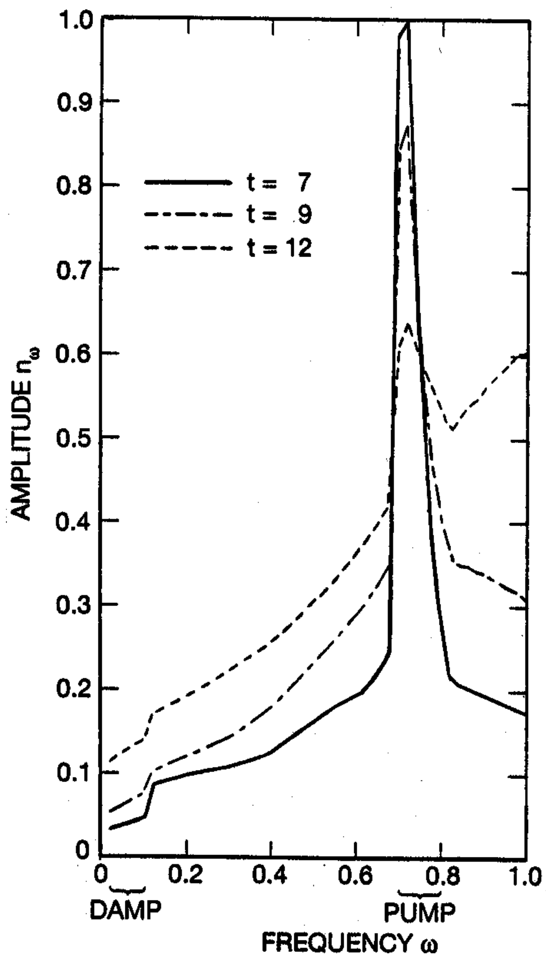

Fig. 3.10. Numerical simulations [3.12] of the distributions of Langmuir turbulence for different intermediate time.

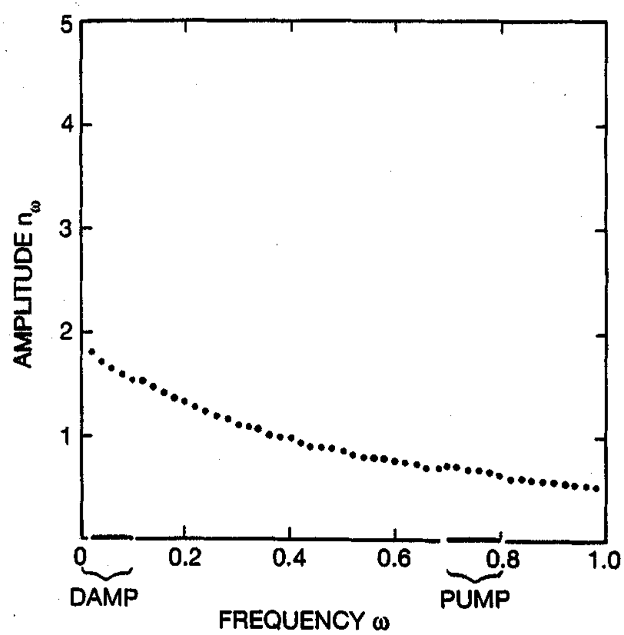

Fig. 3.11. Steady distribution of Langmuir turbulence excited by spectrally narrow pumping

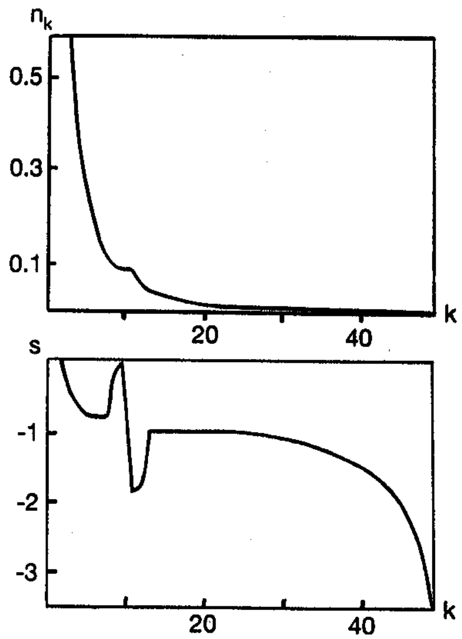

Fig. 3.12. Steady state distribution and its current index for optical turbulence excited by spectrally narrow pump

In particular, setting $F_k = F\big[\delta(k-k_0)+\delta(k+k_0)\big]$, we get a solution in the form of a chain of peaks of decreasing intensity

$$n_k = \frac{\sqrt{2F}}{k}\sum_{j=1}^{\infty}\frac{\delta(k-jk_0)}{4j^2-1}\ , \tag{3.4.12}$$

It is seen that at $j \gg 1$, the number of waves in the jth peak N_j decreases with the growth of the j number according to the Kolmogorov law $N_j \propto j^{-3}$ [for the two-dimensional acoustic turbulence, the Kolmogorov solution (3.2.5) equals $n_k \propto k^{-3}$].

Let us now deal with the excitation of the two-dimensional sound by the narrow increment Γ_k; we shall assume Γ_k to be nonzero in the small neighborhood $\triangle k$ around k_0 ($\triangle k \ll k_0$). If there exists a solution in the form of a chain of sharp peaks on a low background (which will be checked using a numeric experiment, see below Figs. 12-14), then the properties of such distribution may be found using the perturbation theory with the small parameter $\triangle k/k_0$.

As the zeroth approximation, we shall consider interaction of peaks between each other. For the value

$$N_j = \int\limits_{jk_0-\triangle k}^{jk_0+\triangle k} n_k dk$$

presenting the number of waves in the jth peak, we obtain:

$$\begin{aligned} I_j =& \sum_{l=1}^{j} l(j-l)[N_j N_{j-1} - n_j(N_l N_{j-l}] \\ &-2\sum_{l=j}^{\infty} l(l-j)[N_j N_{l-j} - N_l(N_j+N_{l-j})] = -\Gamma_0 N_1 \varDelta_{j1}\ . \end{aligned} \tag{3.4.13}$$

We have assumed here the first peak to be narrower than the source (which will also be justified below by the numeric simulation results), Γ_0 is the value of maximum of the increment. The solution of (3.4.13) evidently follows from (3.4.12) at $F = \Gamma_0 N_1$:

$$N_j = \frac{2\Gamma_0}{3j(4j^2-1)} \ . \tag{3.4.14}$$

Certainly, within the full equation $I(k)+\Gamma_k n_k = \partial n_k/\partial t$ such solution has not external stability, waves outside the peaks (with $k \neq jk_0$) should be excited. The distribution of these waves provides a background whose form nay be found from the approximation following (3.4.13) where for the waves with $k \neq jk_0$ we shall take into account only their interaction with the chain of peaks:

$$\begin{aligned} &-4n(k)\sum_{j=1}^{\infty} k_0 k j N_j + 2\sum_{j=1}^{i} jk_0(k-k_0)N_j n(k-jk_0) \\ &+2\sum_{j=1}^{\infty} k_0 j(jk_0-k)N_j n(jk_0-k) \\ &+2\sum_{j=1}^{\infty} jk_0(jk_0+k)N_j n(jk_0+k) = 0 \ . \end{aligned}$$

Here $i = [k/k_0]$ is an integral part of the k/k_0 ratio. This equation has the solution

$$n(k) = g(k)k^{-1} \ , \tag{3.4.15}$$

where $g(k)$ is a periodic function (with a period k_0) symmetric relative to the points $jk_0/2$. Comparing (3.4.14) and (3.4.15), we see that the spectrum background should decrease with increased k more slowly than the peaks. Thus, the peaks may exist only in an intermediate scale range $k_0 < k < k_*$, where k_* is determined by the ratio of the amplitude of the first peak to that of the background:

$$k_*^2 \simeq \frac{2N_1}{\triangle k_1 n(k_0/2)} = \frac{4\Gamma_0}{9\triangle k_1 n(k_0/2)} \ .$$

Here $\triangle k_1$ is the width of the first peak. In the range $k > k_*$, an ordinary monotonically diminishing Kolmogorov spectrum $n_k \propto k^{-3}$ should be realized.

The spectral background gives rise to additional damping for the waves with $k = jk_0$, with the decrement

$$4jk_*\bar{g} = 4jk_* \int_0^{k_0} g(k)\,dk \ ,$$

due to which, with growth of j, the chain of peaks should increasingly deviate from the Kolmogorov law, and decrease more steeply.

Considering further orders of the perturbation theory and taking into account the terms quadratic in the amplitude of the background, we can obtain other properties of the above solution (deviations from the Kolmogorov law $N_j \propto j^{-3}$, the fine structure of peaks, etc).

The above consideration of the structure of the spectrum generated by a narrow source were verified by numeric simulation due to *Falkovich and Shafarenko* [3.36] who studied the kinetic equation

$$\frac{\partial n(k,t)}{\partial t} = I(k) + \Gamma_k n(k,t) \tag{3.4.16}$$

with a collision integral (3.4.10) and an increment of the form

$$\Gamma_k = \Gamma_0 \exp\left[-\left(\frac{k-k_0}{\triangle k}\right)^2\right] \ . \tag{3.4.17}$$

In order to provide in numeric simulation an efficient energy sink in the region of large k's, one usually sets $n_k \equiv 0$ at $k > k_{\rm d}$ [3.35-36]. As a result, the discretized collision integral becomes

$$\begin{aligned} I(k) = &\sum_{l=1}^{k} l(k-l)\big[n(l)n(k-l) - n(k)n(l) - n(k)n(k-l)\big] \\ &- 2\sum_{l=k}^{k_{\rm d}} l(l-k)\big[n(k)n(l-k) - n(l)n(l-k) - n(l)n(k)\big] \\ &- 2n(k) \sum_{l=k_{\rm d}}^{k_{\rm d}+k} l(l-k)n(l-k) \ . \end{aligned} \tag{3.4.18}$$

The last term in (3.4.18) plays the role of nonlinear damping. It is due to the transfer of waves to the region $k > k_{\rm d}$ at the expense of confluence processes.

In numeric simulations it has been found that, irrespective of the form of the initial condition, the solution of (3.4.16) evolved into the stationary distribution presenting a chain of peaks, which goes with the growth of k over into a monotonically decreasing function. Figure 3.13 illustrates such a steady distribution for $\Gamma_0 = 100$, $k_0 = 20$, $\triangle k = 4$, $k_{\rm d} = 400$.

The dotted straight line in the figure has a slope of -3, i.e., corresponds to the power Kolmogorov spectrum. As we see, the amplitude of the first peak exceeds that of the background at $k \approx k_0/2$ by two orders of magnitude. The half-width of the first peak at a height which is e times as small as the maximal one is approximately equal to $\triangle k_1 \simeq 1$, i.e., it makes up for a quarter of the half-width of the source. The next peaks are broadened according to the law $\triangle k_j \propto j$. The number of waves in the j-th period

$$N_j = \sum_{k=jk_0-k_0/2}^{jk_0+k_0/2} n_k$$

up to $j = 6$ decreases approximately by the Kolmogorov law $N_j \propto j^{-3}$ (for details see [3.36]). As a consequence of these two features ($\triangle k_j \propto j$, $N_j \propto j^{-3}$), the peaks' amplitudes decay according to the law

$$n(k_j) \approx \frac{N_j}{\triangle k_j} \propto j^{-4} \ ,$$

whose index differs from the Kolmogorov law by unity. The background of the spectrum decreases in conformity with (3.4.15), by the law $n(k) \propto k^{-1}$.

Such pre-Kolmogorov asymptotics, in which the relay transfer of energy is effected by the waves concentrated in narrow spectral intervals, is not an exclusive property of the two-dimensional acoustic turbulence. Figure 3.14 shows the stationary distribution generated by the source (3.4.17) for the three-dimensional sound at $\Gamma_0 = 100$, $k_\mathrm{d} = 200$, $k_0 = 20$, $\triangle k = 4$.

The dotted straight line in Fig. 3.14 has a slope of $-11/2$. The background of the distributions decays with the growth of k slower than the peaks.

Figure 3.15 shows the stationary distribution for the capillary waves on deep a fluid with $\Gamma_0 = 100$, ω_d, $\omega_0 = 10$, $\triangle\omega = 2$ (in this case the picture illustrates the situation in the space of eigenfrequencies $\omega_k \propto k^{3/2}$).

In both cases we have a chain of peaks whose amplitude decrease by the power law with indices differing from the Kolmogorov index by unity. Their widths grow according to a linear law (in space of frequencies) which is easily understood. Indeed, let the waves exist in the range $(\omega_1 - \triangle\omega_1,\ \omega_1 + \triangle\omega_1)$. Due to process of confluence $2 \to 1$ the waves appear in the range $(2\omega_1 - 2\triangle\omega_1,\ 2\omega_1 + 2\triangle\omega_1)$ etc.

Evidently, the distributions of such type should in a general case of wave turbulence with the scale-invariant decay dispersion law be generated by a spectrally narrow source.

Having obtained an idea about the structure of the distribution, we return now to discuss the problem of the energy flux absorbed by a wave system emitted a narrow source. Due to narrowness of the first peak, (3.4.6) is replaced by

$$P = \Gamma_0 \omega_0 N_1 k_0^{d-1} \ . \qquad (3.4.19a)$$

We shall estimate the number of waves in the first peak of this equation using Kolmogorov law for N_j

$$N_j \simeq \lambda P^{1/2} k_0 (j k_0)^{-m-d}, \quad N_1 \simeq \lambda P^{1/2} k_0^{1-m-d} \ . \qquad (3.4.19b)$$

Substituting (3.4.19b) into (3.4.19a), we again obtain (3.4.8), which is true irrespective of the width of the source.

The results of numeric modelling reported in detail in [3.36] directly support the dependence $P \propto \Gamma_0^2 k_0^{2h}$. For two-dimensional sound, P does not depend on k_0; for three-dimensional sound, the dependence is almost inverse one $P \propto k_0^{-1}$, $(h = -1/2)$; for the capillary waves, an inverse dependence of the flux on the source frequency is observed $P \propto \omega_0^{-1}$ $(h = -3/4$, $\alpha = 3/2$, $k_0^{2h} = \omega_0^{-1})$.

The flux is indeed proportional to the square of the maximum increment Γ_0. The flux depends only weakly on the integral strength of source $\Gamma = \int \Gamma_k \, dk$. For example, for two-dimensional sound and constant amplitude of Γ_0 the flux slowly decreases with the integral strength of the source: at

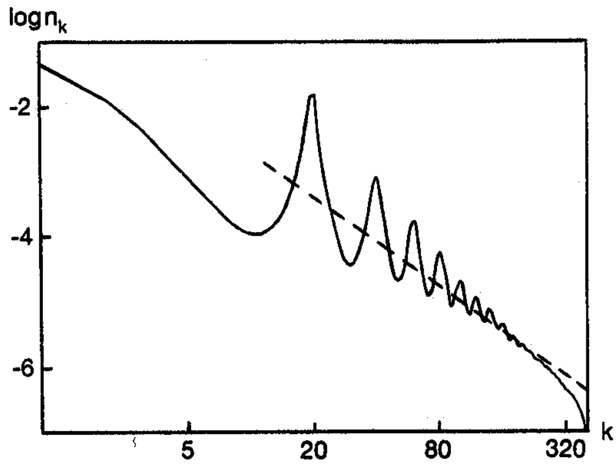

Fig. 3.13. Steady state distribution of two-dimensional sound excited by spectrally narrow pump [3.36]

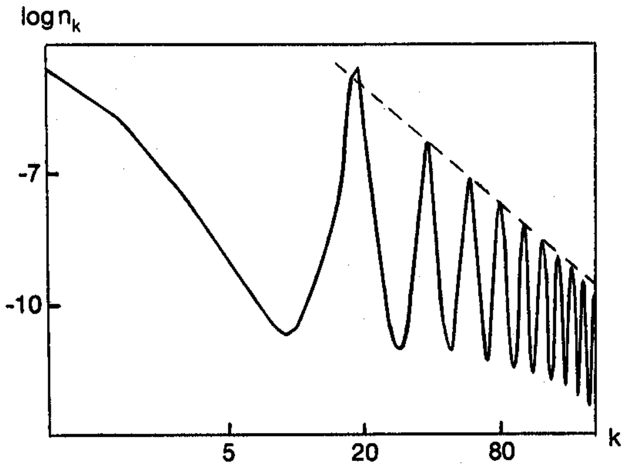

Fig. 3.14. Steady state distribution of three-dimensional sound excited by the spectrally narrow pump [3.36]

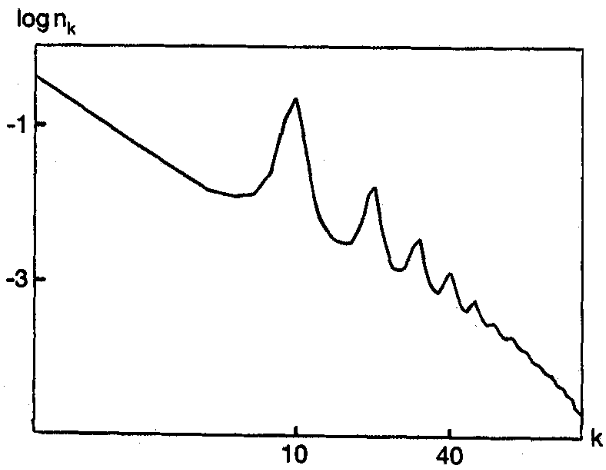

Fig. 3.15. Steady state distribution of capillary waves excited by narrow pump [3.36]

$\triangle k/k_0 = 1/10$ $P = 1534$, at $\triangle k/k_0 = 1/5$ $P = 1442$ and at $\triangle k/k_0 = 2/5$ $P = 1362$. This decrease seems to be due to a smearing of the first peak with broadening of the pump region.

3.4.2 Influence of Dissipation

In considering the dissipation, one naturally encounters three questions:

1. What kind of function Γ_k describing the behavior of wave damping decrement is required for existence of the stationary distribution?
2. Which distortions of the Kolmogorov distribution in the inertial interval are caused by the presence of a remote dissipative region?
3. What is the structure of the stationary spectrum of wave turbulence in the region of strong dissipation?

We have started the discussion of the first question in Sect. 2.3.1. We have proved there that the necessary condition for the existence of the non-equilibrium stationary distribution is the presence of energy damping ($\Gamma_k < 0$) in the region of large k's. Now we shall clarify the further conditions which the asymptotics of the function Γ_k must satisfy at $k \to \infty$ [in addition to negativity] to ensure existence of Kolmogorov distribution in the inertial interval. In the dissipative region, the function Γ_k absorbs part of the flux transmitted by the Kolmogorov spectrum through the inertial interval. Let us substitute the Kolmogorov solution $n_k \propto k^{-s_0}$ into the stationary kinetic equation (2.2.19)

$$\frac{dP_k}{dk} = \Gamma_k E_k = \Gamma_k \pi (2k)^{d-1} \omega_k n_k \ .$$

It is seen that if the dissipative function Γ_k at $k \to \infty$ increases slower that the function

$$k^{\alpha+d-s_0} = k^h \ ,$$

then the dissipation is unable to absorb of the whole flux. According to (3.4.2), the h index specifies location of the energy-containing region of the spectrum. In the next section [see (3.4.23)], we shall see that the characteristic time of nonlinear wave interaction on the background of the Kolmogorov distribution as a function of k behaves as

$$t_{\rm NL} \propto k^{-h} \ .$$

This expression is understood by the following considerations: Let at some k there be a deviation of the distribution from the Kolmogorov one. Then, with k changed by a factor of A, the characteristic evolution time of this perturbation varies by a factor of A^{-h}.

Thus there are two physical explanations for the requirement that the Γ_k function should for $k \to \infty$ increase faster than k^h to ensure existence of the stationary Kolmogorov distribution.On the one hand, this condition implies that the main portion of the energy should be absorbed in the short-wave region. On the other hand, it accounts for the quicker growth (or slower decrease) of the wave damping decrement compared to the inverse time of nonlinear interaction. It is only in the case of an infinite number of modes in a system that the requirement of the vanishing flux P at $k \to \infty$ make sense. If there exists maximal wave number k_m and the corresponding maximal frequency ω_m, then $n(\omega_\mathrm{m})$ is not bound to become zero. Indeed, having determined the flux in the case of the finite and discrete ω-space $\omega_l = l\omega_0,\ l = 1, \ldots, L$ by

$$P_l = \sum_{i=1}^{l} iI_i \ ,$$

we see that $P_1 = P_L = 0$ owing to the energy conservation law. Therefore, in this case, the Γ_l function may be rather arbitrary in the whole ω-space. The only requirement which remains is the condition of entropy withdrawal (2.2.16) or, in the discrete form,

$$\sum_{i=1}^{L} \Gamma_i \le 0 \ .$$

Here summation is carried out over all modes.

In the case of a model system with the ω-space consisting in three points (three spherical harmonics)

$$\begin{aligned}
\frac{dn_1}{dt} =& \Gamma_1 n_1 - 2V_1(n_1 n_2 - n_1 n_3 - n_2 n_3) - 2V_2(n_1^2 - 2n_1 n_2) \\
\frac{dn_2}{dt} =& \Gamma_2 n_2 - 2V_1(n_1 n_2 - n_1 n_3 - n_2 n_3) + V_2(n_1^2 - 2n_1 n_3) \\
\frac{dn_3}{dt} =& \Gamma_3 n_3 + 2V_1(n_1 n_2 - n_1 n_3 - n_2 n_3)
\end{aligned}$$

one can prove that the inequality $\Gamma_1 + \Gamma_2 + \Gamma_3 < 0$ is a necessary and sufficient condition for the existence of at least one (there may be more than one) steady state with positive n_1, n_2, n_3. We leave it for the reader as a small algebraic exercise.

For arbitrary number of modes in the system, it is only in the limit $\sum \Gamma_l \to -0$ that condition (2.2.16) may be proved sufficient. In this case, though separate Γ_l may be rather large, the state of the system is almost in equilibrium. Really, considering (2.2.10), the equation of the rate of entropy variation become in the discrete case

$$\frac{dS}{dt} = \sum_{i,l} U(i,l) \frac{[n_i n_l - n_{i+l}(n_i + n_l)]^2}{n_i n_l n_{i+l}} + \sum_l \Gamma_l \ ,$$

where $U(i,l)$ is the positive function expressed via the square of the interaction coefficient and wave frequency. Hence it appears that at $\sum \varGamma_l \to -0$, each of the square brackets in the first sum should tend to zero. This is only possible for the Rayleigh-Jeans distribution $n_l = A/l$; here l is the coordinate in the ω-space; n_l, the wave density in the k-space taken to be a function of a frequency. To remind, we consider the isotropic case. The stationary distribution may be constructed using perturbation theory: $n_l = A/l + \psi_l + \ldots$. Substituting such a distribution into the discrete analog of the kinetic equation, one can show that $\psi_l \propto \varGamma_l$, $A \propto \left(\sum \varGamma_l^2\right)/\left(\sum \varGamma_l\right)$, the small parameter being $\left(\sum \varGamma_l\right)^2/\sum \varGamma_l^2$. Thus, at $\sum \varGamma_l \to -0$, the effective "temperature" of stationary distribution tends to infinity implying that the characteristic saturation time should also increase.

Numeric simulation of the formation of a steady-state, given a source and the dissipation being distributed in the ω-space, was carried out by Falkovich and Ryzhenkova. Following their paper [3.37], we shall consider at first the discrete kinetic equation describing the capillary waves on deep water. In this case $\alpha = 3/2$, $m = 9/4$, $d = 2$, and the kinetic equation may be written in the form

$$\begin{aligned}\frac{\partial n_k}{\partial t} =& \sum_{l=1}^{k-1} U(k,l)[n_l n_{k-l} - n_k(n_l + n_{k-l})] \\ &+ \sum_{l=k+1}^{L} U(l,k)[n_k n_{l-k} - n_l(n_k + n_{l-k})] + \varGamma_k n_k \;.\end{aligned}$$

Here

$$\begin{aligned}U(k,l) =& k^{8/3} \frac{x^{2/3}(1-x)^{2/3}}{\sqrt{4x^{4/3}(1-x)^{4/3} - [1 - x^{4/3} - (1-x)^{4/3}]^2}} \\ &\times \left\{ \frac{(1-x^{2/3})^2}{(1-x)^{1/3}} + \frac{[1-(1-x)^{2/3}]^2}{x^{1/3}} - [x^{2/3} - (1-x)^{2/3}]^2 \right\}^2\end{aligned}$$

and $x = k/l$. The $\varGamma_k$ function was chosen in the form $\varGamma_k = B\varDelta_{k1} - \sqrt{k}$.

Figure 3.16 illustrates the time dependence of the total energy of a distribution for $L = 100$, $B = 100$, $\sum \varGamma_k = -571.4$. It is seen that at $t \gtrsim 3.8$ the evolution becomes exponential $E(t) = E_0 - E_1 \exp(-t/\triangle t)$. Determining the slope of the curve, one can find the characteristic saturation time $\triangle t$. It is interesting to follow the variation in $\triangle t$ with an increasing number of modes L. The $\varGamma_k$ function grows with k slower than $k^{-h} = k^{3/4}$, therefore there cannot be a stationary distribution in an infinite system. In a finite system, however, the saturation time $\triangle t$ falls off with the growth of L, which is due to the growth of the modulus $|\sum \varGamma_k|$, see Fig. 3.17. It is seen that at $\sum \varGamma_k \to 0$ $\triangle t^{-1} \propto \sum \varGamma_k$.

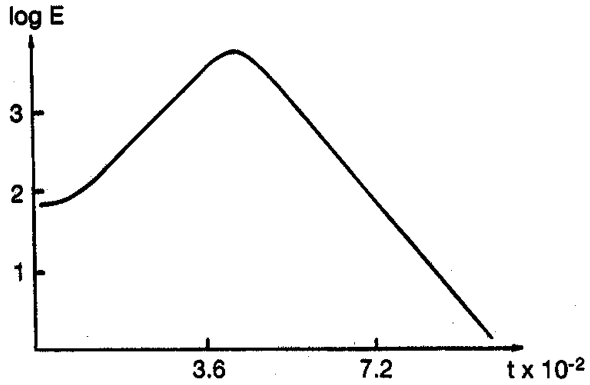

Fig. 3.16. The logarithm of E is given as a function of time t

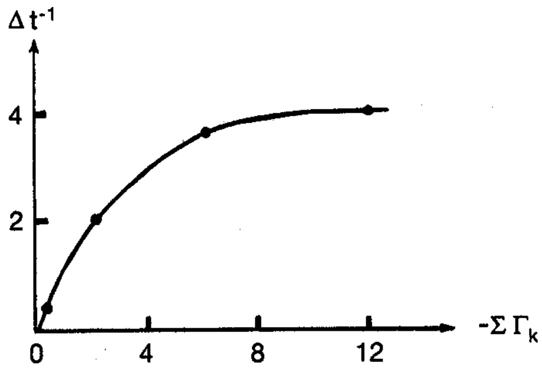

Fig. 3.17. The saturation time as a function of the entropy production rate

The two other wave systems with a decay dispersion law which were considered above, the gravitational-capillary waves on shallow water (two-dimensional sound) and three-dimensional sound, demonstrate in numeric simulation a similar behavior.

Let us now answer the second question posed at the beginning of this subsection. We consider the k-space and assume that starting from some k_{d}, there is a strong wave damping ($\Gamma_k < 0$), leading to a quick reduction of the occupation numbers n_k at $k \gtrsim k_{\mathrm{d}}$. In the region $k_0 \ll k \ll k_{\mathrm{d}}$, where k_0 gives the scale of the source, the stationary distribution should be close to the Kolmogorov distribution $n_k \propto k^{-m-d}$ provided the condition of locality of interaction is satisfied, i.e., if the collision integral converges on the Kolmogorov solution. In Sect. 3.1 we have shown that the collision integral converges on power solutions $n_k \propto k^{-s}$ if the s index falls into the locality interval:

$$m_1 + d - 1 + 2\alpha \equiv s_1 > s > s_2 \equiv 2m - m_1 + d + 1 - 2\alpha$$

and if for $k_1 \ll k$ the asymptotics of an interaction coefficient has a form

$$|V(k, k_1, k_2)|^2 = V^2 k_1^{m_1} k^{2m-m_1} \ , \qquad (3.4.20).$$

The locality interval exists if

$$m_1 - m - 1 + 2\alpha > 0 \ .$$

If this condition is satisfied, the deviation of stationary distribution from a power distribution, induced by the effect of a remote sink, is small and may be found with the help of the perturbation theory with the small parameter k/k_{d}.

According to (3.1.11), the angle-averaged three-wave collision integral equals [except to a constant factor] to

$$I(k) \propto \int_0^\infty \int_0^\infty [R(k,k_1,k_2) - R(k_1,k,k_2) - R(k_2,k,k_1)]\, dk_1 dk_2 \;, \tag{3.4.21}$$
$$R(k,k_1,k_2) = |V_{k12}|^2 (k_1k_2)^{d-1} \Delta_d^{-1} \delta(k^\alpha - k_1^\alpha - k_2^\alpha) \times \Theta(k-k_1)(n_1n_2 - n_kn_1 - n_kn_2) \;.$$

If, in an infinite interval $k \in (0,\infty)$, the Kolmogorov power solution $n_k = Dk^{-m-d}$ is realized, the collision integral is identically equal to zero $I(k) = 0$. The absence of waves at $k > k_{\mathrm{d}}$ (we shall set $n_k \equiv 0$ at $k > k_{\mathrm{d}}$) leads to a slight deviation of the collision integral from zero at $k \ll k_{\mathrm{d}}$:

$$\delta I_1 = 2D^2 \int_{k_{\mathrm{d}}}^\infty |V(k_1,k,k_0)|^2 (k_1k_0)^{d-1} \Delta_d^{-1}(k_1,k,k_0) \times \left[(kk_0)^{-m-d} - (kk_1)^{-m-d} - (k_1k_0)^{-m-d}\right] dk_1 \;. \tag{3.4.22}$$

Here $k_0^\alpha = k_1^\alpha - k^\alpha$. This addition contribution δI_1 owes its origin to the finite character of the inertial interval. For the distribution n_k to be stationary, δI_1 must be compensated for by a contribution δI_2, due to the small deviation of the solution from the power law ($n_k = n_k^0 + \delta n_k$, $\delta n_k \ll n_k^0$ at $k \ll k_{\mathrm{d}}$):

$$\begin{aligned}\delta I_2 &= \hat{L}_k \delta n_k \\ &= 2 \int_0^\infty \int_0^\infty (k_1k_2)^{d-1} \Delta_d^{-1} \Big\{ |V_{k12}|^2 \delta(k^\alpha - k_1^\alpha - k_2^\alpha) \Theta(k-k_1) \\ &\times \left[\delta n_1 \left(n_2^0 - n_k^0\right) - \delta n_k n_1^0\right] - |V_{1k2}|^2 \delta(k_1^\alpha - k^\alpha - k_2^\alpha) \Theta(k_1 - k) \\ &\times \left[\delta n_k \left(n_2^0 - n_1^0\right) + \delta n_1 \left(n_k^0 + n_2^0\right) + \delta n_2 \left(n_k^0 - n_1^0\right)\right] \Big\}\, dk_1 dk_2 \;.\end{aligned} \tag{3.4.23}$$

Here $\hat{L}_k$ is the operator of the kinetic equation linearized with respect to n_k^0. This integral operator is scale-homogeneous $\hat{L}_{\lambda k} = \lambda^{m-\alpha} \hat{L}_k$ with the index equal to $m - \alpha = -h$, see Sect.4.2 for details.

Since the linearized collision integral $\hat{L}_k$ also determines the non-stationary behavior of small perturbations $\partial \delta n(k,t)/\partial t = \hat{L}_k \delta n(k,t)$ [see below (4.1.1)], the characteristic evolution time of such perturbations is proportional to k^h, as indicated in the preceding subsection.

Thus, in order to determine δn_k, we should solve the linear integral inhomogeneous equation

$$\delta I_1 + \delta I_2 = \delta I_1 + \hat{L}_k \delta n_k = 0 \;. \tag{3.4.24}$$

At first we calculate δI_1. Since in (3.4.22) $k_1 > k_\mathrm{d} \gg k$, we shall make use of the asymptotics (3.4.20) of the interaction coefficient and expand the square bracket in (3.4.22) up to the first nonvanishing terms in $(k/k_1)^\alpha$ to obtain

$$\delta I_1 = \frac{2D^2(m+d)V^2}{\alpha(m_1+2\alpha-1-m)} k_\mathrm{d}^{m+1-2\alpha-m_1} k^{m_1-m+\alpha-d-1} \,. \tag{3.4.25}$$

In conformity with the locality condition (3.1.12c), $m_1 + 2\alpha - 1 - n > 0$. Due to homogeneity of the $\hat{L}_k$ operator, the equation

$$\hat{L}_k \delta n_k = k^{-x} \tag{3.4.26}$$

has the power solution

$$\delta n_k = k^{h-x} \left[W(x-h)DV^2\right]^{-1} \,. \tag{3.4.27}$$

where $W(s)$ is a dimensionless integral obtained by substituting $\delta n_k = k^{-s}$ into (3.4.23) and factoring out $DV^2 k^{-h-s}$. The solution of the equation (3.4.26) has the form of (3.4.27) only if the index $s = x - h$ is an element of the locality interval of the collision integral. It is easy to see that the locality interval (s_1, s_2) is the same for the complete (3.1.11) and for the linearized collision integral (3.4.23). In our case, i.e., with (3.4.25) substituted into (3.4.24), $x = m+d+1-m_1-\alpha$, and the quantity $x-h = 2m-m_1+d+1-2\alpha$ coincides with the lower bound of the locality interval s_2 [see (3.1.12a)]. Neglecting the slow logarithmic dependence in the integration, we obtain

$$\delta n_k = Dk^{-m-d} \frac{2(m+d)\ln^{-1}\left(\frac{k_\mathrm{d}}{k}\right)}{\alpha(m_1-m+2\alpha-1)} \left(\frac{k}{k_\mathrm{d}}\right)^{m_1-m+2\alpha-1} \,. \tag{3.4.28}$$

Formula (3.4.28) is valid at $k \ll k_\mathrm{d}$ and shows that the finiteness of the sink scale leads to increased occupation numbers in the inertial interval, since $\delta n_k > 0$. The δn_k value grows with k, i.e., the distribution becomes somewhat more gently sloping. Of course, at $k \simeq k_\mathrm{d}$, a sharp fall-off of n_k should take place, which now cannot be described in terms of the perturbation theory. Thus, the stationary distribution should show the features depicted in Fig. 3.18, where the dashed line corresponds to the Kolmogorov power solution. The dependence of $\lg n_k$ on $\lg k$ should have an inflection point (marked in Fig. 3.18 as k^*), where the index of the solution for the current $s(k) = d\lg n_k/d\lg k$ passes through a minimum. Accumulation of waves at $k \lesssim k^*$ seems to be induced by the "bottle-neck" effect, arising due to a reduction of the flux at $k \gtrsim k^*$ because of a decrease in the occupation numbers at $k \lesssim k_\mathrm{d}$. This picture is confirmed by numeric experiment carried out for the capillary waves on the surface of a deep fluid and for sound [3.37]. Thus, Fig. 3.19a represents the wave-vector dependence of the current index of the stationary solution as obtained in the numeric simulation of the two-dimensional acoustic turbulence. One observes a pronounced minimum $s(k)$

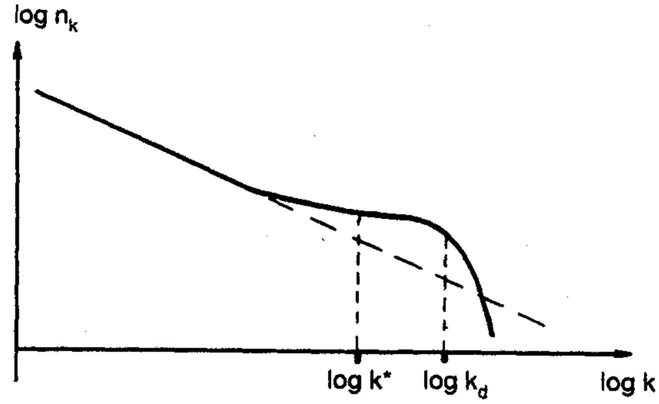

Fig. 3.18. Distortion of the Kolmogorov spectrum caused by damping.

at $k^* = 77$. Using different k_d in the numeric experiment, one can verify that the this effect is associated with the finiteness of the sink scale; the position of minimum of the index is proportional to $k_\mathrm{d} : k^* = Bk_\mathrm{d}$; for the two-dimensional sound and the damping appearing as a jump at $k = k_\mathrm{d}$, we have $B \approx 1/3$. Fig. 3.19b illustrates the behavior of the $s(k)$ index for the three-dimensional sound in a similar situation.

Finally, let us turn to the last question under consideration. We shall discuss the behavior of the stationary turbulence spectrum in the dissipative region at $k \gg k_\mathrm{d}$. We take the damping decrement Γ_k to increase with k faster (or to decrease slower) than the inverse time of wave interaction in the inertial interval (i.e., the k^{-h} function). The implications are that in the dissipative region the occupation numbers should fall of with growing of k faster than by the Kolmogorov law. The character of this fall off is determined by the type of interaction in the dissipative region: they may predominantly interact with each other or with waves from the inertial interval. For waves with strongly different wave numbers, the dependence of their interaction time on k may be found by substituting the asymptotic expression (3.4.20) for the interaction coefficient at $k_1 \ll k$ into the collision integral (3.4.21)

$$t_1^{-1}(k) \propto k^{2m-m_1+1-\alpha} \ . \tag{3.4.29}$$

If the damping decrement increases with k faster than t_1^{-1}, then asymptotic of the distribution at $k \to \infty$ is determined by interaction of waves in the dissipative region with each other. In this case, the exponential "quasi-Planck" spectrum $n_k = D\omega_k^{-b}\exp(-\omega_k/\omega_\mathrm{d})$ is formed. (Similar exponential asymptotics appear in the short-wave region with free evolution of distribution — see Sect. 3.4 below). Really, assuming that the damping decrement Γ_k grows by the power law $\Gamma_k = -Gk^a$ (e.g., for viscosity $a = 2$), and that the main contribution to the collision integral stems from the integration over the region $k_1 \ll k$, we obtain the stationary kinetic equation in the form

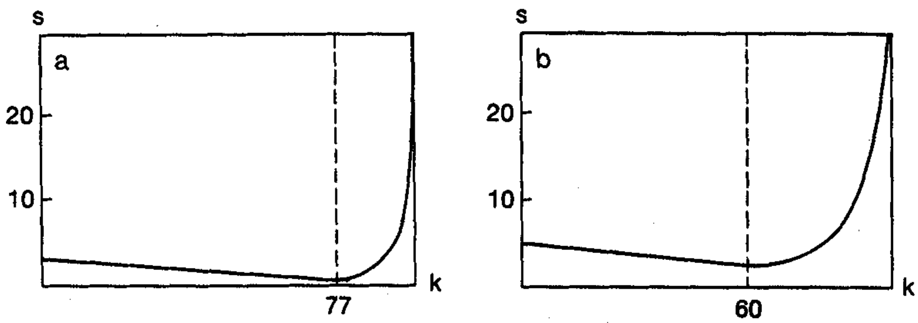

Fig. 3.19. Current index of the steady spectra of acoustic turbulence for a)$d = 2$ and b)$d = 3$

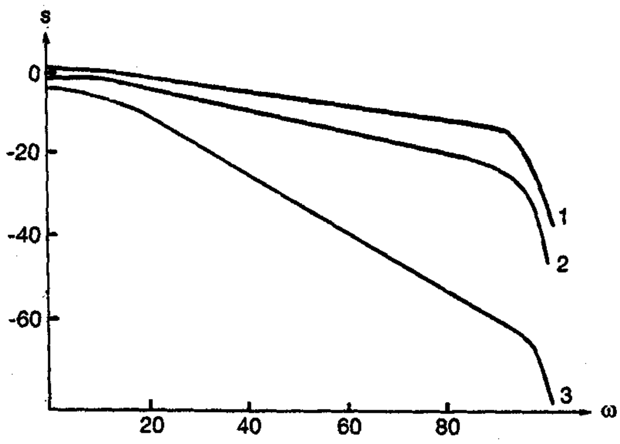

Fig. 3.20. Current index of steady spectrum in the damping region

$$G k^a n_k = D k^{2m-m_1+1-\alpha} n_k \int_{k_d}^{k} k_1^{d-1-\alpha+m_1-b} \times \left[1 - \exp\left(-k_1^{\alpha}/k_d^{\alpha}\right)\right]^2 dk_1 \ . \tag{3.4.30}$$

As the integral in the right-hand side of (3.4.30) is a nondecreasing function of k, a solution of such a form may only exist if the inequality

$$a > 2m - m_1 + 1 - \alpha \ . \tag{3.4.31}$$

is satisfied. For all the three systems under discussion and for viscous damping, this inequality is satisfied. Indeed, the numeric simulation [3.37] shows that for

$$\Gamma_k = G_1 \Delta_{k1} - G_2 k^2$$

in the strong dissipation region the occupation numbers decrease by an exponential law.For the two-dimensional sound and capillary waves on the surface of deep a fluid [3.37] the dependence of the stationary distribution vs. frequency is shown in Figure 3.20. The section of the linear decrease of $s(\omega)$ corresponds to the exponential reduction of $n(\omega)$.

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