4. The Stability Problem and Kolmogorov Spectra

In this chapter we examine those of the solutions obtained in Chap. 3 which are suitable for modeling reality. Obviously, one can expect to observe only Kolmogorov distributions that are stable with regard to perturbations. Sections 4.1 and 4.2 deal with the behavior of distributions slightly differing from Kolmogorov solutions. The reason for the difference may be either a small variation in the boundary conditions (i.e., in the source and in the sink), or immediate modulation of the occupation numbers of the waves. Small perturbations are studied in terms of linear stability theory where the main object is the kinetic equation linearized with respect to the deviation of the resulting distribution from a Kolmogorov one. In Sect. 4.1, the basic properties of the linearized collision integral are considered and the neutrally stable modes, i.e., small steady modulations of the Kolmogorov distributions are obtained. Section 4.2 presents a mathematically correct linear stability theory of the Kolmogorov solutions, formulates the stability criterion and exemplifies instabilities. The last section of this chapter discusses the evolution of distributions which are initially far from Kolmogorov distributions.

4.1 The Linearized Kinetic Equation and Neutrally Stable Modes

4.1.1 The Linearized Collision Term

We shall start with the decay case. Assuming $n(\boldsymbol{k},t) = n(k)+\delta n(\boldsymbol{k},t)$, $\delta n(\boldsymbol{k},t) \ll n(k)$ and linearizing (2.1.12) with respect to the deviation $\delta n(\boldsymbol{k},t)$ we obtain

$$\begin{aligned}\frac{\partial \delta n(\boldsymbol{k},t)}{\partial t} &= \int d\boldsymbol{k}_1 d\boldsymbol{k}_2 \{ \mid V(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) \mid^2 \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2) \\ &\times \delta(\omega_k - \omega_1 - \omega_2)\big[[n(k_2)-n(k)]\delta n(\boldsymbol{k}_1,t) \\ &+ [n(k_1)-n(k)]\delta n(\boldsymbol{k}_2,t) - [n(k_1)+n(k_2)]\delta n(\boldsymbol{k},t)\big] \\ &- 2 \mid V(\boldsymbol{k}_1,\boldsymbol{k},\boldsymbol{k}_2) \mid^2 \delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2)\delta(\omega_1-\omega_k-\omega_2) \\ &\times \big[[n(k)-n(k_1)]\delta n(\boldsymbol{k}_2,t) + [n(k)+n(k_2)]\delta n(\boldsymbol{k}_1,t) \\ &+ [n(k_2)-n(k_1)]\delta n(\boldsymbol{k},t)\big] \} \\ &= \hat{L}_k \delta n(\boldsymbol{k},t) = \int L(\boldsymbol{k},\boldsymbol{k}_1)\delta n(\boldsymbol{k}_1,t)\, d\boldsymbol{k}_1 \ .\end{aligned} \tag{4.1.1}$$

At the end of the preceding section we have already encountered the angle-averaged operator $\hat{L}$, see (3.4.23). If $n(k)$ is a stationary equilibrium distribution $n(k) = T/\omega(k)$, the kernel $L(\boldsymbol{k}, \boldsymbol{k}_1)$ of the integral operator has the important property of being symmetric [4.1]. Namely, one can normalize the function affected by the operator $\delta n(\boldsymbol{k}, t) = f(k)\varphi(\boldsymbol{k}, t)$ (using the function $f(k)$ to be defined below), so that the resulting kernel $M(\boldsymbol{k}, \boldsymbol{k}_1) = L(\boldsymbol{k}, \boldsymbol{k}_1)f(k_1)$ is symmetric

$$M(\boldsymbol{k}, \boldsymbol{k}_1) = M(\boldsymbol{k}_1, \boldsymbol{k}), \tag{4.1.2}$$

and the corresponding operator $\hat{M}$ hermitian. For a proof it is convenient to write the kernel of the renormalized operator (4.1.1) in the form

$$\begin{aligned} M(\boldsymbol{k}, \boldsymbol{k}_1) = \int d\boldsymbol{k}_2 \{ & 2U(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)[n(k_2) - n(k)]f(k_1) \\ & - U(\boldsymbol{k}, \boldsymbol{k}_2, \boldsymbol{k} - \boldsymbol{k}_2)[n(k_2) + n(|\boldsymbol{k} - \boldsymbol{k}_2|)]f(k)\delta(\boldsymbol{k} - \boldsymbol{k}_1) \\ & + 2U(\boldsymbol{k}_2, \boldsymbol{k}, \boldsymbol{k}_2 - \boldsymbol{k})[n(k_2) - n(|\boldsymbol{k}_2 - \boldsymbol{k}|)]f(k)\delta(\boldsymbol{k} - \boldsymbol{k}_1) \\ & + 2U(\boldsymbol{k}_1, \boldsymbol{k}, \boldsymbol{k}_2)[n(k) + n(k_2)]f(k_1) \\ & + 2U(\boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k})[n(k_2) - n(k)]f(k_1)\} \end{aligned} \tag{4.1.3}$$

where the function

$$U(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) = | V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) |^2 \, \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\delta(k^\alpha - k_1^\alpha - k_2^\alpha)$$

is invariant with regard to rearrangements of the second and third arguments. It should be noted that the terms in (4.1.3) containing $\delta(\boldsymbol{k} - \boldsymbol{k}_1)$ are, apparently, symmetric. Let us consider the last term in (4.1.3). If $n(k) \propto \omega^{-1}(k)$, then

$$U(\boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k})f(k_1)\frac{\omega_k - \omega_2}{\omega_k\omega_2}f(k_1) = -U(\boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k})\frac{f(k_1)\omega_1}{\omega_k\omega_2}.$$

Hence, this expression will also be symmetric if we choose $f(k_1) = \omega^{-2}(k_1)$. The first and fourth terms in (4.1.3) go upon the substitution $k \leftrightarrow k_1$ over into each other.

The symmetric character of the linearized four-wave collision term is proved in a similar way:

$$\begin{aligned} \frac{\partial \delta n(\boldsymbol{k}, t)}{\partial t} = \int & |T_{k123}|^2 \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\delta(\omega_k + \omega_1 - \omega_2 - \omega_3) \\ & \times \{[n(k_1)n(k_2) + n(k)n(k_2) - n(k)n(k_1)]\delta n(\boldsymbol{k}_3, t) \\ & + [n(k)n(k_3) + n(k_1)n(k_3) - n(k)n(k_1)]\delta n(\boldsymbol{k}_2, t) \\ & + [n(k_2)n(k_3) - n(k)n(k_2) - n(k)n(k_3)]\delta n(\boldsymbol{k}_1, t) \\ & + [n(k_2)n(k_3) - n(k_1)n(k_2) - n(k_1)n(k_3)]\delta n(\boldsymbol{k}, t)\} \\ & \times d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \\ = \hat{W}_k \delta n(\boldsymbol{k}, t) = \int & W(\boldsymbol{k}, \boldsymbol{k}')\delta n(\boldsymbol{k}', t)d\boldsymbol{k}' \,. \end{aligned} \tag{4.1.4}$$

For the case $n(k) = T/\omega_k$ we should choose again $f(k) = \omega^{-2}(k)$ and the kernel $Q(\boldsymbol{k}, \boldsymbol{k}') = W(\boldsymbol{k}, \boldsymbol{k}')\omega^{-2}(k')$ is also symmetric $Q(\boldsymbol{k}, \boldsymbol{k}') = Q(\boldsymbol{k}', \boldsymbol{k})$. Indeed, because of the symmetry of the interaction coefficients $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = T(\boldsymbol{k}_1, \boldsymbol{k}, \boldsymbol{k}_2, \boldsymbol{k}_3)$ with regard to the substitution $k \leftrightarrow k'$, it transforms each of the four terms in (4.1.4) into itself.

The hermitian character of the operators $\hat{M}_k$ and $\hat{Q}_k$ implies that the eigenvalues are real, i.e., there are no oscillations of the wave system around the equilibrium state. In this context it is useful to recall the theorem proved in Sect. 2.2, it states that distributions evolving according to the kinetic equations are associated with a growing entropy which has its maximum in the equilibrium state. In the next section we shall see that this theorem in liaison with the hermitian character of the linearized collision integral ensures that there is no absolute instability of equilibrium distributions. This prevents us from finding a perturbation which would grow with time in all points of the $\boldsymbol{k}$-space.

If the stationary solution $n(k)$ is not in thermodynamical equilibrium, the operator of the linearized collision integral is nonhermitian as may be directly verified. Therefore, deviations from nonequilibrium stationary distributions may behave absolutely differently. As we shall see in the next section, we may not just observe oscillations of the occupation numbers around the stationary values, but also various instabilities of the Kolmogorov solutions.

There is one more property of the operators (4.1.1, 4) which is common to equilibrium and isotropic Kolmogorov distributions $n(k)$. Owing to the parity of the stationary solution $n(-k) = n(k)$ and the δ-functions, the operators $\hat{L}$ and $\hat{W}$ conserve the parity of the function of k on which they act. This means that application of the operator on the even (odd) function $\delta n(k)$ results in an even (odd) function, respectively.

4.1.2 General Stationary Solutions and Neutrally Stable Modes

As we shall now show, deformations of stationary solutions may neither grow nor decrease. Such stationary additions are called *neutrally stable modes*. They owe their existence to the fact that the general stationary (equilibrium or nonequilibrium) solution depends on several parameters. As a consequence, the neutrally stable modes, i.e., the stationary solutions of the linearized equations (4.1.1, 4) are readily obtained from dimensional analysis.

Let us start with the equilibrium case. For waves with the decay dispersion law, the general equilibrium solution (2.2.13) depends on both integrals of motion of the system (i.e., the ones for energy and momentum)

$$n(\boldsymbol{k}, T, \boldsymbol{u}) = \frac{T}{\omega(k) - (\boldsymbol{k}\boldsymbol{u})} \,.$$

Such a solution is called *the drift equilibrium distribution.* At small momentum of the system ($u \to 0$), this expression may for $(\boldsymbol{k}\boldsymbol{u}) \ll \omega(k)$ be expanded into a series

$$n(\boldsymbol{k},T,\boldsymbol{u}) = \frac{T}{\omega(k)} - \frac{(\boldsymbol{ku})T}{\omega^2(k)} = n_0(k) + \delta n_0(\boldsymbol{k}) \ .$$

Hence, $\delta n_0(\boldsymbol{k}) \propto (\boldsymbol{ku})/\omega^2(k)$ is a stationary solution of (4.1.1). For waves with a nondecay dispersion law there are three integrals of motion (energy, momentum, and action), the general equilibrium solution (2.2.14) has the form

$$n(\boldsymbol{k},T,\boldsymbol{u},\mu) = \frac{T}{\omega(k) - (\boldsymbol{ku}) - \mu} \ ,$$

and the neutrally stable modes are obtained in a similar way

$$\delta n_0(\boldsymbol{k}) \propto (\boldsymbol{ku})/\omega^2(k), \quad \delta n_1(k) \propto \omega^{-2}(k) \ .$$

The general nonequilibrium stationary solution should depend on all fluxes of the integrals of motion. A dimensional analysis shows that the stationary solution for the three-wave kinetic equation may be written in the form

$$\begin{aligned} n(\boldsymbol{k},P,\boldsymbol{R}) &= \lambda P^{1/2} k^{-m-d} f(\xi), \\ \xi &= \frac{(\boldsymbol{Rk})\omega(k)}{Pk^2} . \end{aligned} \tag{4.1.5}$$

Here $P, \boldsymbol{R}$ are the fluxes of energy and momentum, respectively, and λ is the dimensional Kolmogorov constant. Since the medium is assumed to be isotropic the solution depends on the scalar product $(\boldsymbol{Rk})$. The form of the dimensionless function $f(\xi)$ has so far only been established for sound with positive dispersion (see Sect. 5.1 below). In the general case, one can only indicate the asymptotics $f(\xi)$ at $\xi \to 0$ where the solution (4.1.5) should go over to the isotropic Kolmogorov distribution $n_0(k) = \lambda P^{1/2} k^{-m-d}$, therefore at $\xi \to 0$ we have $f(\xi) \to 1$. Assuming $f(\xi)$ to be analytical at zero and expanding (4.1.5) we obtain a stationary anisotropic correction to the isotropic solution

$$\begin{aligned} n(\boldsymbol{k},P,\boldsymbol{R}) &\approx \lambda P^{1/2} k^{-m-d} + f'(0) k^{-m-d} (\boldsymbol{Rk}) \omega(k) P^{-1/2} k^{-2} \\ &= n_0(k) + \delta n(\boldsymbol{k}) \ . \end{aligned} \tag{4.1.6}$$

The solution (4.1.6) was first found by *Kats* and *Kontorovich* [4.2] and is called the *drift Kolmogorov solution*. It should be emphasized that, in contrast to the drift equilibrium distributions, it cannot be derived from the isotropic solution via the "Galilean" substitution $\omega \to \omega - (\boldsymbol{ku})$ [4.2]. We used the quotation marks because this term refers in the given context not to a transition to a moving reference system [in such a transition, wave amplitudes transform according to $c(\boldsymbol{k},t) \to c(\boldsymbol{k},t)\exp[-i(\boldsymbol{ku})t]$ while the simultaneous pair correlators $n(\boldsymbol{k},t)$ do not change at all]. The equilibrium solutions are invariant with regard to the substitution $\omega(k) \to \omega(k) - (\boldsymbol{ku})$ since the integrals of motion enter the entropy extremum condition additively. This invariance has nothing in common with the Galilean invariance. The lack of such an invariance in the nonequilibrium case implies that the Kolmogorov solution does possibly not correspond to an extremum of a functional.

In the nondecay case there are three integrals of motion and it should be possible to give the general nonequilibrium stationary solution as a function of two dimensionless variables

$$\begin{aligned} n(\boldsymbol{k},P,Q,\boldsymbol{R}) &= \lambda_1 P^{1/3} k^{-d-2m/3} F[\omega_k Q/P,\ \omega_k(\boldsymbol{Rk})/Pk^2] \\ &= \lambda_1 P^{1/3} k^{-d-2m/3} F(\eta,\xi) \ . \end{aligned} \tag{4.1.7}$$

Assuming the function $F(\eta,\xi)$ to be analytical in both variables, we can obtain from (4.1.7) a small stationary correction to the solution with an energy flux $n_0(k) = \lambda_1 P^{1/3} k^{-d-2m/3}$. For for the drift solution we thus obtain

$$\begin{aligned} n(\boldsymbol{k},P,\boldsymbol{R}) &\approx n_0(k) + \delta n_0(\boldsymbol{k}) \\ &= \lambda_1 P^{1/3} k^{-d-2m/3} + k^{-d-2m/3-2} (\boldsymbol{Rk}) P^{-2/3} \omega_k \left(\frac{\partial F}{\partial \xi}\right)_{\xi=\eta=0} . \end{aligned} \tag{4.1.8}$$

It is seen that for the drift Kolmogorov corrections to the solution with an energy flux the general formula

$$\frac{\delta n_0(k)}{n_0(k)} \propto \xi = \frac{(\boldsymbol{Rk})\omega_k}{Pk^2} \tag{4.1.9}$$

holds in the decay and nondecay cases [4.3]. If the system does not possess a momentum flux the general solution (4.1.7) goes over into the solution (3.2.25) of the isotropic stationary equation (3.1.23):

$$n(k,P,Q) = \lambda_1 P^{1/3} k^{-d-2m/3} F(\eta), \quad F(\eta) \equiv F(\eta,0) \ . \tag{4.1.10}$$

The asymptotics of $F(\eta)$ may be found from the following considerations: at $Q = 0$ solution (4.1.10) should be transformed to a solution specified only by the energy flux and at $P = 0$ by the wave action flux. Thus, at $\eta = \omega Q/P \to 0$, $F(\eta) \to 1$, and at $\eta \to \infty$, $F(\eta) \to a\eta^{1/3}$, where a is some dimensionless constant. Physically such a solution corresponds to two well separated sources in ω-space. It describes the behavior of the Kolmogorov-like distribution between them with its energy flux in the small-frequency region which for large frequencies goes over to a solution with constant wave action flux. Expanding (4.1.10) for small η, we obtain a stationary addition to the solution with the energy flux. This addition carries the small action flux

$$\begin{aligned} n(k,P,Q) &= \lambda_1 P^{1/3} k^{-d-2m/3} + F'(0) Q P^{-2/3} \omega_k k^{-d-2m/3} \\ &= n_0(k) + \delta n_1(k) \ . \end{aligned} \tag{4.1.11}$$

The solution which transfers the small energy flux against the background of the main distribution with the wave action flux can also be obtained from (4.1.10) by expanding $F(\eta)$ in the $1/\eta$ parameter at $\eta \gg 1$. Another possibility would be to write the general solution (4.1.7) in the form

$$\begin{aligned} n(\boldsymbol{k},P,Q,\boldsymbol{R}) &= \lambda_2 Q^{1/3} k^{-d-2m/3+\alpha/3} G[P/Q\omega_k, (\boldsymbol{R},\boldsymbol{k})/Qk^2] \\ &= \lambda_2 Q^{1/3} k^{-d-2m/3+\alpha/3} G(\varsigma,\vartheta) \ . \end{aligned} \tag{4.1.12}$$

Expanding $G(\varsigma, \vartheta)$ at $\varsigma, \vartheta \to 0$, we obtain then the neutrally stable modes with small energy

$$\frac{\delta n_2(k)}{n_0(k)} \propto \varsigma \propto \omega^{-1}(k) \tag{4.1.13}$$

and momentum

$$\frac{\delta n_3(\boldsymbol{k})}{n_0(k)} \propto \vartheta \propto \frac{(\boldsymbol{R}, \boldsymbol{k})}{k^2} \tag{4.1.14}$$

fluxes against the background of the solution with the wave action flux $n_0(k) \propto k^{-d-2m/3+\alpha/3}$.

With the help of such dimensional and analytical analysis one can try to construct "hybrid" solutions depending on temperature and fluxes. For example, for the three-wave kinetic equation, the general spherically symmetric stationary solution may be written in the form

$$n(k, T, P) = \frac{T}{\omega(k)} g \left[P \left(\frac{\lambda \omega(k)}{T k^{m+d}} \right)^2 \right] . \tag{4.1.15}$$

At $P = 0$, the solution (4.1.15) should go over to the Rayleigh-Jeans distribution, hence $g(0) = 1$. At $T \to 0$ the Kolmogorov distribution $\lambda P^{1/2} k^{-m-d}$ should be obtained, therefore $g(x) \to x^{1/2}$ at $x \to \infty$. Since the dimensionless parameter $x = P(\lambda \omega_k / T k^{m+d})^2$ decreases with ω_k the solution (4.1.15) is close to the Kolmogorov solution at low frequencies. The high-frequency part of the distribution is an equilibrium one. Apparently, one often observes a strong interaction of the high-frequency part of the wave system with an external thermostat. The thermostat is supposed to have an infinite thermal capacity and appears in the form of a sink. Hence, the question about "the temperature of a turbulent medium" (see Sect. 3.1.2) is now answered. Such a "temperature" may be supposed to equal the mean energy of high-frequency motions.

Expanding $g(x)$ in a series up to the first-order terms, for $x \to 0$ we obtain $g(x) = 1 + g_0 x + \ldots$ and

$$n(k, T, P) = \frac{T}{\omega(k)} + \omega(k) k^{-2(m+d)} \frac{g_0 \lambda^2 P}{T} . \tag{4.1.16}$$

For Kolmogorov-like distributions we expand $g(x)$ at $x \to \infty$ in the asymptotic series $g(x) = x^{1/2}(1 + c/x + \ldots)$ to get, at $T \to 0$,

$$n(k, T, P) = \lambda P^{1/2} k^{-m-d} + c \lambda^{-1} T^2 P^{-1/2} \omega_k^{-2} k^{m+d} . \tag{4.1.17}$$

In the same way, one can obtain a nonisotropic correction to to the equilibrium distribution $n_0(k, T) = T/\omega(k)$. This correction carries the small momentum flux

$$\frac{\delta n(k)}{n_0(k, T)} \propto \frac{(\boldsymbol{R}\boldsymbol{k}) \omega_k^3}{k^{-2(m+d+1)} T^{-2}} . \tag{4.1.18}$$

In the nondecay case, we write, similarly to (4.1.15),

$$n(k, T, P) = \frac{T}{\omega(k)} H \left[P \left(\frac{\lambda_1 \omega_k}{T k^{d+2m/3}} \right)^3 \right] \tag{4.1.19}$$

and setting $H(y) = 1 + H_0 y$ at $y \to 0$ and $H(y) = y^{1/3}(1 + c_1/y)$ at $y \to \infty$, we obtain corrections to the equilibrium solution

$$n(k, T, P) = \frac{T}{\omega(k)} + \omega^2(k) k^{-3d-2m} \frac{H_0 \lambda_1^3 P}{T^2} \tag{4.1.20}$$

and to the Kolmogorov solution

$$n(k, T, P) = \lambda_1 P^{1/3} k^{-d-2m/3} + c_1 \lambda_1^{-2} P^{-2/3} T^3 \omega^{-3}(k) k^{2d+4m/3} . \tag{4.1.21}$$

For the solution with temperature and action flux, we have

$$n(k, T, Q) = \frac{T}{\omega(k)} K \left[Q \left(\frac{\lambda_2 \omega_k}{T k^{d+2m/3-\alpha/3}} \right)^3 \right] . \tag{4.1.22}$$

$$n(k, T, Q) = \frac{T}{\omega(k)} + \omega^2(k) k^{-3d-2m+\alpha} \frac{K_0 \lambda_2^3 Q}{T^2} , \tag{4.1.23}$$

$$n(k, T, Q) = \lambda_2 Q^{1/3} k^{-d-2m/3+\alpha/3} + c_2 \lambda_2^{-2} Q^{-2/3} T^3 \omega^{-3}(k) k^{2d+4m/3-2\alpha/3} . \tag{4.1.24}$$

It is important to realize that, as opposed to the case of equilibrium solution, one should directly verify that the modes constructed, see (4.1.6, 16–18) and (4.1.8, 11, 13, 14, 20, 21, 23, 24), are actually the stationary solutions of (4.1.1) and (4.1.4), respectively. The point is that we know in the equilibrium case the explicit form of the general equilibrium solution, whereas in the nonequilibrium case neither the form of the functions f, F, g, G, H, K nor their analyticity properties are known.

It is easy to verify for isotropic corrections (4.1.11, 13, 20, 21, 23, 24). We set $n(k) \propto k^{-s}$, $\delta n(k) \propto k^{-p}$ and, having split up the integral (4.1.4) into four equal parts, we subject three of them to Zakharov transformations similar to (3.1.18)

$$\begin{aligned} k_3' &= k^2/k_3, \quad k_2' = k_1 k/k_3, \quad k_1' = k_2 k/k_3, \quad k = k_3 k/k_3; \\ k_2' &= k^2/k_2, \quad k_1' = k_3 k/k_2, \quad k_3' = k_1 k/k_2, \quad k = k_2 k/k_2; \\ k_1' &= k^2/k_1, \quad k_3' = k_2 k/k_1, \quad k_2' = k_3 k/k_1, \quad k = k_1 k/k_1. \end{aligned} \tag{4.1.25}$$

As a result, the linearized collision integral (4.1.4) becomes:

$$\frac{\partial \delta n(\boldsymbol{k},t)}{\partial t} = \frac{k^\nu}{4}\int |T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_3)|^2 \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)$$
$$\times\, \delta(\omega_k+\omega_1-\omega_2-\omega_3)$$
$$\times \Big\{ \Big[(k_1k_2)^{-s}+(kk_2)^{-s}-(kk_1)^{-s}\Big]k_3^{-p}$$
$$+ \Big[(kk_3)^{-s}+(k_1k_3)^{-s}-(kk_1)^{-s}\Big]k_2^{-p} \tag{4.1.26}$$
$$+ \Big[(k_2k_3)^{-s}-(kk_2)^{-s}-(kk_3)^{-s}\Big]k_1^{-p}$$
$$+ \Big[(k_2k_3)^{-s}-(k_1k_2)^{-s}-(k_1k_3)^{-s}\Big]k^{-p}\Big\}$$
$$\times \left(k^{-\nu}+k_1^{-\nu}-k_2^{-\nu}-k_3^{-\nu}\right)d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 ,$$

where $\nu = 2m+3d-\alpha-2s-p$.

The braced term in (4.1.26) vanishes only for $s=\alpha$ and $p=2\alpha,\alpha$, which corresponds to the equilibrium corrections to the equilibrium solution. To reduce the (rounded) last brackets to zero, one should choose $\nu=0$ or $\nu=-\alpha$ (bearing in mind that $\omega_k \simeq k^\alpha$. For corrections to the equilibrium solution with $s=\alpha$, the choice of $\nu=0$ gives $p=3\alpha-2m-3d$ and $\nu=-\alpha$, $p=2\alpha-2m-3d$, which coincides with (4.1.23) and (4.1.20), respectively. As in the case of stationary additions to the solution with the energy flux for which $s=d+2m/3$, there are also two stationary indices $p=s=d+2m/3$ and $p=\alpha+d+2m/3$. The former coincides with the index of the main solution and refers to a mode with a small change of the energy flux

$$n(k,P) = n_0(k)+\delta n(k) = \lambda P^{1/2}k^{-d-2m/3} + \lambda\frac{\delta P}{2\sqrt{P}}k^{-d-2m/3} \tag{4.1.27}$$

(such modes corresponding to a simple change of the constant exist for all power solutions). The latter incex p corresponds to the correction (4.1.11) with a small wave action flux. The correction with the small temperature is not a stationary solution of (4.1.4). Similarly, for the solution with the wave action flux ($s = d+2m/3-\alpha/3$) we get $p=s$ and $p=d+2m/3+2\alpha/3$ which coincides with (4.1.13) and the mode (4.1.24) does not satisfy the equation.

Likewise, using Zakharov's transformations, one can show that the mode (4.1.16) with a small energy flux is a stationary solution of (4.1.1), while the one with a small temperature (4.1.17) is not. In other words, there are two types of neutrally stable additions to the equilibrium solution, the respective variations of the equilibrium and nonequilibrium parameters. For the Kolmogorov solutions, the universal stationary corrections may only be obtained by varying the fluxes. From the mathematical viewpoint, the fact that the modes (4.1.17, 21, 24) are not stationary solutions of linearized equations, probably indicates nonanalyticity of the functions $g(x)$, $H(x)$, and $K(x)$ at $x\to\infty$. For example, that functions may allow for the expansion of T in noninteger powers for $x\to\infty$. Such powers are be nonuniversal, i.e., they are defined by specific properties of ω_k and V_{k12}.

It remains to check the stationary character of the drift Kolmogorov solutions (4.1.6, 8, 14, 18). They are anisotropic, i.e., they depend not only on the modulus of the wave vector but also on the angles in $\boldsymbol{k}$-space. Therefore Zakharov transformations affecting only the frequencies ω_k do not allow factorization of the collision integral. Elegant transformations which enable one to transform different terms in the linearized collision integrals with $\delta n(\boldsymbol{k}) \propto (\boldsymbol{Rk})$ into each other have been suggested by *Kats* and *Kontorovich* [4.4]. Following [4.2] we shall elaborate on these transformations for the three-wave equation (4.1.1). If

$$n(\boldsymbol{k}) = k^a[1+k^b(\boldsymbol{\kappa R})] ,$$

where $\boldsymbol{\kappa} = \boldsymbol{k}/k$, then the linearized collision integral may be represented as $\hat{L}_k\delta n(\boldsymbol{k}) = (\boldsymbol{\kappa R})I(\boldsymbol{k})$, where

$$I(\boldsymbol{k}) = \int d\boldsymbol{k}_1 d\boldsymbol{k}_2 ([U(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)f(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)$$
$$- U(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k})f(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k})$$
$$- U(\boldsymbol{k}_2,\boldsymbol{k},\boldsymbol{k}_1)f(\boldsymbol{k}_2,\boldsymbol{k},\boldsymbol{k}_1)]\boldsymbol{\kappa}),$$
$$U(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) = |V(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)|^2\delta(\omega_k-\omega_1-\omega_2)\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2), \tag{4.1.28}$$
$$f(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) = -\,\boldsymbol{\kappa}k^{a+b}(k_1^a+k_2^b)+\boldsymbol{\kappa}_1k_1^{a+b}(k_2^a-k^a)$$
$$+\,\boldsymbol{\kappa}_2k_2^{a+b}(k_1^a-k^a) .$$

The wanted transformations should convert the second and third terms of the integral (4.1.28) into the first one (possibly with additional factors). Consequently, it is necessary to transform into each other the surfaces on which the conservation laws given by the δ-functions are valid. This is achieved by applying similarity transformations (involving dilatations and rotations) to the triangles representing the momentum conservation laws. As an example, we show how to transform the third term. Let us provisionally denote the integration variables by k_1', k_2'. Figures 4.1a and 4.1c show similar triangles representing the momentum conservation laws of the first and the third integrals, respectively. The vector k is common to both triangles.

The transformation $\hat{G}_1$ changing triangle "c" into "a" involves two operations:

1) the rotation $\hat{g}_1^{-1}$ of the triangle $\boldsymbol{k}_1'\boldsymbol{k}_2'\boldsymbol{k}$, such that the vector $\hat{g}_1^{-1}\boldsymbol{k}_2'$ is directed along $\boldsymbol{k}$ (see Fig. 4.1b);

2) the dilatation $\hat{\lambda}$ with the coefficient $\lambda_1 = k/k_1$ such that

$$\hat{G}_1^{-1}\boldsymbol{k}_2' = (\hat{\lambda}\hat{g}_1)^{-1}\boldsymbol{k}_2' = \boldsymbol{k} .$$

In the integral (4.1.28) this transformation corresponds to the following substitution of variables:

$$\hat{G}_1 :\ \boldsymbol{k}_2' = (\lambda_1\hat{g}_1)^2\boldsymbol{k}_1,\quad \boldsymbol{k}_1' = \lambda_1\hat{g}_1\boldsymbol{k}_2,\quad \boldsymbol{k} = \lambda_1\hat{g}_1\boldsymbol{k}_1 . \tag{4.1.29a}$$

A similar transformation of the second term (where we also denote the old integration variables by $\boldsymbol{k}_1'$, $\boldsymbol{k}_2'$ and the new ones by $\boldsymbol{k}_1$, $\boldsymbol{k}_2$) has the form

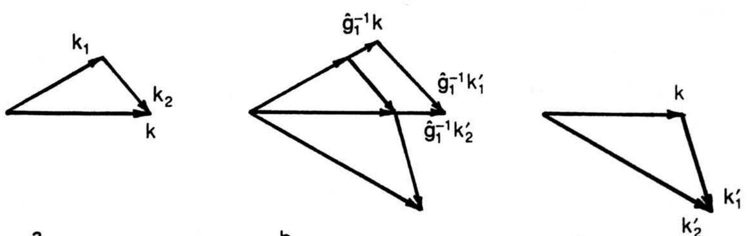

Fig. 4.1. The transformation converting the triangles a and c into each other is illustrated. The triangles represent the conservation laws of energy and momentum

$$\hat{G}_2 : \ \boldsymbol{k}_1' = (\lambda_2\hat{g}_2)^2\boldsymbol{k}_2, \ \boldsymbol{k}_2' = \lambda_2\hat{g}_2\boldsymbol{k}_1, \ \lambda_2 = k/k_2 , \tag{4.1.29b}$$

where $\hat{G}_2 = \lambda_2\hat{g}_2$ transforms $\boldsymbol{k}_2$ to $\boldsymbol{k}$.

Since the transformations (4.1.29) contain dilatations, the second and the third terms in (4.1.28) will acquire factors at the expense of the Jacobian of the transformation and the self-similarity of the functions U and $\boldsymbol{f}$. As a result, the collision integral will become

$$I(\boldsymbol{k}) = \int d\boldsymbol{k}_1 d\boldsymbol{k}_2 U(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)[(\boldsymbol{\kappa f}) - \lambda_1^r(\boldsymbol{\kappa f}_1) - \lambda_2^r(\boldsymbol{\kappa f}_2)]. \tag{4.1.30}$$

Here $r = 2d + 2m - \alpha$ and

$$\begin{aligned} \boldsymbol{\kappa f} &= \boldsymbol{\kappa f}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2), \\ \boldsymbol{\kappa f}_1 &\equiv \boldsymbol{\kappa f}(\hat{G}_1\boldsymbol{k}, \hat{G}_1\boldsymbol{k}_1, \hat{G}_1\boldsymbol{k}_2) \\ &= \lambda_1^{2a+b}(\hat{g}_1\boldsymbol{\kappa}_1)\boldsymbol{f}(k\hat{g}_1\boldsymbol{\kappa}, k_1\hat{g}_1\boldsymbol{\kappa}_1, k_2\hat{g}_1\boldsymbol{\kappa}_2) \\ &= \lambda_1^{2a+b}\boldsymbol{\kappa}_1\boldsymbol{f}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) \ . \end{aligned}$$

The last equality follows from the linearity of f with regard to the wave vectors $\boldsymbol{\kappa}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2$, and the definition of the rotation $\hat{g}_1$ ($\hat{g}_1\boldsymbol{\kappa}_1 = \boldsymbol{\kappa}$), so that $(\boldsymbol{\kappa}, \hat{g}_1\boldsymbol{\kappa}) = (\hat{g}_1\boldsymbol{\kappa}_1, \hat{g}_1\boldsymbol{\kappa}) = (\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)$. Similarly,

$$\boldsymbol{\kappa f}(\hat{G}_2\boldsymbol{k}, \hat{G}_2\boldsymbol{k}_1, \hat{G}_2\boldsymbol{k}_2) = \lambda_2^{2a+b}\boldsymbol{\kappa}_2\boldsymbol{f}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) \ .$$

This gives the integral $I(\boldsymbol{k})$ in factorized form

$$\begin{aligned} I(\boldsymbol{k}) = \int & d\boldsymbol{k}_1 d\boldsymbol{k}_2 U(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)\boldsymbol{f}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) \\ & \times [\boldsymbol{\kappa} - \boldsymbol{\kappa}_1(k/k_1)^{r+2a+b} - \boldsymbol{\kappa}_2(k/k_2)^{r+2a+b}] \ . \end{aligned} \tag{4.1.31}$$

The products of the vectors in these expressions are obviously scalar products. From (4.1.31) it is seen that the integral $I(\boldsymbol{k})$ vanishes for the choice

$$r + 2a + b \equiv 2d + 2m - \alpha + 2a + b = -1 \ . \tag{4.1.32}$$

As in the case of the solution with the momentum flux, the vanishing of the integral is due to the δ-function in the wave vectors. Thus, the Kats-Kontorovich transformations factorize the linearized collision integral for a perturbation $\delta n(\boldsymbol{k})$ proportional to the cosine of an angle between $\boldsymbol{R}$ and $\boldsymbol{k}$. It is readily verified that the corrections found for the decay case (4.1.6, 18) satisfy condition (4.1.32). To conclude that the drift Kolmogorov solutions are neutrally stable modes, we have to verify their locality, i.e., the convergence of the integral (1.1.31). This is done in analogy to the isotropic case (3.1.12). The only particularity consists in the fact that the divergences are reduced by the power of k rather than ω_k [which compresses the locality strip by $2(\alpha - 1)$]. We leave it for the reader to check that for capillary waves on the surface the drift mode is local on deep water (3.1.15b) and not on shallow water (3.1.15a).

The momentum flux direction (towards large or small k) is given by the sign of the derivative of the collision integral with respect to the index of the solution

$$\operatorname{sign} R = -\operatorname{sign}(\partial I/\partial b) \ . \tag{4.1.33}$$

The derivative is calculated at the value of b which satisfies (4.1.32).

The fact that the drift corrections (4.1.8, 14) are the stationary solutions of (4.1.4) is proved in the same way. For factorization of the collision integral, one should use, apart from the dilatations (4.1.25) also rotations transforming similar quadrangles of wave vectors into each other.

Let us note that stationary corrections can also be obtained for the anisotropic spectra introduced in Sect. 3.3. Expanding the general solution (3.3.17) at $\xi \to 0$ one can obtain a stationary drift correction to the spectrum

$$n(p, q, P, R) = P^{1/2}|p|^{-1-u}q^{-2-v} + RP^{-1/2}|p|^{a-u-2}q^{b-v-2}$$

supporting an energy flux. In the opposite limit $\xi \to \infty$, a neutrally stable mode with a small energy flux can be obtained against a background of the spectrum

$$\begin{aligned} n(p, q, R, P) = {} & R^{1/2}|p|^{-u+(a-3)/2}q^{-v-2+b/2} \\ & + PR^{-1/2}|p|^{-u-(a+1)/2}q^{-v-2-b/2} \end{aligned}$$

with a momentum flux. These corrections can be shown to satisfy the linearized kinetic equation. Moreover, the nonlinear kinetic equation (2.5.2) has a stationary solution in the form of a sum of power functions

$$n(p, q) = \sum_{i=1}^{M} c_i p^{-x_i} q^{-y_i}$$

with $M \leq 4$. Such a solution may exist even when different power functions are of the same order [4.5].

Concluding this section, it should be pointed out that the additional contribution of the anisotropic drift correction (4.1.9) to the energy flux distribution

$$\frac{\delta n_0(\boldsymbol{k})}{n_0(k)} \propto \frac{\omega_k}{k} \propto k^{\alpha-1}$$

n the decay case ($\alpha > 1$) grows with k, i.e., while going deeper into the iner-
ial interval. Similarly, in the nondecay case, the contribution of the drift mode (4.1.14) grows from source to sink with respect to the wave action flux distribu-
ion

$$\frac{\delta n_3(\boldsymbol{k})}{n(k)} \propto k^{-1} .$$

Thus the drift Kolmogorov solutions (4.1.9, 14) imply a kind of structural in-
stability of the isotropic Kolmogorov spectrum: even a small anisotropy of the wave source will lead to an essentially anisotropic distribution in the inertial nterval. However, as we shall see in the next section, none of the drift solutions will be established in the case of anisotropic modulation of a wave source. Indeed, apart from the stationary solutions (4.1.9, 14) of the homogeneous equation $\hat{L}_k\delta n_k = 0$ and $\hat{W}_k\delta n_k = 0$ there may also exist solutions of inhomogeneous equations $\hat{L}_k\delta n(\boldsymbol{k}) = \delta\gamma(\boldsymbol{k})n_0(k)$ and $\hat{W}_k\delta n(\boldsymbol{k}) = \delta\gamma(\boldsymbol{k})n_0(k)$ (here $\delta\gamma$ is the anisotropic part of the source assumed to be small), which decrease while going further into the inertial interval $[\delta n(\boldsymbol{k})/n_0(k) \to 0]$. In order to clarify which distribution is generated by a weakly anisotropic source, it is necessary to solve the initial value problem. As will be shown in the next section, only those drift solutions may be observed in the inertial interval that transfer momentum flux into the same direction as the flux of the main integral of motion (of energy or action) – *the Falkovich criterion* [4.6]. It should be noted that this criterion is the natural generalization of the Frisch and Fournier criterion for isotropic solutions (see Sect. 3.1).

4.2 Stability Problem for Kolmogorov Spectra of Weak Turbulence

The proponents of the Kolmogorov spectrum concept in the hydrodynamics of incompressible fluids supposed this spectrum to be stable. An equivalent statement known as the "hypothesis of local isotropy of turbulence" asserts that in the step-by-step transfer of energy over scales, the turbulence spectrum becomes isotropic. In other words, it is usually supposed that the anisotropic spectrum, being determined by external anisotropic pumping in the region of small wave numbers, is in the inertial interval replaced by the isotropic Kolmogorov spectrum (see e.g. [4.7]). The concept of local isotropy for the small scales was introduced by *Taylor* [4.8]. In this section we shall show that an opposite situation may arise, at least for weak turbulence. Namely, the degree of anisotropy of the distribution may be small close to the source and increase further away in the inertial interval. This phenomenon is, in effect, one of the variants of a "self-organization

process" or of the emergence of structures in nonlinear systems and may be considered as a kind of structural instability of the isotropic Kolmogorov spectrum. The possibility of such a kind of instabilities was first indicated by *L'vov* and *Falkovich* [4.9]. A general stability theory (including analyses of different types of instabilities and of various physical systems) was developed by *Balk* and *Zakharov* [4.10] and *Falkovich* and *Shafarenko* [4.11] observed the phenomenon in numerical simulations. We view this instability to be of an "interval" type. This name is associated with the fact that this instability owes its existence to the large inertial interval and has thus an asymptotic character: perturbations increase the more dramatically the larger the inertial interval. With interval instability, the perturbations grow by a power law while k goes into the inertial interval, so that the turbulent medium generates a universal (i.e., determined only by the properties of the medium itself), ordered structure in the region of large or small scales (in the remaining range, a Kolmogorov-like spectrum is realized).

Empirical Approach. Before delving into rigorous mathematical theory, let us try to guess a criterion for the structural stability of the Kolmogorov spectra on the basis of plausible physical reasoning. We consider a perturbation in the form of an angular l-harmonic Y_l. If the linearized kinetic equation has a stationary solution (neutrally stable mode)

$$\frac{\delta n(\boldsymbol{k})}{n(k)} = Y_l(\Omega)k^{-p} ,$$

then it also conserves the integral of motion

$$I_l = \int Y_l(\Omega)k^{p+h-1}\frac{\delta n(\boldsymbol{k})}{n(k)}\, dk d\Omega ,$$

whose constant flux is transported by that mode.

Let the spectrum in question carry a positive flux. Then the inertial interval is in the small-scale region. The harmonic may affect the stability of the spectrum in the inertial interval if $p < 0$. It is natural to assume that the harmonic can be generated by external anisotropic pumping if the flux of I_l is directed towards the damping region. The sign of the flux is defined by the derivative (4.1.33) of the linearized collision integral with respect to the exponent of the solution. Introducing the dimensionless collision integral $W_l(s)$ calculated with $\delta n(\boldsymbol{k}) = Y_l k^{-s} n(k)$ we thus obtain the instability criterion $W_l'(p_l) > 0$. Similarly one should in the case of negative main flux require $p_l > 0$ and $W_l'(p_l) < 0$ as a criterion for the existence of instability. Thus we get the simple physical criterion of structural instability connected with neutrally stable modes [4.6]:

$$-\text{sign}\, p_l = \text{sign}\, W_0'(0) = \text{sign}\, W_l'(p_l) . \qquad (4.2.1a)$$

Here $W_o'(0) = I'(\nu)$ is the derivative of the linearized collision integral with $\delta n(k) = n(k)k^{-s}$ which for $s = 0$ coincides with the derivative of the complete collision integral $\partial I/\partial x$ at x being equal to Kolmogorov index ν. Thus $W_0'(0)$ defines the flux of the main integral.

Equally simple we can obtain an instability criterion for the free evolution of a perturbation of the spectrum (without any additional pumping). Requiring the integral of motion I_l to be conserved while an l-harmonic perturbation evolves in k-space, we obtain

$$\frac{\delta n(\boldsymbol{k})}{n(k)} \propto k^{p+h} . \tag{4.2.1b}$$

In this case the role of p is played by the quantity $p+h$. A necessary condition for the existence of an instability is $\operatorname{sign}(p_l + h) = -\operatorname{sign} W_0'(0)$. Indeed, the ultraviolet instability (for a positive main flux) occurs for $p_l + h > 0$, etc.

Introduction to Stability Theory. Instability studies are usually reduced to deriving a complete set of eigenfunctions of the linearized operator and to investigating the eigenvalues determining the time evolution of the eigenfunctions. In our case, the operators $\hat{L}_{\boldsymbol{k}}$ and $\hat{W}_{\boldsymbol{k}}$ are scale-invariant, since the functions $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$, $\omega(k)$ and $n(k)$ possess this property. A natural set of functions consists of power functions k^s with different s. In this connection we encounter the first mathematical difficulty: such functions grow either at $k \to 0$ or at $k \to \infty$ giving rise to divergences of different terms in the collision integral. For the sake of convenience we will in this section use the variable $x = \ln k$ rather than k. The scale invariance of the operators $\hat{L}_{\boldsymbol{k}}, \hat{W}_{\boldsymbol{k}}$ will make it possible to use the more customary Fourier representation with eigenfunctions in the form of exponents $k^s \to \exp(sx)$. Unfortunately, our integral equation is not of the convolution type, since the operators $\hat{L}_{\boldsymbol{k}}$, $\hat{W}_{\boldsymbol{k}}$ have nonzero h-indices. So the linearized kinetic equation in the variables x may be written in the form

$$\frac{\partial}{\partial t}\delta n(x,t) = e^{-hx} \int_{-\infty}^{\infty} U(x-x')\delta n(x',t)\, dx' ,$$

which after Fourier transformation turns into an equation of the Carleman type (and not into an algebraic one)

$$\lambda \Psi(s+h) = W(s)\Psi(s) .$$

A rather developed theory of Carleman-type equations has been formulated by the mathematician *Cherskii* [4.12–13] for kernels of the form

$$U(x) = U_0\delta(x) + u(x) ,$$

where U_0 is a constant and $u(x)$ an ordinary integrable function. Unfortunately, it is in general not possible to represent the kernels of linearized collision integrals in such a form. This is due to the fact that in the kinetic equation separate integrals diverge, and the regularity of the whole expression is an effect of mutual cancellation of divergences. Indeed, the constant N equal to the sum of integrals appearing in (4.1.1) or (4.1.4) as factors at $\delta n(k)$ may, e.g., prove to be infinite. Thus, another mathematical difficulty is the singularity of the kernel of the integral operator. To overcome it and to obtain rigorous results the method of generalized functions should be applied carefully.

Finally, a last difficulty is associated with the fact that physical considerations are insufficient for choosing the boundary conditions to be imposed on perturbations. As we shall see, the correct conditions are obtained by considering the Cauchy initial value problem and the transition to the limit at $t \to \infty$ and by demanding this problem to be correctly posed for the chosen class of functions. Thus, we will not only have to analyze the eigenvalues and stationary solutions of the linearized equation, but we will also have to solve the associated initial value problem. We shall follow a similar way as Landau in the solution of the description of Langmuir wave damping in plasmas [4.14].

The mentioned mathematical difficulties explain the somewhat higher level of mathematical sophistication adopted in this section. However, as a reward quite beautiful results are obtained. In this section we shall fully classify possible types of behavior of a weakly turbulent medium in the vicinity of the Kolmogorov spectrum, show how one can efficiently identify the type of the system, describe the asymptotics of its behavior for large times ($t \to \infty$) and in the limits of small ($k \to \infty$) and large ($k \to 0$) scales.

The central result of the stability theory is a verifiable stability criterion for Kolmogorov spectra in the case of weak turbulence. This criterion reduces the examination of the stability of a Kolmogorov spectrum to calculating several integers (rotations of certain analytical functions around the imaginary axes; explicit formulas will be given). For the stability of the Kolmogorov spectrum it is necessary and sufficient that all these integers be equal to zero.

In this section we shall also try to reach a more profound understanding of locality of the Kolmogorov spectra. It appeared that, despite the locality of the Kolmogorov spectrum in the above sense, i.e., the convergence of the collision integral on the spectrum, the evolution of the distribution $n(\boldsymbol{k}, t)$ weakly deviating from the Kolmogorov spectrum may possibly not be determined by the interaction of waves only of scales of the same order and may considerably depend on the conditions at the ends of the inertia interval. This phenomenon was called the evolution nonlocality of Kolmogorov spectra; we shall describe the necessary and sufficient conditions of the evolution locality.

The general results of this chapter will be applied in the next one to the analysis of the turbulence of capillary waves, gravitational waves, Langmuir turbulence in plasmas and acoustic turbulence.

Our studies into the stability of Kolmogorov solutions were strongly stimulated by a desire to explain the experimentally observed anomalous angle narrowness of the spectrum of wind-stimulated undulation on the ocean surface.

The available technique enables one to study the stability of isotropic turbulence spectra of gravitational waves in the framework of the kinetic equation and to find that the spectra are stable. This probably suggests that for the description of wind-induced undulation it is insufficient to take into account only the interaction of gravitational waves with each other.

In the case of turbulent capillary waves, the Kolmogorov spectrum proved to e unstable with respect to anisotropic perturbations having only a first angular armonic. In acoustic turbulence, the number of angular harmonics, with respect o which the Kolmogorov spectrum is unstable, is inversely proportional to the alue $\sqrt{\varepsilon} = \sqrt{\alpha - 1}$ (see the Sect. 5.1). In both cases the instability proves to be f the hard interval type and shows itself in small scales.

The results formulated in this section for the Kolmogorov spectra of weak urbulence, which are the exact solutions of the kinetic equations for waves, may e extended to studies of the Kolmogorov spectra which are the exact solutions f other kinetic equations (Boltzmann equation, polymerization equation etc).

.2.1 Perturbations of the Kolmogorov Spectrum

tatement of the Stability Problem. Inside the inertial interval there may be xternal effects or dissipation which is small by the very definition of the inertial nterval (as compared to the values of source and sink forming the Kolmogorov pectrum). Consideration of these factors leads to an additional term of the form $(\boldsymbol{k},t)n_k^0$ on the right-hand-side of (4.1.1, 4). Assuming the initial solution to be f power type

$$n_k^0 = Rk^{-\nu}, \tag{4.2.2}$$

e get, for the relative part of the perturbation,

$$A(\boldsymbol{k},t) = \delta n(\boldsymbol{k},t)/n_k^0, \tag{4.2.3}$$

n equation of the form

$$\frac{\partial A(\boldsymbol{k},t)}{\partial t} = \hat{L}_k A(\boldsymbol{k},t) + \gamma(\boldsymbol{k},t). \tag{4.2.4}$$

One can study the stability of the Kolmogorov spectrum with respect to initial erturbations in terms of this equation (one should then set $\gamma = 0$) and external ction ($\gamma \neq 0$), respectively.

he Evolution Equation and its Reduction to the Carleman Equation. Let s expand the function A in the Fourier series into an orthonormal system of ngular harmonics $Y_l(\zeta)$ (ζ is a point on the sphere $\Omega = \{\zeta \in R^d/|\zeta = 1\}$):

$$A(\boldsymbol{k},t) = \sum_l A_l(k,t)Y_l(\zeta), \quad A_l(k,t) = \int_\Omega A(\boldsymbol{k},t)Y_l^*(\zeta)\,D\zeta.$$

Iere $D\zeta$ is a surface element on the sphere, $d\boldsymbol{k} = k^{d-1}dkD\zeta$. For two-imensional media ($d=2$): $Y_l = 2\pi^{-1/2}e^{il\varphi}$, $l = 0, \pm 1, \pm 2, \ldots$, $D\zeta = d\varphi$. In e case of three-dimensional media ($d=3$), the $Y_l(\zeta)$-functions are the normal-ed ordinary spheric functions $Y_l^j(\theta,\varphi)$, $l = 0,1,2,\ldots$, $j = 0, \pm 1, \pm 2, \ldots, \pm l$, $D\zeta = \sin\theta\,d\theta\,d\varphi$. For different functions A_l we have uncoupled equations of the orm (the l-index is omitted):

$$\frac{\partial A}{\partial t} = \hat{L}[A] + \chi, \tag{4.2.5}$$

where $\chi = \chi(k,t) = \int_\Omega \gamma(\boldsymbol{k},t)Y^*(\zeta)\,D\zeta$, the linear operator $\hat{L}$ (assuming scale invariance of the medium) is homogeneous with a certain power $(-h)$:

$$\hat{L}[f \cdot \varepsilon](k) = \varepsilon^{-h}\hat{L}[f](\varepsilon k)$$

[$f = f(k)$ is an arbitrary function and ε is an arbitrary positive number]. It is convenient to go over from the variable k to the variable $x = \ln k$ by introducing the notation

$$F(x,t) = A(k,t), \quad \phi(x,t) = \chi(k,t) \quad (k = e^x).$$

Then (4.2.5) reads

$$\frac{\partial F(x,t)}{\partial t} = e^{-hx}[U(x) * F(x,t)] + \phi(x,t) \tag{4.2.6}$$

where $U(x)$ is a generalized function determined by the form of the operator $\hat{L}$ and $*$ denotes the convolution

$$U(x) * F(x,t) = \int_{-\infty}^{+\infty} U(x-x')F(x',t)\,dx'.$$

The operator $\hat{L}$ is given by an integral of a sum of several expressions containing the function $A(k,t)$ with different arguments k, see (4.1.1) or (4.1.4). If this integral could be divided into the sum of integrals over these expressions then the generalized function $U(x)$ would have a form

$$U(x) = U_0\delta(x) + u(x) \tag{4.2.7}$$

where U_0 is a constant and $u(x)$ an ordinary integrable function. In the general case, when the integral specifying the operator $\hat{L}$ may not be divided into the sum of several integrals with the function A occurring only once in each of the integrals (the indivivual integrals diverge but the total integral converges due to a cancellation of the divergences of the different terms), then the generalized function $U(x)$ are regularizations of rather diverse singular functions, see [4.15].

In physically interesting situations, the kernel $U(x)$ exponentially tends to zero at $|x| \to \infty$:

$$U(x) = \begin{cases} O\left(e^{-ax}\right), & x \to -\infty, \\ O\left(e^{-bx}\right), & x \to +\infty. \end{cases} \tag{4.2.8}$$

Here $a < b$.

We shall study the Cauchy problem of the evolution equation (4.2.6) with the initial condition

$$F(x,0) = \phi_0(x). \tag{4.2.9}$$

The solution of (4.2.6) should be sought in such a (possibly wider) class of functions that the convolution in (4.2.6) is determined. Such a class is constituted by the functions $f(x)$ which at $x \to +\infty$ grow slower than $\exp(-ax)$ and at $x \to -\infty$ slower than $\exp(-bx)$

$$f(x) = \begin{cases} O\left(e^{-\sigma_1 x}\right), & x \to +\infty \quad \sigma_1 > a \ ; \\ O\left(e^{-\sigma_2 x}\right), & x \to -\infty \quad \sigma_2 < b \ . \end{cases} \tag{4.2.10}$$

Let us denote this class of functions by $\mathcal{L}(a, b)$. We shall require that the solutions $F(x, t)$ of the evolution equation (4.2.6) are elements of the space $\mathcal{L}(a, b)$ at every fixed t. The class $\mathcal{L}(a, b)$ is analogous to the class of solutions treated by the classic Wiener-Hopf theory, see [4.15–16]. The exponential decrease of $F(x)$ corresponds to a power decrease of $\delta n(k)$. The quantities a, b coincide for zero spherical harmonics in the decay case with the boundaries of the locality interval s_1, s_2 determined in Sect. 3.1. For the remaining harmonics we have $a = s_2 + \alpha - 1$, $b = s_1 + 1 - \alpha$. Subjecting the evolution equation (4.2.6) to the Laplace time transformation

$$F(x) = F_\lambda(x) = \int_0^\infty F(x, t) e^{-\lambda t}\, dt$$

leads to

$$\lambda F(x) = e^{-hx}[U(x) * F(x)] + \Phi(x) \tag{4.2.11}$$

where

$$\Phi(x) = \Phi_\lambda(x) = \phi_0(x) + \int_0^\infty \phi(x, t) e^{-\lambda t}\, dt \ .$$

The functions $\phi_0(x)$ and $\phi(x, t)$ and consequently also $\Phi(x)$ may be considered to be finite functions of the variables x. The solution of (4.2.11), like those of the evolution equation (4.2.6), should be regarded in the class $\mathcal{L}(a, b)$.

If one formally Fourier transforms (4.2.11)

$$G(s) = \int_{-\infty}^{+\infty} F(x) e^{sx}\, dx \tag{4.2.12}$$

(whereby the convolution is converted into a product and multiplication by the exponential function into a translation) one obtains an equation of the Carleman type (see [4.12–13])

$$\lambda G(s + h) = W(s) G(s) + \Psi(s + h), \tag{4.2.13}$$

where $W(s)$ and $\Psi(s)$ are the Fourier images of the functions $U(x)$ and $\Phi(x)$, respectively.

For $h = 0$ it is easy to solve (4.2.13) and, hence, (4.2.6). We shall consider this case in Sect. 5.1.2 for two-dimensional acoustic turbulence. In this section we shall everywhere use $h \neq 0$.

The Fourier transformation in terms of the variable x corresponds to the Mellin transformation in the initial variable k

$$G(s) = \int_0^\infty F(k) k^{s-1}\, dk.$$

The function

$$W(s) = \int_{-\infty}^{+\infty} U(x) e^{sx}\, dx, \quad a < \mathrm{Re}\, s < b,$$

will play a key role in our further considerations. It is the image of an operator $\hat{L}_0 = \exp(hx)\hat{L}$ in the Mellin transformation and will be called the Mellin function. It is convenient to derive explicit expressions for Mellin functions by using

$$\begin{aligned} W(s)\delta(r - s) &= \frac{1}{2\pi} \int_0^\infty k^{r+h} \hat{L}[(k')^{-s}](k)\, dk/k \\ &= \frac{1}{2\pi} \int_0^\infty k^{r+h} Y^*(\zeta) \hat{L}[(k')^{-s} Y(\zeta')](k, \zeta)\, D\zeta\, dk/k \ . \end{aligned} \tag{4.2.14}$$

If we write in (4.2.14) the operator $\hat{L}$ in an explicit form, we obtain an integral in which the integration is to be performed over the variables k, k_1, k_2 in the decay case or k, k_1, k_2, k_3 in the nondecay case. This allows for a proper symmetrization of the integrand. Having chosen the variable ξ over which to integrate from zero to ∞ (for example, $\xi = k$), we can represent the integral (4.2.14) in the form

$$\frac{1}{2\pi} \int_0^\infty M(r, s, \xi) \xi^{r-s}\, d\xi/\xi, \tag{4.2.15}$$

where $M(r, s, \xi)\xi^{r-s}$ is the result of the remaining integrations. $M(r, s, \xi)$ is homogeneous and of zeroth power in ξ and does not explicitly depend on ξ. Taking

$$\frac{1}{2\pi} \int_0^\infty \xi^{r-s} d\xi/\xi = \delta(r - s)$$

into account we obtain that $W(s) = M(s, s, \xi) = M(s, s, 1)$. Formula (4.2.14) should be used to obtain the Mellin functions for any kinetic equation. But when kinetic equations for waves are considered, this formula with the variable $\xi = k$ [see (4.2.15)] leads to the following expressions:

(i) in the decay case to

$$\begin{aligned} W(s) = R \int & 2\pi |V(k, k_1, k_2)|^2 \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \delta(\omega_k - \omega_1 - \omega_2) \\ & \times (k k_1 k_2)^{d-\nu} \{[(k^{-s} Y + k_1^{-s} Y_1 + k_2^{-s} Y_2)(k^\nu - k_1^\nu - k_2^\nu) \\ & - (k^{\nu-s} Y - k_1^{\nu-s} Y_1 - k_2^{\nu-s} Y_2)](k^{\mu+s} Y^* - k_1^{\mu+s} Y_1^* \\ & - k_2^{\mu+s} Y_2^*)\}\, D\zeta D\zeta_1 D\zeta_2 \frac{dk_1\, dk_2}{k_1 k_2}, \end{aligned} \tag{4.2.16a}$$

(ii) in the nondecay case to

$$W(s) = R^2 \int |T_{k123}|^2 \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\delta(\omega_k + \omega_1 - \omega_2 - \omega_3)$$
$$\times (kk_1k_2k_3)^{d-\nu}\{[(k^{-s}Y + k_1^{-s}Y_1 + k_2^{-s}Y_2 + k_3^{-s}Y_3)$$
$$\times (k^\nu + k_1^\nu - k_2^\nu - k_3^\nu) - (k^{\nu-s}Y + k_1^{\nu-s}Y_1 \tag{4.2.16b}$$
$$- k_2^{\nu-s}Y_2 - k_3^{\nu-s}Y_3)](k^{\mu+s}Y^* + k_1^{\mu+s}Y_1^*$$
$$+ k_2^{\mu+s}Y_2^* - k_3^{\mu+s}Y_3^*)\}\pi D\zeta D\zeta_1 D\zeta_2 D\zeta_3 \frac{dk_1 dk_2 dk_3}{k_1 k_2 k_3}.$$

Here $\mu = h + \nu - d$.

In the nondecay case we can use the representation

$$\delta(\omega + \omega_1 - \omega_2 - \omega_3) = \int_0^\infty \delta(y^\alpha - \omega - \omega_1)\delta(y^\alpha - \omega_2 - \omega_3)\alpha y^\alpha\, dy/y$$

to obtain a more symmetric expression for the Mellin function. Substituting this expression into (4.2.14) and specifying the variable $\xi = y$ [see (4.2.15)] we obtain

$$W(s) = R^2\alpha \int \pi |T_{k123}|^2 \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\delta(1 - \omega - \omega_1)\delta(1 - \omega_2 - \omega_3)$$
$$\times \{[(k^{-s}Y + k_1^{-s}Y_1 + k_2^{-s}Y_2 + k_3^{-s}Y_3)(k^\nu + k_1^\nu - k_2^\nu - k_3^\nu)$$
$$- (k^{\nu-s}Y + k_1^{\nu-s}Y_1 - k_2^{\nu-s}Y_2 - k_3^{\nu-s}Y_3)] \tag{4.2.16c}$$
$$\times (k^{\mu+s}Y^* + k_1^{\mu+s}Y_1^* - k_2^{\mu+s}Y_2 - k_3^{\mu+s}Y_3)\}(kk_1k_2k_3)^{d-\nu}$$
$$\times D\zeta D\zeta_1 D\zeta_2 D\zeta_3 \frac{dk\,dk_1 dk_2 dk_3}{kk_1k_2k_3}.$$

The integral (4.2.16) contains [to a linear approximation] all information about the behavior of the perturbations of the Kolmogorov spectra. That is, each of the Mellin functions $W(s)$ determines the behavior of the particular perturbation having the form of the angular harmonic contained in the integral (4.2.16) defining this Mellin function. The integrals (4.2.16a, b) are independent of the k-value [so that it may be chosen arbitrarily, for example, $k = 1$]. They are homogeneous functions of the variable k [of the zeroth order]. Thus is is possible to find the constant h (determining the scaling index of the $\hat{L}$-operator):

(i) in the decay case [see (4.1.1)],

$$h = \alpha - 2m - d + \nu, \tag{4.2.17a}$$

(ii) in the nondecay case [see (4.1.4)],

$$h = \alpha - 2m - 2d + 2\nu. \tag{4.2.17b}$$

It is much more convenient to handle the analytical functions $W(s)$ [for which explicit symmetrical integral representations (4.2.16) are available] than to deal with the generalized functions $U(x)$ or the $\hat{L}$-operators. Therefore, all conditions and statements referring to the evolution equation (4.2.5, 6) will be formulated in terms of the Mellin function $W(s)$.

To facilitate the further treatment we shall specify the following condition. The strip $\{s \in \mathbb{C}/\mathrm{Re}\, s \in I\}$, where $\mathbb{C}$ is the space of complex numbers and I is some interval or section, will be denoted by ΠI. We shall say that some function $g(s)$ in the strip ΠI is polynomially bounded on infinity if for any interval $K \in I$ there is such a number $j(K)$ that $g(s) = O(|s|^j)$, $s \Longrightarrow \infty$, $s \in \Pi K$.

The Mellin functions $W(s)$ (arising when considering the kinetic equations) have the following three properties.

1) The function $W(s)$ is analytical in some strip $\Pi(a,b)$ (on the straight lines $\mathrm{Re}\, s = a$ and $\mathrm{Re}\, s = b$ it has singularities).
2) The $W(s)$ and $1/W(s)$-functions in the $\Pi(a,b)$ strip are polynomially bounded on the infinity.
3) The value of the $W(s)$-function at $|\mathrm{Im}\, s| \to \infty$ becomes asymptotically real negative; to be more exact, for any interval $K \subset (a,b)$:

$$\arg[-W(s)] = O(1/s), \quad s \to \infty, \quad s \in \Pi K\,, \tag{4.2.18}$$

that is

$$\frac{\mathrm{Im}\, W(s)}{\mathrm{Re}\, W(s)} = O\left(\frac{1}{s}\right), \quad \mathrm{Re}\, W(s) < 0, \quad s \to \infty, \quad s \in \Pi K\,.$$

The first property and the polynomial constraint of the function $W(s)$ imply that the generalized function $U(x)$ satisfies the condition (4.2.8). If the function $U(x)$ were regular, the Mellin function $W(s)$ would have tended to zero at $|\mathrm{Im}\, s| \to \infty$, but since the generalized function $U(x)$ is a regularized singular function, the Mellin function may grow without bound at $|\mathrm{Im}\, s| \to \infty$, but not faster than the polynomial. The third property and the polynomial constraint of the $1/W(s)$-function are explained in the following way. The expression in the braces in (4.2.16) may be represented as a sum of two expressions, one of which is independent of s, and another (depending on s) is fast oscillating at large $|\mathrm{Im}\, s|$ because of the presence of the functions k^s; upon integration of (4.2.16) this expression will give values with different signs which will "quench" each other. Provided that the μ, ν indices correspond to the thermodynamic or Kolmogorov spectrum, the remaining, nonoscillating part in the braces in (4.2.16) is on the resonant manifold [specified by the δ-functions in (4.2.16)] transformed into the expression

$$-\left(k^{\mu+\nu} + k_1^{\mu+\nu} + k_2^{\mu+\nu} + k_3^{\mu+\nu}\right) \tag{4.2.19}$$

(in the decay case $k_3 \equiv 0$) which is negatively determined.

For the function $W(s)$ satisfying the conditions 1-3, the *function of rotation* $\kappa(\sigma)$ may be specified, which is important for all further treatment and has the properties mentioned below. We shall define rotations of the function $W(s)$ around a straight line $\mathrm{Re}\, s = \sigma$ [$\sigma \in (a,b)$] as a complete increment of the argument of a complex value $W(s)$ (with s moving from $\sigma - i\infty$ to $\sigma + i\infty$ along the straight line $\mathrm{Re}\, s = \sigma$) divided by 2π and denote it as $\kappa(\sigma)$. The $\kappa(\sigma)$-function is defined on the whole interval (a,b), except the points which are the

(ii) in the nondecay case to

$$
\begin{aligned}
W(s) = R^2 \int & |T_{k123}|^2 \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\delta(\omega_k + \omega_1 - \omega_2 - \omega_3) \\
& \times (kk_1k_2k_3)^{d-\nu}\{[(k^{-s}Y + k_1^{-s}Y_1 + k_2^{-s}Y_2 + k_3^{-s}Y_3) \\
& \times (k^\nu + k_1^\nu - k_2^\nu - k_3^\nu) - (k^{\nu-s}Y + k_1^{\nu-s}Y_1 \\
& - k_2^{\nu-s}Y_2 - k_3^{\nu-s}Y_3)](k^{\mu+s}Y^* + k_1^{\mu+s}Y_1^* \\
& + k_2^{\mu+s}Y_2^* - k_3^{\mu+s}Y_3^*)\}\pi D\zeta D\zeta_1 D\zeta_2 D\zeta_3 \frac{dk_1 dk_2 dk_3}{k_1k_2k_3}.
\end{aligned}
\tag{4.2.16b}
$$

Here $\mu = h + \nu - d$.

In the nondecay case we can use the representation

$$
\delta(\omega + \omega_1 - \omega_2 - \omega_3) = \int_0^\infty \delta(y^\alpha - \omega - \omega_1)\delta(y^\alpha - \omega_2 - \omega_3)\alpha y^\alpha \, dy/y
$$

to obtain a more symmetric expression for the Mellin function. Substituting this expression into (4.2.14) and specifying the variable $\xi = y$ [see (4.2.15)] we obtain

$$
\begin{aligned}
W(s) = R^2\alpha \int & \pi |T_{k123}|^2 \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\delta(1 - \omega - \omega_1)\delta(1 - \omega_2 - \omega_3) \\
& \times \{[(k^{-s}Y + k_1^{-s}Y_1 + k_2^{-s}Y_2 + k_3^{-s}Y_3)(k^\nu + k_1^\nu - k_2^\nu - k_3^\nu) \\
& - (k^{\nu-s}Y + k_1^{\nu-s}Y_1 - k_2^{\nu-s}Y_2 - k_3^{\nu-s}Y_3)] \\
& \times (k^{\mu+s}Y^* + k_1^{\mu+s}Y_1^* - k_2^{\mu+s}Y_2 - k_3^{\mu+s}Y_3)\}(kk_1k_2k_3)^{d-\nu} \\
& \times D\zeta D\zeta_1 D\zeta_2 D\zeta_3 \frac{dk dk_1 dk_2 dk_3}{kk_1k_2k_3}.
\end{aligned}
\tag{4.2.16c}
$$

The integral (4.2.16) contains [to a linear approximation] all information about the behavior of the perturbations of the Kolmogorov spectra. That is, each of the Mellin functions $W(s)$ determines the behavior of the particular perturbation having the form of the angular harmonic contained in the integral (4.2.16) defining this Mellin function. The integrals (4.2.16a, b) are independent of the k-value [so that it may be chosen arbitrarily, for example, $k = 1$]. They are homogeneous functions of the variable k [of the zeroth order]. Thus is is possible to find the constant h (determining the scaling index of the $\hat{L}$-operator):

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To facilitate the further treatment we shall specify the following condition. The strip $\{s \in \mathbb{C}/\mathrm{Re}\, s \in I\}$, where $\mathbb{C}$ is the space of complex numbers and I is some interval or section, will be denoted by ΠI. We shall say that some function $g(s)$ in the strip ΠI is polynomially bounded on infinity if for any interval $K \in I$ there is such a number $j(K)$ that $g(s) = O(|s|^j)$, $s =\!\!\longrightarrow \infty$, $s \in \Pi K$.

The Mellin functions $W(s)$ (arising when considering the kinetic equations) have the following three properties.

1) The function $W(s)$ is analytical in some strip $\Pi(a, b)$ (on the straight lines $\mathrm{Re}\, s = a$ and $\mathrm{Re}\, s = b$ it has singularities).
2) The $W(s)$ and $1/W(s)$-functions in the $\Pi(a, b)$ strip are polynomially bounded on the infinity.
3) The value of the $W(s)$-function at $|\mathrm{Im}\, s| \to \infty$ becomes asymptotically real negative; to be more exact, for any interval $K \subset (a, b)$:

$$
\arg[-W(s)] = O(1/s), \quad s \to \infty, \quad s \in \Pi K, \tag{4.2.18}
$$

that is

$$
\frac{\mathrm{Im}\, W(s)}{\mathrm{Re}\, W(s)} = O\left(\frac{1}{s}\right), \quad \mathrm{Re}\, W(s) < 0, \quad s \to \infty, \quad s \in \Pi K .
$$

The first property and the polynomial constraint of the function $W(s)$ imply that the generalized function $U(x)$ satisfies the condition (4.2.8). If the function $U(x)$ were regular, the Mellin function $W(s)$ would have tended to zero at $|\mathrm{Im}\, s| \to \infty$, but since the generalized function $U(x)$ is a regularized singular function, the Mellin function may grow without bound at $|\mathrm{Im}\, s| \to \infty$, but not faster than the polynomial. The third property and the polynomial constraint of the $1/W(s)$-function are explained in the following way. The expression in the braces in (4.2.16) may be represented as a sum of two expressions, one of which is independent of s, and another (depending on s) is fast oscillating at large $|\mathrm{Im}\, s|$ because of the presence of the functions k^s; upon integration of (4.2.16) this expression will give values with different signs which will "quench" each other. Provided that the μ, ν indices correspond to the thermodynamic or Kolmogorov spectrum, the remaining, nonoscillating part in the braces in (4.2.16) is on the resonant manifold [specified by the δ-functions in (4.2.16)] transformed into the expression

$$
-\left(k^{\mu+\nu} + k_1^{\mu+\nu} + k_2^{\mu+\nu} + k_3^{\mu+\nu}\right) \tag{4.2.19}
$$

(in the decay case $k_3 \equiv 0$) which is negatively determined.

For the function $W(s)$ satisfying the conditions 1-3, the *function of rotation* $\kappa(\sigma)$ may be specified, which is important for all further treatment and has the properties mentioned below. We shall define rotations of the function $W(s)$ around a straight line $\mathrm{Re}\, s = \sigma$ [$\sigma \in (a, b)$] as a complete increment of the argument of a complex value $W(s)$ (with s moving from $\sigma - i\infty$ to $\sigma + i\infty$ along the straight line $\mathrm{Re}\, s = \sigma$) divided by 2π and denote it as $\kappa(\sigma)$. The $\kappa(\sigma)$-function is defined on the whole interval (a, b), except the points which are the real parts of zeros of the $W(s)$-function; it takes only integer values on and does not monotonically decrease in (a, b); it assumes each of its values in the whole interval rather and not just at a single point; if $\sigma_1, \sigma_2\,(\sigma_2 > \sigma_1)$ are points of its definition domain, then the difference $\kappa(\sigma_2) - \kappa(\sigma_1)$ is equal to the number of zeros of the Mellin function $W(s)$ in the strip $\Pi(\sigma_1, \sigma_2)$. We shall see below that the rotation $\kappa(\sigma)$ is the basic characteristic of the evolution equations (4.2.5, 6).

To be specific we shall consider below only $h > 0$. It is readily seen that this not really a limitation: the case with $h < 0$ reduces to the one with $h > 0$ if one performs in (4.2.5) or (4.2.6) the substitution $k \to 1/k$ or $x \to -x$, respectively; i.e., the function $W(s)$ is replaced by $W(-s)$.

Lemma. Having formally carried out the Fourier transformation in (4.2.11), we arrived at (4.2.13). The solutions of the latter are readily seen to have a structure with:

1. The general solution of the nonhomogeneous equation (4.2.13) is a sum of the particular solution of this equation and the general solution of the homogeneous equation

$$\lambda G(s+h) = W(s)G(s). \tag{4.2.20}$$

2. If $G_0(s)$ is a particular solution of the homogeneous equation (4.2.20), then the general solution of that equation has the form

$$G(s) = G_0(s)M(s), \tag{4.2.21}$$

where $M(s)$ is an arbitrary periodic function with period h, i.e., $M(s+h) = M(s)$.

Whence it is seen that (4.2.13) has many "extra" solutions. Besides, it is clear that the Fourier transformation is, generally speaking, not applicable to functions in the space $\mathcal{L}(a, b)$ [in which (4.2.11) is formulated], since they may simultaneously grow exponentially at $x \to +\infty$ and at $x \to -\infty$. In this section we shall verify the validity of the Fourier transformation in (4.2.11) and indicate the class of functions for which (4.2.13) should be considered to be equivalent to (4.2.11) in the class $\mathcal{L}(a, b)$.

First of all we would like to mention the following property of (4.2.11). Let $F(x)$ be a solution of (4.2.11) with $\lambda \neq 0$. Now, if the function $F(x)$ grows at $x \to +\infty$ not faster than $\exp(-\sigma x)$ with $\sigma \in (a, b)$, then this function grows at $x \to +\infty$ not faster than $\exp[-(\sigma + h)x]$. More precisely we have

$$F(x) = O\left(e^{-\sigma x}\right) \Longrightarrow F(x) = O\left(e^{-(\sigma+h)x}\right), \quad (x \to +\infty,\ a < \sigma < b) \tag{4.2.22}$$

Indeed, using (4.2.8), we obtain from the first estimate (4.2.22)

$$U * F = \int U(x - x')F(x')\,dx' = O\left(e^{-\sigma x}\right), \quad x \to +\infty, \tag{4.2.23}$$

and, consequently, owing to (4.2.11) at $\lambda \neq 0$ we have another estimate (4.2.22).

The property (4.2.22) makes it possible to "improve", by virtue of (4.2.11), the *a priori* characteristics of the solutions of this equation. Let $F(x)$ be a solution of (4.2.11) with $\lambda \neq 0$. Because of $F \in \mathcal{L}(a, b)$ there are numbers $\sigma_1, \sigma_2 \in (a, b)$ with

$$F(x) = \begin{cases} O\left(e^{-\sigma_1 x}\right) & \text{at } x \to +\infty, \\ O\left(e^{-\sigma_2 x}\right) & \text{at } x \to -\infty \end{cases}$$

see (4.2.10). Hence, we have by virtue of (4.2.22) $F(x) = O\big[\exp[-(\sigma_1 + h)x]\big]$, $x \to +\infty$. Now, for $\sigma_1 + h < b$ we can again make use of the property (4.2.22) to obtain $F(x) = O\big[\exp[-(\sigma_1 + 2h)x]\big]$ for $x \to +\infty$. Via repeated application of such a procedure one can show that $F(x) = O\big[\exp[-(b + h)x]\big]$, $x \to +\infty$. Thus, for $a < \varrho < b$

$$F(x) = \begin{cases} O\left(e^{-\varrho x}\right) & \text{at } x \to -\infty, \\ O\left(e^{-(b+h)x}\right) & \text{at } x \to +\infty \end{cases}$$

with $\varrho = \sigma_2$. From (4.2.23) it follows that the Fourier transformation (4.2.12) is applicable to $F(x)$ at $s \in \Pi(\varrho, b + h)$. The Fourier transformation $G(s)$ of this function is analytical and polynomially bounded in the strip $\Pi(\varrho, b + h)$. It satisfies (4.2.13) in the strip $\Pi(\varrho, b)$.

For a finite $\Phi(x)$ the function $\Psi(s)$ is an integer function and is polynomially bounded in the strip $\Pi(-\infty, +\infty)$.

The function $G(s)$ may be redefined in the strip $\Pi(a, b + h)$; for that purpose (4.2.13) must be rewritten in the form

$$G(s) = \frac{\lambda G(s+h) - \Psi(s+h)}{W(s)}. \tag{4.2.24}$$

Knowing the values of $G(s)$ in the strip $\Pi(\varrho, b+h)$, we can calculate by the aid of (4.2.24) its values in the strip $\Pi(\varrho - h, b)$, then in the strip $\Pi(\varrho - 2h,\ b - h)$, etc. Owing to the properties 1)–2) of the Mellin function $W(s)$, $G(s)$ thus redefined in the strip $\Pi(a, b + h)$ is meromorphic and polynomially bounded at infinity; it satisfies (4.2.13) in the strip $\Pi(a, b)$.

Let $\mathcal{M}I$ (where I is some interval) denote the space of such functions of a complex variable which i) are meromorphic and polynomially bounded at infinity in the strip $\Pi(a, b + h)$ and ii) are analytical in the strip ΠI.

We have shown that only those solutions $G(s)$ of (4.2.13) should be considered which belong to the space $\mathcal{M}(\varrho, b + h)$ at some $\varrho \in (a, b)$. These solutions satisfy (4.2.13) in the whole strip $\Pi(a, b)$. The quantity ϱ determines the rate of decrease (or growth) of the corresponding solutions $F(x)$ of (4.2.11) at $x \to -\infty$. The fact that the solution $G(s)$ belongs to the class $\mathcal{M}(\varrho, b + h)$ implies that the corresponding solution $F(x)$ satisfies the condition (4.2.23).

Remark 4.1. The arguments given in this section are not mathematically rigorous. If, for example, $U(x) = \delta''(x)$ then we have $U * F = F''$ and the property (4.2.23) turns out to be invalid in the form formulated here. The situation is similar when the generalized function $U(x)$ is a regularization of a singular function. However, all the foregoing and subsequent arguments may be made absolutely rigorous by using the method of generalized functions. For example, condition (4.2.8) is rigorously formulated in the following way: $U(x)\exp(\sigma x) \in S'$ for any $\sigma \in (a,b)$; S' is a space of generalized slow-growing functions (see [4.15]). It should be noted that properties 1) to 3) of the Mellin functions were formulated precisely; they go well with the generalized functions method.

Analysis of the Cauchy Problem for the Evolution Equation. It may be shown that for (4.2.13), the following alternative exists. Let the parameter λ in this equation be different from a negative number or zero and let ϱ be some number out of the interval (a,b). Then

A. If $\kappa(\varrho+0) = 0$, then (4.2.13) has always a (unique) solution in the class $\mathcal{M}(\varrho, b+h)$.

B. If $\kappa(\varrho+0) < 0$, then for the solution of (4.2.13) to be in the class $\mathcal{M}(\varrho, b+h)$ it is necessary and sufficient that the function $\Psi(s)$ should satisfy some conditions (of the type of equations). The number of the conditions is equal to $|\kappa(\varrho+0)|$ and these conditions are different for different λ.

C. If $\kappa(\varrho+0) > 0$, the solution of (4.2.13) in the class $\mathcal{M}(\varrho, b+h)$ depends on arbitrary constants whose number is equal to $|\kappa(\varrho+0)|$.

The properties 1)–3) of the Mellin functions cover all situations of note thus allowing for a complete investigation of the Carleman equation (4.2.13) and the proof of the above-formulated alternative similar to that described in [4.12–13].

Below the given alternative is proved in two steps. At first, a "basic" function is constructed which is a partial solution of the homogeneous equation, see (4.2.27). Then the inhomogeneous equation (4.2.13) is examined using the basic function. This will be done below [see (4.2.27) and later on]. We shall solve (4.2.13) in the space $\mathcal{M}(\varrho, b+h)$ at $\kappa(\varrho+0) = 0$ and thus prove statement A. To avoid extensive mathemtical complexities, we shall not give the proof of the complete alternative.

This ABC-alternative allows one to analyze the Cauchy problem for (4.2.6) with the initial condition (4.2.9).

If the rotation function $\kappa(\sigma)$ is negative in the whole interval (a,b), the Cauchy problem (4.2.6, 9) either has no solution at all or its solutions grow "too quickly" with time, so that the Laplace transformation in time is inapplicable. [Equation (4.2.6), for example, might have solutions which become infinite within a finite time.] Indeed, if (4.2.11) had solutions in the space $\mathcal{L}(a,b)$, then (4.2.13) would be solvable in the class $\mathcal{M}(\varrho, b+h)$ for $\varrho \in (a,b)$. But since $\kappa(\varrho) < 0$, it is necessary that the $|\kappa(\varrho)|$ conditions (of the type of equalities) for the function $\Phi(x)$ or ultimately, the functions ϕ_0, ϕ should be satisfied. These conditions must be satisfied at all λ with sufficiently large real parts and are therefore rather rigorous (besides, these conditions are rather specific in their form, and it is difficult to assign a physical meaning to them). These conditions will be violated for practically all functions ϕ_0, ϕ.

If only in one point $\varrho \in (a,b)$ the rotation function assumes a positive value, the solution of the Cauchy problem (4.2.6, 9) is not unique. Using the inverse Fourier and Laplace transformations to solve the homogeneous equation (4.2.20) in the class $\mathcal{M}(\varrho, b+h)$, one can construct a nontrivial solution (not identically zero) $F_0(x,t)$ of the Cauchy problem (4.2.6, 9) with the functions $\phi_0 \equiv 0$, $\phi \equiv 0$ satisfying the condition:

$$F_0(x,t) = \begin{cases} O\left(e^{-\varrho x}\right) & \text{at } x \to -\infty, \\ O\left(e^{-(b+h)x}\right) & \text{at } x \to +\infty. \end{cases} \tag{4.2.25}$$

The mentioned nonuniqueness of the Cauchy problem (4.2.6, 9) suggests one should impose boundary conditions on the evolution equation (4.2.6) to consider (4.2.9) in a narrower space [belonging to the space $\mathcal{L}(a,b)$] where the Cauchy problem for this equation will have a unique solution. The choice of suitable boundary conditions is ambiguous: physical considerations only are insufficient for such a choice so that additional mathematical arguments are required.

At first glance it seems natural from the physical viewpoint to demand that the solution of (4.2.6) at every fixed t should be bounded on the whole straight line $-\infty < x < +\infty$. It appears, however, that this is not a good choice; given such boundary conditions the Cauchy problem (4.2.6, 9) may have many solutions or may have no solutions at all depending on the particulars of the situation.

We shall proceed in a different way. Since, from the physical viewpoint, we are concerned with the solutions of the Cauchy problem where the initial conditions are given by finite functions $\phi_0(x)$ we shall impose boundary conditions according to the following rule. The solution $F(x,t)$ should be chosen from a set of solutions of the Cauchy problem (4.2.6, 9) and the solution $F(x)$ from a set of solutions of (4.2.11) whose magnitude tends to zero as quickly as possible (or grows as slowly as possible) at $|x| \to \infty$. The solution $G(s)$ of (4.2.13) corresponding to this choice is an element of the space $\mathcal{M}(\beta,\gamma)$ with the widest interval (β,γ).

It is clear that this boundary condition leads to a unique solution of (4.2.11), provided the rotation function $\kappa(\sigma)$ tends to zero in the interval (a,b). Indeed, let (σ_-,σ_+) be a zero rotation interval:

$$\kappa(\sigma)\begin{cases} <0 & \text{at } a<\sigma<\sigma_-, \\ =0 & \text{at } \sigma_- <\sigma<\sigma_+, \\ >0 & \text{at } \sigma_+ <\sigma<b\,. \end{cases}$$

Then (in line with the alternative formulated at the beginning of this subsection) (4.2.13) has a unique solution $G(s)$ belonging to the space $\mathcal{M}(\varrho, b+h)$ at any $\varrho \in (\sigma_-,\sigma_+)$, and all the remaining solutions of this equation belong to the space $\mathcal{M}(\varrho, b+h)$ at $\varrho \ge \sigma_+$. Consequently, (4.2.11) has a unique solution $F(x)$ satisfying the condition (4.2.23) at $\varrho \in (\sigma_-,\sigma_+)$; this solution tends to zero faster than all other solutions (or grows slower than the rest ones) at $x \to -\infty$.

Let $\mathcal{L}_\varrho$ denote the space of functions $f(x)$ for which $f\exp(\varrho x)$ is limited by a constant both at $x \to -\infty$ and $x \to +\infty$. Conversely, $\mathcal{L}_\varrho$ is a "space of functions with a weight $\exp(-\varrho x)$". It is readily understood that at $\varrho \in (a, b)$, the condition (4.2.23) for the solution $F(x)$ of (4.2.11) is equivalent to the condition $F \in \mathcal{L}_\varrho$.

Thus we have shown that the suggested boundary condition is equivalent to considering (4.2.6, 11) in the space $\mathcal{L}_\varrho$ at some ϱ from the zero rotation interval (σ_-, σ_+). In this space the solution of the Cauchy problem (4.2.6, 9) always exists, is unique and (as will be seen later on) stable against perturbations of the initial data and external disturbances: if the functions $|\phi_0 \exp(sx)|$ and $|\phi \exp(sx)|$ are small, the function $F \exp(sx)$ will also be small at $t > 0$. Whence follows the correctness of the Cauchy problem (4.2.6, 9) in the space $\mathcal{L}_\varrho$.

There are three alternatives for realizing a nonvanishing rotation function $\kappa(\varrho)$ in the interval (a, b).

First, the function κ may be negative in the whole interval (a, b) as discussed above.

Secondly, the function κ may be positive in the whole interval (a, b). In this case the Cauchy problem (4.2.6, 9) has a nonunique solution in the space $\mathcal{L}_\varrho$ at any $\varrho \in (a, b)$, see (4.2.25).

Finally, the function κ may be different from zero and take on both negative and positive values:

$$\kappa \begin{cases} < 0 & \text{at } a < \sigma < \sigma_0 \,, \\ > 0 & \text{at } \sigma_0 < \sigma < b \end{cases} \tag{4.2.26}$$

where σ_0 is a number out of the interval (a, b). On the straight line $\mathrm{Re}\, s = \sigma_0$ there must be at least two zeros of the Mellin function $W(s)$.

One can show that in all cases where there is no zero rotation, the adopted boundary condition does not allow to obtain the correct formulation of the Cauchy problem for (4.2.6) and it is not possible to find a physically sensible space in which the Cauchy problem (4.2.6, 9) is correct. In the next subsection 4.2.2 we shall clarify the physical meaning of this incorrectness and find the physical pictures corresponding to the above three cases.

Basic Function. The special solution $B(s)$ of the auxiliary homogeneous equation

$$-B(s+h) = W(s)B(s) \tag{4.2.27}$$

is of great importance for deriving a solution of the Carleman equation (4.2.13) and the evolution equation (4.2.9). We will call this solution the *basic function.* Let there be an interval of zero rotation (σ_-, σ_+). We shall define the basic function $B(s)$ as a solution of (4.2.27) having the following properties:

i) it is meromorphic in the strip $\Pi(\sigma_-, \sigma_+ + h)$,
ii) it has neither zeros nor poles in the strip $\Pi(a, b+h)$,
iii) the functions $B(s)$ and $1/B(s)$ are polynomially bounded on the infinity in the strip $\Pi(a, b+h)$.

Later on we shall only need the fact the basic function exists.

For an arbitrary Mellin function $W(s)$ the basic function $B(s)$ at $s \in \Pi(\sigma_-, \sigma_+ + h)$ is defined by the following formulas:

$$w(s) = \ln[-W(s)] \; ; \tag{4.2.28a}$$

$$\mathcal{R}(s) = \frac{\pi}{2ih^2} \int\limits_{\sigma-\varepsilon-i\infty}^{\sigma-\varepsilon+i\infty} \frac{w(r)}{\sin^2[\pi(s-r)/h]}\, dr \,, \tag{4.2.28b}$$

$$\sigma = \mathrm{Re}\, s, \; \sigma_< \sigma - \varepsilon < \sigma_+ + h, \; 0 < \varepsilon < h \, ;$$

$$\mathcal{P}(s) = \int \mathcal{R}(s) ds \; ; \tag{4.2.28c}$$

$$B(s) = \exp \mathcal{P}(s) \; . \tag{4.2.28d}$$

In (4.2.28a), that continuous branch of the logarithm is chosen for which $\mathrm{Im}\, w(s) \to 0$ at $\mathrm{Im}\, s \to \pm\infty$. In the strip $\Pi(a, b+h)$, the function $B(s)$ is by virtue of (4.2.27) through its values in the strip $\Pi(\sigma_-, \sigma_+ + h)$.

Starting from (4.2.28) and taking the logarithm of (4.2.27), we obtain:

$$\mathcal{P}(s+h) - \mathcal{P}(s) = w(s), \quad s \in \Pi(\sigma_-, \sigma_+) \; . \tag{4.2.29}$$

The integral (4.2.28b) does not depend on the choice of ε and determines the $\mathcal{R}(s)$-function which is analytical in the strip $\Pi(\sigma_-, \sigma_+ + h)$ (for every s from this strip one can select such an ε that $0 < \varepsilon < h$, $\sigma_- < \mathrm{Re}\, s - \varepsilon < \sigma_+ + h$). By direct substitution it can be verified that the function (4.2.28b) satisfies

$$\mathcal{R}(s+h) - \mathcal{R}(s) = w'(s), \quad s \in \Pi(\sigma_-, \sigma_+) \; . \tag{4.2.30}$$

Comparing (4.2.30) and (4.2.29), we see that as a solution $\mathcal{P}(s)$ of (4.2.29) one can take the integral (4.2.28c) of the function (4.2.28b). Then (4.2.28d) satisfies (4.2.27) in the strip $\Pi(\sigma_-, \sigma_+)$, is analytical, and different from zero in the strip $\Pi(\sigma_-, \sigma_+ + h)$. The last thing that is left to be done is to show that the function $B(s)$ and $1/B(s)$ are polynomially bounded in the strip $\Pi(\sigma_-, \sigma_+ + h)$. We do not give the proof here since it is very tedious. We shall only remark that it relies heavily on the properties 1)–3) of the Mellin functions, in particular, on the estimate (4.2.18).

Solution of the Cauchy Problem. Let there exist a zero rotation interval (σ_-, σ_+). We shall derive a solution of the Cauchy problem (4.2.6, 9) which satisfies the suggested boundary condition, i.e., which belongs to the space $\mathcal{L}_\varrho$ for $\varrho \in (\sigma_-, \sigma_+)$.

The solution $F(x)$ of (4.2.11) in the space $\mathcal{L}_\varrho$ corresponds to the solutions $G(s)$ of (4.2.13) in the space $\mathcal{M}(\varrho, b{+}h)$. According to (4.2.24), if $G \in \mathcal{M}(\varrho, b{+}h)$ and $\sigma_- < \varrho < \sigma_+$, then $G \in \mathcal{M}(\sigma_-, b+h)$. Let us seek the solution of (4.2.13) in the form

$$G(s) = B(s)g(s) \; .$$

For the g-function we have a difference equation with constant coefficients:

$$\lambda g(s) + g(s-h) = Q(s) = \frac{\Psi(s)}{B(s)}, \quad s \in \Pi(a, b+h) \,, \tag{4.2.31}$$

which may be solved using the Fourier transformation. Owing to the properties of the basic function $B(s)$we have (i) $Q \in \mathcal{M}(\sigma_-, \sigma_+ + h)$ and (ii) (4.2.13) in the class $\mathcal{M}(\sigma_-, b+h)$ is equivalent to (4.2.31) in the class $\mathcal{M}(\sigma_-, \sigma_+ + h)$. Whence follows the existence and uniqueness of functions $P(x)$, $f(x)$ ensuring

$$Q(s) = \int_{-\infty}^{+\infty} P(x)e^{sx}\,dx,\ g(s) = \int_{-\infty}^{+\infty} f(x)e^{sx}\,dx,\ s \in \Pi(\sigma_-, \sigma_+ + h)\ .$$

Substituting these expressions into (4.2.31), we obtain

$$\lambda f(x) + f(x)e^{-hx} = P(x)$$

leading to

$$f(x) = \frac{P(x)}{\lambda + e^{-hx}}\ .$$

Consequently (at $\lambda > 0$), the solution $F(x)$ of (4.2.11) in the space $\mathcal{L}_\varrho$ $(\sigma_- < \varrho < \sigma_+)$ exists, is unique and defined by

$$\Psi(s) = \int_{-\infty}^{+\infty} \Phi(x)e^{-sx}\,dx,\ Q(s) = \frac{\Psi(s)}{B(s)}\ ,$$

$$P(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} Q(s)e^{-sx}\,dx,$$

$$f(x) = \frac{P(x)}{\lambda + e^{-hx}},\ g(s) = \int_{-\infty}^{+\infty} f(x)e^{sx}\,dx\ , \qquad (4.2.32)$$

$$G(s) = B(s)g(s),\ F(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} G(s)e^{-sx}\,ds\ ,$$

where $\sigma = \mathrm{Re}\, s \in (\sigma_-, \sigma_+ + h)$. The formulas (4.2.32) may be written in the form

$$F = \mathbb{Z}^{-1}[\lambda + \exp(-hx)]^{-1}\mathbb{Z}[\Phi]\ , \qquad (4.2.33a)$$

where $\mathbb{Z}$, $\mathbb{Z}^{-1}$ are the mutually inverse convolution operators with generalized functions $z_1(x)$, $z_2(x)$ whose Fourier images are $1/B(s)$ and $B(s)$, respectively,

$$\mathbb{Z}f = z_1 * f,\ \frac{1}{B(s)} = \int_{-\infty}^{+\infty} z_1(x)e^{sx}\,dx,$$

$$\mathbb{Z}^{-1}f = z_2 * f,\ B(s) = \int_{-\infty}^{+\infty} z_2(x)e^{sx}\,dx\ , \qquad (4.2.33b)$$

where $s \in \Pi(\sigma_-, \sigma_+ + h)$; evidently, $z_1 * z_2 = \delta(x)$. The function $1/[\lambda + \exp(-hx)]$ is understood in (4.2.33) as an multiplication operator acting on this function.

get a solution of the Cauchy problem (4.2.6, 9). If there is no external action ($\phi = 0$), the function $\Phi = \phi_0$ does not depend on λ and

$$F(x,t) = T(t)\phi_0(x) \quad (\phi \equiv 0),$$

where

$$T(t) = \mathbb{Z}^{-1}\exp(-te{-}hx)\,\mathbb{Z} \quad \text{for } t \geq 0. \qquad (4.2.34)$$

Here we took advantage of the fact that the operator $\mathbb{Z}$ is independent of λ, and

$$\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{e^{\lambda t}}{\lambda + e^{-hx}}\,d\lambda = \exp[-t\exp(-hx)]$$

at any $\delta > 0$.

For an arbitrary external action, the solution of the Cauchy problem (4.2.6, 9) has the form

$$F(x,t) = T(t)\phi_0(x) + \int_0^t T(t-\tau)\phi(x,\tau)d\tau. \qquad (4.2.35)$$

Remark 4.2. In the study of the evolution equation (4.2.6) for an incorrect Cauchy problem of this equation, it is of interest to establish mathematically rigorously the correctness of this Cauchy problem, provided there exists a zero rotation interval. This may be done using the method of generalized functions. Let S'_ϱ be a space of such generalized functions $f(x)$ that $f(x)\exp(\varrho x)$ is a slowly growing generalized function (cf. the definition of the space $\mathcal{L}_\varrho$). The correctness of the Cauchy problem (4.2.6), (4.2.9) and the fact that its solutions are actually determined by (4.2.34–35) are ensured by the following.

Theorem. A family of operators (4.2.34) forms a semigroup [4.17] in the space S'_ϱ at any $\varrho \in (\sigma_-, \sigma_+ + h)$; the operator

$$f \longmapsto e^{-hx}[U * f] \qquad (4.2.36)$$

is an infinitesimal generating operator of this semigroup in the space S'_ϱ at any $\varrho \in (\sigma_-, \sigma_+ + h)$. (For the relationship between correctness and semigroups see, e.g., [4.15]). An evident consequence of this theorem is the stability of solutions of (4.2.6) (relative to initial perturbations) in the topology of space S'_ϱ at any $\varrho \in (\sigma_-, \sigma_+ + h)$.

4.2.2 Behavior of Kolmogorov-Like Turbulent Distributions. Stability Criterion

Based on the results of the preceding subsection, we shall examine the behavior of the solutions of the evolution equation (4.2.6) which describes the evolution of perturbations of the Kolmogorov spectrum having the form of an arbitrary angular harmonic. The character of this behavior depends mainly on the details of the rotation function κ. Let us first consider a "regular" situation in which there exists a zero rotation interval (σ_-, σ_+). In this case the solution $F(x,t)$ of the Cauchy problem (4.2.6, 9) is given by (4.2.34–35) whose form is determined by the basic function $B(s)$, see (4.2.33b).

Zeros and Poles of the Basic Function. The asymptotics of solutions of the Cauchy problem (4.2.6, 9) (at $t \to \infty$ or $|x| \to \infty$) are determined by the zeros and poles of the basic function $B(s)$. $B(s)$ has neither zeros nor poles in the strip $\Pi(\sigma_-, \sigma_+ + h)$. Hence, it follows from (4.2.27) that its zeros and poles in the strip $\Pi(a, b+h)$ are characterized as follows: if p is a zero of the Mellin function $W(s)$ lying on the right of the strip $\Pi(\sigma_-, \sigma_+)$ [i.e., $p \in \Pi(\sigma_+, b)$], then the $B(s)$-function has zeros in all points of the form

$$p+h,\ p+2h,\ p+3h, \dots \tag{4.2.37}$$

located in the strip $\Pi(a, b+h)$. All zeros of a sequence like (4.2.37) have the same multiplicity equal to that of the zero p of the function $W(s)$. If the function $W(s)$ has a zero q on the left of the strip $\Pi(\sigma_-, \sigma_+)$ [i.e., $q \in \Pi(a, \sigma_-)$], then the function $B(s)$ has zeros in all points

$$q,\ q-h,\ q-2h, \dots , \tag{4.2.38}$$

on the strip $\Pi(a, b+h)$, with all poles of such a sequence (4.2.38) having the same multiplicity equal to the one of the zero at q. The function $B(s)$ has no other zeros and poles in the strip $\Pi(a, b+h)$, in particular, no zeros in the strip $\Pi(a, \sigma_+ + h)$ and no poles in the strip $\Pi(\sigma_-, b+h)$

For the sake of simplicity of the form of the asymptotic expansion, we shall later on consider all zeros and poles of the $W(s)$, $B(s)$-functions to be of first order.

Asymptotics at $|x| \to \infty$. Using (4.2.24), it is easy to see that the solution $G(s)$ of (4.2.13) in the class $\mathcal{M}(\varrho, b+h)$ $(\sigma_- < \varrho < \sigma_+)$ may have poles in the strip $\Pi(a, b+h)$ only in points of the form (4.2.37) in which the basic function $B(s)$ possesses poles. The poles of the Fourier image $G(s)$ of the solution $F(x,t)$ of (4.2.6) have the same points as $B(s)$. Therefore, the asymptotic behavior of the solution $F(x,t)$ at $|x| \to \infty$ is:

$$F(x,t) \approx \begin{cases} \sum_q K_q(t) e^{-qx} & \text{at } x \to -\infty , \\ O\left(e^{-(b+h)x}\right) & \text{at } x \to +\infty , \end{cases} \tag{4.2.39}$$

$$K_q(t) = \operatorname{res} G_t(q) ,$$

where the summation extends over the set of poles q of the basic function $B(s)$ of the series (4.2.38). The main terms in the sum (4.2.39) are those in which the values of q have the largest real part. Therefore

$$F(x,t) \sim \sum_{\operatorname{Re} q = \sigma_-} K_q(t) e^{-qx}, \quad x \to -\infty , \tag{4.2.40}$$

where the summation includes the set of zeros q of the Mellin function $W(s)$ lying on the line $\operatorname{Re} s = \sigma_-$.

Free Evolution of Initial Perturbations. Let us describe at first the behavior of the solution $F(x,t)$ of the evolution equation (4.2.6) without external action ($\phi \equiv 0$). These solutions are determined by

$$F(x,z) = Z^{-1}\left[\Theta_t(x) P(x)\right] \tag{4.2.41}$$

where $P(x) = \mathbb{Z}\phi_0(x)$ is a time-independent function and

$$\Theta_t(x) = \exp[-t \exp(-hx)] , \tag{4.2.42}$$

see (4.2.34–35). The function (4.2.42) has the property

$$\Theta_{t\tau}(x) = \Theta_\tau\left(x - \frac{\ln t}{h}\right) . \tag{4.2.43}$$

The Fourier transforms $\psi_0(s)$, $Q(s)$ of the respective functions $\phi_0(x)$, $P(x)$ are related by $Q(s) = \psi_0(s)/B(S)$. Since $\psi_0(S)$ is an integer function, the poles of $Q(s)$ are determined by the zeros of the basic function $B(s)$. Therefore

$$P(x) \approx \begin{cases} O\left(e^{-ax}\right) & \text{at } x \to -\infty , \\ \sum_p -\psi_0(p)/B'(p) & \text{at } x \to +\infty , \end{cases} \tag{4.2.44}$$

where summation is effected over the set of zeros p of the function $B(S)$ being a combination of the sequences (4.2.37). Since the function (4.2.42) is "practically equal to zero" at $x \ll \ln t/h$, we have at sufficiently large t in conformity with (4.2.41, 44)

$$F(x,t) \approx \sum_p -\frac{\psi_0(p)}{B'(p)} Z^{-1}\left[\Theta_t(x) e^{-px}\right], \ (t \to \infty) \tag{4.2.45}$$

where the summation is performed over the set of the zeros p of the basic function $B(s)$ being a combination of the sequences (4.2.37). The main terms in the sum (4.2.45) are those in which the zeros p have the smallest real parts. Hence,

$$F(x,t) \sim \sum_{\operatorname{Re} p = \sigma_+} -\frac{\psi_0(p+h)}{B'(p+h)} Z^{-1}\left[\Theta_t(x) e^{-(p+h)x}\right], \quad (t \to \infty) , \tag{4.2.46}$$

where the summation is performed over the set of zeros p of the Mellin function $W(s)$ lying on the straight line $\operatorname{Re} s = \sigma_+$.

Since the function $\Theta_t(s)$ satisfies (4.2.43) we know that according to (4.2.46), the perturbation $F(x,t)$ at large t represents a superposition of several "running waves"

$$F_p(x,t) = -\frac{\psi_0(p+h)}{B'(p+h)} Z^{-1}\left[\Theta_t(x) e^{-(p+h)x}\right] \tag{4.2.47}$$

possessing the self-similarity

$$F_p(x,t\tau) = F_p\left(x - \frac{\ln t}{h}, \tau\right) t^{-(p+h)/h}, \quad (t > 0, \tau > 0) . \tag{4.2.48}$$

The form of any such "wave" is universal, it does not depend on the initial conditions but is entirely determined by the characteristics of the medium; the initial conditions affect only the "amplitudes" of these "waves". From (4.2.48) we see that the "wave" (4.2.47) travels in the positive direction of the x axis according to the law const $+ \ln t/h$ (at $h < 0$ the "wave" moves into the negative direction). If the function (4.2.47) assumes at the moment τ in the point ξ a certain value $C_p = F_p(\xi, \tau)$, then at the moment $t > \tau$ at the point $x = \xi + (1/h)\ln(t/\tau)$ it will have the value

$$F_p(x,t) = C_p \left(\frac{t}{\tau}\right)^{-(p+h)/h} = C_p e^{-(p+h)x} .$$

Therefore, one can say that the "wave amplitude" changes according to the law $t^{-(p+h)/h}$ and that the "wave" has the "envelope" const $\cdot \exp[-(\sigma_+ + h)x]$ with $\mathrm{Re}\, p = \sigma_+$). If the zero p is a real number ($p = \sigma_+$), the "wave amplitude" changes monotonically; if the zero p contains a nonzero imaginary part, the "wave amplitude" oscillates with time. The sum (4.2.46) of the "waves" represents a "wave" evolving according to the logarithmic law const $+ \ln t/h$ with the "envelope" const $\cdot \exp[-(\sigma_+ + h)x]$. However, the amplitude of the "complete wave" may show a much more complex behavior in time [the point is that zeros p of the function $W(s)$ located on the straight line $\mathrm{Re}\, s = \sigma_+$ may have imaginary parts whose values are rather different]. At every fixed moment of time t the behavior of the perturbation $F(x,t)$ at $|x| \to \infty$ is determined by the asymptotics (4.2.39–40). Figures 4.2 a,b,c ($t_1 < t_2 < t_3$) schematically show the evolution of the perturbation $F(x,t)$ at different positions of the interval $(\sigma_-, \sigma_+ + h)$ relative to the point $\sigma = 0$.

If $\sigma_- < 0 < \sigma_+ + h$, then any small initial perturbation will give rise to deviations from the Kolmogorov distribution that remain small throughout the whole spectrum and tend to zero as time progresses (Fig. 4.2). Thus the Kolmogorov spectrum proves to be stable against small perturbations. (A rigorous confirmation of this statement follows from the theorem given in Remark 4.2.) If $\sigma_+ + h < 0$, such a perturbation of the Kolmogorov spectrum increases for large x (Fig. 4.2b); the turbulent medium develops the above-mentioned "running structure", see (4.2.45–48). Such an instability is usually referred to as a convective instability. If $\sigma_- > 0$, a small initial perturbation leads to a large deviation from the Kolmogorov spectrum at large negative x (Fig. 4.2c); the form of this deviation is also universal, see (4.2.39–40).

Thus if the interval $(\sigma_-, \sigma_+ + h)$ does not contain the point $\sigma = 0$, the Kolmogorov spectrum is unstable with regard to initial perturbations. But the character of this instability is unusual. Indeed, since in real systems the inertial interval (k_0, k_d) is always finite ($0 < k_0 < k_d < \infty$), the perturbations of the Kolmogorov

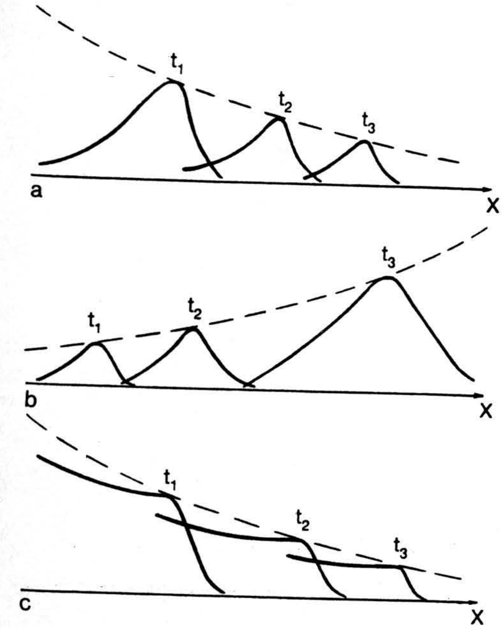

Fig. 4.2. The behavior ($t_1 < t_2 < t_3$) of the perturbation for different positions of the zero rotation interval: **a)** $\sigma_- < 0, \sigma_+ + h$, **b)** $\sigma_+ + h < 0$, **c)** $\sigma_- > 0$. The left slope is proportional to $\exp(-\sigma_- x)$, the right one to $\exp[-(b+h)x]$. The dotted line corresponds to $\exp[-(\sigma_+ + h)x]$

spectrum cannot increase infinitely as a result of this instability (see Fig. 4.2b, c) and, consequently, the Kolmogorov spectrum is in fact stable with respect to infinitely small perturbations. The above instability of the Kolmogorov spectrum is of asymptotic character: the perturbation of the Kolmogorov spectrum may increase arbitrarily strongly if the inertial interval is sufficiently large ($k_d/k_0 \gg 1$). Such an instability having an asymptotic meaning and originating from the existence of a large inertial interval will be referred to as *interval instability*.

The two qualitatively different interval instabilities are characterized by (i) $\sigma_+ + h < 0$ and (ii) $\sigma_- > 0$. In the first case the Kolmogorov spectrum perturbations grow gradually [at large t the perturbation value grows by a power law proportion to $t^{-(\sigma_+ + h)/h}$, see Fig. 4.2b], and in the second case the small initial perturbation leads almost instantly to a strong deviation from the Kolmogorov spectrum [in the range of small wave numbers] (see Fig. 4.2c). But in reality this "instancy" is also the result of the infinity of the inertial interval. Within a finite interval, all interaction times are finite. In the former case we shall call the instability *soft interval instability* and in the latter the *hard interval instability*.

Evolution of Perturbations Under External Action. Under the influence of a constant external action $\phi = \phi(x)$ in the system (4.26) the solution of the Cauchy problem (4.2.6, 9) is according to (4.2.34, 35) determined by

$$F(x,t) = Z^{-1}\Theta_t(x)Z[\varrho_0] + Z^{-1}(1 - \Theta_t(x))e^{hx}Z[\phi] . \tag{4.2.49}$$

At $t \to \infty$ this solution tends to

$$F_\infty(x) = Z^{-1} e^{hx} Z[\phi] = \frac{1}{2\pi i} \int_{\varrho - i\infty}^{\varrho + i\infty} -\frac{\psi(s+h)}{W(s)}\, ds\,, \qquad (4.2.50)$$

$$\sigma_- < \varrho < \sigma_+\,,$$

where $\psi(s)$ is the Fourier image of $\phi(x)$. The function (4.2.50) is, evidently, the stationary solution of (4.2.6). In general (4.2.6) has many stationary solutions. In particular, further stationary solutions of this equation may be obtained with the help of the integral from (4.2.50) provided the parameter ϱ is chosen not to be from the (σ_-, σ_+) interval. If the interval (a, b) contains the point $\sigma = 0$ then there always exists a stationary solution of (4.2.6), which is bounded on the whole line $-\infty < x < +\infty$. However, only the solution (4.2.50) may be a bounded solution of the Cauchy problem for (4.2.9); all other stationary solutions have no relation to the evolution equation (4.2.6).

The asymptotic behavior of the solution (4.2.50) at large $|x|$ is

$$F_\infty(x) \approx \begin{cases} \sum_q -[\psi(q+h)/W'(q)] e^{-qx} & \text{at } x \to -\infty\,, \\ \sum_p -[\psi(p+h)/W'(p)] e^{-px} & \text{at } x \to +\infty\,, \end{cases} \qquad (4.2.51)$$

where q (or p) goes through many zeros of the Mellin function $W(s)$ situated on the left (or right, respectively) of the band $\Pi(\sigma_-, \sigma_+)$. In the upper sum of (4.2.51), the main terms are those in which $\mathrm{Re}\, q = \sigma_-$, in the lower one the ones with $\mathrm{Re}\, p = \sigma_+$.

The character of the evolution of the solution (4.2.49) to the limiting stationary solution (4.2.50) is schematically indicated in Figs. 4.3a, b, c ($t_1 < t_2 < t_3$) for different positions of the zero rotation interval (σ_-, σ_+) relative to the point $\sigma = 0$. The external action feeds in the perturbation of the Kolmogorov spectrum with the perturbation front expanding according to the logarithmic law $x_b = \mathrm{const} + \ln t/h$.

If $\sigma_- < 0 < \sigma_+$, the Kolmogorov spectrum is stable against weak external actions: at all wave numbers the turbulence spectrum differs only slightly from the Kolmogorov spectrum (Fig. 4.3a). When the zero rotational interval (σ_-, σ_+) does not contain the point $\sigma = 0$, there is an interval instability of the Kolmogorov spectrum against external effects. As a result of this instability, a stationary distribution is formed which strongly differs from the Kolmogorov spectrum either at $x \to +\infty$ (if $\sigma_+ < 0$, see Fig. 4.2b) or at $x \to -\infty$ (if $\sigma_- > h$, see Fig. 4.3c); in the remaining part of the inertial interval the state of the turbulent medium is Kolmogorov-like. The form of the resulting large deviation from the Kolmogorov spectrum is universal and according to (4.2.15) it is determined by the zeros of the Mellin function $W(s)$. Thus, a turbulent medium generates an ordered stationary structure. Since the zeros of the function $W(s)$ located on the straight lines $\mathrm{Re}\, s = \sigma_\pm$ may have nonzero imaginary parts, this structure may have rather a complex (nonmonotonic) form. When $\sigma_+ < 0$, the structure appears gradually, covering an increasingly larger region [at $t \to \infty$, the magnitude of the structure grows by a power law proportional to $t^{-\sigma_+/h}$, see Fig. 4.3b]

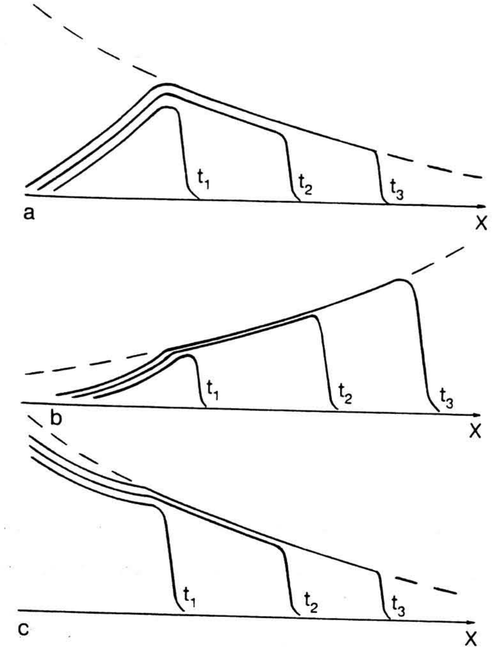

Fig. 4.3. Temporal behavior ($t_1 < t_2 < t_3$) of the perturbation under external pumping; **a)–c)** illustrate different positions of the zero rotation interval: **a)** $\sigma_- < 0 < \sigma_+$, **b)** $\sigma_+ < 0$, **c)** $\sigma_- > 0$. The dotted line depicts $\exp(-\sigma_+ x)$

and, consequently, the interval instability is soft. This instability results in the *soft generation of a stationary structure*. When $\sigma_- > 0$ a stationary structure [exponentially growing at $x \to -\infty$, see Fig. 4.3c] is formed within a finite time. We have a hard interval instability leading to the *hard generation of the stationary structure*.

As seen from comparison of (4.2.47) with (4.2.50) and of Fig. 4.2 with Fig. 4.3, the stationary solution stability condition with regard to external actions is more strict than the stability condition with regard to initial perturbations. In the former case, a necessary condition for stability is that the point $\sigma = 0$ corresponding to the Kolmogorov solution index falls within the zero rotational interval (σ_-, σ_+). In the latter case, it is sufficient that the point $\sigma = 0$ falls beyond a wider interval $(\sigma_-, \sigma_+ + h)$ at $h > 0$ or $(\sigma_- + h, \sigma_+)$ at $h < 0$. The physical reason for this difference is, of course, the fact that external actions constantly generate perturbations of the distribution.

The difference in the behavior of perturbations exhibited in Figs. 4.2, 3 may also be explained with the help of conservation laws. Indeed, the boundaries of the zero rotation interval σ_-, σ_+ are specified by the zeros of the Mellin function $W_l(s)$. But every zero $W_l(p) = 0$ implies the presence of an integral of motion of the form

$$I_l = \int Y_l(\Omega) k^{p+h-1} A(\boldsymbol{k})\, dk\, d\Omega \qquad (4.2.52)$$

(we shall take p to be a real quantity) in the linearized kinetic equation. Besides, the presence of a zero of the Mellin function implies the presence of a stationary solution of the form

$$A_l(\boldsymbol{k}) = Y_l(\Omega)k^{-p},$$

transferring the constant flux of the integral I_l (examples of such solutions are the neutrally stable modes derived in Sect. 4.1 for $l = 0, 1$). It is readily understood that the behavior of the perturbations shown in Fig. 4.2 corresponds to conservation of the integral of motion I_l, while Fig. 4.3 illustrates the formation of the power asymptotics with a constant flux of this integral.

Remark 4.3. One can have the impression that the exponentially growing asymptotics at $|x| \to \infty$ [occurring for $\kappa(0) \neq 0$, see (4.2.51, 40)] imply the inapplicability of the linear approximation (4.2.4). As a matter of fact, the use of the linear approximation in such situations is based on the fact that in real systems the inertial interval (k_0, k_d) is finite $(0 < k_0 < k_d < \infty)$. We suppose that real systems do not have a spectrum like $n(\boldsymbol{k}, t) = n_k^0[1 + A(\boldsymbol{k}, t)]$ as determined by (4.2.4–6), but to have a distribution which is close to this spectrum within the inertial interval [and is very different from it for $k \ll k_0$ and $k \gg k_d$] – in a similar way as supposed for the Kolmogorov spectrum itself (4.2.2). Then applicability of the linear approximation requires $|F| \ll 1$ only inside the inertial interval.

Evolution Locality and Nonlocality of the Kolmogorov Spectra. We shall discuss the case that the locality interval contains the zero rotation interval. In this case, all integrals in (4.2.5, 6) converge, i.e., the evolution of perturbations having the form of the respective harmonics is determined only by the interaction of waves with scales of the same order. It seems natural to call this property the *evolution locality of the spectrum* (as opposed to the locality for which the collision integral converges on the Kolmogorov spectrum, see Sect. 3.1).

The nonexistence of a zero rotation interval within the locality interval may be shown to imply nonlocality of the evolution. In other words, the dependence of the behavior of the perturbation on conditions at the ends of the inertia interval. To understand this feature, it is convenient to discuss the continuous transition from systems with the interval (σ_-, σ_+) within (a, b) to systems having no zero rotation interval at all. Such a transition may be realized in three ways: by contraction of the interval (σ_-, σ_+) to the left or right end or to an inner point of the interval (a, b).

Let, for example, the interval (σ_-, σ_+) be contracted to the right limit of the locality interval $\sigma_\pm \to b$ then the rotation function is negative for most values within (a, b). At any moment of time $t > 0$ the perturbation $F(x, t)$ grows at $x \to -\infty$ proportionally to $\exp(-\sigma_- x)$, see Fig. 4.3. Consequently, the convolution $U * F$ is "located" on the boundary of the divergence so that the dominant role is played by the interaction with the left end of the inertia interval, i.e., with small k. Thus, if the rotation function is negative over the whole locality interval, the behavior of the perturbation arising at the moment $t = 0$ will immediately (at all $t > 0$) depend strongly on the conditions at the left end of the inertial interval. We may consider this as a hard evolution nonlocality.

Let us now discuss the transition to a rotation function that is positive over the whole interval (a, b). If $\sigma_\pm \to a$, the external action should give rise to a perturbation growing at $x \to +\infty$ proportional to $\exp(-\sigma_+ x)$, see Fig. 4.3b. The perturbation front expands according to the law x_{fr} = const + $(\ln t)/h$. Consequently, if (k_0, k_d) is the inertial interval and the quantity k_1 characterizes the scale of the external action $(k_0 \ll k_1 \ll k_d)$, then after a time of the order of $(k_d/k_1)^h$ the behavior of the perturbation will considerably depend on the conditions at the right end of the inertial interval.

Let us recall that above we set $h > 0$. If $h < 0$, in both cases described above, small and large scales exchange roles [i.e., the behavior of the perturbations depends on the conditions at the right end of the inertial interval for the negative rotation function, etc.].

If there is no convergence strip for integral (4.2.16) then the system has no evolution locality at all. An example for such a situation is given by shallow-water capillary waves [see (1.2.40)]: in the isotropic case the width of the locality interval equals $s_1 - s_2 = 2$ while it is for even angular harmonics given by $s_1 - s_2 - 2(\alpha - 1) = 0$. Thus, the evolution of the perturbation in the form of an even angular harmonic depends on the conditions at the ends of the inertial interval.

Strong Instability of Kolmogorov Spectra. Let us finally consider the third case in which there is no zero rotation. It is defined by (4.2.26) and is not related to nonlocality. It may even occur when the carrier of the function $U(x)$ is concentrated in a single point $x = 0$ [the convolution with the function $U(x)$ yields in this case the differential operator].

It is clear that with the help of an arbitrarily small perturbation, one can go over from the function $U(x)$ to $U_\varepsilon(x)$ in which the Mellin function $W_\varepsilon(s)$ has a zero rotation interval $(\sigma_-^\varepsilon, \sigma_+^\varepsilon)$ such that $\varepsilon \to 0$, $\sigma_\pm^\varepsilon \to \sigma_0$, $W_\varepsilon \to W_0$. Equation (4.2.6) with the function $U_\varepsilon(x)$ should be considered in the space $\mathcal{L}_\varrho$ at $\varrho \in (\sigma_-^\varepsilon, \sigma_-^\varepsilon)$ in which the Cauchy problem for this equation always has one and only one solution $F_\varepsilon(x, t)$.

The incorrectness of the (unperturbed) Cauchy problem (4.2.6, 9) in the case under discussion is easily understood from the fact that the limit of the function $F_\varepsilon(x, t)$ at $\varepsilon \to 0$ largely depends on the family $\{U_\varepsilon(x), \varepsilon \to 0\}$ within which we approach $U(x)$; that holds in particular for the asymptotic of the limiting function at $x \to -\infty$.

Since $\sigma_\pm \to \sigma_0$ at $\varepsilon \to 0$, it would be natural to consider (4.2.6) in the space $\mathcal{L}_{\sigma 0}$. However, the set of eigenvalues of the operator (4.2.36) in that space covers the entire complex plane. Therefore, we can consider the Kolmogorov spectrum to be strongly unstable (to perturbations having the form of the angular harmonic in question). In contrast to interval instability, the perturbations of the Kolmogorov spectrum grow in this case with time throughout the whole inertial interval. Under the influence of constant external actions, no stationary solution is formed and an essentially nonstationary regime, "secondary turbulence", may result. It should be noted that a more consistent formulation of the problem of the stability of the Kolmogorov spectrum should be as follows. First of all, the kinetic equation should be supplemented by terms describing the isotropic pumping and damping regions and a stationary solution of this equation should be found that is close to the Kolmogorov spectrum in an interval (k_1, k_2); outside this interval the solution may strongly differ from the Kolmogorov spectrum. Then the kinetic equation must be linearized in the vicinity of this stationary solution and expanded in angular harmonics. For a particular angular harmonic the expansion leads to an evolution equation of the form (4.2.5) for which the Cauchy problem always has one and only one solution; the operator $\hat{L}$ is no longer homogeneous. Having examined the behavior of the solutions of this equation, one should clarify the changes that occur when the ranges of the source and sink in k-space go to zero or to infinity and examine the behavior of the perturbations established in the interval (k_1, k_2). Finally, one should analyze in which situations this behavior is independent of the specific type of the source and sink.

This program for examining the stability of Kolmogorov spectra turns out to be too complex. Currently there exists no strict proof of the fact that in general the kinetic equation with a source and sink has a stationary solution close to the Kolmogorov spectrum in some interval.

As seen above, the strongly unstable equation (4.2.6) may become stable as the result of an arbitrary small variation in the medium characteristics (the interval instability is not an absolute instability). Such a sharp transition from strong instability to stability occurs only in the limit when both the range of a source and the range of a sink tend to zero or to infinity and the inertial interval becomes infinitely large ($k_0 \to 0$, $k_d \to \infty$). With a finite interval the Kolmogorov spectrum may also be unstable in the case of a rather small zero rotation interval (σ_-, σ_+) with $(\sigma_+ - \sigma_- \approx 0)$; the perturbations of the Kolmogorov spectrum will exponentially grow with time in the whole inertia interval. In the case of a small zero rotation interval the stability of the Kolmogorov spectrum established above has the following asymptotic meaning: no matter how small the value of $\sigma_+ - \sigma_-$ is, it is always possible to find a sufficiently large inertial interval in which the Kolmogorov spectrum is stable (with regard to perturbations having the form of a respective angular harmonic). If there exists no zero rotation interval then the increment of the instability of the Kolmogorov spectrum is finite within a finite inertial interval (k_0, k_d). The increment tends to infinity at $k_0 \to 0$, $k_d \to \infty$.

The Kolmogorov Spectrum. Stability Criterion. The treatment of the preceding subsections allows us to clarify the conditions under which the state of a turbulent medium is in the whole inertial interval under various perturbations and at any moment of time close to the Kolmogorov distribution. Thus we arrive at the stability criterion for the Kolmogorov spectrum obtained by *Balk* and *Zakharov* [3.7]:

The Kolmogorov spectrum is stable against disturbances having the form of the angular harmonic $Y(\zeta)$ if and only if $\kappa(0) = 0$, i.e., if the rotation of the Mellin function $W(s)$ corresponding on the imaginary axis to the harmonic $Y(\zeta)$ is defined and equal to zero. (If the zero rotation interval exists and the point $\sigma = 0$ is its boundary, the Kolmogorov spectrum is indifferently stable against perturbations of the form of the respective angular harmonic.)

When $\kappa(0) > 0$, the instability of the Kolmogorov spectrum is strongest for large k and when $\kappa(0) < 0$, for small k. The interval instability and evolution nonlocality are soft if the quantity $h\kappa(0)$ is positive and hard if it is negative.

It should be noted that for different angular harmonics the behavior of the perturbations may be of different types and may have different asymptotics, so that the overall perturbation of the Kolmogorov spectrum may be rather diverse since it is a superposition of perturbations corresponding to all angular harmonics.

It is readily seen that for the order of angular harmonic $Y(\zeta)$ tending to infinity, the value of the integral (4.2.16) specifying the Mellin functions becomes real and negative, just as at $|\mathrm{Im}\, s| \to \infty$, see (4.2.18). Consequently, for angular harmonics of sufficiently high order l, the quantity $\kappa(0)$ is always zero, with the zero rotation interval extending at $l \to \infty$ over the whole interval (a, b). Hence, the Kolmogorov spectrum is always stable with regard to perturbatins in

form of higher angular harmonics. The first few angular harmonics determine the behavior of a turbulent medium near the Kolmogorov spectrum.

Examination of the instability of the Kolmogorov spectrum involves verification of a finite number of conditions of the form $\kappa(0) = 0$ which may be conveniently checked using a computer. Since the calculation of the value of $\kappa(0)$ should necessarily yield an integer, the use of a computer including an estimate of the error would even yield a rigorous mathematical proof of the stability status of the Kolmogorov spectrum.

It is worthwhile to draw attention to the following three general symmetries of the Mellin functions.

The fact that the kinetic equation is real implies that the Mellin functions W_Y and W_{Y^*} corresponding to the angular harmonics Y and Y^*, respectively, satisfy

$$W_Y(s) = W^*_{Y^*}(s^*)$$

and that their rotation functions coincide

$$\kappa_Y(0) \equiv \kappa_{Y^*}(0) \, .$$

For three-dimensional media the Mellin functions $W_l^j(s)$ corresponding to angular harmonics $Y_l^j(\zeta)$, $j = -l, \ldots, l$ of the same order l are identically equal

$$W_l^j(s) = \frac{1}{2l+1} \sum_{n=-l}^{l} W_l^n(s) \, . \tag{4.2.53}$$

If a continuous isotropic medium is also mirror-symmetric then the interaction coefficient is invariant with regard to reflections in k-space and the Mellin functions satisfy

$$W(s^*) = W^*(s) \, . \tag{4.2.54}$$

This equation is always satisfied in the decay three-dimensional case, because in three dimensions it is always possible to accomplish reflections of three vectors $\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2$ by the aid of rotations.

From (4.2.54) it follows that the zeros of the function $W(s)$ are either real or form pairs of complex conjugate numbers. Hence, the third case for the nonexistence of zero rotation (4.2.26) may be the case of general position (on the line $\mathrm{Re}\, s = \sigma_0$, there may be a pair of complex conjugated zeros of the Mellin function).

When the function $W(s)$ obeys (4.2.54), one can formulate a rather simple sufficient condition of the instability of the Kolmogorov spectrum

$$W(0) > 0 \, . \tag{4.2.55}$$

Indeed, it follows from (4.2.54–55) that the rotation $\kappa(0)$ is inevitably odd.

The criterion for the instability of the Kolmogorov solution with regard to isotropic perturbations is formulated in a different way. For the zero harmonic, $W_0(0)$ is always zero and the rotation function $\kappa_0(0)$ is not defined. To obtain the wanted criterion, one should slightly shift the vertical axis on which κ is calculated to the right or to the left depending on the region in which the Kolmogorov solution is realized, i.e., for small or large k, respectively. For example, a sufficient condition for the instability of the short-wave spectrum with regard to isotropic perturbations is the inequality $W_0(\varepsilon) > 0$ for the small negative ε. It is readily seen that this is equivalent to the condition $W_0'(0) < 0$. Thus we arrive again at the Fournier-Frisch criterion described in Sect. 3.1: solutions with the "wrong" sign of the flux are unstable.

4.2.3 Physical Examples

As we have seen in the preceding subsection, the behavior of perturbations is determined by position of the zeros of the Mellin functions $W_l(s)$. However, every zero $W_l(p) = 0$, for example, for real p corresponds to the stationary power-type solution $A(\boldsymbol{k}) = \delta n(\boldsymbol{k})/n_k^0 = Y_l k^{-p}$ of the linearized kinetic equation. For zeroth and first angular harmonics, the stationary power solutions [the neutrally stable modes (4.1.9, 11, 13, 14, 16, 18, 20, 23)] were derived in Section 4.1. These solutions are universal, i.e., they do not depend on the particular form of the interaction coefficient, but are entirely determined by the indices. All these modes correspond to small fluxes of the integrals of motion (4.2.52). The locality of a neutrally stable mode implies that p is an element of the analyticity strip of the corresponding function $W_l(s)$.

The zeros corresponding to universal modes, like all other zeros of Mellin functions, determine the terms of the asymptotic expansions of the Kolmogorov spectrum, see (4.2.39, 45, 51). The role of an individual term depends considerably on the position of the corresponding zero relative to the zero rotation strip $\Pi(\sigma_-, \sigma_+)$: it matters whether the location is on the right or left of this strip, on its boundary or far away from it. In particular, in the case of the instability of the Kolmogorov spectrum, the main and fastest growing correction to the spectrum is entirely determined by the particular zero p of $W(s)$ for which either $\mathrm{Re}\, p = \sigma_- > 0$ holds or $\mathrm{Re}\, p = \sigma_+ > 0$.

Let us consider at first the stability problem of Rayleigh-Jeans spectra that are in thermodynamic equilibrium. If the power solution (4.2.2) is a thermodynamic spectrum, then it may easily be seen from (4.2.16) that the Mellin functions $W(s)$ have the properties

$$W(r+s) = W(r-s), \quad W(r+i\omega) < 0, \quad \text{where} \quad r = \frac{\nu-\mu}{2} = \frac{d-h}{2} .$$

That condition just presents the H-theorem for a given harmonic of the linearized kinetic equation. It follows that the analyticity strip $\Pi(a,b)$ of the Mellin function $W(s)$ is symmetric with regard to the line $\mathrm{Re}\, s = r$; on this line the rotation of the function $W(s)$ is zero. Consequently, there exists a zero rotation interval

symmetric relative to the point r. However, this interval does not necessarily include the point $s = 0$. Thus, the H-theorem does not ensure the stability of the equilibrium distribution: the initially small perturbation can grow in the process of evolution. The existence of a zero rotation interval implies that the instability of thermodynamic spectra is not strong, i.e., the perturbations cannot grow in the whole k-space. If interval instability exists, it will inevitably occur at large scales if $r > 0$ or at small scales if $r < 0$. Interval instability will take place, for example, when the zero p corresponding to a neutrally stable mode is located between the points $s = 0$ and $s = r$. Let us consider, for example, the decay case. For physically interesting media, the index of the Kolmogorov spectrum is generally larger than that of the thermodynamic spectrum. Therefore the quantity $r = d + m - \alpha$ is positive and, consequently, the interval instability can manifest itself only at large scales. In the decay case it is also easy to show that the zero $p_0 = 2(m+d-\alpha) = 2r$ corresponding to the isotropic mode (4.1.16) is not located between the points $s = 0$ and $s = r$ and cannot lead to interval instability. For the first angular harmonic the situation is different: since $\alpha > 1$ we know that out of the two zeros $p_1 = \alpha - 1$, $p_2 = 2r + 1 - \alpha$ symmetric with regard to the point r, one is always located in the interval $(0, r)$. This is in general an equilibrium zero p_1 (in all cases considered we have $p_1 < p_2$). Hence, the spectrum T/ω_k will show interval instability in the region of small k, provided that this zero falls also within the analyticity strip. The physical meaning of this instability is rather simple. An external source generates a perturbation in the form of the first angular harmonic having a nonzero momentum. The instability corresponds to the rearrangement process of the distribution T/ω_k to $T/[\omega_k - (\boldsymbol{k}\boldsymbol{u})]$ with the nonzero momentum. For example, for deep-water capillary waves ($m = 9/4$, $d = 2\,\alpha = 3/2$), we have $0 < p_1 < r$ and $h = -7/2 < 0$, i.e., the perturbations of the equilibrium spectrum are shifted to small k, with the part of the perturbation due to the first harmonic growing in magnitude. Thus we have a soft interval instability. Since $p_1 + h < 0$, the spectrum under consideration is stable with regard to initial perturbations.

We can discuss the nondecay case in a similar way. For the solution $T/\omega_k \propto k^\alpha$ with zero chemical potential, the presence of the mode (4.1.20) with a small energy flux P cannot lead to instability, because $p_0(P) = 2m + 3d - 3\alpha = 2r$. However, there are two more isotropic, neutrally stable modes whose indices are symmetric relative to the point r: the equilibrium one with $p_0(\mu) = \alpha$ and the nonequilibrium mode (4.1.23) with $p_0(Q) = 2r - \alpha$. At $r > \alpha$, the instability may be associated with the former and at $\alpha > r > 0$ with the latter. At $r < 0$, the Rayleigh-Jeans spectrum is stable with regard to isotropic perturbations, but we may have interval instability with respect to perturbations in the form of the first angular harmonic, as one of the zeros $p_1 = \alpha - 1$, $p_2 = 2r + 1 - \alpha$ will inevitably be found in the interval $(0, r)$. As we see, all these instabilities are associated with the conservation laws and correspond to structural rearrangements of the distributions after adding to them the initially nonexistent integral of motion.

Let us discuss now possible instabilities of the Kolmogorov spectrum that are associated with the universal neutrally stable modes. In particular, we shall consider formation of these modes under perturbations of a source.

As usual, we shall start from the decay case. In an isotropic general system, there are no other integrals of motion besides the one of energy; therefore, the isotropic neutrally stable modes are not formed (except for the trivial one $\delta n_k/n_k^0$ =const corresponding to energy flux variations). In all the examples the Mellin function $W_0(s)$ has a single zero $W_0(0) = 0$ in the locality strip, and the Kolmogorov spectra are indifferently stable with respect to isotropic perturbations. We shall note, however, that this statement has not been proved in the general case.

With regard to the first angular harmonic, there is a drift mode (4.1.9) corresponding to constant momentum flux. Its index $p_1 = 1 - \alpha < 0$ gives the zero of the Mellin function $W_1(p_1) = 0$ located on the left of the point $s = 0$. Consequently, the instability associated with this zero may be observed in the region of large k, i.e., just in the inertial interval [we suppose as usual $\nu = m + d > \alpha$ which, according to (3.1.13), corresponds to a positive energy flux and to a source at small k]. If the perturbations of the source have the form of the first angular harmonic, a drift mode may be formed (i.e., determine the spectrum perturbation asymptotics at $k \to \infty$), if p_1 is the zero closest to the point $s = 0$. Besides, in conformity with (4.2.55), the condition $W_1(0) > 0$ should be satisfied. From this follows the necessity of the condition

$$W_1'(p_1) = W_1'(1 - \alpha) > 0 \,. \tag{4.2.56}$$

But, as we have seen in Sects. 3.1.3 and 4.1, the derivative of the collision integral with regard to the index of the stationary-state solution, specifies the sign of the corresponding flux. Thus, the condition (4.2.56) implies that in the decay case the drift mode is formed only for positive momentum flux, i.e., it has the same direction as the energy flux of the main solution. Physically this condition seems to be quite natural, as fluxes should be directed towards the damping region.

This criterion (first formulated by *Falkovich* [4.6]) is also valid for any of the universal steady-state modes (4.1.9, 11, 13, 14, 16, 18, 20, 23): a neutrally stable mode is formed and leads to structural instability of the Kolmogorov spectrum only when the flux of the integral of motion transferred by it has the same direction as the flux of the main integral of motion. Indeed, in mirror-symmetric media, for the mode (4.1.11) dominating at large k, a sufficient instability condition $W_0(0) > 0$ is provided by the inequality $W_0'(-\alpha) > 0$, i.e., by the positive character of the small-wave action flux. On the other hand, the modes (4.1.13–14) can lead to structural instability of a spectrum with an action flux if the fluxes transferred by those modes are directed towards small k: $W_0'(\alpha) < 0$, $W_1'(1) < 0$.

Let us consider some examples. The integrals (4.2.16) determining the Mellin function are rather complex and cannot be calculated analytically. However, they may be calculated on a computer. Since it is sufficient to find only the Mellin function rotations (being integers), these computations can be rather inaccurate. In [4.10] one can find a transformation of the integrals (4.2.16) to a form suitable for machine computations.

Let us start with the turbulence of gravitational waves on the surface of a deep incompressible fluid. The appropriate dispersion law is given by (1.1.42) and the

four-wave interaction coefficient, by (1.2.42), so that $\alpha = 1/2$, $m = 3$, $d = 2$. There are two Kolmogorov solutions, one with action flux towards the region of large scales (3.1.27) and the other one with energy flux towards the region of small scales (3.1.28).

Calculations with formula (A.4.10) show that for both spectra the values of all Mellin functions $W_l(s)$ on the imaginary axis ($s = i\omega$, $-\infty < \omega < \infty$) are located in the left half-plane. [The calculations were performed for $l = 0, 1, \ldots, 29$; with growing l, the values of the function $W_l(i\omega)$ for $-\infty < \omega < \infty$ are displaced further into the left half-plane.] Consequently, rotations of all Mellin functions around the imaginary axis vanish and the Kolmogorov spectra (3.1.27–28) are stable. One can also directly verify that in this case the neutrally stable modes (4.1.11, 13–14) transfer backwards fluxes of action, energy and momentum, respectively, that are small compared to the fluxes of the main integrals of motion. Consequently those nodes could not be realized. In real situations at very small k ($k \simeq k_0$) and at very large k ($k \simeq k_d$), the medium gives rise to damping (sometimes the damping may be due to the nonsteady state of the spectrum in the region of small or large k); pumping is observed in an intermediate range $k \simeq k_1$. If the pumping and damping regions have sufficiently strong differences in their scales $k_0 \ll k_1 \ll k_d$, then in a stationary state all the energy pumped into the system should be transferred to the region of large k and all the wave action pumped in, to the region of small k, see (3.1.26). Thus (at least for weak anisotropic perturbations), the Kolmogorov spectrum (3.1.28) supporting an energy flux should be formed in the inertial interval $k_1 \ll k \ll k_d$ and the Kolmogorov spectrum (3.1.27) supporting a wave action flux in the inertial interval $k_0 \ll k \ll k_1$.

A typical example for a system with a decay law and weak turbulence are turbulent capillary waves (on the surface of a deep incompressible fluid). The dispersion law of these waves and the interaction coefficient are determined by (1.2.40). Thus we have $\alpha = 3/2$, $m = 9/4$, $d = 2$. Numerical computation shows that the rotations $\kappa_l(0)$ of the Mellin functions $W_l(s)$ are zero for all $m \neq \pm 1$ and $\kappa_{\pm 1}(0) = 1$. Hence, the spectrum (3.1.15b) is unstable with regard to perturbations having the form of the first angular harmonic. Numerical computation also yields $\kappa_{\pm 1}(-2/3) = 0$. This leads to the following conclusions. First, the instability of the first angular harmonic is of the interval type and is hard since $h = -3/4 < 0$; this instability manifests itself in the region of large k. Second, in the strip $\Pi(-2/3, 0)$ the function $W_1(s)$ has a single zero which occurs at $s = 1 - \alpha = -1/2$ and corresponds to the universal mode (4.1.9). In this case the direction of the (small) momentum flux coincides with the one of the energy flux. Consequently, in a system of capillary waves under weak constant anisotropic action, a spectrum of the form (4.2.51) should be established:

$$n(\boldsymbol{k}) \simeq \lambda P^{1/2} k^{-17/4} \begin{cases} 1, & k \to 0 \,, \\ \text{const}\sqrt{k}\cos\theta, & k \to \infty \,. \end{cases}$$

Thus, the anisotropy extends into the region of large k and the hypothesis about local isotropy does not hold for capillary wave turbulence; small fluctuations at large scales will lead to large fluctuations at small scales.

It is interesting to have a look at the structure of the stationary turbulence spectrum in the region of k-space in which the anisotropy is no longer small. The structural instability of the isotropic spectrum is associated with the stationary mode (4.1.9) transferring the momentum flux. Therefore it is natural to suppose that in the short-wave region a universal stationary distribution should be given by the fluxes of both conserved values P and R and have the form (4.1.5)

$$n(\boldsymbol{k}, P, \boldsymbol{R}) = \lambda P^{1/2} k^{-m-d} f(\xi), \quad \xi = \frac{(\boldsymbol{R}\boldsymbol{k})\omega_k}{Pk^2} .$$

In Sect. 4.1 we discussed the properties of such a solution at $\xi \to 0$, which is almost isotropic with $f(0) = 1$. One can make the hypothesis that at $\xi \to +\infty$ (i.e., at $k \to \infty$ and $\cos\theta > 0$) the distribution should be determined only by the momentum flux. A necessary for this is $f(\xi) \propto \sqrt{\xi}$. Finally, we assume $f(\xi) \to 0$ at $\xi \to -\infty$ to hold. Let us now describe in brief the properties of such a hypothetical solution. In the direction of the vector $\boldsymbol{R}$, i.e., the pumping has a maximum (i.e., at $\theta = 0$), the occupation numbers should decrease slower than for the isotropic Kolmogorov solution $n(\boldsymbol{k}) \propto k^{-15/4}$. In transversal directions $\theta = \pm\pi/2$, the decrease of $n(\boldsymbol{k})$ with increasing k coincides with the behavior of the isotropic case (3.1.15b): $n(\boldsymbol{k}) \propto k^{-17/4}$. Most of the waves are found in the right part of the hemisphere with $|\theta| < \pi/2$. It should be noted that in the only case in which $f(\xi)$ was determined unambiguously [for sound turbulence, see below (5.1.12, 14)], its properties proved to be identical to the above case: $f(\xi) \propto \sqrt{\xi}$ at $\xi \to +\infty$ and $f(\xi) \to 0$ at $\xi \to -\infty$.

It should be remarked that such an instability does not occur for all media with a decay dispersion law. In general, the vaules of $\kappa_l(0)$ can for $l = 0, 1, \ldots$ assume rather diverse sets of values. Thus we may consider a model with the same values of the parameters α, m, d as for capillary waves where we have $\kappa_1(0) = 1$, $\kappa_7(0) = 3$ and for l other than 1 and 7 the value $\kappa_l(0) = 0$.

An important example of nondecay weak turbulence is the turbulence of Langmuir waves in plasmas. The dispersion law of these waves (1.3.3) may be considered to be scale-invariant with $\alpha = 2$. If the main nonlinear process of plasmon interaction is the exchange of virtual ion sound oscillations, the interaction coefficient is given by (1.3.14) and has the scaling index $m = 0$.

In this situation, the only local power spectrum is a Kolmogorov spectrum with the wave action flux Q. In two dimensions, the computer calculation has shown this Kolmogorov spectrum to be unstable with regard to perturbations having the form of angular harmonics with $l = 0, \pm 1$; the other harmonics are stable. As we see, in this case the Kolmogorov solution is unstable with respect to isotropic perturbations.

If the dynamic equations (1.1.14) depict the nonlinear Schrödinger equation (1.4.24) written in momentum representation, then $\alpha = 2$, $m = 0$ and the interaction coefficient is a constant. In this case the computations show the only local Kolmogorov spectrum (with wave action flux) at $d = 2$ to be unstable with regard to the same angular harmonics as in the case of Langmuir turbulence.

Both these instabilities are associated with universal modes (4.1.13–14). Curiously, for both modes the (energy and momentum) fluxes are positive, i.e., are directed towards large k. However, for the initial spectra the wave action flux is also positive since $n_0(k) \propto \omega^{-2/3}$ and according to (3.1.22) we have sign $Q > 0$. It is the "wrong" direction of the flux Q that gives in these two cases rise to the spectrum instability of isotropic media according to the Frisch-Fournier criterion (see Sect. 3.1.3).

In all cases considered, the instabilities of the Kolmogorov spectrum occur only for the zero and first angular harmonic and are associated with universal modes carrying small fluxes of the integrals of motion. These structural instabilities obviously describe the rearrangement processes of single-flux distributions to multi-flux distributions as for equilibrium systems. In this case, the asymptotics of stationary turbulent distributions at $k \to 0, \infty$ are determined by the directions of the fluxes of the integrals of motion. One can also assume that in general cases, structural instabilities of single-flux spectra may be associated only with the universal modes derived in Sect. 4.1 which carry the fluxes of energy, momentum or wave action. We thus come to the more sophisticated form of the universality hypothesis: in the inertial interval, a stationary spectrum should be defined by those fluxes of the integrals of motion which are directed from the source to the sink. Such spectra are universal since they depend on the fluxes only, being independent of the fine structure of the source. However, a universal spectrum is not necessarily an isotropic one. If the momentum flux is directed from the pumping to the damping region, then the stationary spectrum is anisotropic and the isotropy hypothesis is incorrect while the universality hypothesis may still hold.

In degenerate cases (for example, with additional integrals of motion) the structural instabilities of Kolmogorov spectra could be connected with angular harmonics larger than zero or unity. In the Sect. 5.1 we shall discuss structural instabilities of isotropic turbulence spectra for small-dispersion waves. Such wave systems are close to a degenerate system of nondispersive waves obeying a linear dispersion law that is intermediate between the decay and nondecay cases.

4.3 Nonstationary Processes and the Formation of Kolmogorov Spectra

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In this section we shall discuss the nonstationary behavior of weakly turbulent distributions. We shall be concerned with both free evolution regimes (in the absence of external sources) and the formation of stable Kolmogorov distributions after pumping is switched on. As we shall see now, the character of the evolution dramatically depends on the sign of the index $h = \alpha + d - s_0$ dealt with in Sect. 3.4–4.2. Indeed, the index of the collision integral linearized against the background of the Kolmogorov solution is equal to $-h$. This means that if at some k the distribution deviates from a stationary one then the typical time for variations of the occupation numbers is

$$t_{NL}^{-1} \propto k^{-h} . \tag{4.3.1}$$

If there are no external effects to be considered then the energy should be conserved so that the characteristic time of the system may be evaluated from the kinetic equation

$$t_{NL}^{-1} \propto k^{-2h} E \tag{4.3.2}$$

where $E = \int \omega_k n_k dk$ is the total energy of the distribution. Thus, in both cases the process of wave transfer, e.g., to large k is accelerated or slowed down depending on the sign of h. It is appropriate to draw attention to the point that the quantity h shows also at which end of the Kolmogorov distribution transporting an energy flux, the major part of the energy of the turbulence is concentrated (see Sect. 3.4.1)

$$E = \int \omega_k n_k^0 dk \propto k^h . \tag{4.3.3}$$

For example, for $h > 0$ most of the energy of the Kolmogorov spectrum is confined to the region of large k. As we can see from (4.3.1, 2), the motion of the distribution slows in this case down as it moves towards larger wave numbers while its evolution into the opposite direction (containing little or no energy) is an accelerated process.

In Sect. 4.3.1 we shall first discuss the nonstationary behavior of weakly turbulent distributions of waves with a decay dispersion law and $s_0 = m + d$ and $h = \alpha - m$, see (3.4.3). Since h is equal to the difference between the frequency index and the index of the interaction coefficient, its sign indicates which coefficient of the Hamiltonian grows quicker with k, the one responsible for linear phenomena or for the interaction. The dimensional analysis of Sect 4.3.1 will show that distributions initially localized in the long-wave region evolve on their way towards large k in a self-similar manner. After a long-wave source has been switched on, the stationary Kolmogorov distribution is formed with the help of a self-similar relaxation front.

Wave systems with the opposite sign of h, i.e., with $h < 0$ are characterized by an evolution of the front of spectrum formation according to an explosion law, i.e., it approaches infinity within a finite time.

A strict analytical proof of the explosive character of pumping is given in Sect. 4.3.2 for the particular case of weak three-dimensional sound turbulence ($h = -1/2$). The idea of the proof is to consider the dynamics of the moments of the distribution function in k-space. Proceeding from the kinetic equation, one can prove that if initially some moments were finite (i.e., the distribution was decreasing fast) then they will become infinite within a finite time, which corresponds to the formation of the power asymptotics at $k \to \infty$. For the intermediate case $h = 0$ of two-dimensional acoustic turbulence, the same section gives analytical proofs of the nonexplosive character of the evolution. Section 4.3.2 gives the results of numerical simulations which vividly support the ideas displayed in Sects. 4.3.1, 2.

4.3.1 Analysis of Self-Similar Substitutions

We start with the three-wave kinetic equation (2.1.12). For the sake of simplicity we assume the distributions to be isotropic. In this case, there is only one integral of motion, the energy. As usual we shall consider $m + d > \alpha$, i.e., after its formation, the Kolmogorov distribution transfers energy to the short-wave region, see (3.1.13). We shall discuss two physically different statements of the problem: 1) extension of the Kolmogorov distribution into the region of large k; 2) decaying turbulence: the free evolution of the initially long-wave packet which should in an attempt to arrive at a equilibrium distribution be spread out over the entire k-space.

It would be natural to assume that far from the source or from the initial localization site of a wave packet, the evolution will after some time become self-similar. Let us discuss possible self-similar substitutions for the three-wave kinetic equation. We shall seek the solution of the nonstationary equation (2.1.12) in the form

$$n(k,t) = t^{-q} f(kt^{-p}) = t^{-q} f(\xi) . \tag{4.3.4}$$

We assume the variable ξ to be dimensionless, measure k in units of k_0 (where k_0 is the initial location of the source or packet) and t in units of $t_N = V_0^2 k_0^{2m+d} n^2(k_0)/\omega(k_0)$. In here t_N is the characteristic time of nonlinear wave interaction in the region $k \simeq k_0$ and V_0 the dimensional constant of the interaction coefficient, see (3.1.7a)

Substituting (4.3.4) into the three-wave kinetic equation

$$\frac{\partial n(k,t)}{\partial t} = I(k) , \tag{4.3.5}$$

we obtain

$$-(gf + p\xi f') = I(\xi) t^{p(2m-\alpha+d)-q+1} .$$

It follows that a solution of the form (4.3.4) may only exist if the condition

$$p(2m - \alpha + d) - q + 1 = 0 \tag{4.3.6}$$

is satisfied.

To obtain another relationship between the parameters p and q, we should consider cases 1) and 2) separately:

1) In the region between the source and the relaxation front, the quasi-stationary Kolmogorov distribution $n_k^0 \propto k^{-m-d} = k^{-s_0}$ should be formed. This means that at $\xi \to 0$ we should have $f(\xi) \propto \xi^{-m-d}$ and since we are dealing with a stationary state,

$$p(m + d) = q . \tag{4.3.7}$$

From (4.3.6, 7), we find $q = (m + d)/(\alpha - m) = s_0/h$ and get

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which is consistent with the estimate (4.3.1). The boundary of the Kolmogorov distribution corresponds to $\xi \simeq 1$ and moves in the k-space according to the law $k_b \propto t^p = t^{1/h}$. We encountered this law already when considering the motion of small perturbations against the background of the stationary spectrum. It is obvious that the solution (4.3.4) describes in this case the evolution of the Kolmogorov spectrum towards large k only for $h > 0$. The same conclusion will be arrived at by considering the kinetic equation in the self-similar variables

$$-(qf + p\xi f') = I(\xi) \tag{4.3.9}$$

where $I(\xi)$ is the three-wave collision integral $I(k)$ (3.1.11) in which k has been replaced by ξ. Substituting $f(\xi) \propto \xi^{-m-d}$ into (4.3.9), we see that $I(\xi)/f(\xi) \propto \xi^{-h}$, i.e., the $I(\xi)$ term prevails in the region $\xi \ll 1$ [and $f(\xi)$ has the Kolmogorov asymptotics there] at $h > 0$. The front velocity measured on the logarithmical scale

$$\frac{d \ln[k_b(t)/k_b(0)]}{dt} \propto (ht)^{-1}$$

decreases with time, in line with the notion that the expansion process of the distributions with the expansion of the energy-containing region is a slowing-down process.

At $h \to 0$, the front velocity dramatically increases to be infinite at $h = 0$. This indicates that at $h < 0$ the formation rate of the Kolmogorov spectrum is so high (to be more exact, it increases with time so quickly) that the relaxation front reaches infinity within a finite time. To obtain at $h < 0$ a self-similar relaxation front moving towards large k, we replace the t in (4.3.4) by $\tau = t_0 - t$:

$$n(k,t) = \tau^{-q} f(k\tau^{-p}) . \tag{4.3.10}$$

Equations (4.3.6–8) for p and q will regain their previous form, but now the right boundary of the Kolmogorov distribution with $k_b\tau^{-p} \simeq 1$ evolves according to the explosion law $k_b \propto \tau^{1/h}$ and reaches infinity within the finite time t_0 determined by the initial distribution (see below). Since for a system with $h < 0$ the energy of the Kolmogorov distribution is localized in the long-wave region, the relaxation of the stationary spectrum in the interval from a finite k to ∞ demands redistribution of the finite energy and takes finite time.

It is of interest to clarify the behavior of the energy accumulated in the self-similar part of the distribution. For the solutions (4.3.4) and (4.3.10) we have

$$E(t) = t^{2ph-1} \int_0^\infty f(\xi) \xi^{\alpha+d-1} d\xi , \tag{4.3.11a}$$

$$E(t) = \tau^{2ph-1} \int_0^\infty f(\xi) \xi^{\alpha+d-1} d\xi , \tag{4.3.11b}$$

respectively. $p = 1/h$ leads to $E \propto t$ at $h > 0$ and $E \propto \tau = t_0 - t$ at $h < 0$. The linear growth of the energy in systems with $h > 0$ means, that a self-similar solution is formed when the occupation numbers of the waves associated with the source do not change any longer and a constant energy flux has been established. In this case, the main portion of the energy is concentrated in the self-similar region. However, at $h < 0$ the portion of energy contained in the solution (4.3.10) decreases as the self-similar wave moves towards the short-wave region.

Thus, very different time scales are realized for the universal relaxation regimes of the Kolmogorov spectra with $h > 0$ or $h < 0$. At $h > 0$, the self-similar wave (4.3.4) is formed for a time much larger than the typical stabilization time for the occupation numbers in the pumping region. For $h < 0$, the self-similar wave is realized for a small period of time ($t_0 - t \ll t_0$), which is too small for the occupation numbers of long waves to undergo any essential changes.

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2) Now we consider the free evolution starting from the system with $h < 0$. Since the wave (4.3.10) is self-accelerating

$$\frac{dk_b/dt}{k_b} \propto 1/\tau ,$$

the behavior of the short-wave part of the distribution should be insensitive to the presence or absence of a long-wave source that changes the occupation numbers not faster than exponentially. Hence, for times smaller than t_0, the free expansion of a turbulent distribution towards large k proceeds in the same way as in the case of the source: the self-similar wave $n(k,t) = \tau^{-q} f(k\tau^{-p}) = \tau^{-q} f(\xi)$ with $q = s_0/h, p = 1/h$ moves according to an "explosion" or power law, leaving the Kolmogorov distribution behind, $f(\xi) \propto \xi^{-s_0}$ at $\xi \ll 1$. Ahead of the relaxation front, the occupation numbers should rapidly decrease with growing ξ (and, accordingly, k). In [4.18] it has been assumed that if waves in the short-wave region of the distribution interact mainly with each other, then the (quasi-Planck) asymptotics are exponential $n_k \propto \exp[-(\omega_k/\omega_b)] = \exp[-(k/k_b)^\alpha]$. Let us look for the conditions under which this is possible. Going over to the variable $\eta = \xi^\alpha$, we write the kinetic equation (4.3.9) in the form

$$\begin{aligned} -[qf(\eta) + \alpha p \eta f'(\eta)] = & \int_0^\eta [\eta_1(\eta - \eta_1)]^{d/\alpha - 1} \eta^{2m/\alpha} f_1^2(\eta/\eta_1) \\ & \times \Delta_d^{-1} [f(\eta_1) f(\eta - \eta_1) - f(\eta) f(\eta_1) \\ & - f(\eta) f(\eta - \eta_1)] d\eta_1 \\ & - 2 \int_\eta^\infty [\eta_1(\eta_1 - \eta)]^{d/\alpha - 1} \eta_1^{2m/\alpha} f_1^2(\eta/\eta_1) \Delta_d^{-1} \\ & \times [f(\eta) f(\eta_1 - \eta) - f(\eta_1) f(\eta_1 - \eta) \\ & - f(\eta_1) f(\eta)]\, d\eta_1 \,. \end{aligned} \tag{4.3.12}$$

Here f_1 is the structural function of the interaction coefficient (3.1.7c). Expressed in terms of the frequency ratio $x = \omega_1/\omega$ it has the properties $f_1(x) = f_1(1-x)$ and $f_1(x) \propto x^{m_1/\alpha}$ at $x \to 0$. The quantity Δ_d^{-1} is a result of angle averaging of the δ-function of wave vectors, see its definition in Sect. 3.1. Let us consider (4.3.12) at $\eta \gg 1$ and set $q = s_0/h$, $p = 1/h$, $f(\eta) = \eta^{-b}\exp(-\eta)$. Using the asymptotics of the function $f_1(x)$ at $x \to 0$, we obtain from (4.3.12)

$$\begin{aligned} \alpha\eta^{1-b} e^{-\eta} = & -2he^{-\eta}\eta^{(1-\alpha+2m-2m_1)/\alpha - b} \\ & \times \int_1^\eta \eta_1^{(d-1-\alpha+2m_1)/\alpha - b}(1 - e^{-\eta_1})^2 d\eta_1 \,. \end{aligned}$$

Hence, such asymptotics can only exist if the inequality

$$m_1 - m + \alpha \geq \frac{1}{2} \tag{4.3.13}$$

is satisfied. For example, for capillary waves on a deep fluid this condition is satisfied ($h = \frac{3}{4}$, $\alpha = \frac{3}{2}$, $m = \frac{9}{4}$, $m_1 = \frac{7}{2}$) while it is violated for three-dimensional sound ($h = -\frac{1}{2}$, $\alpha = 1$, $m = \frac{3}{2}$, $m_1 = \frac{1}{2}$). The question with regard to the possibility of an analytical construction of the self-similar asymptotics in the region ($\xi \geq 1$) if condition (4.3.13) is violated remains open. Numerical simulations carried out for three-dimensional sound (see Fig. 4.4 below) show that here again n_k is ahead of the wave front quickly diminished with the growth of k (i.e., quicker than by the Kolmogorov law).

The total energy is conserved, since for $t < t_0$ the asymptotics of the distribution at $k \to \infty$ fall off steeper than the Kolmogorov one. What happens at $t \geq t_0$, when the right boundary of the Kolmogorov distribution will reach infinity? The energy should start to decrease, because then $n_k \propto k^{-s_0}$ at $k \to \infty$ and the energy flux at $k = \infty$ is nonzero. Naturally, any realistic system has a sink at finite k_m so that the energy will start to decrease somewhat earlier than at t_0, namely, when the relaxation front reaches the sink. It would be natural to assume that after a large enough period of time ($t \gg t_0$), the decrease of the distribution would start to proceed according to the second self-similar regime with the energy decaying according to a power law. Such a behavior should be described by a self-similar solution of the form (4.3.4). The relationship between q and p is given by (4.3.6) and the energy decreases by the law (4.3.11a). To determine the index p, we have to solve the nonlinear eigenvalue problem (4.3.9) with the additional requirement $f(\xi) \geq 0$ at $0 < \xi < \infty$. Starting from the condition ensuring the decrease of the energy, one can impose on p the limitation

$$p > (2h)^{-1} \,.$$

Since we consider negative h, this inequality allows p to be positive or negative. The solutions with positive p describe a distribution moving to the right and those with negative p move to the left. We can, however, *suppose* that, since the sink (be it at $k = k_m$ or $k = \infty$) is stationary, the distribution on the whole should be diminished without being in motion, i.e., $p = 0$ and $E \propto p^{-1}$. The law depicting the energy decrease may also be obtained from a sequence of estimates based on the same assumption as above to yield $dE/dt \propto P \propto E^2 \Rightarrow E \propto t^{-1}$. The results of numerical simulations (see below Fig. 4.9) are consistent with our supposition. Thus, at $h < 0$ the behavior is bi-self-similar.

As time increases, the free evolution of systems with $h > 0$ should for large k approach a self-similar solution of the type (4.3.4). But now the energy should be conserved, therefore the index p is determined unambiguously by $p = (2h)^{-1}$ which is consistent with the estimate (4.3.2). The positiveness of p implies that the distribution moves towards the short-wave region. Such a solution has no Kolmogorov asymptotics at all, since the energy flux vanishes at $k \to 0$ and $k \to \infty$.

The intermediate case with $h = 0$ is degenerate since it has zero measure in the space of conceivable wave systems. We shall study it now to obtain a more complete general picture and to account for two systems observed in nature, gravitational-capillary (1.2.39a), (3.1.3) and capillary (1.2.39b, c) waves on shallow water. As a rule, there is a different type of self-similarity in the degenerate points separating regions of deviating behavior ($h < 0$ and $h > 0$). Indeed, according to (4.3.1, 2) the condition $h = 0$ implies the independence of the characteristic interaction time from the wave number. For this reason, the velocity of the relaxation front measured on a logarithmical scale, should be constant which implies a self-similarity of the exponential rather than of the power type.

$$n(k,t) = \exp[-qt] f(k\exp[-pt]) = \exp[-qt] f(\xi) \qquad (4.3.14)$$

For (4.3.14) to be a solution of (4.3.5), one should have $q = p(2m+d-\alpha) = p(m+d) = ps_0$. In this case the energy

$$E = \exp[(ps_0 - q)t] \int_0^\infty f(\xi)\xi^{\alpha+d-1}d\xi = \int_0^\infty f(\xi)\xi^{\alpha+d-1}d\xi$$

remains unchanged, therefore such a self-similar solution corresponds to the free evolution regime. Having reached the sink, the evolution of the self-similar front (4.3.14) may switch over to the regime (4.3.11) in which the energy is diminished inversely proportional to time [see (4.3.11a) and Fig. 4.10 below].

According to the character of their evolution, wave systems with a decay dispersion law are divided into two classes: one with $\alpha > m$ ($h > 0$) and the other with $\alpha < m$ ($h < 0$).

In the nondecay case, the classification also includes two Kolmogorov solutions: $n_1(k) \propto k^{-s_1}$, $s_1 = d + 2m/3$, with the energy flux towards the region of large k and $n_2(k) \propto k^{-s_2}$, $s_2 = d + 2m/3 - \alpha/3$, with the wave action flux towards the long-wave region. Accordingly, there are two significant indices, h_1 and h_2, determining the position of the energy-containing region of the n_1 solution and the region where the main part of wave action of the n_2 solution is concentrated:

$$E = \int \omega_k n_1(k)d\boldsymbol{k} \propto k^{\alpha+d-s_1} = k^{\alpha-2m/3} \equiv k^{h_1}, \qquad (4.3.15a)$$

$$N = \int n_2(k)d\boldsymbol{k} \propto k^{d-s_2} = k^{(\alpha-2m)/3} \equiv k^{h_2}. \qquad (4.3.15b)$$

Since we consider, as a rule, systems with $\alpha > 0$, we usually have $h_1 > h_2$. Consequently, in the space of systems there are three regions of parameters corresponding to evolutions of weakly turbulent distributions with diverse characters: a) $h_1, h_2 > 0$; b) $h_1, h_2 < 0$; c) $h_1 > 0 > h_2$.

In the presence of a source, the Kolmogorov solution $n_1(k)$ at $k \to \infty$ is established by a self-similar front of the form (4.3.4) for $h_1 > 0$ and an explosion front (4.3.10) for $h_1 < 0$, while the $n_2(k)$ solution at $k \to 0$ is established by the wave (4.3.4) for $h_2 < 0$ and by (4.3.10) for $h_2 > 0$. In all these cases we have $p = 1/h_i$, $i = 1, 2$.

We shall now briefly outline the different cases of free evolution of turbulent distributions. In case a), the long-wave Kolmogorov asymptotics corresponding to a constant action flux is accomplished via an explosive power law: $n(k,t) = (t_0 - t)^q f[k(t_0-t)^{-p}]$, $f(x) \propto x^{-s_2}$ at $x \gg 1$, $p = 1/h_2$, $q = ps_2$. Then, at $t > t_0$, energy conservation leads to a self-similarity regime with (4.3.4) and

$q = p(\alpha + d)$, $p = (3h_1)^{-1}$. A time progresses, this distribution is shifted towards the short-wave region. Specifying the position of the energy-containing region in ω-space by $\omega_E = E/N$, energy is seen to be pumped over towards large ω due to the fact that N is decreasing while E is conserved. In case b), the short-wave Kolmogorov asymptotics corresponding to a constant energy flux arise according to the explosion law (4.3.10). Then we observe the gradual formation of a wave which propagates into the long-wave region according to the self-similar law (4.3.4) with parameters $q = pd$, $p = (3h_2)^{-1}$ corresponding to conservation of the wave action integral. The maximum of the energy distribution ω_E decreases with time. Finally, in the case c) both integrals, E and N, should conserve, precluding the existence of two-parameter self-similar solutions (4.3.4) or (4.3.10), therefore the evolution is non-self-similar.

For example, in the case of gravitational waves on deep water we have $\alpha = 1/2$, $m = 3$, i.e., we are dealing with case b), since $h_1 = -3/2 < 0$, $h_2 = -11/6 < 0$. Thus, in the short-wave region a Kolmogorov spectrum with an energy flux is formed according to the explosion law $n(k,t) = (t_0 - t)^{8/3} f[k(t_0 - t)^{2/3}]$. The long-wave spectrum with an action flux is formed by the decelerating relaxation front $n(k,t) = t^{23/11} f(kt^{6/11})$. The boundary frequency of such a spectrum moves according to $\omega \propto t^{-3/11}$. What about the attenuation of waving after the wind (or pumping) abates? The distribution has Kolmogorov asymptotics with constant energy flux at $k \to \infty$, so the energy should not be conserved. The evolution of decaying turbulence should reach the self-similar regime $n(k,t) = t^{4/11} f(kt^{2/11})$. Thus, in the isotropic case, the mean waving frequency decreases according to $\omega_E \propto t^{-1/11}$.

4.3.2 Method of Moments

The treatment of the previous section was based on plausible arguments rather than rigorous proofs. Indeed, we left not only the transition of arbitrary distributions to the regime (4.3.4) or (4.3.10) without substantiation, but did not even provide a proof for the existence of self-similar solutions of such form.

However, it turns out that the most interesting property of evolution, the explosive character of the formation of the Kolmogorov asymptotics, may be strictly established for a particular case of weak sound turbulence. The idea of the proof is quite simple and was suggested by *Falkovich* [4.19]. It is based on consideration of the dynamics of the moments of the distribution $n(k,t)$

$$M_i(t) = \int k^i n(k,t) d\boldsymbol{k}.$$

Indeed, if at $k \to \infty$ the (power) asymptotics $n(k) \propto k^{-s}$ are established during a finite time, then the moments M_i with $i > s - d$ should become infinite. (A similar train of thought has also been used in the discussion of approximate models of hydrodynamic turbulence [4.20]).

The evolution of three-dimensional isotropic distributions of weak sound turbulence is described by (3.2.3). Going over to dimensionless ...

assumes the form

$$\frac{\partial n(k,t)}{\partial t} = 4\pi \int_0^k (k-k_1)^2[n(k_1)n(k-k_1) - n(k)n(k_1) - n(k)n(k-k_1)]k_1^2 dk_1 - 8\pi \int_k^\infty (k_1-k)^2[n(k)n(k_1-k) - n(k_1)n(k_1-k) - n(k_1)n(k)]k_1^2\, dk_1 \; . \tag{4.3.16}$$

In the weak dispersion limit when $\omega_k \approx k$, the energy is the first moment of the distribution function

$$E = M_1 = \int kn(k,t)d\mathbf{k} = 4\pi \int_0^\infty k^3 n(k,t)\, dk$$

It is conserved if, at $k \to \infty$, the function $n(k,t)$ decreases faster than by the Kolmogorov law (3.2.5) $n(k) \propto k^{-9/2}$. In our case of weak sound turbulence we have $\alpha = 1$, $m = \frac{3}{2}$, $h = -\frac{1}{2}$.

Let us consider the behavior of other moments of the distribution function. For the zero moment, the total number of waves

$$N(t) = 4\pi \int_0^\infty n(k,t)k^2\, dk$$

is readily calculated from (4.3.16)

$$\begin{aligned}\frac{dN}{dt} &= -(4\pi)^2 \int_0^\infty k^2 dk \int_k^\infty k_1^2(k-k_1)^2[n(k)n(k_1-k) - n(k_1)n(k) - n(k_1)n(k-k_1)]dk_1 \\ &= (4\pi)^2 \int_0^\infty k^2 n(k)\, dk \int_0^\infty k_1^2 n(k_1)\left[(k-k_1)^2 - (k+k_1)^2\right] dk_1 \\ &= -4E^2 \; .\end{aligned} \tag{4.3.17}$$

In deriving this equation, we rearranged the integration limits, which is only correct if the integral $\int_0^\infty k^3 n(k)dk$ converge. Thus, (4.3.17) is valid, if at $k \to \infty$ the quantity $n(k)$ diminishes quicker than k^4. We shall note here that we assume that there are no singularities of $n(k)$ at $k = 0$.

We see from (4.3.17) that an initial distribution that rapidly decreases at $k \to \infty$ and that has finite and nonzero N and E, cannot remain unchanged for arbitrary long times.

The existence of the equality (4.3.17) prompts the following picture of evolution: during the time $t_0 = N(0)/4E^2$, the initial long-wave packet will spread out over the k-space, N will vanish and the energy-containing scale $k_E = E/N$ will become infinite. This scenario implies that all the energy is pumped over to infinity (to large k) within a finite period of time.

However, the evolution is actually different. Let us consider the behavior of the second moment $M_2 = 4\pi \int_0^\infty k^2 n(k)k^2 dk$:

$$\frac{dM_2}{dt} = 6M_2^2 + 8\pi E \int_0^\infty k^3 n(k)k^2 dk + 32\pi^2 \int_0^\infty u^2 n(u)du \int_0^u v^4(u-v)^2 n(v)dv \geq 8M_2^2 \; . \tag{4.3.18}$$

Here we have used the simple inequality

$$\int_0^\infty k^5 n(k)dk \int_0^\infty k^3 n(k)dk \geq \left(\int_0^\infty k^4 n(k)dk\right)^2 .$$

The inequality (4.3.18) is valid if the integral $M \propto \int_0^\infty k^4 n(k)dk$ converges, i.e., if $n(k)$ decays faster than k^{-5}.

The equation $dx/dt = 8x^2$ has an explosive solution $x(t) = x(0)/\left(1-8x(0)t\right)$. Consequently, it follows from (4.3.18) that M_2 should become infinite during the time $t \leq t_1 = [8M_2(0)]^{-1}$. It is easy to show $t_1^{-1} = 8M_2(0) \geq 2t_0^{-1} = 8E^2/N(0)$ to hold. In other words, the "explosion" occuring when the second moment M_2 becomes infinite, takes place at least twice as quickly as the N vanish. We shall see below (see Figs. 4.4–6) that as M_2 increases, the Kolmogorov asymptotic $k^{-9/2}$ is explosively formed in the region of large k.

On the distribution $n(k) \propto k^{-9/2}$, the energy is contained in the region of small k: $\varepsilon \propto k^{-3/2}$. Therefore, during the relaxation time of the Kolmogorov asymptotics, only a small part of the initial energy of the turbulence is located in the region of large k. The effect of explosive formation of the Kolmogorov distribution in the short-wave region may be compared to a weak collapse (see [4.21]) when the value of the integral of motion captured into a singularity region (in our case, into $k \to \infty$) tends to zero.

Thus, the evolution of a weakly turbulent acoustic distribution should exhibit two stages. During the first, "explosive" stage energy is conserved, the number of waves decreases by a linear law and the second moment increases explosively. As a result, a finite energy flux will set on at $k \to \infty$, the total energy will start to diminish. Then at $t \gg t_1$, the self-similar regime of the type (4.3.4) is established with $p = 0 - n(k,t) = t^{-1}f(k)$ and with the energy decreasing by the power law $E \propto t^{-1}$. The short-wave asymptotic in this case is of the Kolmogorov type $n(k) \propto k^{-9/2}$, so that (4.3.17) remains valid, i.e., the total number of waves decreases monotonically (though this time not by the linear law, see Fig. 3.9 below).

We also note that, as the averaged interaction coefficient in (4.3.16) is proportional to k^2, consideration of moments above the second one will not change this picture: their derivatives may be expressed in terms of lower-order moments.

From (4.3.16) we see that the inverse of the time of nonlinear interaction $t_{NL}^{-1} \propto kE$ [see (4.3.2)] grows with k slower than the dispersion correction to the frequency $\delta\omega = a^2 k^3$. This means that the applicability criterion of the weakly turbulent approximation, $\delta\omega t_{NL} \gg 1$, will be satisfied increasingly better as the distributions move towards large k. Consequently, the formation of the power-type Kolmogorov asymptotics $n(k) \propto k^{-9/2}$ in an infinite k-space will proceed up to the absorption region or to $k \simeq a^{-1}$ where the dispersion will cease to be small and (4.3.16) will become inapplicable.

It should also be noted that the presence of an external wave source, i.e., the addition of positive terms F_k (external force) or $\gamma_k n_k$ (the instability increment) to the right-hand side of (4.3.16) does not violate the inequality (4.3.18). Therefore, the stabilization of the stationary Kolmogorov spectrum should terminate at a finite time.

4.3.3 Numerical Simulations

Any realistic experiment, be it with the real or a model system or numerical, deals with a finite number of modes. This is due either to finiteness of the system and the discreteness of the medium or to strong damping of higher harmonics. With these considerations in mind, *Falkovich* and *Shafarenko* [4.22] carried out numerical simulations of (4.3.16) in two variants:

(i) for a closed system of L modes,

$$\begin{aligned}\frac{\partial n(k,t)}{\partial t} &= \sum_{l=1}^{k} l^2(k-l)^2\{n(l)n(k-l) - n(k)[n(l)+n(k-l)]\} \\ &\quad - 2\sum_{l=k}^{L} l^2(l-k)^2\{n(k)n(l-k) \\ &\quad - n(l)[n(k)+n(l-k)]\} = W_k ,\end{aligned} \tag{4.3.19a}$$

(ii) for an open system with wave drift [we set $n(k) \equiv 0$ at $k > L$],

$$\frac{\partial n(k,t)}{\partial t} = W_k - 2n(k)\sum_{l=L}^{L+k} l^2(l-k)^2 n(l-k) . \tag{4.3.19b}$$

The last term in (4.3.19b) implies that due to confluence processes waves drift into the region with $k > L$. It plays the role of nonlinear damping and provides an efficient energy sink, see (3.4.18).

The evolution of the closed system should lead to the equilibrium distribution [$n(k) = T/k$ at $k \gg 1$] with (4.3.19a) retaining the total energy

$$E = \sum_{k=1}^{L} k^3 n(k) .$$

For the total number of waves $N = \sum_{k=1}^{L} k^2 n(k)$ we have from (4.3.19a):

$$\frac{\partial N}{\partial t} = -4E^2 + \sum_{k=1}^{L} k^2 n(k) \sum_{l=L-k}^{L} l^2(k+l)^2 n(l) .$$

The last term is an "overlap integral" covering the regions $(1, L/2)$, $(L/2, L)$. Consequently, the law describing the decrease of N will deviate more and more from a linear rule as the waves are distributed over the whole interval,

In the open system (4.3.19b) the energy monotonically decreases

$$\frac{\partial E}{\partial t} = -2\sum_{k=1}^{L} k^3 n(k) \sum_{L-k}^{L} l^2(k-l)^2 n(l)$$

due to similar "overlap integrals".

Thus, if a wave packet was initially concentrated in the region $k \ll L$, its evolution will be the same for (4.3.19a) and for (4.3.19b), respectively, until the finiteness of k-space manifests itself. In a closed system, the waves will be accumulated near the right; in an open one the occupation numbers will decrease (as compared to the evolution in an infinite system) because of nonlinear damping. For those moments of time and regions of k-space for which the solutions (4.3.19a) and (4.3.19b) are close to each other, one can expect numerical experiments to be a good simulation of the behavior of waves in an infinite medium. In these simulations, the initial distribution was chosen to be localized in the region of small k

$$n(k,0) = \exp[-k^2/k_0^2] = \exp[-k^2/10]$$

and the equations (4.3.19) were solved numerically. The time derivative was approximated by the first difference

$$\frac{\partial n(k,t)}{\partial t} \approx \frac{n(k,t+\Delta t) - n(k,t)}{\Delta t}$$

where Δt was made sufficiently small $(5 \cdot 10^{-5} \div 2 \cdot 10^{-8})$ to ensure stability of the numerical procedure. The number of modes was chosen to be 200, 400, and 1000.

Figure 4.4 presents the evolution of $n(k,t)$ up to the time when the solutions (4.3.19a) and (4.3.19b) start to deviate from each other. At the moment $t^* = 5.3 \cdot 10^{-4}$ the Kolmogorov distribution transmitting the constant energy flux to

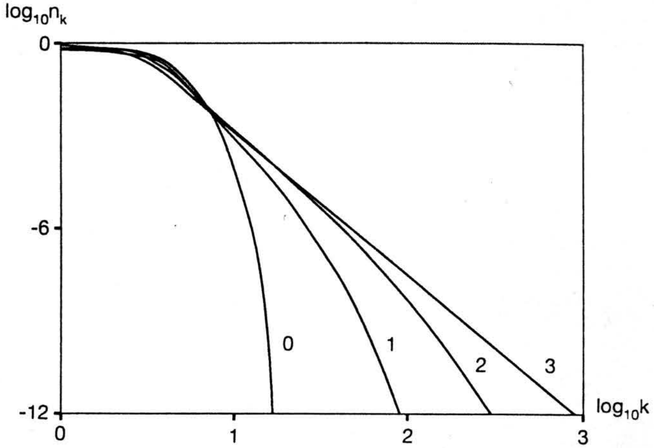

Fig. 4.4. The evolution of $n(k,t)$ is illustrated. The curves 0 to 4 represent $\ln n(k)$ at the moments of time 0, 10^{-4}, $3\cdot10^{-4}$ and $5.3\cdot10^{-4}$, respectively. The dotted line depicts the Kolmogorov power law $n(k)^0 = Ak^{-9/2}$

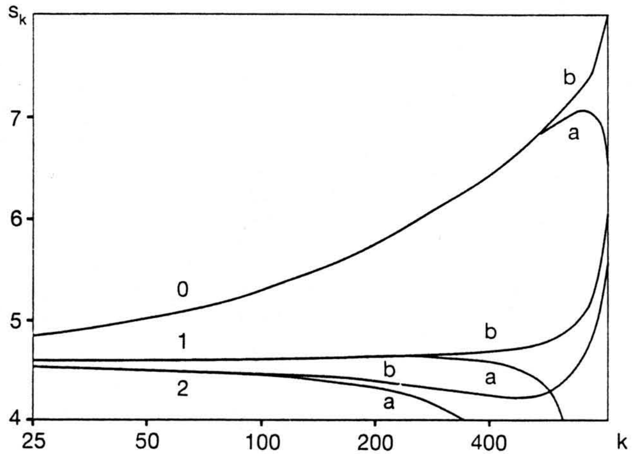

Fig. 4.5. The dependence of the local index of the spectrum on k is shown for different moments of time. The curves labeled 0, 1, and 2 correspond to the times $t = 4\cdot10^{-3}$, $t = 5.3\cdot10^{-4}$, and $t = 6\cdot10^{-4}$, respectively

large k is established almost over the whole interval of 1000 points. In detail this is illustrated in Fig. 4.5.

$$s(k) \equiv -\frac{\ln\big(n(k)/n(k+1)\big)}{\ln(k/k+1)} .$$

The labels "a" and "b" indicate that the corresponding curve are based on (4.3.19a) and (4.3.19b), respectively. An appreciable difference is noted between the behaviors of open and closed system when starting the evolution from $t \simeq t^*$. In the closed system, the distribution quickly becomes smoother [$s(k)$ decreases] and starting from the right end of the interval an equilibrium spectrum with $s(k) \equiv 1$ is formed.

Figure 4.6 shows the time dependence of all the three moments of the distribution function.

One can see that there are two well-separated evolution stages. Approximating the dependence $M_2^{-1}(t)$ by a dashed line we get $t^* = 5.32\cdot10^{-4} \approx [8.7M(0)]^{-1}$. The fact that $M_2^{-1}(t)$ does not go down to zero and the energy starts to decrease earlier than the time $t = t^*$ is a result of the finiteness of the system, see also Fig. 4.7.

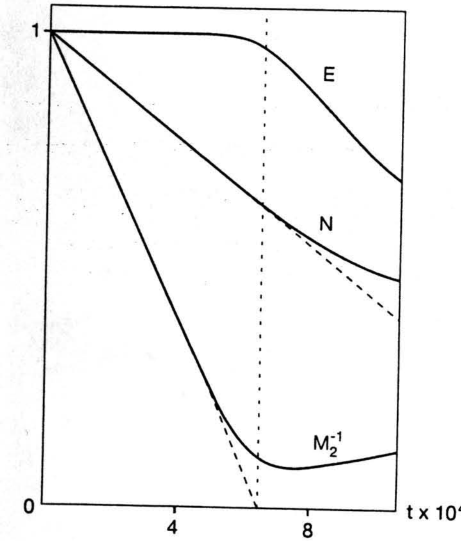

Fig. 4.6. The three moments of the distribution function are shown for (4.3.19b), $L = 1000$

Let us now verify the supposition formulated in Sect. 4.3.1, it indicates that the self-accelerated formation of Kolmogorov asymptotics involves the region $k \gg k_0$ at $\tau = t^* - t \ll t^*$. Equation (4.3.16) allows for a self-similar explosive-type substitution

$$n(k,t) = \tau^{-5p-1} f(k\tau^{-p}) = \tau^{-5p-1} f(z) \qquad (4.3.20)$$

One can expect the formation of self-similar asymptotics, provided that in the region $k \gg k_0$ the occupation numbers change much quicker than at $k \simeq k_0$. Then the self-similar part of the solution can be matched with the energy-containing region $k \simeq k_0$ via the quasi-stationary intermediate asymptotics. For these asymptotics to be of the Kolmogorov type, it is necessary that at $z \ll 1$ we have $f \propto z^{-9/2}$. From the steady state condition we get $p = -2$. Thus, the boundary of the region containing the Kolmogorov distribution should be specified by the condition $z \simeq 1$ and should become infinite during the finite time $k_b \propto (t^* - t)^{-2}$. The numerical simulations give a qualitative confirmation of such a behavior [4.22].

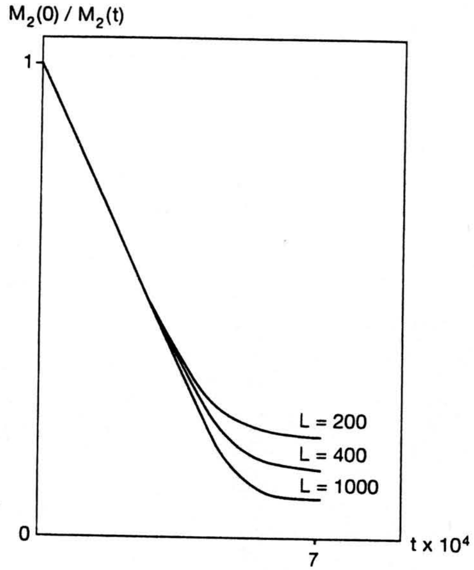

Fig. 4.7. The behavior of $M_2^{-1}(t)$ is illustrated for different L-modes

At $t \gg t^*$, the energy $E(t)$ approaches the power law $E \propto t^{-\beta}$ as illustrated in Fig. 4.8 showing the time dependence of

$$\beta = \frac{\partial \ln E}{\partial \ln t} \ .$$

According to our presumption, β should tend to unity with time which turns out to be consistent with Fig. 4.8.

Concluding this section, we shall consider two-dimensional weak sound turbulence [(3.2.3) at $d = 2$] as an example of a system with $h = 0$. The isotropic kinetic equation has a kernel which is polynomial in k, k_1

$$\begin{aligned} \frac{\partial n(k,t)}{\partial t} = 2\pi & \int_0^k (k - k_1)[n(k_1)n(k - k_1) \\ & - n(k)n(k_1) - n(k)n(k - k_1]k_1 dk_1 \\ - 4\pi & \int_k^\infty (k_1 - k)[n(k)n(k_1 - k) \\ & - n(k_1)n(k_1 - k) - n(k_1)n(k)]k_1 dk_1 \ . \end{aligned} \tag{4.3.21}$$

Since the averaged interaction coefficient is proportional to the first power of k, it is reasonable to consider the behavior of the zeroth and first moments of the distribution function $n(k)$. The first moment $\int_0^\infty kn(k)kdk = E$ is the energy which is conserved if there is no external damping. For the zeroth moment (the total number of waves), we can obtain from (4.3.21) [4.19] the expression

$$\begin{aligned} \frac{dN}{dt} &= 2\int_0^\infty kn(k)dk \int_0^\infty k'(k+k')n(k+k')dk' - 2NE \\ &= -\int_0^\infty dk \left(\int_k^\infty k'n(k')dk' \right)^2 \ . \end{aligned}$$

Now we see that $dN/dt \leq 0$ holds, i.e., the energy-containing scale $k_E = E/N$ of an arbitrary distribution is monotonically shifted to large k. On the other hand, $dN/dt \geq -2NE$, i.e., $N(t)$ does not decrease faster than by an exponential law. *Falkovich* and *Shafarenko* [4.22] examined the evolution of $n(k,t)$ in terms of a discrete open system (3.4.18) corresponding to (4.3.21) up to times when the energy is decreased by almost a factor of three with regard to the initial value. From Fig. 4.9 (with $L = 400$) one can see that there are two stages of evolution. During the first stage the energy is conserved and the Kolmogorov distribution forms until the absorption region is reached [in the given case $n(k) \propto k^{-3}$, see (3.2.5)].

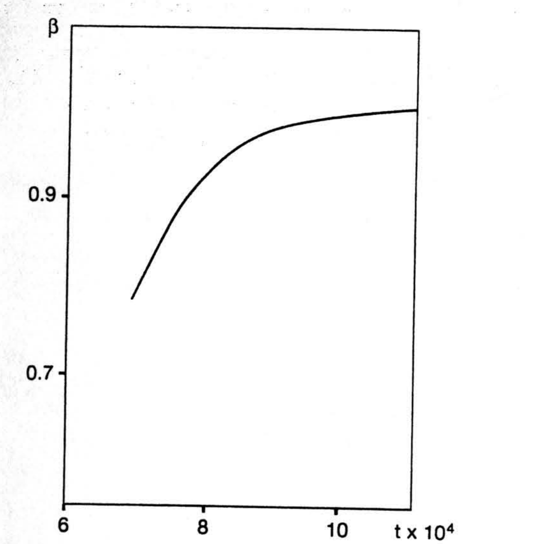

Fig. 4.8. The time dependence of β is illustrated

The time t_1 required to form a constant energy dissipation rate, grows with the size of the system approximately logarithmically ($L = 50$, $t_1 \simeq 5.70$; $L = 100$, $t_1 \simeq 8.05$; $L = 200$, $t_1 \simeq 10.70$; $L = 400$, $t_1 \simeq 13.49$), which is in agreement with (4.3.14). At $t \gg t_1$ the self-similar regime (4.3.4) is established and the energy decreases like t^{-1}. Such a regime persists until the energy concentrated in the region without external damping (approximately $k \leq L/2$) becomes comparable to the energy in the interval $k \geq L/2$. At this stage, the energy dissipation rate starts to decrease.

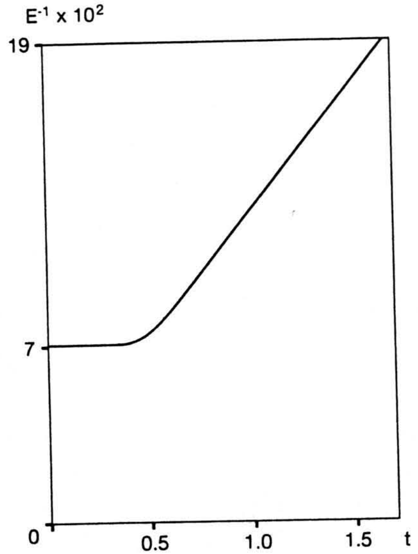

Fig. 4.9. Different stages of the evolution are depicted

5. Physical Applications

... a time to gather stones together

Ecclesiastes

Now it is time for our theory to bear fruit and for us to harvest. In this chapter we shall recollect the results concerning specific physical systems and shall obtain some new results by specializing general approaches from the previous chapters. We shall try to describe most hitherto known facts with regard to the spectra of developed weak turbulence. The general model of wave turbulence adopted in this book is based on the detailed consideration of the wave-wave interaction. The interaction with the external environment was described by the function $\Gamma(\boldsymbol{k})$ specifiying the decrement of wave attenuation or the growth-rate of wave instability. Nature is naturally more complex (for example, the interaction of water waves with wind and currents or wave-particle interactions in plasmas). Nevertheless, we believe the following fomulas and interpretations to provide a good basis for the further development of the theory of wave turbulence in various systems each of which might require a separate monograph for an adequate detailed description.

5.1 Weak Acoustic Turbulence

In this section we shall discuss wave turbulence with a near-sound dispersion law. Plenty of physical systems belong to this type. In spatially homogeneous media according to the Goldsone theorem, the wave frequency $\omega(k)$ should vanish together with the wave number k. In most cases the frequency expansion at small k starts from the first (i.e., linear) term. So the large-scale perturbations produce acoustic waves in solids, fluids, gases, and plasmas. The dispersion is supposed to be weak but sufficiently large to justify applicability of the kinetic equation and to be greater than dissipation. The magnitude of the nonlinear interaction coefficient and the sign of the dispersive frequency addition are different for various media. As repeatedly mentioned above, the properties of acoustic turbulence strongly depend on the sign of dispersion and the dimension of the space. Indeed, for positive dispersion three-wave interactions are allowed, while for negative dispersion one should take four-wave processes into account. Different kinetic equations are to be used in these cases. As far as the space dimension is concerned, we shall mention here only one fact to demonstrate the large difference between two- and three-dimensional cases. Considering acoustic turbulence