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Wave Turbulence Under Parametric Excitation

Applications to Magnets

With 69 Figures

3 Spin Waves (Magnons) in Magnetically Ordered Dielectrics

The previous two chapters were introductory. Chapter 1 treated the dynamic processes of nonlinear wave interactions within the scope of the classical Hamiltonian formalism. At that stage we refrained from analyzing the nature of particular waves (electromagnetic waves, sound waves, spin waves etc.), their polarization and the type of medium in which they propagate. All the information required for the study of nonlinear dynamics and kinetics of specific wave types under certain physical conditions and nonlinear media is contained in the Hamiltonian function of the waves, i.e. in the dispersion laws $\omega_j(\boldsymbol{k})$ and functions $V(q)$ and $T(p)$ describing the amplitudes of interaction. Therefore, in order to study the nonlinear behavior of spin waves (magnons) in magnets within the frame of the classical Hamiltonian method one should first of all obtain (or know) their Hamiltonian function, namely the law of dispersion of spin waves, and the amplitudes of their interaction. This chapter mainly presents the summarized results of calculation of the Hamiltonian functions of spin waves accompanied by short comments.

Sect. 3.1 deals with the magnons in ferromagnets, Sect. 3.2 treats the magnons in antiferromagnets. At the first stage of the study of the nonlinear magnons it is sufficient to read Sects. 3.1, 2 of this chapter. One can skip the details of the "internal structure" of the magnets which are given in Sects. 3.3, 4 and required for obtaining the Hamiltonian coefficients. For general information on magnets required for the further understanding of the physics of the considered nonlinear phenomena the reader is referred to the previous chapter. It must be emphasized that we can (as has been already mentioned) employ the Hamiltonian of a magnetic in the physics of nonlinear magnons irrespective of its origin because in the Hamiltonian approach the following two problems are quite independent. The first one is the investigation of the nonlinear behavior of a particular wave type (e.g. spin waves) through its known Hamiltonian function and the second one is the investigation of different classes of magnetodielectrics and possible types of interaction in order to obtain the Hamiltonian function of the spin subsystem.

Readers interested primarily in the problems of nonlinear physics of different waves of which spin waves are only one of the interesting examples can simply skip the Sects. 3.3-4 devoted to the calculation of the spin Hamiltonian. Likewise expert cooks can ignore the details of butchering or grape picking if they are satisfied with the quality of the foodstuffs. But the great chefs are not content with a mere standard, they seek perfection and know how to use the finest flavors depending on the way, time and place of the food product primary procession. It is for such "chefs" of physics that the last part of this chapter is intended.

It will be of interest also for the researchers using nonlinear processes as a method of studying the magnetodielectrics. Sections 3.3.1 gives and comments upon the "verbal" derivation of the equations of motion for magnetization known as "Bloch equations". Section 3.3.2 presents the relation between the observable medium variables (magnetization) and canonical variables. In these variables the Bloch equations are re-expressed as canonical Hamiltonian equations, and the expression for the energy of the spin subsystem of a magnetic (obtained in Sect. 2.3) becomes a Hamiltonian function. Afterwards it is formally simple to calculate the coefficients of the expansion of this function into a power series of normal canonical variables (i.e. to obtain the dispersion laws of the spin waves and the coefficients of their interaction Hamiltonian), although actually it is a rather awkward procedure. The general outline of these calculations is given in Sect. 3.4.

The readers who will set off through the obstacle race of this chapter with its heap of facts and formulae and get to the end must have their reward. First of all they will be proud to have won the battle with the chapter. Secondly, they may acquire a new viewpoint on the spin waves in magnetodielectrics as an interesting and relatively simple subject of inquiry in the physics of nonlinear wave phenomena. And they will attain an ultimate understanding of the fact that things are actually much more complicated than they seem. For example, the applicability range of the described approach to the nonlinear magnons under finite temperatures is yet to be found. We do not know the exact degree of error in obtaining the amplitudes of the interaction $V(q)$ and $T(p)$ at $T \simeq T_c/2$. There are other facts that are not clear. The detailed treatment of those fine and complicated problems is yet to be carried out. We can now only express our certainty that the simple Hamiltonian method of describing the dynamics of magnetodielectrics will reveal much necessary information about nonlinear magnons.

3.1 Hamiltonian of Magnons in Ferromagnets (FM)

As it has already been mentioned in Chap. 2 Yttrium Iron Garnet (YIG), which is a cubic ferromagnet, serves as a classic object of experimental studies in nonlinear dynamics and kinetics of magnons. At low temperatures the relative motions of magnetic sublattices (optical magnons) are not excited and YIG can be considered a ferromagnet. Therefore we shall begin our study with the spin Hamiltonian in ferromagnets with cubic symmetry. In

typical experiments the external magnetic field $\boldsymbol{H}$ is spatially homogeneous. Therefore we will study this particular case. Secondly, we can assume the sample shape to be ellipsoid. In this situation the magnetic field inside of sample $\boldsymbol{H}_1$ is also spatially homogeneous [3.1]:

$$H_{1.i} = H_1 - 4\pi \sum_j N_{ij} M_j \ . \tag{3.1.1}$$

Here $\boldsymbol{M}$ is the magnetization and N_{ij} is the tensor of demagnetizing coefficients; its trace is equal to unity. Most practicable are samples shaped as a sphere, long cylinder or thin discs. For the sphere, long (along the z-axis) cylinder and for the thin (along the z-axis) disc we have respectively

$$N_{ij} = \frac{1}{3}\delta ij \ , \tag{3.1.2}$$

$$N_z = N_{zz} = 0 \ , \quad N_{ij} = \frac{1}{2}\delta_{ij} \ , \quad i,j = x,y \ , \tag{3.1.3}$$

$$N_z = N_{zz} = 1 \ , \quad N_{ij} = 0 \ , \quad i,j = x,y \ . \tag{3.1.4}$$

And, finally, let us confine ourselves to the case when the field $\boldsymbol{H}$ is oriented along one of the principal axes of the ellipsoid which, in turn, coincides with one of the axes of symmetry of the crystal sample: 4-fold - [100], 3-fold [111] or 2-fold [110] axes. It is sufficient for the magnetization $\boldsymbol{M}$ in unexcited ferromagnets to be parallel to $\boldsymbol{H}$.

3.1.1 Spectrum of Magnons in Cubic Ferromagnets

The frequency of the magnons (obtained in Sect. 3.4 with the help of information from Sects. 1.1, 2) are:

$$\omega^2(\boldsymbol{k}) = A^2(\boldsymbol{k}) - |B(\boldsymbol{k})|^2 \ , \tag{3.1.5}$$

$$\begin{aligned} A(\boldsymbol{k}) &= \omega_{\mathrm{H}} - \omega_{\mathrm{M}} N_z + \alpha\omega_{\mathrm{a}} + \omega_{\mathrm{ex}}(ak)^2 + \frac{1}{2}\omega_{\mathrm{M}} \sin^2\Theta \ , \\ B(\boldsymbol{k}) &= \frac{1}{2}\omega_{\mathrm{M}} \sin^2\Theta \exp(2i\varphi) + \beta\omega_{\mathrm{a}} \ . \end{aligned} \tag{3.1.6}$$

Here Θ and φ are polar and azimuthal angles of the vector $\boldsymbol{k}$ in the spherical coordinates with z-axis oriented along $\boldsymbol{M}$, and the angle taken φ, which is respect to the direction [100] in the plane perpendicular to $\boldsymbol{M}$, ω_{M} is the circular frequency of the precession of the magnetic moment in the external field $\boldsymbol{H}$

$$\omega_{\mathrm{H}} = gH \ , \tag{3.1.7}$$

where $g = \mu_{\mathrm{b}}/\hbar$ is the magneto-mechanical ratio for the electron; μ_{b} is Bohr magneton; $g \simeq 2\pi \cdot 2.8$ MHz/Oe is a dimensional value. It must not be confused with the dimensionless g-factor approximately equal to 2 for an electron. The frequencies ω_M and ω_H characterize the magnetic dipole–dipole interaction and crystallographic anisotropy

$$\omega_M = 4\pi g M \ , \quad \omega_a = g H_a \ , \quad H_a = K_4 M^3 \ . \tag{3.1.8}$$

The field of anisotropy H_a is proportional to the conventional constant of the cubic anisotropy of the fourth order K_4, specified by (2.3.5). Coefficients α and β depend on the orientation of the magnetic field $\boldsymbol{H}$ (and the coinciding magnetization direction) with respect to the crystallographic axes (Table 3.1). The frequency ω_{ex} characterizes the exchange interaction, the order of magnitude $\hbar\omega_{ex}$ approximates T_c, where T_c is the temperature of magnetic ordering, usually called the Curie temperature, and a is the lattice constant. The exchange frequency ω_{ex} is proportional to the constant of nonhomogeneous exchange α_{ij} given by the formula (2.3.6). In cubic ferromagnets:

$$\alpha_{ik} = \alpha\delta_{ik} \ , \quad \omega_{ex} a^2 = 2gM\alpha \ . \tag{3.1.9}$$

Table 3.1. Coefficients characterizing the anisotropy energy contribution to the frequency of the magnons (3.1.5) and to the interaction amplitudes (3.1.25)

Magnetization orientation	α	β	δ
[100]	1	0	-9
[111]	-2/3	0	6
[110]	-1/2	-3/2	9/2

The specially selected notation of (3.1.5) makes it convenient to compare the contributions of different interactions to the spin wave frequency $\omega(\boldsymbol{k})$ (and later also to the amplitudes of H_{int}). The frequency $\omega_M = gH_M$ and magnetic field $H_M = 4\pi M$ can be used as convenient characteristic scales. For YIG at room temperature $4\pi M = 1.75$ kOe, $H_{ex} = \omega_{ex}$ kOe, $H_a = 0.084$ kOe. Analyzing experimental results it is useful to remember that the linear frequency associated with the field $H_a = 1$ kOe is $f = \omega/2\pi$, equal to $2.8 \cdot 10^9$ Hz, the circular frequency associated with the field is $1.76 \cdot 10^{10}$ s^{-1} and the associated temperature is 0.13 K. For the selected scale the cubic anisotropy is obviously small: $\omega_a/\omega_M = 0.05$. It is small also in some other cubic ferromagnets. In the first approximation such crystals can be assumed to be isotropic. This, however, does not imply that the crystallographic anisotropy can be completely neglected. Later some striking manifestations of nonlinear dynamics of magnons will be shown to be direct results of the presence of anisotropy.

The exchange frequency ω_{ex} significantly exceeds ω_M. For YIG $\omega_{ex}/\omega_M \simeq 220$. Formula (3.1.5) shows that the characteristic value of the wave vector

k_M can be determined at which the contributions of the dipole–dipole and exchange interactions become equal

$$\omega_{ex}(ak_M)^2 = \omega_M \ .$$

For YIG $ak_M = 0.07$ and correspondingly $k_M = 7 \cdot 10^5$ cm^{-1}. In usual experiments on parametric magnon excitation the pumping frequency approximates 10 or 36 GHz (1 GHz = 10^9 Hz), in this case the wave vector of the magnons may vary depending on the value of H, from $5 \cdot 10^3$ to $5 \cdot 10^3$ cm^{-1}. Consequently, both dipole–dipole and the exchange interactions are significant for us. The $\boldsymbol{k}$-dependence (3.1.5) of the magnon frequency $\omega(\boldsymbol{k})$ is rather complicated. It may be graphically shown (Fig. 3.1) as a family of surfaces with a constant frequency. Under $\omega(\boldsymbol{k}) \gg \omega_M$ their shape is approximately spherical:

$$\omega(\boldsymbol{k}) = \omega_H - \omega_M N_z + \omega_{ex}(ak)^2 + \frac{1}{2}\omega_M \sin^2\Theta \ . \tag{3.1.10}$$

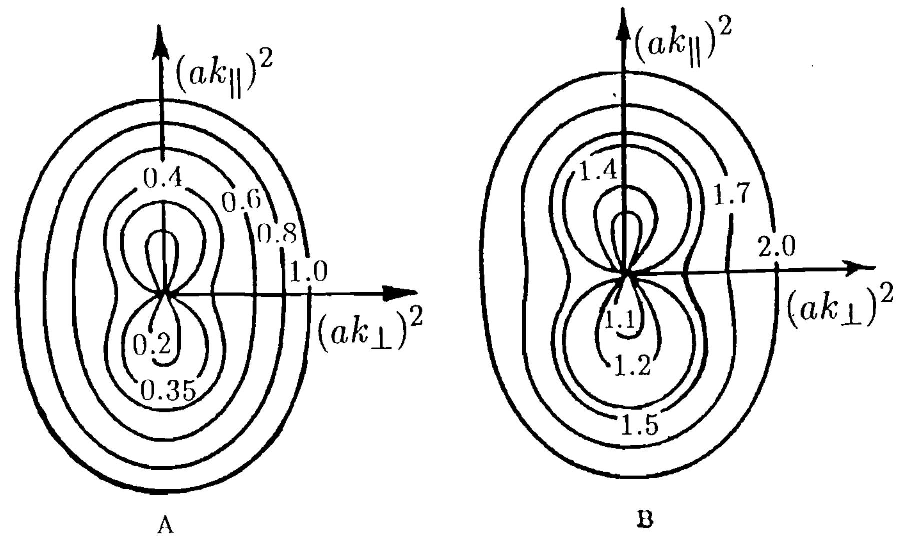

Fig. 3.1. Family of surfaces of constant frequency for an isotropic ferromagnet: the internal field $H_1 = H - 4\pi N_z M$ was chosen equal to one tenth of $H_M = 4\pi M$ (*a*); $H_1 = H_M$ (*b*). On the y-axis the value $(ak_{||})$, on the x-axis the value $(ak_\perp)$. The numbers on the curves designate the ratio $\omega(\boldsymbol{k})/\omega_M$

Under $\omega(\boldsymbol{k}) \simeq \omega_M$ a necking arises at the equator of the surface $\omega(\boldsymbol{k}) =$ const and it resembles a dumb-bell. Under further decrease of k this surface resembles two rounded cones with contacting points. Under $k \leq k_M$, as can be seen, the dependence of $\omega(\boldsymbol{k})$ on the direction $\boldsymbol{k}$ at $k \to 0$ plays an important role. This nonanalyticity is due to the long-range influence of the dipole–dipole interaction.

Concluding the analysis of the dispersion law of magnons (3.1.5) we should like to make two more remarks. First, there is a gap in the $\omega(\boldsymbol{k})$-dependence. The minimum frequency characterizes magnons with $k \to 0$, $\Theta = 0$. Neglecting the anisotropy

$$\omega_{\min} = \omega_H - \omega_M N_z = g(H - 4\pi N_z M) . \tag{3.1.11}$$

As can be seen from (3.1.1), this frequency corresponds to the precession of the magnetic moment in the internal field $\boldsymbol{H}_1$: $\omega_M = gH_1$. Under $H < 4\pi N_z M$ formula (3.1.11) is no longer valid. It follows from (3.1.5) in this case that $\omega^2(\boldsymbol{k})$ becomes negative under $k \to 0$, $\Theta = 0$ and, consequently, the frequency of magnons becomes purely imaginary. This in turn implies an exponential increase of amplitudes, since $a(\boldsymbol{k}, t) \sim \exp[-i\omega(\boldsymbol{k})t]$. Therefore, under $H < 4\pi N_z M$ the homogeneously magnetized state of the ferromagnet proves to be unstable and the sample is broken into domains.

The second remark concerns the applicability of (3.1.5). The formula holds true only when $kL \gg 1$, where L is the characteristic size of the sample. Under smaller k the very concept of a wave vector becomes meaningless, since the eigen-oscillation modes of the sample are no longer plane travelling waves. In ferromagnets such modes are traditionally called *Walker modes* (of magnetization oscillation). They are a solution of the magnetostatic equation

$$\operatorname{div}(\boldsymbol{H} + 4\pi \boldsymbol{M}) = 0 , \tag{3.1.12}$$

with corresponding boundary conditions on the surface of the sample [3.1, 2]. The simplest Walker mode is a homogeneous precession of the magnetization. It can be said that homogeneous precession is magnons with $\boldsymbol{k} = 0$. Its frequency ω_0 when the crystallographic anisotropy is neglected is given by

$$\omega_0^2 = \left[\omega_H - \omega_M N_z + \frac{1}{2}\omega_M(N + x + N_y)\right]^2 - \left[\frac{1}{2}\omega_M(N_x - N_y)\right] . \tag{3.1.13}$$

An especially simple expression can be obtained for a sphere: $\omega_0 = gH$. Expressions for the frequencies of other Walker modes can be found in [3.1].

3.1.2 Amplitudes of Three- and Four-Magnon Interaction

Earlier in Sect. 1.2.4 we estimated the amplitudes of four-wave interactions $T(\mathbf{12}, \mathbf{34})$ for the Heisenberg ferromagnet whose energy is given by the exchange Hamiltonian

$$T_{ex}(\mathbf{12}, \mathbf{34}) \simeq \omega_{ex}(ak)^2 g/2M \tag{3.1.14}$$

at $k_1 \simeq k_2 \simeq k_3 \simeq k_4 \simeq k$. This formula has been derived from (1.2.22) by substituting for α via ω_{ex} in accordance with (3.1.9).

Now let us find the contribution to the interaction Hamiltonian resulting from the magnetic dipole–dipole interaction. The energy of this interaction is determined only by the magnetization $\boldsymbol{M}$. In addition to $\boldsymbol{M}$, the dipole–dipole Hamiltonian can include the dimensional parameter g, specifying the dynamics of the electron spin system in the magnetic field; this parameter is the ratio of the magnetic moment of the electron to the mechanical moment (1.2.19). Using the two dimensional values M and g a value with dimensionality of frequency can be constructed uniquely: $[gM] = \mathrm{s}^{-1}$ (here, as in Sect. 1.2, square brackets denote the dimensionality of the enclosed value). As can be seen from (3.1.7) the frequency ω_M characterizing the value of the magnetic dipole–dipole interaction differs from this combination by a numeric factor 4π. This factor may be due to the fact that the oscillation of the magnetic moment with the amplitude $\boldsymbol{m}$ generates a magnetic field $\boldsymbol{h}$ with the amplitude $4\pi\boldsymbol{m}$.

Likewise by dimensional analysis we can estemate the contribution of the dipole–dipole interactions to the amplitudes of the 3-magnon interaction $V_M(\mathbf{1}, \mathbf{23})$ and to the 4-magnon interaction $T_M(\mathbf{12}, \mathbf{34})$. Out of the values M and g the combinations with dimensionality V and T can be constructed: $[V_M] = [g^{1/2}M^{1/2}]$, $[T_M] = [g^2]$. Hence the dimensional estimates for these amplitudes are

$$V_M \simeq \sqrt{gM} \to \omega_M\sqrt{g/2M} , \tag{3.1.15}$$

$$T_M \simeq g^2 \to \omega_M(g/2M) = 2\pi g^2 . \tag{3.1.16}$$

Behind the arrows more exact estimates of the interaction amplitudes are presented with the numerical factors obtained by calculations (Sect. 3.1.3).

Expressions (3.1.14) and (3.1.16) for T_{ex} and T_M on comparison manifest their similar structure: $T \simeq \omega_{eff}(g/2M)$ where ω_{eff} is the effective frequency of interactions equal to $\omega_{ex}(ak)^2$ for the exchange interaction and to ω_M for the magnetic dipole–dipole interaction. Evidently, the contribution of the energy of the crystallographic anisotropy (2.3.5) to the interaction Hamiltonian must be proportional to the constant of the anisotropy K_4 (2.3.5) or to the frequency of anisotropy ω_a (3.1.8). Therefore the expressions for the contribution of the crystallographic anisotropy energy V_a and T_a to the amplitudes of 3- and 4-magnon interactions V and T can be obtained from (3.1.15) and (3.1.16) by the substitution $\omega_M \to \omega_a$:

$$V_a \simeq \omega_a gM \to 0 , \tag{3.1.17a}$$

$$T_a \simeq \omega_a(g/M) \to \delta\omega_a(g/2M) . \tag{3.1.17b}$$

On the right of the arrow are the results of calculating these amplitudes. The interactions of the anisotropy do not generally result in 3-magnon processes so $V_a = 0$. The numerical factor δ depends on the orientation of the equilibrium magnetization with respect to the crystallographic axes. The coefficients δ for the three most symmetric orientations are presented in Table 3.1.

The obtained estimates of the interaction amplitudes are sufficient to calculate the magnon characteristics, showing the integral dependence on the amplitudes of the $\mathcal{H}_{\text{int}}$, e.g. temperature dependence of the magnetization, frequency and attenuation decrement of magnons. However, other nonlinear characteristics of the spin dynamics, which are discussed below, may depend significantly on the fine properties of the amplitudes of the $\mathcal{H}_{\text{int}}$, i.e., the details of their angular dependences, etc. Therefore we have no way out but to calculate $\mathcal{H}_{\text{int}}$ (not the most pleasant of procedures) taking into account all more or less significant interactions - exchange, dipole interactions, anisotropy, etc. The scheme of these calculations is given in Sects. 1.1, 2.3 and 3.4. Here only the results are presented.

3.1.3 Three-Magnon Hamiltonian

As the calculations show, the amplitudes of the Hamiltonian $\mathcal{H}_{\text{int}}$ given by the formula (1.1.25) have the following form:

$$\begin{aligned} V_{1,23} =& \frac{1}{2}[(V_3+V_2)u_1u_2u_3 + (V_2^*+V_3^*)v_1^*v_2v_3 \\ &+ v_1^*(V_2u_2v_3+V_3v_2u_3) + u_1(V_2^*v_2u_3+V_3^*u_2v_3)] \\ &+ (V_1v_1^*+V_1^*u_1)(u_2v_3+v_1u_3) , \end{aligned} \tag{3.1.18}$$

$$\begin{aligned} U_{123} =& \frac{1}{2}[V_1u_1(v_2u_3+u_2v_3) + V_2u_2(u_1v_3+v_1u_3) \\ &+ V_2u_3(v_1u_2+u_1v_2) + V_1^*v_1(v_2u_3+u_1v_2) \\ &+ V_2^*v_2(u_1v_3+v_1u_3) + V_3^*v_3(u_1v_2+v_1u_2)] , \end{aligned} \tag{3.1.19}$$

$$\begin{aligned} &V(\boldsymbol{k}) = -Vk_z(k_x+ik_v)/k^2 = -V\sin\Theta_k\cos\Theta_k\exp(i\varphi_k) , \\ &V = \omega_{\text{M}}(g/2M) , \quad V_j = V(\boldsymbol{k}_j) . \end{aligned} \tag{3.1.20}$$

Here u_k and v_k are the coefficients of the linear canonical diagonalizing transforms (1.1.19) where the frequency of the magnons $\omega(\boldsymbol{k})$ and coefficients $A(\boldsymbol{k})$ and $B(\boldsymbol{k})$ are given by (3.1.5, 6). The order of magnitude of $\tilde{V}(\mathbf{1},\mathbf{23})$ has been estimated above (3.1.15). Nevertheless the exact expressions (3.1.18–20) are essential for our further study. These give the angular dependences $\tilde{V}(\mathbf{1},\mathbf{23})$, and, in particular, reflect the fact that $\tilde{V}(\mathbf{1},\mathbf{23})$ vanishes if all the wave vectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$ are orientated along or across $\boldsymbol{M}$, i.e. $\Theta(\boldsymbol{k})=0$ or $\Theta(\boldsymbol{k})=\pi/2$.

3.1.4 Four-Magnon Interaction Hamiltonian

The general expression for the amplitudes of the $\mathcal{H}_4$ (1.1.26) is too complicated to be presented here. For this reason we shall study several limiting cases which will be of further use. Firstly we shall consider the situation when ω_{M} is small as compared to ω_{H} or $\omega_{\text{ex}}(ak)^2$. Here $A(\boldsymbol{k}) \gg |B(\boldsymbol{k})|$, thus the u,v-transformation in $\mathcal{H}_4$ is not necessary and one can take $a(\boldsymbol{k})=b(\boldsymbol{k})$. Then

$$\begin{aligned} W_{12,34} =& 2(E_{12}+E_{34}) + (C_{13}+C_{14}+C_{23}+C_{24}) \\ &- \frac{1}{2}(D_1+D_2+D_3+D_4) + W_{\text{a}} , \end{aligned} \tag{3.1.21}$$

$$\begin{aligned} &E_{kk'} = -\omega_{\text{ex}}a^2\boldsymbol{k}\cdot\boldsymbol{k}'g/2M , \quad C_{kk'} = C(\boldsymbol{k}-\boldsymbol{k}') , \\ &C(\boldsymbol{k}) = (g/M)\omega_{\text{M}}(k_z/k)^2 , \quad C(0) = \omega_{\text{M}}N_zg/M , \\ &D_k = |B_k|g/M = (g/M)\omega_{\text{M}}(k_+/k)^2 , \quad k_+ = k_x+ik_y , \\ &W_{\text{a}} = \delta\omega_{\text{a}}g/2M . \end{aligned} \tag{3.1.22}$$

Values for δ are given in Table 3.1.

$$G_{1,234} = -g(B_2+B_3+B_4)/3M . \tag{3.1.23}$$

Abandoning the assumption $A(\boldsymbol{k}) \gg |B(\boldsymbol{k})|$ we shall write the expressions for $W(\mathbf{12},\mathbf{34})$ in the case when the wave vectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$ satisfy the particular relations:

$$\begin{aligned} S_W(\boldsymbol{k},\boldsymbol{k}') =& W(\boldsymbol{k},-\boldsymbol{k};\boldsymbol{k}',-\boldsymbol{k}')/2 \\ =& W_1\left[u^2(\boldsymbol{k})u^2(\boldsymbol{k}')+v^2(\boldsymbol{k})v^2(\boldsymbol{k}')\right] + 4W_2u(\boldsymbol{k})u(\boldsymbol{k}')v(\boldsymbol{k})v(\boldsymbol{k}') \\ &- \frac{g}{2M}\Big\{v(\boldsymbol{k})u(\boldsymbol{k}')\big[u(\boldsymbol{k})u(\boldsymbol{k}')[B^*(\boldsymbol{k})+2B^*(\boldsymbol{k}')] \\ &+ v(\boldsymbol{k})v^*(\boldsymbol{k}')[B(\boldsymbol{k}')+2B(\boldsymbol{k})]\big] \\ &+ v(\boldsymbol{k}')u(\boldsymbol{k})\big[u(\boldsymbol{k})u(\boldsymbol{k}')\big(2B(\boldsymbol{k})+B(\boldsymbol{k}')\big) \\ &+ v(\boldsymbol{k})v(\boldsymbol{k}')\big(2B(\boldsymbol{k}')+B(\boldsymbol{k})\big)\big]\Big\} , \end{aligned} \tag{3.1.24a}$$

$$\begin{aligned} T_W(\boldsymbol{k},\boldsymbol{k}') =& \big[W(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{k},\boldsymbol{k}')+W(\boldsymbol{k},-\boldsymbol{k}';\boldsymbol{k},-\boldsymbol{k}')\big]/4 \\ =& 2W_1\operatorname{Re}\{u(\boldsymbol{k})u(\boldsymbol{k}')v(\boldsymbol{k})v(\boldsymbol{k}')\} \\ &+ W_2\left[u(\boldsymbol{k})^2+|v(\boldsymbol{k})|^2\right]\left[(u(\boldsymbol{k}))^2+|v(\boldsymbol{k}')|^2\right] \\ &- \frac{g}{2M}\left[(u(\boldsymbol{k})^2+|v(\boldsymbol{k})|^2\right]\operatorname{Re}\Big\{\left[B(\boldsymbol{k})+2B(\boldsymbol{k}')\right]v^*(\boldsymbol{k}')u(\boldsymbol{k})\Big\} \\ &+ (u(\boldsymbol{k}')^2+|v(\boldsymbol{k}')|^2)\operatorname{Re}\{[2B(\boldsymbol{k})+B(\boldsymbol{k}')]v^*(\boldsymbol{k})u(\boldsymbol{k}')\}] , \end{aligned} \tag{3.1.24b}$$

$$\begin{aligned} &W_1 = [\omega_{\text{ex}}(ak)^2-\omega_{\text{M}}]g/2M + W_{\text{a}} , \\ &W_2 = \omega_{\text{M}}(N_z-1)g/2M + W_{\text{a}} . \end{aligned} \tag{3.1.25}$$

Besides, an essential contribution to the amplitudes of the four-magnon interaction is made by the amplitudes of the 3-wave interaction in the second order of perturbation theory. The general expression for this contribution is given in Sect. 1.1.4. Now we are interested only in its particular cases at $\boldsymbol{k}_2 = -\boldsymbol{k}_1$, $\boldsymbol{k}_4 = -\boldsymbol{k}_3$ and $\boldsymbol{k}_2 = \boldsymbol{k}_3$, $\boldsymbol{k}_1 = \boldsymbol{k}_4$:

$$\begin{aligned} S_{12} =& S_W(\boldsymbol{k}_1,\boldsymbol{k}_2) - \frac{2U^*(\boldsymbol{k}_0,\boldsymbol{k}_1,-\boldsymbol{k}_1)U(\boldsymbol{k}_0,\boldsymbol{k}_2,-\boldsymbol{k}_2)}{\omega(\boldsymbol{k}_0)+2\omega(\boldsymbol{k})} \\ &- \frac{2\mathrm{Re}\{V^*(\boldsymbol{k}_0,\boldsymbol{k}_1,-\boldsymbol{k}_1)V(\boldsymbol{k}_0,\boldsymbol{k}_2,-\boldsymbol{k}_2)\}}{\omega(\boldsymbol{k}_0)-2\omega(\boldsymbol{k})} \\ &- \frac{4V(\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_5)V^*(\boldsymbol{k}_2;\boldsymbol{k}_1,-\boldsymbol{k}_5)}{\omega(\boldsymbol{k}_5)} \\ &- \frac{4V(\boldsymbol{k}_1;-\boldsymbol{k}_2,\boldsymbol{k}_6)V^*(\boldsymbol{k}_2;\boldsymbol{k}_1,-\boldsymbol{k}_6)}{\omega(\boldsymbol{k}_6)}\,, \end{aligned} \tag{3.1.26}$$

$$\begin{aligned} T_{12} =& T_W(\boldsymbol{k}_1,\boldsymbol{k}_2) - \frac{|U(\boldsymbol{k}_2,\boldsymbol{k}_1,-\boldsymbol{k}_6)|^2}{\omega(\boldsymbol{k}_6)+2\omega(\boldsymbol{k}_1)} \\ &- \frac{|U(-\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_5)|^2}{\omega(\boldsymbol{k}_5)+2\omega(\boldsymbol{k})} - \frac{|V(\boldsymbol{k}_6;\boldsymbol{k}_1,\boldsymbol{k}_2)|^2}{\omega(\boldsymbol{k}_6)-2\omega(\boldsymbol{k})} \\ &- \frac{|V(\boldsymbol{k}_5;-\boldsymbol{k}_1,\boldsymbol{k}_2)|^2}{\omega(\boldsymbol{k}_5)-2\omega(\boldsymbol{k})} - \frac{4\mathrm{Re}\{V(\boldsymbol{k}_1;\boldsymbol{k}_1,0)V(\boldsymbol{k}_2;\boldsymbol{k}_2,0\}}{\omega(\boldsymbol{k}_0)} \\ &- \frac{2[|V(\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_5)|^2+|V(\boldsymbol{k}_2;\boldsymbol{k}_1,-\boldsymbol{k}_5)|^2]}{\omega(\boldsymbol{k}_5)} \\ &- \frac{2[|V(\boldsymbol{k}_1;-\boldsymbol{k}_2,\boldsymbol{k}_6)|^2+|V(\boldsymbol{k}_2;-\boldsymbol{k}_1,\boldsymbol{k}_6)|^2]}{\omega(\boldsymbol{k}_6)}\,. \end{aligned} \tag{3.1.27}$$

Here $\boldsymbol{k}_5 = \boldsymbol{k}_1 - \boldsymbol{k}_2$, $\boldsymbol{k}_6 = \boldsymbol{k}_1 + \boldsymbol{k}_2$, ω_0 is the frequency of the homogeneous precession and it is assumed that $\omega(\boldsymbol{k}_1) = \omega(\boldsymbol{k}_2) = \omega(\boldsymbol{k})$. The second and third terms in (3.1.27) are due to the interaction of magnon pairs with $\pm\boldsymbol{k}_1$ and $\pm\boldsymbol{k}_2$ via "virtual" homogeneous precession; the last two terms are caused by the interaction via "virtual" magnons with $\boldsymbol{k} = \boldsymbol{k}_1 \pm \boldsymbol{k}_2$. The order of magnitude of the terms (3.1.26-27) is the same as that of the direct contribution of the dipole–dipole interaction to W, i.e. similar to the order of magnitude of πg^2 in full accordance with the estimation of (3.1.16). However, in an important specific case when $\boldsymbol{k}_1 \perp \boldsymbol{M}$ and $\boldsymbol{k}_2 \perp \boldsymbol{M}$ (i.e. $\Theta(\boldsymbol{k}_1) = \Theta(\boldsymbol{k}_2) = \pi/2$) terms of this order of magnitude make no contribution to $S(\boldsymbol{k},\boldsymbol{k}')$ and $T(\boldsymbol{k},\boldsymbol{k}')$ at all.

It must be noted that in (3.1.24–28) the frequency of the anisotropy ω_a is multiplied by the numerical factor 9 – 11 and at the same time ω_M is divided by two. Thus, the relative contribution of ω_a compared to ω_M becomes 20 times as great which leads to a strong crystallographic anisotropy of the four-wave interaction in "almost isotropic ferromagnets" with a small ratio $\omega_\mathrm{a}/\omega_\mathrm{M}$. Note that in the classical ferrimagnet YIG the ratio $\omega_\mathrm{a}/\omega_\mathrm{M}$ is about 1/20.

The expression for W includes the demagnetization factor N_z (z is the magnetization direction). As a result W is dependent on the shape of the sample and magnetization direction. Thus, for a normally magnetized disc $N_z = 1$; for a tangentially magnetized disc $N_z = 0$. All of the above-mentioned offers experimenters a rare opportunity to change at will the amplitudes of the interaction Hamiltonian of the magnons. For the benefit of the theoreticians cooperating with the experimenters one may even provide the knobs of the installation with special inscriptions notifying what particular coefficients of the Hamiltonian are being changed.

3.2 Hamiltonian Function of Magnons in Antiferromagnets

3.2.1 Magnon Spectrum in Antiferromagnets (AFM)

The simplest AFMs have two magnetic sublattices and accordingly two branches of magnon spectra. The quadratic part of the Hamiltonian is of a standard form (1.1.18):

$$\mathcal{H}_2 \doteq \sum_{\boldsymbol{k}}[\omega(\boldsymbol{k})a^*(\boldsymbol{k})a(\boldsymbol{k}) + \Omega(\boldsymbol{k})b^*(\boldsymbol{k})b(\boldsymbol{k})]\,. \tag{3.2.1}$$

It is given in order to define the designations of magnon frequencies in the two branches of spectra $\omega(\boldsymbol{k})$ and $\Omega(\boldsymbol{k})$ and of the normal canonical variables $a(\boldsymbol{k})$, $b(\boldsymbol{k})$ where the quadratic part of the Hamiltonian is diagonal. In uniaxial antiferromagnets with anisotropy of the "easy axis" type (AFM EA) the field of crystallographic anisotropy $\boldsymbol{H}_\mathrm{a}$ tends to maintain magnetization parallel to this axis (it is usually called the z-axis).

By analogy with ferromagnets the frequencies of magnons with $k \to 0$ may be conjectured to correspond to the frequencies of the precession of sublattices magnetization in the field $\boldsymbol{H}_\mathrm{a}$, i.e. $\omega_0 = \Omega_0 = \omega_\mathrm{a}$, where

$$\omega_\mathrm{a} = gH_\mathrm{a}\,. \tag{3.2.2}$$

This, however, is wrong. Upward oriented magnetization of one sublattice $\boldsymbol{M}_1$ is influenced by the field of anisotropy $\boldsymbol{H}_{a1}$, also directed upwards: $\boldsymbol{H}_{a1} = \boldsymbol{H}_1$. Another field ($\boldsymbol{H}_{a2} = -\boldsymbol{H}_\mathrm{a}$) acts upon the other sublattice with $\boldsymbol{M}_2 = -\boldsymbol{M}_1$ directed downward. As a result the sublattices "desire" to be oppositely precessed. In this case antiparallelism of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ is inevitably violated, impeded by a strong exchange interaction between the sublattices. As a result it turns out that [3.3]

$$\omega_0^2 = \Omega_0^2 = (\omega_\mathrm{a} + \omega_\mathrm{ex})^2 - \omega_\mathrm{ex}^2 \simeq 2\omega_\mathrm{a}\omega_\mathrm{ex}\,, \tag{3.2.3}$$

where the frequency $\omega_{ex} = gH_{ex}$ characterizes the value of the antiferromagnetic exchange between sublattices. Sometimes the dimensionless "exchange constant" B acts as such a characteristic: $H_{ex} = BM_{ex}$.

If the external magnetic field is applied to the AFM EA along the main axis, the equivalence of the sublattices bringing about the coincidence of the frequencies ω_0 and Ω_0 is violated. Here

$$\begin{aligned} \Omega_0 &= \sqrt{2\omega_a\omega_{ex}} + \omega_H \ , \\ \omega_0 &= \sqrt{2\omega_a\omega_{ex}} - \omega_H \ , \quad \omega_H = gH \ . \end{aligned} \tag{3.2.4}$$

In the critical field $H_{cr} = \sqrt{2H_aH_{ex}}$ the frequency ω_0 becomes zero and the instability of magnons increases, which leads to a changed ground state, i.e. to the overturning of the sublattices [3.1].

In the uniaxial antiferromagnets with the anisotropy of the "easy plane" type (AFM EP) the anisotropy develops the moments $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ into the plane perpendicular to the axis $\boldsymbol{c}$. Since the moments in this plane can oscillate almost freely one of the magnon frequencies under $k \to 0$ proves to be small - of the order of gH. Calculations following the below described procedure show

$$\begin{aligned} \omega^2(\boldsymbol{k}) =& \omega_H(\omega_H + \omega_D) + 2\omega_{ex}(\omega_n + \omega_{ph}) + v_1^2k_z^2 + v_2^2k^2 \ , \\ \Omega^2(\boldsymbol{k}) =& 2\omega_a\omega_{ex} + \omega_D(\omega_H + \omega_D) \\ &+ 2\omega_{ex}(\omega_n + \omega_{ph}) + v_1^2k_z^2 + v_2^2k^2 \ . \end{aligned} \tag{3.2.5}$$

Here $\boldsymbol{H}$ is perpendicular to the direction of the main axis; v_1 and v_2 are the longitudinal and lateral velocities of magnons with order of magnitude $v_{1,2} \simeq \omega_{ex}a$, where a is the lattice constant; $\omega_D = gH_D$, H_D is the Dzyaloshinsky field due to the specific relativistic interaction between the sublattices. The value of H_D is usually of the order of several kOe. In some AFM because of symmetry $H_D = 0$.

Table 3.2. Phenomenological constants determining magnon frequencies in antiferromagnets with the anisotropy of the "easy plane" type

Crystal	H_{ex}, kOe	H_a, kOe	H_D, kOe	H_Δ^2, (kOe)^2
$MnCO_3$	320	3.04	4.4	$5.8/T$(K)
$CsMnF_3$	350	2.48	0	$6.4/T$(K)+0.3
$FeBO_3$	3000	5.3	100	4.9

The two terms in (3.2.5) $\omega_{ex}\omega_n$ and $\omega_{ex}\omega_{ph}$ are due to the smallness of respective electron exchange with nuclear spins and crystal lattice. However, in (3.2.5) these frequencies are multiplied by the greatest frequencies ω_{ex}, and the contribution of these small interactions to frequency $\omega(\boldsymbol{k})$ may

appear quite significant. This phenomenon in AFM is traditionally named the *exchange amplification of small interactions* AFM. The frequency ω_n corresponds to the precession of the magnetic moment of an electron in an effective magnetic field $\boldsymbol{H}$ due to hyperfine interaction of electrons with the nucleus

$$\omega_n = gH_n = A < I > /\hbar \ . \tag{3.2.6}$$

A designates the constant of the superfine interaction, $< I_z >$ is the mean value of the nuclear spin inversely proportional to the temperature. For manganese nuclei, for instance, $H_\Delta \simeq (10/T)$ Oe and "the nuclear gap" $H_\Delta = \sqrt{2H_{ex}H_n}$ is of the order of several kOe under $T = 4$ K (Table 3.2). The question may arise as to the cause of the gap in the magnon spectrum, which is equal to gH_Δ, if the problem in this case has an axial symmetry with respect to the rotation of the movements of sublattices in the easy plane. The answer is as follows: The frequency of the nuclear spin precession ω_n is much smaller than the frequencies of the electron subsystem precession gH_Δ, gH_D... Therefore, nuclear spins do not partake in the movement of the electron spins. They are arranged along the mean values of magnetizations of the sublattices. Therefore, the axial symmetry of the crystal is broken which gives rise to effective magnetic fields $\boldsymbol{H}_{n1} = \boldsymbol{H}_n$ and $\boldsymbol{H}_{n2} = -\boldsymbol{H}_n$ acting respectively on the sublattices $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$. These fields by (3.2.3) for AFM EA bring about electron spin precession with frequency $\sqrt{2\omega_{ex}\omega_n}$. The same is true about magnetostriction leading to the uniaxial deformation of the crystal which fails to follow the precession of the sublattice moments. This deformation in its turn results in the emergence of effective fields $\boldsymbol{H}_{ph1} = -\boldsymbol{H}_{ph2} = \boldsymbol{H}_{ph}$ in the easy plane. The field $\boldsymbol{H}_{ph}$ is small; it is proportional to the square of the magnetostriction constant. The exchange amplification may result in a considerable contribution of the magnetostriction to the magnon frequency.

In the cubic AFM (e.g. in $RbMnF_3$) the expressions for magnon frequencies have the form

$$\begin{aligned} \omega^2(\boldsymbol{k}) =& 3\omega_a\omega_{ex} + \omega_\Delta^2 + v^2k^2 \ , \quad \omega_\Delta = gH_\Delta \ , \\ \Omega^2(\boldsymbol{k}) =& \omega^2(\boldsymbol{k}) - (3/2)\omega_a\omega_{ex} + \omega_\Delta^2 + v^2k^2 \ . \end{aligned} \tag{3.2.7}$$

It is assumed here that $\boldsymbol{H}\|[100]$ and its value is greater than the overturn field $\sqrt{(3/2)H_aH_{ex}}$.

In all the expressions for magnon frequencies in AFM the dipole–dipole interaction has not been allowed for. The external magnetic field and relativistic interactions result in the small magnetic moment in AFM $\boldsymbol{M} = \boldsymbol{M}_1 + \boldsymbol{M}_2$. Accordingly, the relative contribution of the dipole–dipole interaction to the magnon frequencies as a rule proves to be H_{ex}/H times smaller; this factor amounts to $10^3 \div 10^4$ for most AFM. However this interaction can prove essential when studying fine nonlinear properties of antiferromagnetic magnons.

3.2.2 Interaction Hamiltonian in "Easy Plane" Antiferromagnets

Experiments on nonlinear properties of magnons in the "easy plane" antiferromagnets (AFM with the anisotropy of "the easy-plane" type) and in cubic AFM are of particular interest due to low-lying magnon branches whose frequencies are in the convenient range below 50 gHz. The calculation of the Hamiltonian is the simplest for AFM EP where the second branch of magnons lies much higher than the other one: $\Omega(\boldsymbol{k}) \gg \omega(\boldsymbol{k})$. For AFM EP we shall write the results of such calculations for the part of $\mathcal{H}_{\text{int}}$ of great interest for us:

$$\begin{aligned}\mathcal{H}_3 = &\sum_{1=2+3} [\frac{1}{2}V_q^{(1)} b_1 a_2^* a_3^* + V_q^{(2)} a_1 b_2^* a_3^* + \text{c.c.}] \\ &+\frac{1}{2}\sum_{1+2+3=0} [U_q b_1^* a_2^* a_3^* + \text{c.c.}] + \ \dots\ , \quad q = (\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\ ,\end{aligned} \tag{3.2.8}$$

$$\mathcal{H}_4 = \frac{1}{2}\sum_{1+2=3+4} W_p a_1^* a_2^* a_3^* a_4\ , \quad p = (k_1, k_2, k_3, k_4)\ , \tag{3.2.9}$$

$$\begin{aligned}V_q^{(1)} &= -\frac{\sqrt{g\omega_{\text{ex}}}}{4\sqrt{M\omega_2\omega_3\Omega_1}}\omega_{\text{H}}(\Omega_1 + \omega_2 + \omega_3)\ , \\ V_q^{(2)} &= -\frac{\sqrt{g\omega_{\text{ex}}}}{4\sqrt{M\omega_1\omega_3\Omega_2}}\omega_{\text{H}}(\omega_1 - \Omega_2 + \omega_3)\ , \\ U_q &= -\frac{\sqrt{g\omega_{\text{ex}}}}{4\sqrt{M\omega_2\omega_3\Omega_1}}\omega_{\text{H}}(\Omega_1 - \omega_2 - \omega_3)\ ,\end{aligned} \tag{3.2.10}$$

$$\begin{aligned}W_p =& \frac{9\omega_{\text{ex}}}{4M\sqrt{\omega_1\omega_2\omega_3\omega_4}}[\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 \\ &-\omega_{1-3}^2 - \omega_{2-3}^2 - \omega_{1+2}^2 - \omega_{3+4}^2]\ .\end{aligned} \tag{3.2.11}$$

As before, we used the shorthand notation

$$\Omega_j = \Omega(\boldsymbol{k}_j)\ , \quad \omega_j = \omega(\boldsymbol{k}_j)\ , \quad \omega_{i-j} = \omega(\boldsymbol{k}_i - \boldsymbol{k}_j)\ .$$

It should be recalled that for describing scattering processes of the $2 \to 2$ type not only the Hamiltonian $\mathcal{H}_4$ is to be taken into consideration, but also the Hamiltonian $\mathcal{H}_3$ in the second order of perturbation theory. It can be shown that the contribution of the processes with the participation of the virtual waves from the upper branch is as great as others in $\omega_{\text{ex}}/\omega(\boldsymbol{k})$ times. The Hamiltonian of the processes is explicitly written in (3.2.8). Using (1.1.32) we obtain the expression for the effective amplitudes of 4-magnon processes

$$\begin{aligned}T_{\text{p}} =& W_{\text{p}} - \frac{V^{(1)}(\mathbf{5};\mathbf{1},\mathbf{2})V^{(1)*}(\mathbf{5};\mathbf{3},\mathbf{4})}{\omega(\boldsymbol{k}_1+\boldsymbol{k}_2) - 2\omega(\boldsymbol{k})} \\ &+ \frac{U(-\mathbf{5},\mathbf{2},\mathbf{1})U^*(-\mathbf{5},\mathbf{3},\mathbf{4})}{\omega(\boldsymbol{k}_5) + 2\omega(\boldsymbol{k})} \\ &+ \frac{V^{(2)}(\mathbf{1};\mathbf{6},\mathbf{3})V^{(2)*}(\mathbf{4};\mathbf{6},-\mathbf{2}) + V^{(2)*}(\mathbf{3};-\mathbf{6},\mathbf{1})V^{(2)}(\mathbf{2};\mathbf{6},\mathbf{4})}{\omega(\boldsymbol{k}_1 - \boldsymbol{k}_3)} \\ &+ \frac{V^{(2)}(\mathbf{2};\mathbf{7},\mathbf{1})V^{(2)*}(\mathbf{4};\mathbf{6},\mathbf{2}) + V^{(2)*}(\mathbf{3};\mathbf{6},-\mathbf{1})V^{(2)}(\mathbf{1};-\mathbf{7},\mathbf{2})}{\omega(2\boldsymbol{k})}\ .\end{aligned} \tag{3.2.12}$$

In this expression $\boldsymbol{k}_5 = \boldsymbol{k}_1 + \boldsymbol{k}_2$, $\boldsymbol{k}_6 = \boldsymbol{k}_1 - \boldsymbol{k}_3$, $\boldsymbol{k}_7 = \boldsymbol{k}_2 - \boldsymbol{k}_1$. From (3.2.10–12) the diagonal contribution to T_p due to $\mathcal{H}_3$ is of the same order of magnitude as W_p. The general expression for T is rather cumbersome. We shall write it for the most interesting case when $\omega(\mathbf{1}) = \omega(\mathbf{2}) = \omega(\boldsymbol{k})$ and $\boldsymbol{k}_1 = \boldsymbol{k}_3$, $\boldsymbol{k}_2 = \boldsymbol{k}_4$ or $\boldsymbol{k}_1 + \boldsymbol{k}_2 = 0$:

$$\begin{aligned}&S(\boldsymbol{k},\boldsymbol{k}') = T(\boldsymbol{k},-\boldsymbol{k};\boldsymbol{k}',-\boldsymbol{k}')/2\ , \quad T(\boldsymbol{k},\boldsymbol{k}') = T(\boldsymbol{k}\boldsymbol{k}',\boldsymbol{k}\boldsymbol{k}')/2\ , \\ &S(\boldsymbol{k},\boldsymbol{k}') = T(\boldsymbol{k},\boldsymbol{k}') = \\ &\qquad -\frac{g^2\omega_{\text{ex}}}{8M\omega^2(\boldsymbol{k})}\left[\omega_0^2 + \omega_{\text{H}}^2\frac{3\Omega_0^2 - 4\omega^2(\boldsymbol{k})}{\Omega_0^2 - 4\omega^2(\boldsymbol{k})}\right]\ .\end{aligned} \tag{3.2.13}$$

3.2.3 Nuclear Magnons in "Easy Plane" Antiferromagnets

The concept of *Nuclear Magnons* (NM) was first put forward by *de Gennes* et al. [3.3]. NM are the collective oscillations of spins of nuclei, e.g. Mn nuclei in AFM $MnCO_3$. There is, of course, no direct exchange interaction between the nuclear spins and their magnetic dipole interaction is negligibly small because of the smallness of the magnetic moment of the nuclei. Nuclear spin motion becomes collective as a result of the indirect exchange interaction described by *Suhl* and *Nakamura* in 1958. It arises in the second order of perturbation theory in the hyperfine interaction (SFI) of nuclear spins with electron spins. The spin deflection of the i-th nucleus deflects under SFI the electron spin of the i-th atom; this perturbation propagates in the electron spin system and, in its turn, acts on other nuclear spins through SFI. The resulting correlation of the nuclear spin motion is the nuclear magnon. A detailed presentation of theoretical concept and experimental data on NM is given, for example, by *Tulin* [3.4].

In two-sublattice AFM EP there are two branches of electron magnons $\omega_e(\boldsymbol{k})$ and $\Omega_e(\boldsymbol{k})$, and, respectively, two NM branches $\omega_{\text{n}}(\boldsymbol{k})$ and $\Omega_{\text{n}}(\boldsymbol{k})$:

$$\omega_{\text{n}}^2(\boldsymbol{k}) = \Omega_{\text{n}}^2(0)[1 - \Delta^2/\omega_e^2(\boldsymbol{k})]\ , \tag{3.2.14}$$

$$\Omega_{\text{n}}^2(\boldsymbol{k}) = \Omega_{\text{n}}^2(0)[1 - \Delta^2/\Omega_e^2(\boldsymbol{k})]\ . \tag{3.2.15}$$

$\omega_{\text{n}}^2(0)$ here designates the frequency of the magnetic resonance in the hyperfine field of the electron ($\Omega_{\text{n}}(0)/2\pi$ for AFM $CsMnF_3$, $MnCO_3$ and $RbMnF_3$

equals, respectively, 666 MHz, 640 MHz and 686 MHz), $\Delta^2 = 2\omega_{\mathrm{ex}}\omega_{\mathrm{n}}$, where ω_{ex} is the frequency of the inter-sublattice exchange and ω_{n} is the precession frequency of the electron magnetic moment in the effective magnetic field due to SFI and to the ordering of nuclear spins

$$\omega_{\mathrm{n}} = gH_{\mathrm{n}} = A < I_z > /\hbar \simeq AH/T .$$

The value Δ is the nuclear gap (which is amplified by exchange) in the spectrum of electron magnons (see (3.2.6) and Table 3.2).

The Hamiltonian of NM interaction is of no use for us and will therefore not be described here.

3.3 Comments at the Road Fork

Dear reader, dear colleague! Before we go further, let us take a short rest and look back: in Chap. 1 we got to know the classical Hamiltonian formalism for nonlinear waves of arbitrary nature and wrote the Hamiltonian equations of motion for the weakly nonlinear waves in almost conservative media. We understood that all the required information (and almost nothing superfluous) is contained in the Hamiltonian of the nonlinear wave system. The second chapter briefly outlined the general information on the physics of magnetodielectrics useful when discussing the nonlinear behavior of their spin system. As for the Hamiltonian function of the magnons in ferro-, antiferro- and ferrimagnets it was presented in Sects. 3.1–3 in a form sufficient for further consideration and to start the study of the nonlinear dynamics of magnons in Chap. 4. Thus, the end of the present Chapter (Sect. 3.4) can be skipped by those readers who are not interested in the recipes of the magnetic "cuisine". We invite others to consider in the following section the equations of motion for the spin waves and to study the simplest way of calculating the magnon Hamiltonian.

3.4 Calculation of Magnon Hamiltonian

3.4.1 Equation of Motion of Magnetic Moment

To derive consistently equations of motion of the magnetic moment $\boldsymbol{M}(\boldsymbol{r},t)$ at finite temperatures is an extremely complicated task even for the long-wave variations. To solve this problem we would have to determine the temperature dependences of the amplitudes of interaction, magnon frequencies, to find the magnon damping and to obtain a host of other useful data. If we had made this attempt strictly and consistently we would have got stuck in the very beginning and given up our ultimate aim, i.e. to obtain a clear

and simple method for describing nonlinear dynamics of the magnons under low and intermediate temperatures. Consequently, we shall make several assumptions difficult to check but simple and highly "natural" and therefore traditional. Let us cite from [3.1], p.44: "Owing to strong exchange interaction between the spins of individual atoms of the ferromagnet its magnetic moment is in good approximation "rigid" if only the ferromagnetic temperature is sufficiently low". In other words, the modulus of the magnetic moment density vector can, although very weakly, depend on time. Therefore, for the first approximation the time variation in the density of the magnetic moment must to be of precession character, i.e. must obey the law

$$\partial \boldsymbol{M}(\boldsymbol{r},t)/\partial t = g[\boldsymbol{H}_{\mathrm{eff}}(\boldsymbol{r},t) \times \boldsymbol{M}(\boldsymbol{r},t)] , \tag{3.4.1}$$

where $\boldsymbol{H}_{\mathrm{eff}}(\boldsymbol{r},t)$ is a certain vector which will be henceforth referred to as the *effective magnetic field.* Likewise, at the accepted level of rigor, we can assert that the equations of motion (3.4.1) must be conservative, i.e. conserve the total energy of the ferromagnet W. Indeed, if in our model there is nothing but the magnons described by (3.4.1), their total energy must be conserved. Since W is a functional of magnetization, we have that

$$\frac{dW}{dt} = \int \frac{\delta W}{\delta \boldsymbol{M}(\boldsymbol{r},t)} \cdot \frac{\partial \boldsymbol{M}(\boldsymbol{r},t)}{\partial t} d\boldsymbol{r} .$$

Substituting here $\partial \boldsymbol{M}/\partial t$ from the equations of motion (3.4.1), we have

$$\frac{dW}{dt} = g \int \frac{\delta W}{\delta \boldsymbol{M}} \cdot [\boldsymbol{M} \times \boldsymbol{H}_{\mathrm{eff}}]) d\boldsymbol{r} .$$

For this integral to be zero under any dependence $\boldsymbol{M}(\boldsymbol{r})$, the vector $\boldsymbol{H}_{\mathrm{eff}}$ must be parallel to the vector $\delta W/\delta \boldsymbol{M}$. The proportionality coefficient can be found by considering some simple situation with the present equation of motion, for example, the uniform precession of the magnetic moment in a homogeneous field $\boldsymbol{H}$. Then W is the energy of the interaction with the external field

$$W = -\int \boldsymbol{H} \cdot \boldsymbol{M}(\boldsymbol{r}) d\boldsymbol{r} . \tag{3.4.2}$$

Calculating the functional derivative by the rule (A 1.2) we obtain

$$\delta W/\delta \boldsymbol{M} = -\boldsymbol{H} . \tag{3.4.3}$$

It implies that the sought-for factor is minus one

$$\boldsymbol{H}_{\mathrm{eff}} = -\delta W/\delta \boldsymbol{M} . \tag{3.4.4}$$

Therefore we obtain the equations of motions

$$\frac{\partial \boldsymbol{M}(\boldsymbol{r},t)}{\partial t} = g[\frac{\delta W}{\delta \boldsymbol{M}} \times \boldsymbol{M}] \,. \tag{3.4.5}$$

Certainly, the easy attitude to manipulations leading to (3.4.5), an attitude I share with the authors of [3.1], by no means guarantees their correctness. All of us took great pains to sweep under the carpet many delicate and complicated problems. Some of them will be discussed later. Now we shall proceed from (3.4.5) assuming that they give a good first approximation for the description of the magnons (spin waves).

3.4.2 Canonical Variables for Spin Waves in Ferromagnets (FM)

In (3.4.5) we shall pass to the circular variables

$$M_\pm = M_x \pm iM_y \,, \quad M_z^2 = M_0^2 - M_+M_- \,,$$

$$\partial M_+(\boldsymbol{r},t)/\partial t = 2igM_z(\boldsymbol{r},t)(\delta W/\delta M_-) \,. \tag{3.4.6}$$

The z-axis direction should be selected along the equilibrium value of the magnetization. Then at small variations in amplitudes of the magnetic moment the M_+-values will be small and M_z will approach the length of the vector M, i.e. M_0. Comparison of (1.1.7) and (3.4.6) shows that these equations, in the approximation linear in M, have the Hamiltonian form if the following values a and a^* are taken as canonical variables:

$$M_+/\sqrt{2gM_0} \,, \quad M_-/\sqrt{2gM_0} \,.$$

Therefore, the canonical variables should better be sought in the following form:

$$M_+ = af(a^*a)\sqrt{2gM_0} \,, \quad M_- = a^*f(a^*a)\sqrt{2gM_0} \,, \tag{3.4.7}$$

where $f(0) = 1$. Substituting (3.4.7) into (3.4.6) we obtain an equation for $a(\boldsymbol{r},t)$ which must coincide with the canonical one (1.1.7). The resulting differential equation for $f(|a|^2)$

$$f^2(x) + 2xf(x)[df(x)/dx] = 2g\sqrt{M_0^2 - xf^2(x)}$$

has a unique solution satisfying the condition $f(0) = 1$,

$$f(x) = \sqrt{1 - gx/2M_0} \,.$$

Thus we have expressed the "natural variables" of ferromagnets $\boldsymbol{M}$ in terms of the canonical variables

$$M_+ = a\sqrt{2gM_0(1 - ga^*a/2M_0)} \,,$$

$$M_- = M_+^* \,, \quad M_z = M_0 - ga^*a \,. \tag{3.4.8}$$

The energy of the ferromagnet W expressed in terms of the canonical variables becomes the Hamiltonian function (Hamiltonian) $\mathcal{H}(a^*,a)$. It can be seen easily that the transformation (3.4.8) is a classical analog of the Holdstein-Primakoff transformation [3.1]. They were first employed in 1960 for analyzing the nonlinear dynamics of magnons by *Schlömann* et al. [3.5].

This choice of the canonical variables is certainly not the only possible one. Thus, it can be shown (see Problem 3.1) that for (3.4.5) there exist other canonical variables and in terms of these variables the vector $\boldsymbol{M}$ could be expressed as follows

$$M_z + iM_x = M_0\sqrt{1 + g|b^* - b|^2/2M_0}\exp\{i(b^* + b)\sqrt{g/2M_0}\} \,,$$

$$M_y = i(b^* - b)\sqrt{gM_0/2} \,. \tag{3.4.9}$$

These formulations are a classical analogue of the representation of the spin operator in terms of Bose-operators suggested by *Bar'yakhtar* and *Yablonsky* [3.6].

Comparison of (3.4.8) and (3.4.9) shows that the variables of Holstein-Primakoff a, a^* and Bar'yakhtar and Yablonsky b, b^* coincide in the linear approximation. The advantages of the various representations the magnetization in terms of canonical variables for the solution of specific problems will be discussed later. Now we only want to emphasize once more that there is a wide choice of variables in which the equations of motion have a canonical form.

3.4.3 Calculation of Frequencies and Interaction Amplitudes of Waves

Now we are ready to obtain the frequencies of the magnons and the amplitudes of the three- and four-magnon interactions in ferro-, antiferro- and ferrimagnets. Summing up the above, we may say that it is a problem which can be solved in 7 steps.

The first step is to find the canonical variables. The solution of this problem is not formalized and to solve it is to a certain extent an art. However, for a wide scope of interesting physical situations the canonical variables have already been obtained. For references see Sect. 1.3.3. For spin waves in a magnetic the canonical variables have been presented in (3.4.8 9).

The second step is to obtain the Hamiltonian function (Hamiltonian) of the system. As a rule this Hamiltonian is the energy of the system expressed in terms of the canonical variables. This refers also to the spin system of the magnetodielectrics. Phenomenological expression for the energy of the antiferromagnets includes additional terms of relativistic origin. It will be given later (3.4.10).

The third step is to find an equilibrium state and to expand the Hamiltonian in a power series in the canonical variables $a_j(\boldsymbol{r})$, $a_j(\boldsymbol{r})$, describing deviations from the state.

The fourth step is almost as simple. We must pass to the $\boldsymbol{k}$-representation (to the variables $a_j(\boldsymbol{k})$) by means of the Fourier-transform (1.1.12). After that, however, do not fail to divide the Hamiltonian by the volume of the sample.

The fifth step is to find the normal canonical variables $b_j(\boldsymbol{k})$, where the quadratic part of the Hamiltonian is diagonal. Generally, this problem can be solved via a linear canonical u, v-transformation (1.1.17). For the ferromagnets which have only one magnon branch the expression for the coefficients $u(\boldsymbol{k})$ and $v(\boldsymbol{k})$ is given by (1.1.21). For two-sublattice antiferromagnets (where there are two branches of magnons) the problem is reduced to obtaining the diagonalizing u, v-transformation for the Hamiltonian $[A, B]$-matrix with the dimensions 4×4. This transformation is also a 4×4 matrix and to obtain it is not trivial. A little trick to achieve this will be described later. In a twenty-sublattice ferrimagnet YIG there are twenty magnon branches. Therefore the Hamiltonian $[A, B]$-matrix, generally speaking, has the dimensionality 40×40 and the u, v-matrix diagonalizing it must have the same dimensionality. Although the dimensionality of both matrices can be reduced to 20×20 (due to the collinearity of the sublattices), the "direct" obtaining of the $[u, v]$-matrix is technically impossible. However, in [3.7] we show that in this problem there are latent numerically small parameters that make it possible to obtain analytically with acceptable accuracy both the diagonalizing $[u, v]$-matrix and the dispersion laws of all the twenty branches of the spin waves in YIG. The exact procedure of that is given in [3.7].

The sixth step is to change over in the interaction Hamiltonian from the initial canonical variables $a_j(\boldsymbol{k})$ to the normal canonical variables $b_j(\boldsymbol{k})$. For multisublattice magnets this procedure, though generally simple, becomes very cumbersome. Additional problems of computation due to the cancellation of big terms in the final expression arise for antiferromagnets.

The seventh and the last step is to change over the interaction Hamiltonian via nonlinear canonical transformation (1.1.31) from the variables $b_j(\boldsymbol{k})$ to the variables $c_j(\boldsymbol{k})$ where there are no non-resonance terms in the three-magnon Hamiltonian. This step is required for calculation of the contribution of the three-magnon processes to the four-magnon Hamiltonian in the second order of the perturbation theory.

The successive performance of all the 7 steps for ferromagnets presents no difficulties. The results of these calculations are presented in Sect. 3.1.

On calculating the frequencies of magnons and the amplitudes of their interaction in AFM EP one can proceed from the following phenomenological expression for the energy in a two-sublattice model

$$W = \frac{1}{2}\int\{[BM^2 + \alpha_1|\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{r}}|^2 + \alpha_2|\frac{\partial \boldsymbol{M}}{\partial \boldsymbol{r}}|^2] + bM_z^2 + aL_z^2 - 2\beta(\boldsymbol{n}\cdot[\boldsymbol{M}\times\boldsymbol{L}]) - 2\boldsymbol{M}\cdot\boldsymbol{H}\}\,d\boldsymbol{r}\ . \tag{3.4.10}$$

Here $\boldsymbol{n} = \boldsymbol{r}/r$, $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ designate the magnetization of sublattices, $M_1 = M_2 = M_0$, $\boldsymbol{M} = \boldsymbol{M}_1 + \boldsymbol{M}_2$, $\boldsymbol{L} = \boldsymbol{M}_1 - \boldsymbol{M}_2$, $\alpha_{1,2} > 0$ are exchange constants, $\beta > 0$ denotes the Dzyaloshinsky constant [3.1], $a > 0$ and b are the anisotropy constants.

It is convenient to proceed as follows: first find the configuration of the moments $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ in the ground state from the minimum energy condition; second, in two axillary coordinate systems oriented along the respective magnetizations $\boldsymbol{M}_j$ $(j = 1, 2)$ pass to the canonical variables (3.4.8) $c_j(\boldsymbol{r})$ and $c_j^*(\boldsymbol{r})$ $(j = 1, 2)$ in each sublattice as in FM. Third, to change over to Fourier-representation: $c(\boldsymbol{r}) \to c(\boldsymbol{k})$, and go to the symmetric variables $d_{1,2} = (c_1 \pm c_2)/2$. In this case the quadratic part of the Hamiltonian will be quasi-diagonal, the variables d_1 and d_1^* will describe one magnon branch and d_1 and d_1^* will describe the other one; then through the u, v-transformation in each branch the quadratic parts of the Hamiltonian must be diagonalized; and, finally, the expressions for $\omega(\boldsymbol{k})$, $\Omega(\boldsymbol{k})$ and $\mathcal{H}_{\rm int}$ will be obtained in the variables described in Sect. 3.2.

It is essential in the u, v-transformation that $u > 1$, $|v| > 1$and at the same time $|u|^2 - |v|^2 = 1$. It will lead in $\mathcal{H}_{\rm int}$ to the cancellation of the greatest terms proportional to Bu^4, Bu^2v^2, Bv^4. Therefore it is necessary to do the cumbersome calculations very accurately, retaining in the initial expression for the energy small terms whose contribution after multiplication by u^4, v^4 and u^2v^2 may prove essential. There exists the opinion that the magnetic dipole interaction can thus result in an anomalously big contribution to $\mathcal{H}_4$. However, the calculations by *Lutovinov* and *Safonov* [146] have shown that this does not take place.

Problem

Show that for the Bloch equation (3.4.5) the canonical variables coordinate and momentum are

$$q = M_x/g \ , \quad p = \varphi = \arctan(M_x/M_z) \ .$$

Hint: Directly differentiating allowing for the bond $M^2(p,q) = M_0^2$ make sure that (3.4.5) in the variables (3.5.1) are reduced to the form (1.1.6). Further, following the procedure described in Sect. 1.1 one may pass from p, q to the complex variables b, b^*. The vector $\boldsymbol{M}$ via these variables will be represented by (3.4.9).

4 Nonlinear Dynamics of Narrow Packets of Spin Waves

Chapter 1 contained the introduction to the general theory of nonlinear wave dynamics within the classical Hamiltonian formalism. "General" here implies that this theory describes nonlinear processes irrespective of the nature of waves and the type of the nonlinear media where their propagation takes place. Clearly, this approach, alongside with the evident advantages has certain limitations. To avoid them one must use the concrete laws of waves dispersion as well as the $\boldsymbol{k}$-dependences of the interaction Hamiltonian amplitudes providing specific information about a practical problem. In this chapter the attempt is made to overcome such limitations on the description of the nonlinear dynamics of spin waves in magnetically ordered dielectrics. In this attempt we shall proceed from the general theory developed in Chap. 1 and use the magnon Hamiltonians calculated in Chap. 3.

Section 4.1 describes elementary interaction processes involving three and four magnons in ferromagnets. This section illustrates the general theory of the three- and four-wave processes developed in Sects. 1.4.1, 2 and 1.5.1. Section 4.2 is much more independent. Here, the theory of wave self-focusing (an elementary introduction to this theory is given in Sects. 1.5.3, 4) is presented in connection with magnetoelastic waves in antiferromagnets. This problem is very interesting from the experimental point of view and rather peculiar from the theoretical angle. In Sect. 4.3 various methods of parametric magnon excitation in ferro- and antiferromagnets are described. They have all been given thorough experimental study.

4.1 Elementary Processes of Spin Wave Interaction

4.1.1 Three-Magnon Processes

Two simple examples of the behavior of magnetodielectrics will be considered here to illustrate the general theory of three-wave processes.

The first example is the confluence of two magnons in an isotropic ferromagnetic. Let two spin waves with wave vectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and canonical