Towards a Theory of Anomalous Scaling in Turbulence

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In this short note we present a brief references-assisted overview of our recent progress in understanding the universal statistics of fully developed turbulence, with a stress on anomalous scaling.

Modern concepts about high Re number turbulence started to evolve with Richardson's insightful contributions [1] which contained the famous "poem" that paraphrased J. Swift: "*Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity -in the molecular sense*". In this way Richardson conveyed an image of the creation of turbulence by large scale forcing, setting up a cascade of energy transfers to smaller and smaller scales by the nonlinearities of fluid motion, until the energy dissipates at small scales by viscous mechanisms. This picture led in time to innumerable "cascade models" that tried to capture the statistical physics of turbulence by assuming some thing or other about the cascade process. Possibly the most ingenious attempt is due to Kolmogorov who in 1941 [2] proposed the idea of universality (turning the study of small scale turbulence from mechanics to fundamental physics) based on the notion of the inertial range. The idea is that for very large values of Re there is a wide separation between the scale of forcing and the typical scale at which dissipative processes become important. In this "inertial" range the only parameter is the mean energy flux per unit time per unit mass, denoted as $\bar\epsilon$, and the value of the kinematic viscosity is irrelevant. This assumption allows one to develop dimensional analysis that leads to detailed predictions about the statistical physics of turbulence. This is important; no one in their right mind is interested in the full solution of the turbulent velocity field at all points in space-time. The interest is in the statistical properties of the turbulent flow. For example, quantities that were considered by Kolmogorov and later attracted decades of experimental and theoretical attention (see for example [3–7]) are the structure functions of velocity differences, denoted as $S_n(R) = \langle|\mathbf{u}(\mathbf{r}+\mathbf{R},t)-\mathbf{u}(\mathbf{r},t)|^n\rangle$ where $\langle\dots\rangle$ stands for a suitably defined ensemble average. If one considers R to be in the inertial range, i.e $\eta \ll R \ll L$, the only combination of $\bar\epsilon$ and R that gives the right dimension for S_n is $(\bar\epsilon R)^{n/3}$. In particular for $n = 2$ this is the famous Kolmogorov "2/3" law which in Fourier representation is also known as the "-5/3" law. This ideology that one extracts universal properties by focusing on statistical quantities can be applied also to the correlations of gradients of the velocity field. An important example is the rate $\epsilon(\mathbf{r},t)$ at which energy is dissipated into heat due to viscous damping. This rate is roughly $\nu|\nabla\mathbf{u}(\mathbf{r},t)|^2$. The often-studied correlation function of $\hat\epsilon(\mathbf{r},t) = \epsilon(\mathbf{r},t) - \bar\epsilon$ is $K_{\epsilon\epsilon}(R) = \langle\hat\epsilon(\mathbf{r}+\mathbf{R},t)\hat\epsilon(\mathbf{r},t)\rangle$. Within the Kolmogorov theory one estimates $K_{\epsilon\epsilon}(R) \simeq \nu^2\bar\epsilon^{4/3}R^{-8/3}$.

Experimental measurements show that Kolmogorov was remarkably close to the truth. The major aspect of his predictions, i.e that the statistical quantities depend on the length scale R as power laws is corroborated by experiments. On the other hand, the predicted exponents seem not to be exactly realized. For example, the correlation $K_{\epsilon\epsilon}(R)$ decays according to a power law, $K_{\epsilon\epsilon}(R) \sim R^{-\mu}$ for $\eta \ll R \ll L$, with μ having a numerical value of $0.2-0.3$ instead of 8/3 [5]. The structure functions also behave as power laws, $S_n(R) \simeq R^{\zeta_n}$, but the numerical values of ζ_n deviate progressively from $n/3$ when n increases [4,6]. Something fundamental seems to be missing.

Theoretical studies of the universal small scale structure of turbulence can be classified broadly into two main classes. Firstly there is a large body of phenomenological models that attempt to achieve agreement with experiments on the basis of some ad hoc assumption on the nature of the cascade or the statistics of the turbulent fields [7]. Since these models are not derived on the basis of the equations of fluid mechanics, one is always left with uncertainties about the validity or relevance of these models. The second class of approaches is based on the equations of fluid mechanics. Typically one acknowledges the fact that fluid mechanics is a (classical) field theory and resorts to field theoretic methods in order to compute statistical quantities. Even though there had been a continuous effort of almost 50 years in this direction, the analytic derivation of the scaling laws for $K_{\epsilon\epsilon}(R)$ and $S_n(R)$ from the Navier-Stokes equations and the calculation of the numerical value of the scaling exponents μ and ζ_n have been among the most elusive goals of theoretical research. Why did it turn out to be so difficult?

To understand the difficulties, we need to elaborate a little on the nature of the field theoretic approach [13]. Suppose that we want to calculate the average response of a turbulent fluid at some point $\mathbf{r}_0$ to forcing at point $\mathbf{r}_1$. The field theoretic approach allows us to consider this response as an infinite sum of all the following processes: firstly there is the direct response at point $\mathbf{r}_0$ due to the forcing at $\mathbf{r}_1$. This response is instantaneous if we assume that the fluid is incompressible (and therefore the speed of sound is infinite). Then there is the process of forcing at $\mathbf{r}_1$ with a response at an intermediate point $\mathbf{r}_2$, which then acts as a forcing for the response at $\mathbf{r}_0$. This intermediate process can take time, and we need to integrate over all the possible positions of point $\mathbf{r}_2$ and all times. This is the second-order term in perturbation theory. Then we can force at $\mathbf{r}_1$, the response at $\mathbf{r}_2$ acting as a forcing for $\mathbf{r}_3$ and the response at $\mathbf{r}_3$ forces a response at $\mathbf{r}_0$. We need to integrate over all possible intermediate positions $\mathbf{r}_2$ and $\mathbf{r}_3$ and all the intermediate times. This is the third-order term in perturbation theory. And so on. The actual response is the infinite sum of all these contributions. In applying this field theoretical method one encounters three main difficulties:

(A) The theory has no small parameter. The usual tack is to develop the theory perturbatively around the linear part of the equation of motion. The expansion parameter is then Re, and since we are interested in Re$\gg 1$, naive perturbation expansions are badly divergent. In other words the size of the various processes described above increases as $(\mathrm{Re})^n$ with the number n of intermediate points in space-time.

(B) The theory suffers from two types of nonlinear interactions. Both are hidden in the nonlinear term $\mathbf{u}\cdot\nabla\mathbf{u}$ in the Navier-Stokes equations. The larger of the two is the sweeping of small "eddies" by all the larger ones. Physically this sweeping couples any given scale of motion to all the larger scales scales. Technically in perturbation theory these interactions cause infrared divergences of the integrals; the divergences become stronger with the order of of the contribution. This may lead to a belief that the theory of turbulence is not renormalizable. However, sweeping is just a kinematic effect that does not lead to energy redistribution between scales, and one may hope that if the effect of sweeping is taken care of in a consistent fashion a renormalizable theory might emerge. This redistribution of energy results from the second type of interaction, that stems from the shear and torsion effects that are sizeable only if they couple fluid motions of comparable scales. The second type of nonlinearity is smaller in size but crucial in consequence, and it may certainly lead to a scale-invariant theory.

(C) Nonlocality of interaction in $\mathbf{r}$ space. The effect of the pressure in the incompressible approximation is to instantaneously couple distant points.

Indeed, these difficulties seemed to complicate the application of field theoretic methods to such a degree that a consensus appeared to the effect that it is impossible to gain valuable insight into the universal properties of turbulence along these lines, even thought they proved so fruitful in other field theories. The present authors think differently, and in the rest of this paper we will explain why.

The first task of a successful theory of turbulence is to overcome the existence of the interwoven nonlinear effects with different scale dependence that rule out a scale invariant formulation even in the inertial range. This is not achieved by directly applying a formal field-theoretical tool to the Navier-Stokes equations. It does not matter whether one uses standard diagrammatics [8], path integral formulation, renormalization group [9] ϵ-expansion, large N-limit [10] or one's formal method of choice. One needs to take care of the particular nature of hydrodynamic turbulence as embodied in difficulty (B) *first*, and *then* proceed using formal tools.

The removal of the effects of sweeping is based on Richardson's remark that universality in turbulence is expected for the statistics of velocity *differences across a length scale* R rather than for the statistics of the Eulerian field itself. The Eulerian fields are dominated by the large scale winds that are not universal. This remark was developed by Kraichnan who attempted to cast the field theoretic approach in terms of Lagrangian paths that automatically remove the large scale contributions [11]. Kraichnan's approach was fundamentally correct, and gave rise to important and influential insights in the description of turbulence, but did not provide a convenient technical way to consider all the orders of perturbation theory. The theory was not "diagrammatic" in the sense that there were no transparent rules governing how to consider a arbitrarily high term in the perturbation theory. Only low order truncations were considered.

A way to overcome difficulty (B) was suggested by Belinicher and L'vov [12] who introduced a novel transformation that allowed on one hand the elimination of the sweeping that leads to infrared divergences, and on the other hand produced a diagrammatic theory for the universal statistics. The use of this transformation allowed the examination of the structure functions of velocity differences $S_n(R)$ to all orders in perturbation theory. Of course, difficulty (A) remains; the perturbation series still diverges rapidly for large values of Re, but now standard field theoretic methods can be used to resum infinite subsets of contributions (Dyson line resummation) to achieve a new perturbation theory in terms of renormalized quantities [14]. In particular the damping coefficient in the renormalized theory changes from the bare kinematic viscosity to the eddy viscosity. The result is that the effective expansion parameter is no longer Re but an expansion parameter of O(1). Of course, such a series may still diverge as a whole. Notwithstanding it is crucial to examine first the order-by-order properties of series of this type.

qSuch an examination leads to a major surprise: every term in this perturbation theory remains finite when the outer scale L goes to ∞ and the inner scale η goes to 0 [14]. The meaning of this is that the perturbative theory for S_n does not indicate the existence of a renormalization length that can be used in order to describe anomalous scaling. In other areas of theoretical physics in which anomalous scaling has been found the perturbative series indicates this by exhibiting (resummable) logarithmic divergences that must be tamed by truncating the integrals at some renormalization length. Hydrodynamic turbulence is different. The nonlinear Belinicher-L'vov transformation changes the underlying linear theory such that the line resummation results in full renormalization of the perturbative scheme [12,14]. The physical meaning of this result is two-fold. Firstly it means that in terms of the reposnse at $\mathbf{r}_0$ to forcing at $\mathbf{r}_1$ as discussed above there exists a "ball of locality" of radius $R = |\mathbf{r}_1 - \mathbf{r}_0|$ such that if the intermediate points $\mathbf{r}_2 \ldots \mathbf{r}_n$ are taken far outside it, the contributions to the final result die out. In addition if the intermediate points come too em close to one other, the contributions of these processes remain finite. The main contribution comes when all the intermediate points are separated from one other by a distance of the order of R. This is the perturbative justification of the Richardson-Kolmogorov cascade picture in which widely separated scales do not interact.

But the main question still remains: how does a renormalization scale appear in the statistical theory of turbulence?

The first answer comes from considering the statistics of gradient fields rather than the velocity differences themselves [18,15]. As indicated above the discrepancy in the exponent μ between the experimental value and the naive expectation is large. This gives hope that the explanation must lie close to the surface. Indeed, considering the perturbative scheme for $K_{\epsilon\epsilon}$ leads immediately to the discovery of logarithmic ultraviolet divergences with some ultraviolet viscous scale $\tilde{\eta}$ acting as the renormalization scale. The summation of this infinite series results in a factor $(R/\tilde{\eta})^{2\Delta}$ with some anomalous exponent Δ which is, generally speaking, of the order of unity. This changes the exponent μ from $8/3$ to $8/3 - 2\Delta$. The physical explanation of this renormalization [18,19] is the effect of the multi-step interaction of two R-separated small eddies of scale $\tilde{\eta}$ with a large eddy of scale R via an infinite set of eddies of intermediate scales. At this point it is important to understand what is the numerical value of the anomalous exponent Δ. It can be shown using elementary inequalities that $\Delta \leq 2 - \zeta_2$. The structure of the theory is however qualitatively different for Δ smaller than $2 - \zeta_2$ (referred to as the "subcritical scenario" [19]) and for $\Delta = 2 - \zeta_2$ (which is the "critical scenario" [14,15].

In the subcritical scenario the perturbation series for the structure functions can be shown to converge, with the sum of the series being of the order of each one of the terms. In this scenario no new length scale enters in the limit $\mathrm{Re} \to \infty$, and the K41 exponents ζ_n are expected to be exact. However, we showed in [14,15] that the subcritical scenario is *not* realized, and the theory of turbulence follows the critical scenario. The technical step leading to this is the derivation of an exact sum rule that forces Δ to attain precisely the critical value $2 - \zeta_2$. Physically this sum rule is a consequence of the existence of a universal nonequilibrium stationary state that supports an energy flux from large to small scales. The criticality of Δ has far-reaching implications for the theory of the structure functions. Now the series for the structure functions $S_n(R)$ diverge when the outer scale L approaches ∞ as powers of L, $(L/R)^{\delta_n}$. The anomalous exponents δ_n are the deviations of the exponents of $S_n(R)$ from their K41 value. This is a very delicate and important point. It is not simple to see why a series each of whose terms converges when $L \to \infty$ still diverges as a whole. One can understand this as in resumming the series of a desired function one finds an inhomogeneous equation whose inhomogeneous solutions are indeed L-independent. However, the equation possesses also homogeneous solutions which are inherently nonperturbative in nature and may have anomalous scaling. Homogeneous solutions must be matched with the boundary conditions, and this is the way that the outer scale appears in the theory. The next step in the theoretical development is to understand how to compute the anomalous exponents δ_n.

The divergence of the perturbation theory for $S_n(R)$ with $L \to \infty$ forces us to seek a nonperturbative handle on the theory. One finds this in the idea that there exists always a global balance between energy input and dissipation, which may be turned into a nonperturbative constraint on each n-th order structure function [16]. Using the Navier-Stokes equations one derives the set of equations of motion

$$\frac{\partial S_n(R,t)}{\partial t} + D_n(R,t) = \nu J_n(R,t) \ , \tag{1}$$

where D_n and J_n stem from the nonlinear and the viscous terms in the Navier Stokes equations respectively. In the stationary state the time derivative vanishes and one has the balance equation $D_n(R) = \nu J_n(R)$. The evaluation of $D_n(R)$ requires dealing with the difficulty (C) of the nonlocality of the interaction, but it does not pose conceptual difficulties. It was shown [16] that it is of the order of dS_{n+1}/dR. On the other hand, the evaluation of $J_n(R)$ raises a number of very interesting issues whose resolution lies at the heart of the universal scaling properties of turbulence. Presently not all of these issues have been resolved, and we briefly mention here some ground on which progress has been made.

The term $J_n(R)$ consists of a correlation of $\nabla^2 u$ with $n-2$ velocity differences across a scale R. The question is how to evaluate such a quantity in terms of the usual structure functions $S_n(R)$. Recall that a gradient of a field is the difference in the field values at two points divided by the separation when the latter goes to zero. In going to zero one necessarily crosses the dissipative scale. It becomes therefore crucial to formulate so-called "fusion rules" for turbulence, which govern the properties of n-point correlation functions when two or more coordinates "fuse" together. The answer is [20] that when p coordinates are separated by a small distance r, and the remaining $n-p$ coordinates are separated by a large distance R, then the r dependence is like that of $S_p(r)$, i.e. r^{ζ_p}. This is true until r crosses the dissipative scale. Assuming that below the dissipative scale derivatives exist and the fields are smooth, one can estimate gradients at the end of the smooth range by dividing differences across the dissipative scale over that scale. The question is, what is the dissipative scale? Is there just one dissipative scale, or is there a multiplicity of such scales, depending on the function one is studying? For example, when does the above n-point correlator become differentiable as a function of r? Is that typical scale the same as the one exhibited by $S_p(r)$ itself, or does it depend on n and on the remaining distances R?

The answer is that there is a multiplicity of crosover scales. For the n-point correlator discussed above we denote the dissipative scale as $\eta(p,n,R)$, and it depends on each of its arguments [20,17]. In particular it depends on the inertial range variables R and this dependence must be known when one attempts to determine the scaling exponents ζ_n of the structure functions. In brief, this line of thoughts leads to the scaling relation $\mu = 2 - \zeta_6$ (in close agreement with the experimental values) and the prediction that although the ζ_n are not K41, they are nevertheless linear in n for large n.

In summary, it appears that there are four conceptual steps in the construction of a theory of the universal anomalous statistics of turbulence on the basis of the Navier-Stokes equations. First one needs to take care of the sweeping interactions that mask the scale invariant theory [12,14]. After doing so the perturbation expansion converges order by order, and K41 scaling of the velocity structure functions is found as a perturbative solution. Secondly one understands the appearance of the inner renormalization scale in the theory of the correlation functions of the gradient fields [18,15]. This step is similar to critical phenomena and it leads to a similarly rich theory of anomalous behaviour of the gradient fields. Only the tip of the iceberg was considered above. The third step is the understanding of the criticality of the theory, which results in the divergence of the diagrammatic series for the structure functions [16]. This sheds light on the appearance of the outer renormalization scale in the theory of turbulence. Last but not least is the formulation of the fusion rules and the multiplicity of the dissipative scales which should eventually result in a satisfactory description of all the scaling properties [17]. The road ahead is not fully charted, but it seems that some of the conceptual difficulties have been surmounted. We believe that the crucial building blocks of the theory are now available, and they begin to delineate the structure of the theory.

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