Simple mathematical model of climate variability

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Satellite view of the Hurricane Bonnie, wind speed $>$ 1000 Km/H

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Outline:

- Introduction
  - Weather Forecast and Climate
  - Resonances: from Simple Pendulum to Planetary Waves
  - Diophantine Equation for Interaction Triads of Planetary Waves

- Energy Exchange in Planetary Triads
  - Meteorologically Relevant Triads
  - Jacobian elliptic functions for Periods of Atmospheric Variability
  - Conservation Laws and Periods of the Variability

- Conclusion: Trinity of Meteorology, Physics and Mathematics

Weather Forecast Required:

Weather Forecast Problem

• Our days weather forecast is based on numerical solution of the (Navier-Stokes) equations for the wind velocity, (partial differential) equations for heat and humidity transfer, etc. on a grid with horizontal resolution about $(100 \div 200)$ Km and two-three vertical slices at best. With modern computers it gives reasonable forecast up to $(7 \div 10)$ days.

• To increase the resolution twice one needs to increase the compute power (speed and memory) in about 10 times. Unfortunately this allows one to improve the forecast only on one-two days.

• The main problem is the "butterfly" effect: very small variations in the wind velocity (say, caused by a flight of butterfly or by uncertainties in initial and boundary conditions) increases in time exponentially, like unstable behavior of a cone, staying on a peak. The result of the instability is full loss of the predicability on long-time scales (more then $7 \div 10$ days).

–One needs essentially improve an understanding of basic physical processes in the Atmosphere and Ocean and their mathematical description.

This talk is devoted to the nature of

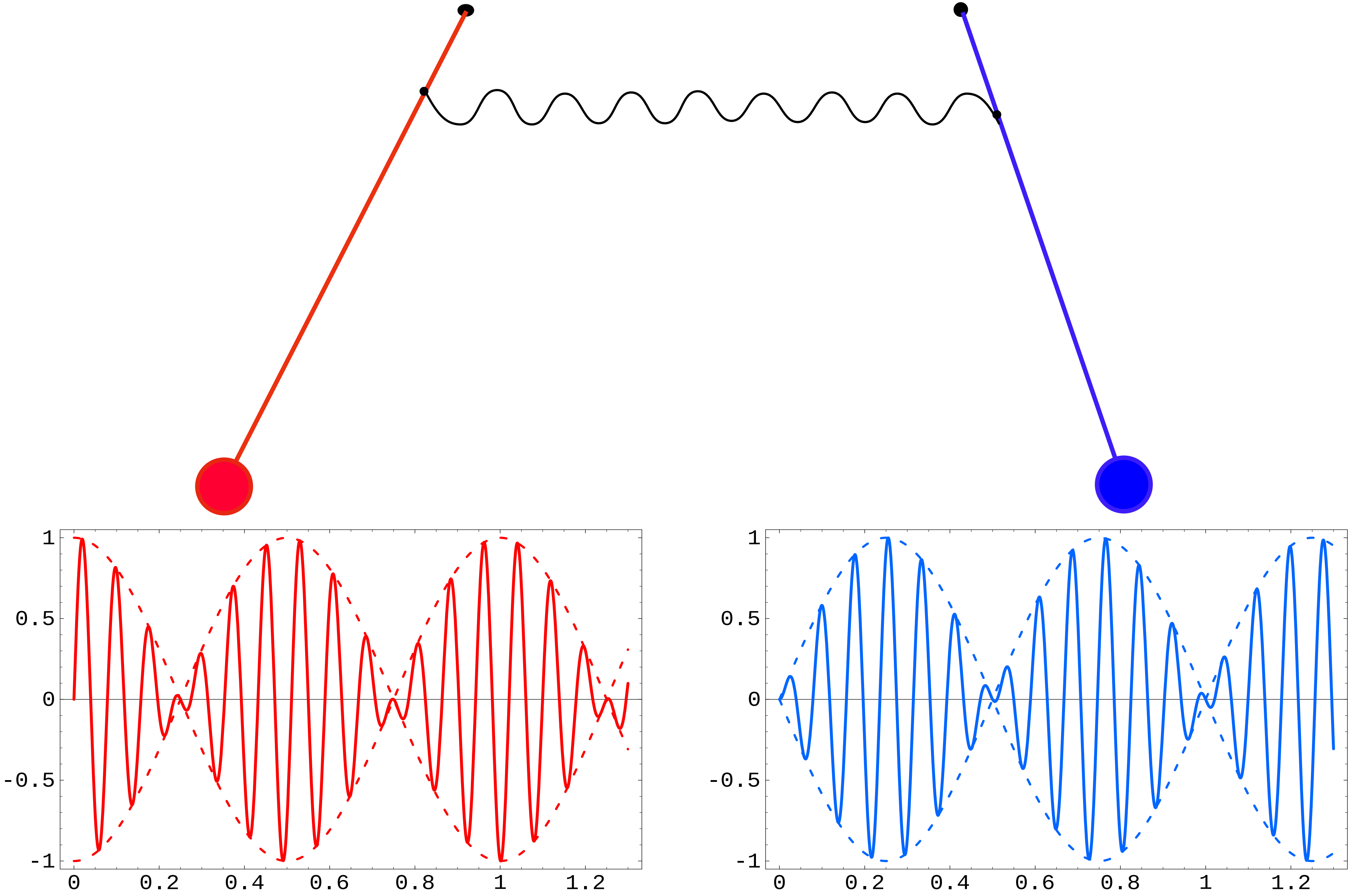
Intra-seasoning Oscillations (IOs) in atmospheric activity

(wind speed, atmospheric pressure, etc.) with periods 10-100 days, that persist beyond the life times individual weather disturbances, i.e., beyond about a week.

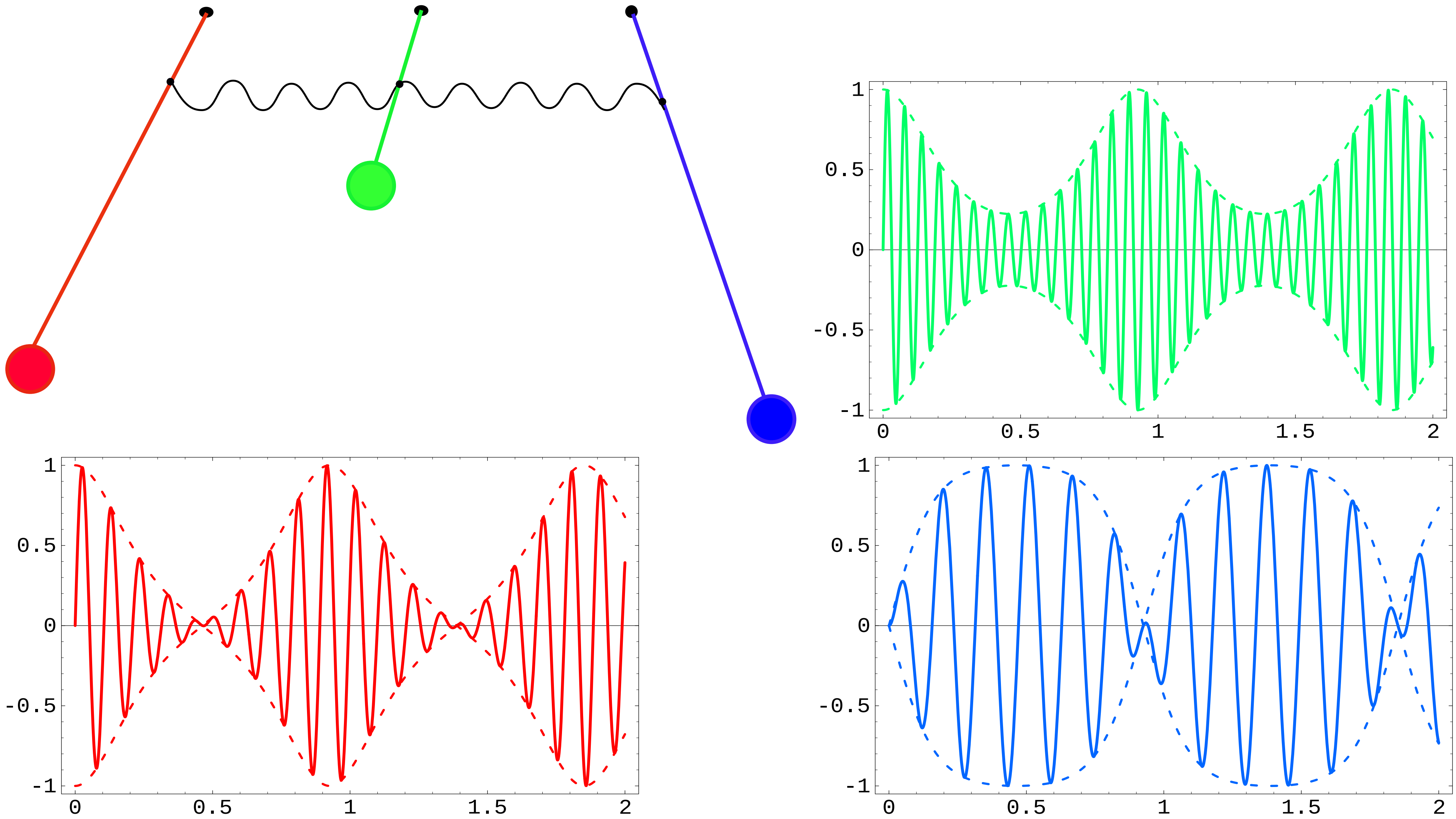
Many aspects of the IOs remain unexplained, *e.g.* the reason of IOs in the North Hemisphere is supposed to be topography and no reason is given for IOs in the South Hemisphere, there is no known way to predict the appearance of IOs, etc.

We propose a totally different approach. Rejecting the assumption that the process is dominated by a single mode, **we study** wave systems which naturally have the period of desired order - namely, **triads of resonantly interacting planetary waves**.

Energy exchange between coupled pendulums: $\omega_1 = \omega_2$



Energy exchange between three pendulums: $\omega_3 = \omega_1 + \omega_2$



Pendulum $\Rightarrow$ Sea waves $\Rightarrow$Atm.-Rossby waves$\Rightarrow$Planetary Waves

$$\omega_L = \sqrt{\frac{g}{L}} \Rightarrow \omega_\lambda = \sqrt{\frac{2\pi g}{\lambda}} \Rightarrow \omega_{\vec{\lambda}} = \frac{2\Omega\cos\phi}{R}\,\frac{\lambda_x^2+\lambda_y^2}{\lambda_x} \Rightarrow \omega_{\ell,m} = 2\Omega\cos\phi\,\frac{m}{\ell(\ell+1)}$$

Here ω_L – frequency of the pendulum oscillations, L is the pendulum length, g is the gravity accelerations

$\omega_\lambda\,,\omega_{\vec{\lambda}}$ and $\omega_{\ell,m}$ are frequencies of corresponding waves,

$\Omega = 2\pi/(24\text{h})$ is the Earth rotation frequency, $R = 6,400$km is the Earth radius, ϕ is the latitude, $\lambda_{x,y}$ are projections of the Rossby wave length,

integers ℓ and $(\ell - m)$ are numbers of planetary-wave periods in the northern and eastern directions, formally – the eigen-numbers of the spherical functions $Y_{\ell,m}(\theta,\varphi)$, describing air-velocity field of the planetary wave:

$$\begin{aligned} V(\theta,\varphi,t) &= Y_{\ell,m}(\theta,\varphi)\sin(\omega_{\ell,m}t)\,, \\ Y_{\ell,m}(\theta,\varphi) &\equiv \cos(\varphi)\,P_{\ell,m}(\cos\theta)\,, \end{aligned}$$

θ,φ – spherical coordinates, $P_{\ell,m}(\cos\theta)$ – associated Legendre polynomials

Resonance conditions for interaction triads of Planetary waves

Time-resonance: $\omega_{\ell_1,m_1} + \omega_{\ell_2,m_2} = \omega_{\ell_3,m_3} \;\Rightarrow$

$$\frac{m_1}{\ell_1(\ell_1+1)} + \frac{m_2}{\ell_2(\ell_2+1)} = \frac{m_3}{\ell_3(\ell_3+1)}, \quad (1a)$$

2D Spatial resonance: $m_1 + m_2 = m_3\,,$ $(1b)$

$$|\ell_1 - \ell_2| < \ell_3 < \ell_1 + \ell_2\,, \quad (1c)$$

Non-zero interaction: $\ell_1 + \ell_2 + \ell_3 \quad \text{is odd}\,,$ $(1d)$

$$\ell_i \neq \ell_j \quad \forall i \neq j \quad m_j \leq \ell_j\,, \quad (1e)$$

are Diophantine equations for 6 integers $\ell_j\,,\ m_j$ with $j = 1, 2, 3$.

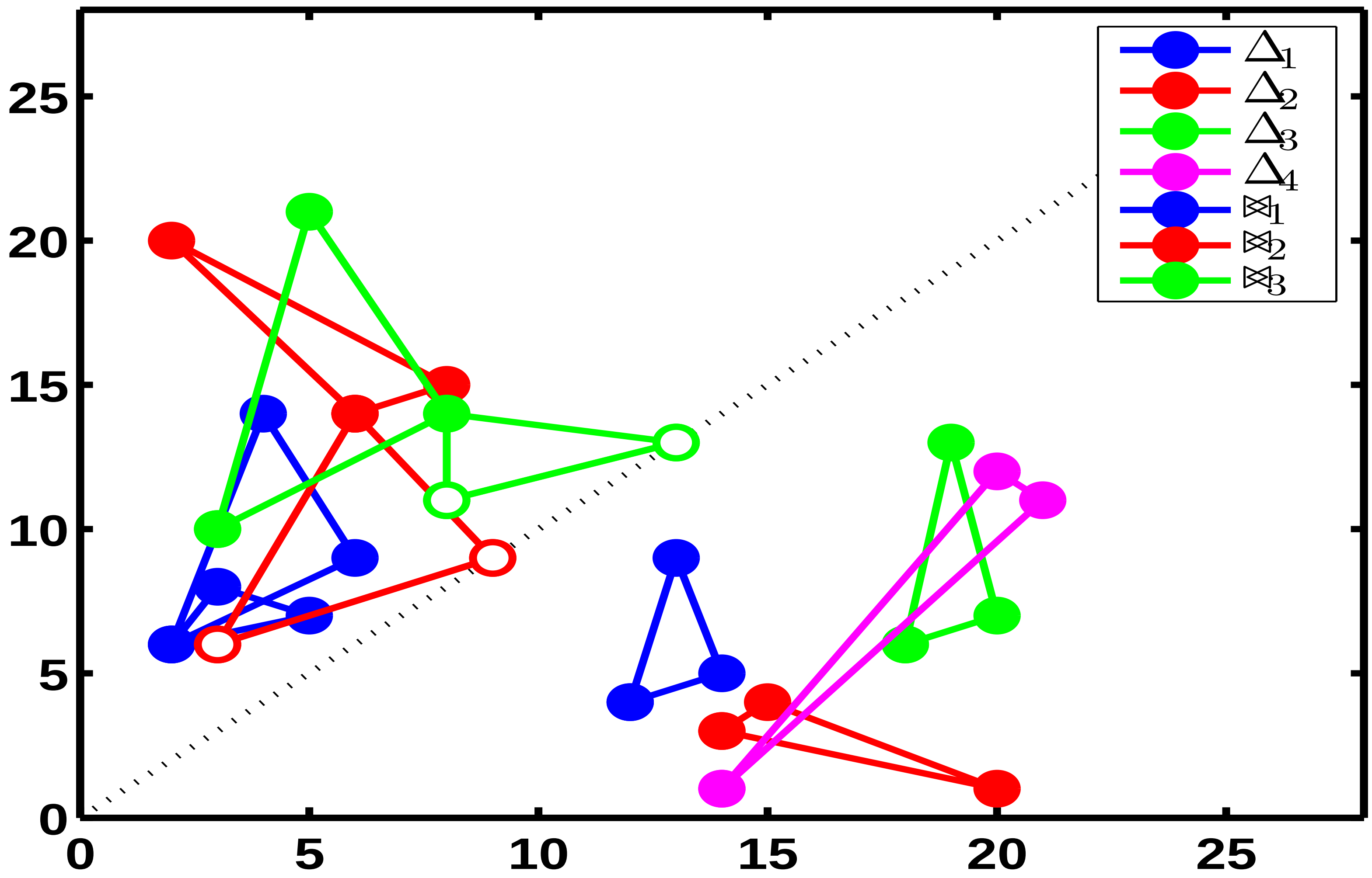
In the meteorological relevant domain $\ell_j \leq 21$ we have found only four solutions of the Diophantine equations (1):

$$\ell_1 = 12\,, m_1 = 4\,, \quad \ell_2 = 14\,, m_2 = 5\,, \quad \ell_3 = 13\,, m_3 = 9\,;$$
$$\ell_1 = 14\,, m_1 = 3\,, \quad \ell_2 = 20\,, m_2 = 1\,, \quad \ell_3 = 15\,, m_3 = 4\,;$$
$$\ell_1 = 18\,, m_1 = 6\,, \quad \ell_2 = 20\,, m_2 = 7\,, \quad \ell_3 = 19\,, m_3 = 13\,;$$
$$\ell_1 = 14\,, m_1 = 1\,, \quad \ell_2 = 21\,, m_2 = 11\,, \quad \ell_3 = 20\,, m_3 = 13\,;$$

responsible for only 4 ISOLATED TRIADS of planetary waves.

We also found three "butterflies" i.e. groups of two triads, connected by a common mode, consisting of five interacting planetary waves, and one group of six connected triads consisting from thirteen interacting waves.

A solution $\ell_j\,,\ m_j\,,\ j=1,2,3$ can be shown on the (ℓ,m) plane as a triangle. For the clarity we placed below isolated triads below the diagonal $\ell=m$ and all "butterflies" - over it.



Equations of motion for amplitudes A_j in triads of planetary waves:

$$\begin{cases} N_1\, dA_1/dt = 2\, Z\, (N_3 - N_2)\, A_3 A_2^*, \\ N_2\, dA_2/dt = 2\, Z\, (N_1 - N_3)\, A_1^* A_3, \\ N_3\, dA_3/dt = 2\, Z\, (N_2 - N_1)\, A_1 A_2\,, \end{cases} \tag{1}$$

Here $Z = Z(\ell_j) \simeq 20 \div 30\,,\ N_j \equiv \ell_j(\ell_j + 1)$.

Conservation of energy E and enstrophy H:

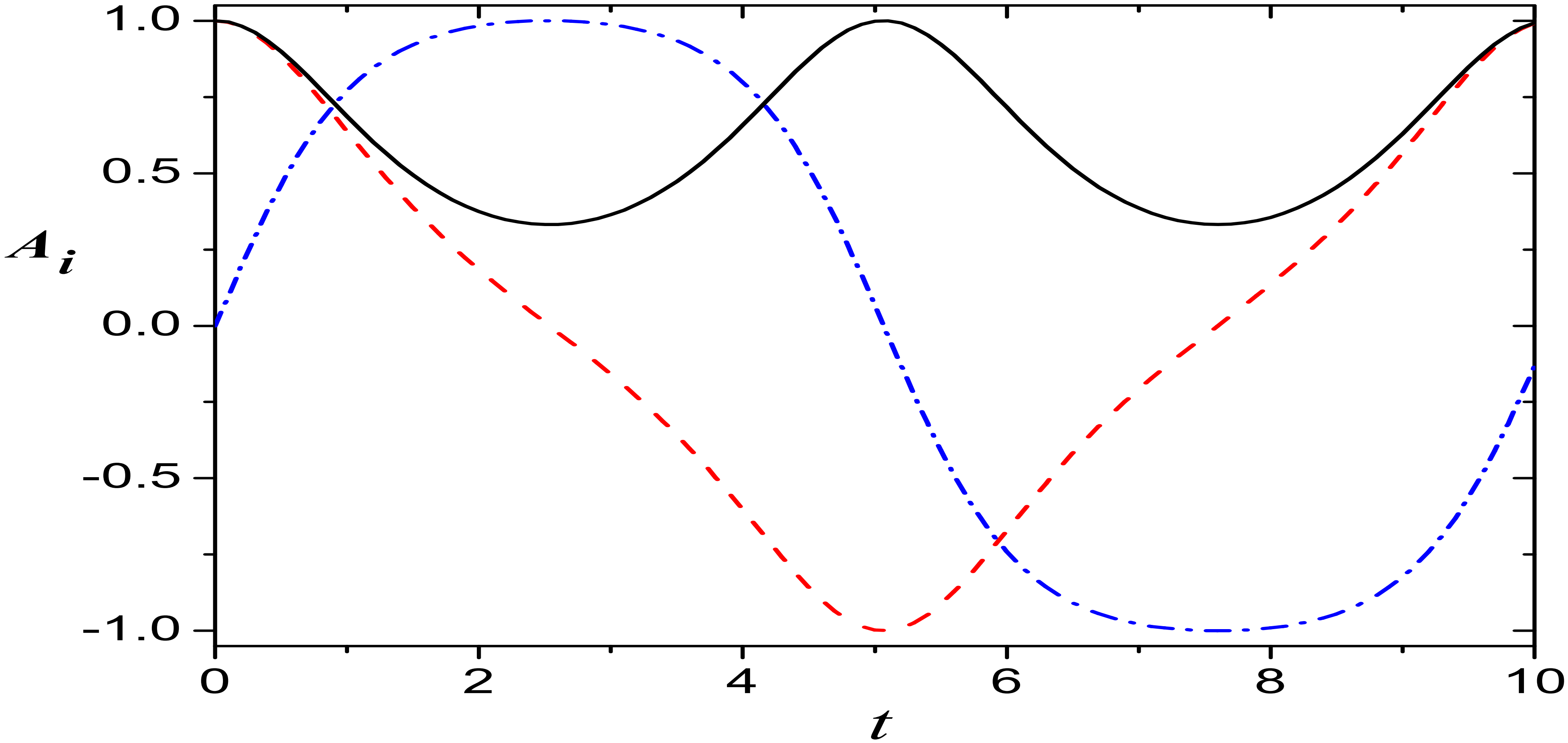
$$\begin{aligned} E &= E_1 + E_2 + E_3 = \text{const.}\,, \qquad E_i \equiv N_i |A_i|^2\,, \\ H &= N_1 E_1 + N_2 E_2 + N_3 E_3 = \text{const.} \end{aligned}$$

General solution of triad equations (2):

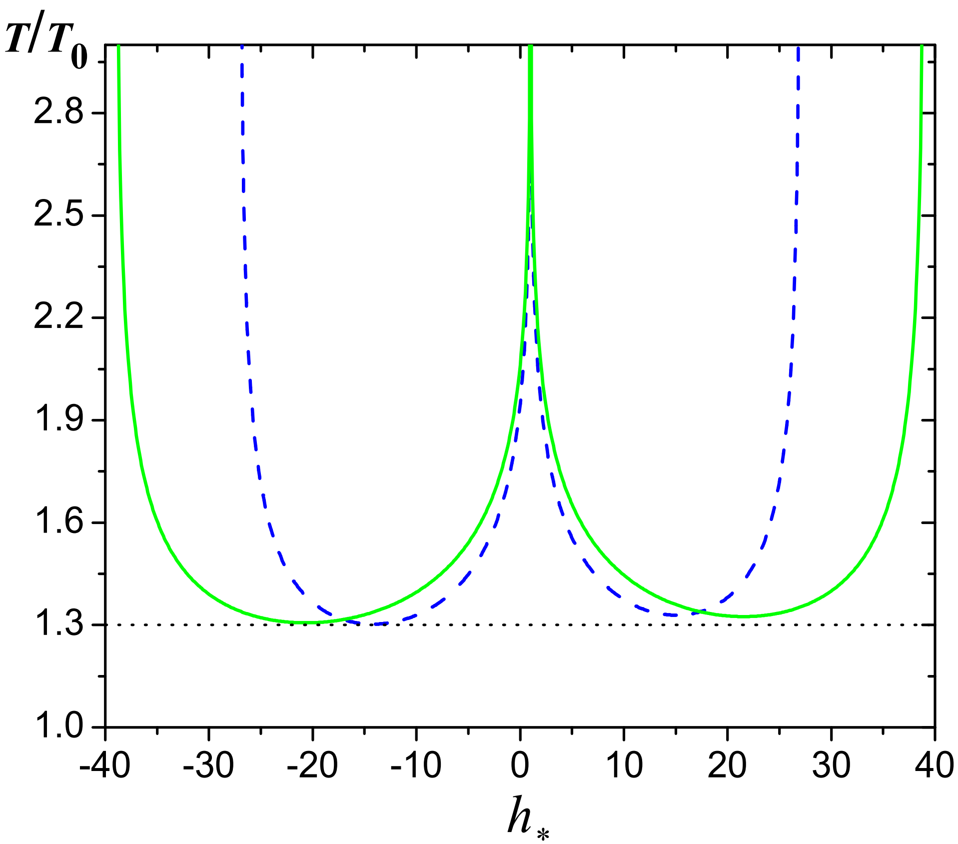
$$A_1 = A_{1,0}\, \text{cn}\,(\tau)\,, \quad A_2 = A_{2,0}\, \text{sn}\,(\tau)\,, \quad A_3 = A_{3,0}\, \text{dn}\,(\tau)\,,$$

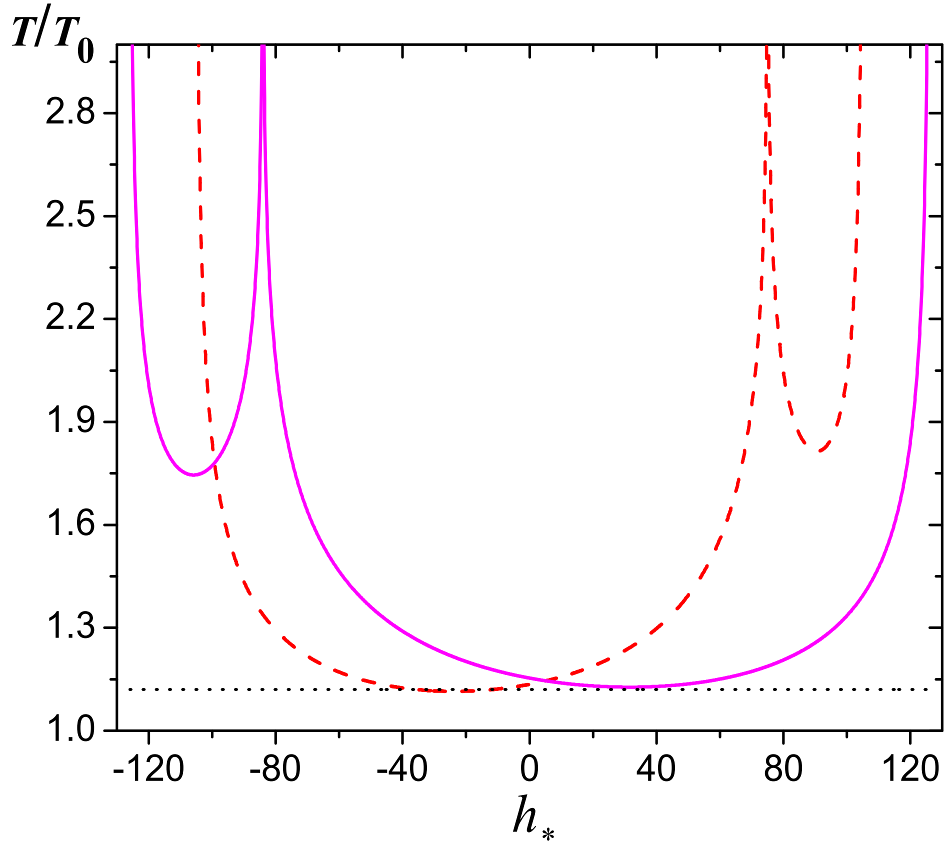
where the Jacobian elliptic functions cn(τ), sn(τ) and dn(τ) are periodic with the period $4K, 4K$ and $2K$, with $K(\mu)$ a normalized complete elliptic integral of the first order with modulus μ:

$$K(\mu) = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \mu \sin^2\theta}}\,, \qquad \mu^2 \equiv \mu(E, H) \leq 1\ .$$



Time dependence of A_1, A_2 and A_3 [denoted by green solid, red dashed and blue dot-dashed lines accordingly] of the first triad with $\mu = 0.89$, corresponding to observed data.





Dependence of the triad period on the energy distribution between modes, characterized by parameter $h_* \equiv H/E - (N_2 - N_1)/2$.

Left: Blue–dashed and green-solid lines corresponds to 1st and 3rd triad.

Right: Red-dashed and magenta-solid lines 2nd and 4th triad.

Intra-seasonal Oscillations as Resonant Triads of Planetary waves

Our model answers to the most questions stated by meteorologists:

–What is the cause of Intra-seasonal Oscillations in South Hemisphere?

The same as in the North Hemisphere, resonant triads.

–Why the period of the oscillations in North Hemisphere is given as 40 days by some researchers and 20-30 days - by others?

Because the period depends on H/E.

– How do the tropical and mid-latitude oscillations interact?

For example, via connected triads with different geological locations: butterflies, the six-triad group.

Conclusion:

Our approach has three cornerstones:

1. Physical idea that intra-seasonal oscillations are caused by the energy exchange inside of isolated triads of planetary waves. Analysis of corresponding equations of motion results in highly nontrivial Diophantine equations for three pairs of eigen-numbers, characterizing velocity field of the planetary waves;

2. Mathematical analysis of obtained Diophantine equations that results in their solution: four methodologically relevant triads are found.

3. Mathematical analysis of solution of resulting equations for triads in the form of Jacobian elliptic functions resulting in explicit formulae for their period in terms of the conservation laws.

Our model provides main robust features of important meteorological phenomenon:

– Intra-seasonal oscillations of atmospherical activity

– Their locations,

– Magnitudes of the periods, etc.

The same approach can be applied for other important problems:

- The role of isolated group of four gravity surface waves in the dynamics of a stormy sea,
- The role of triads of the Kelvin-Helmholts waves in the Ocean dynamics,
- The role of triads of the shift waves in plasma instabilities in Tokamacs – energy reactors for thermo-nuclear synthesis, etc.

Corresponding Diophantine equations – by request.

Related problem of atmospheric turbulent boundary layer over a cold sea surface or over a stormy sea requires analysis and solution of a system of ($20 \div 36$) algebraic polynomial equations. Looking for a help.