Phenomenology of Wall Bounded Turbulence: Minimal Models

In collaboration with :

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The goal is physically transparent and as simple as possible but still adequate description of main statistical characteristics of wall-bounded turbulence, that accounts for basic physical processes and symmetries in classical examples of wall bounded turbulence:

- Large Re turbulent plain flow (Channel, Pipe and Couette flows, flow over incline plane in the gravity field (modeling river flows), atmospheric Turbulent Boundary Layer (TBL), *etc*;
- Time and spacial developing (TD & SD) TBL, TBL with Stable temperature and heavy-particle Stratifications,*etc*.

Outline: Phenomenology of Wall Bounded Turbulence: Minimal Models

- **First Story: Large Re Neutrally-Stratified (NS) Channel Flow**
  - Formulation and Analysis of the "NS Minimal Model" (NS-MM)
  - DNS, LES & Experiment in comparison with the NS-MM
  - First Summary: Strength & Limitations of the NS-MM
- **Second Story: TBL with Stable temperature Stratification**
  - Minimal Model for Stably Stratified TBL (SS-MM)
  - (ℓ/L)-dependence of mean velocity & temperature profiles, of the heat fluxes and profiles of kinetic & "temperature" energies
  - Second summary: Fail of the down-gradient approximation, internal waves and more...
- **Third and last Story: Time-Developing (TD) TBL**
  - TD-MM for TD TBL and its Factorized asymptotical solution
  - The velocity and energy profiles in TD-TBL and the Energy
  - Third and last summary: What did we learn?

• Introduction: our goal and approach

The goal is physically transparent and s simple description of main statistical characteristics of wall-bounded turbulence, that accounts for basic physical processes and symmetries in Large Re turbulent plain flow (Channel, Pipe and Couette flows, flow over incline plane in the gravity field (modeling river flows), atmospheric Turbulent Boundary Layer (TBL), *etc*; in Time and spacial developing (TD & SD) TBL, TBL with stable temperature and heavy-particle stratifications,*etc*.

Suggested **Minimal Models** (MMs) as a rule are versions of algebraic Reynolds stress models, designed to describe in the plain geometry the mean flow and statistics of turbulence on the level of all relevant simultaneous, one-point second-order (cross)-correlation functions of velocity, temperature, particle concentrations, etc.
If required (for developing TBL, for the core of the channel flow, etc.) it includes nonlinear spacial fluxes of energy in differential approximations

First Story: Large Re Neutrally-Stratified (NS) Channel Flow

– Algebraic MM for Neutral and Stable temperature Stratifications

– Governing equations for velocity $\boldsymbol{\mathcal{U}}$ and potential temperature Θ

$$(\partial_t + \boldsymbol{\mathcal{U}} \cdot \nabla)\boldsymbol{\mathcal{U}} = -\frac{\nabla p}{\rho_b} - \boldsymbol{g}\,\beta\,\Theta + \nu\,\Delta\boldsymbol{\mathcal{U}}\,,\ \ \nabla \cdot \boldsymbol{\mathcal{U}} = 0\,,\ \ \text{(Boussinesq Appr.)}$$

$$(\partial_t + \boldsymbol{\mathcal{U}} \cdot \nabla)\,\Theta = \chi\,\Delta\,\Theta, \qquad \Theta \equiv T_b\{\exp\left[(\mathcal{S} - \mathcal{S}_b)/c_p\right] - 1\}\ . \tag{1}$$

$\mathcal{S}(\boldsymbol{r}, t)$ – "current" entropy, $\mathcal{S}_b$ – entropy of isentropic atmosphere (BRS).

– Exact balance equations for 2nd order correlators $(D_t \equiv \partial_t + \boldsymbol{U} \cdot \nabla)$

$$\begin{aligned}
\mathcal{P}_{ij} - \mathcal{C}_{ij} + \mathcal{R}_{ij} - \varepsilon_{ij} - \partial_k T_{ijk} &= D_t \tau_{ij}\,, \qquad \tau_{ij} \equiv \langle u_i u_j \rangle\,, \\
\mathcal{A}_i + \mathcal{B}_i - \epsilon_i - \partial_j T_{ij} &= D_t F_i\,, \qquad F_i \equiv \langle u_i\,\theta \rangle, \\
-\boldsymbol{F} \cdot \nabla\Theta - \varepsilon - \nabla \cdot \boldsymbol{T} &= D_t E_\theta\,, \qquad E_\theta \equiv \frac{1}{2}\left\langle \theta^2 \right\rangle\ .
\end{aligned} \tag{2}$$

$$\mathcal{P}_{ij} \equiv -\tau_{ik}\frac{\partial U_j}{\partial x_k} - \tau_{jk}\frac{\partial U_i}{\partial x_k}\,,\ \ \text{Energy Production tensor}\,; \tag{3}$$

$$\mathcal{C}_{ij} \equiv -g\beta(F_i\,\delta_{j\,z} + F_j\,\delta_{i\,z}), \quad \text{Tensor of Conversion } E_K \Rightarrow E_\theta;$$
$$\mathcal{A}_i \equiv A_i^{S\Theta} + A_i^{SU} + A_i^{E\theta}, \quad \text{Heat-flux production:} \tag{4}$$
$$A_i^{S\Theta} \equiv -\tau_{ij}\,\frac{\partial\Theta}{\partial x_j}, \qquad A_i^{SU} \equiv -\boldsymbol{F}\cdot\boldsymbol{\nabla}\,U_i\,, \qquad A_i^{E\theta} \equiv 2g\beta\,E_\theta\,\delta_{i\,z}\,;$$

Pressure-rate-of-strain traceless tensor:

$$\mathcal{R}_{ij} = \left\langle \frac{\widetilde{p}}{\rho_b}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\rangle = R_{ij}^{RI} + R_{ij}^{IP} + R_{ij}^{IC} \tag{5}$$

Poisson's equation for fluctuating pressure:

$$\Delta\widetilde{p} = -\nabla_i\nabla_j\left(u_iu_j - \left\langle u_iu_j\right\rangle + U_iu_j + U_ju_i\right) + g\beta\nabla_z\theta \quad \Rightarrow$$

$$R_{ij}^{RI} \simeq -\gamma_{RI}\left(\tau_{ii} - \frac{2E_K}{3}\right)\delta_{ij} - \widetilde{\gamma}_{RI}\tau_{ij}\left(1-\delta_{ij}\right), \text{ “Return-to-Isotropy” },$$

$$R_{ij}^{IP} \simeq -C_{IP}\left(\mathcal{P}_{ij} - \frac{\mathrm{Tr}\left\{\mathcal{P}_{ij}\right\}}{3}\,\delta_{ij}\right), \quad \text{“Isotropization-of-Production”},$$

$$R_{ij}^{IC} \simeq -C_{IC}\left(\mathcal{C}_{ij} - \frac{\mathrm{Tr}\left\{\mathcal{C}_{ij}\right\}}{3}\,\delta_{ij}\right), \quad \text{“Isotropization-of-Conversion”};$$

Pressure-temperature-gradient vector:

$$\mathcal{B}_i \equiv \left\langle \frac{\widetilde{p}}{\rho_b} \frac{\partial \theta}{\partial x_i} \right\rangle = B_i^{RD} + B_i^{SU} + B_i^{E\theta}, \tag{6}$$

$$B_i^{RD} \approx -\gamma_{RD} F_i , \quad \text{Renormalization of Dissipation } \epsilon_i ,$$

$$B_i^{SU} \approx (1 - C_{SU})(\boldsymbol{F} \cdot \boldsymbol{\nabla})U_i , \quad \text{Renormalization of Production } A_i^{SU} ,$$

$$B_i^{E\theta} \approx -2\, g\beta\, (C_{E\theta} + 1) E_\theta\, \delta_{i\,z} , \quad \text{Renormalization of Production } A_i^{E\theta} .$$

The dissipation rates of

$$\varepsilon_{ij} \equiv 2\,\nu \left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right\rangle \approx \left(\nu \frac{a^2}{\ell^2} + \gamma_{uu} \right) E_K \delta_{ij} + \left[\nu \frac{\widetilde{a}^2}{\ell^2} + 0 \right] \tau_{ij}(1 - \delta_{ij}) , \quad \text{Reynolds-stress} ;$$

$$\epsilon_i \equiv (\nu + \chi) \left\langle \frac{\partial \theta}{\partial x_k} \frac{\partial u_i}{\partial x_k} \right\rangle \approx \left[(\nu + \chi) \frac{a_1^2}{\ell^2} + 0 \right] F_i , \quad \text{Heat-flux} ; \tag{7}$$

$$\varepsilon \equiv \chi\, |\boldsymbol{\nabla} \theta|^2 \approx \left(\chi \frac{a_2^2}{\ell^2} + \gamma_{\theta\theta} \right) E_\theta , \quad \text{"Temperature-energy"} ;$$

Nonlinear dissipation frequencies γ's can be estimated as:

$$\gamma_{uu} \sim \frac{\langle u\, u\, u \rangle}{\ell\, \langle u\, u \rangle} \quad \gamma_{\theta\,\theta} \sim \frac{\langle \theta\, \theta\, u \rangle}{\ell\, \langle \theta\, \theta \rangle} , \quad \ell\text{- outer scale, } a\text{'s - constants;} \tag{8}$$

T_{ijk}, T_{ij}, $\boldsymbol{T}$ describe spacial transport, which currently will be neglected.

– "K-41" closure for the relaxation frequencies:

$$\gamma_{uu} = c_{uu}\frac{\sqrt{E_K}}{\ell}\,,\ \gamma_{RI} = C_{RI}\gamma_{uu}\,,\ \widetilde{\gamma}_{RI} = \widetilde{C}_{RI}\gamma_{RI}\,, \gamma_{\theta\theta} = C_{\theta\theta}\gamma_{uu}\,,\ \gamma_{RD} = C_{u\theta}\gamma_{uu}\,,$$

E_K – kinetic energy, ℓ – outer scale of turbulence (distance to the wall,

c_{uu}, C, $\widetilde{C}$ – fitting constants

- NS-MM: Minimal Model for Neutral Stratification $\Theta, \theta \to 0$

has five fitting constants: c_{uu}, C_{RI}, $\widetilde{C}_{RI}$ a and $\widetilde{a}$

$$\begin{aligned}
-\tau_{xy}(y) + \nu_0 S(y) &= P(y)\,, \\
[\Gamma_{uu} + 3\widetilde{\gamma}_{RI}]\tau_{xx} &= \widetilde{\gamma}_{RI}\tau_{xy} - 2S\tau_{xy}, \quad [\Gamma_{uu} + 3\widetilde{\gamma}_{RI}]\tau_{yy} = \widetilde{\gamma}_{RI}\tau_{xy}\,, \\
[\Gamma_{uu} + 3\widetilde{\gamma}_{RI}]\tau_{zz} &= \widetilde{\gamma}_{RI}\tau_{xy}\,, \qquad\qquad \widetilde{\Gamma}_{RI}\tau_{xy} = -S\tau_{yy}\ .
\end{aligned}$$

Here $\Gamma_{uu} \equiv \gamma_{uu} + \nu_0\dfrac{a^2}{y^2}$ and $\widetilde{\Gamma}_{RI} \equiv \widetilde{\gamma}_{RI} + \nu_0\dfrac{\widetilde{a}^2}{y^2}$.

These Eqs. were solved analytically with accuracy of few %

• **Reminder: Wall units and characteristic regions of TBL**

$u_\tau \equiv \sqrt{\mathcal{P}_0(t)}\,, \quad \ell_\tau \equiv \frac{\nu}{u_\tau}\,,$

$\tau \equiv \frac{\nu}{\mathcal{P}}\,,$

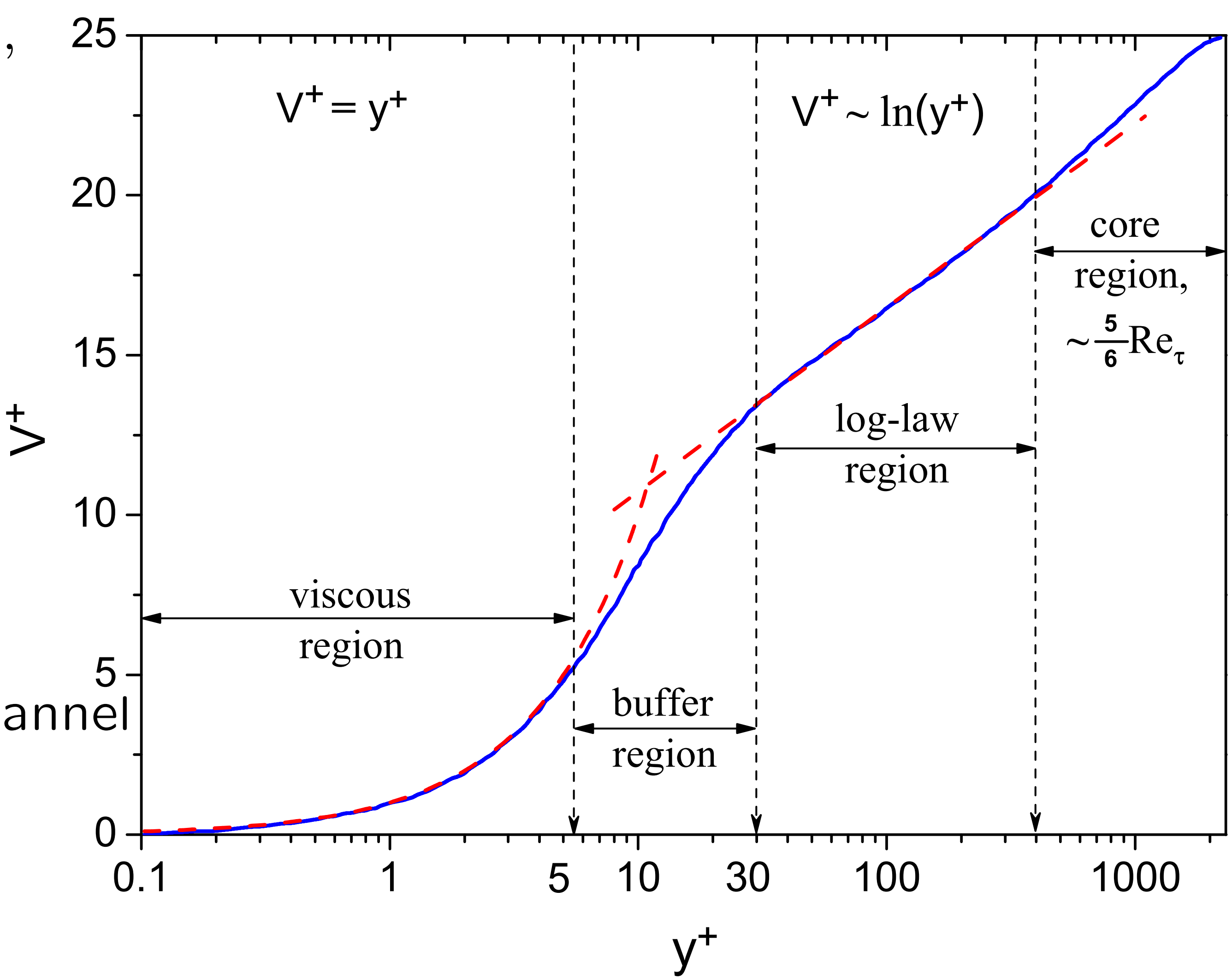
$V^+ \equiv \frac{V}{u_\tau}\,, \quad K^+ \equiv \frac{K}{u_\tau^2}\,,$

$z^+ \equiv \frac{z}{\ell_\tau}\,, \quad t^+ \equiv \frac{t}{\tau}\,, \dots$

Friction Reynolds # :

$\mathrm{Re}_\lambda \equiv L^+(t),$

L -half width of the channel



Von-Karman law, red dash straight line $(- - -)$, $V^+(y^+) = \frac{1}{\kappa} \ln y^+ + B\,,$

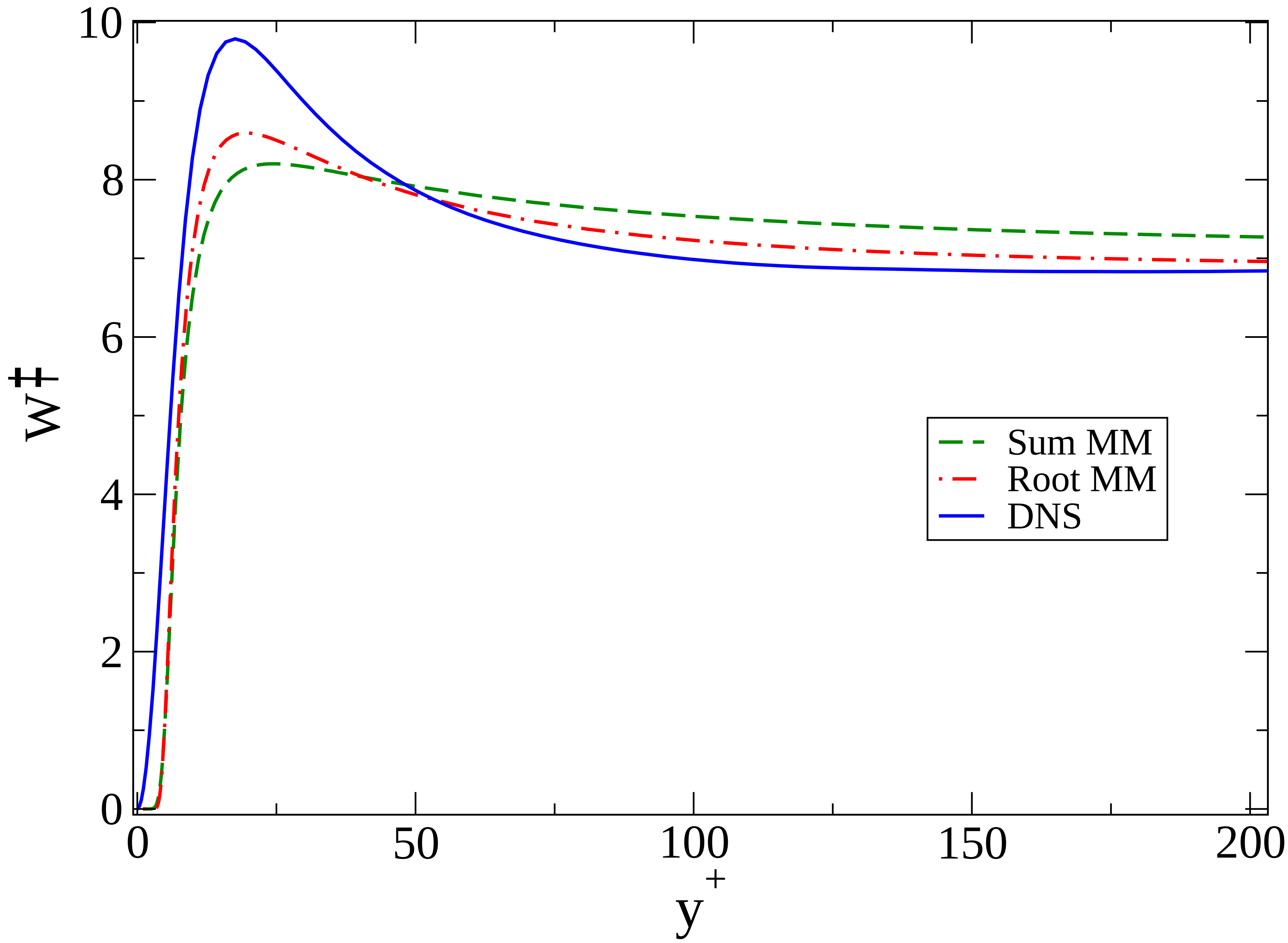
$\kappa \approx 0.436, \quad B \approx 5.2.$

– Anisotropy of the outer layer: DNS, LES & Experiment vs MM result:

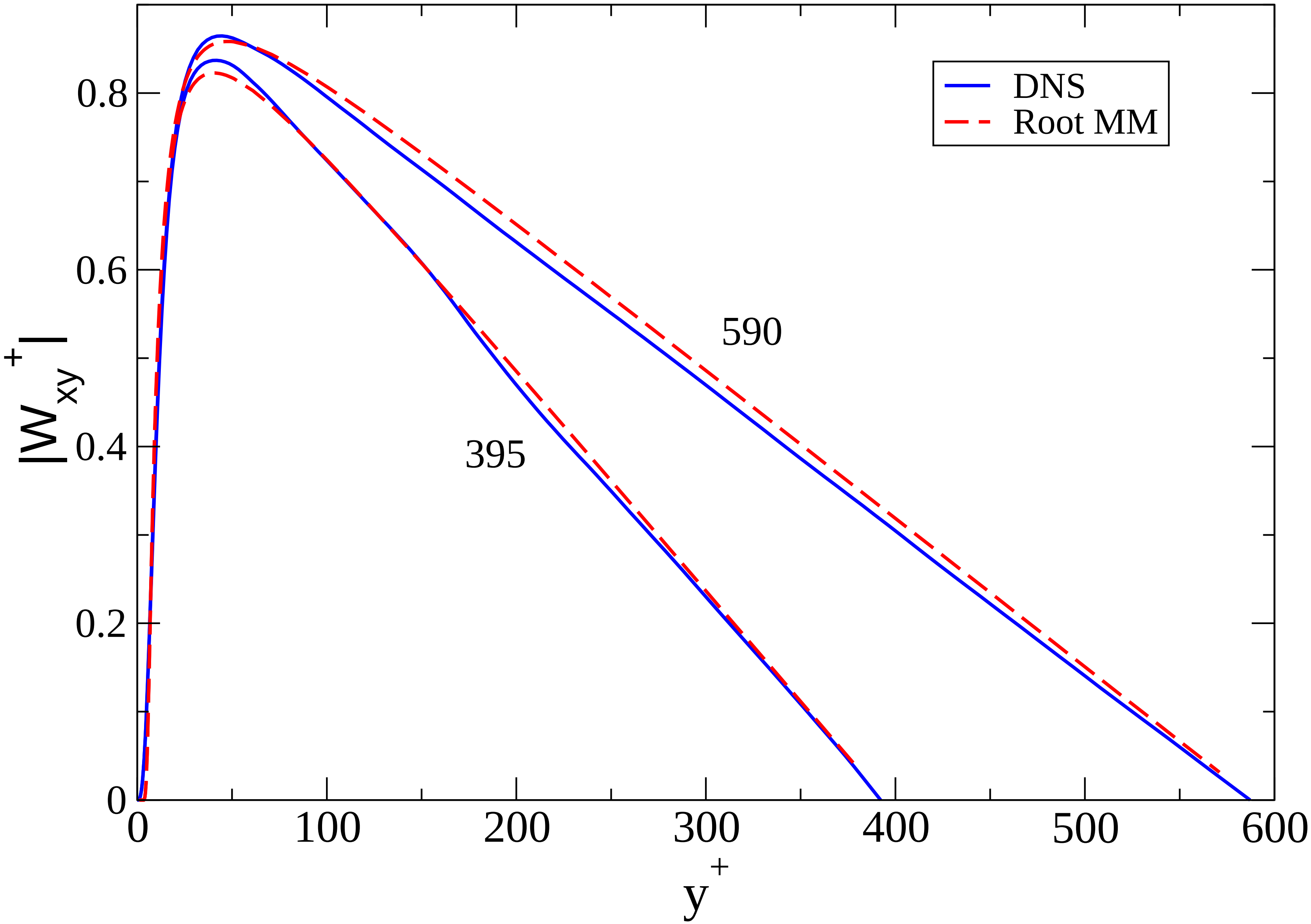
Asymptotic values of the relative kinetic energies $\widetilde{\tau}_{ii}$ in the log-law region $y^{\ddagger} > 200$ taken from DNS by Moser, R. G., Kim, J., and Mansour, N. N.: (1999) with $\mathrm{Re}_\lambda = 395, 590$) LES by Carlo M. Casciola (2004) and experiment by A. Agrawal, L. Djenidi and R.A. Antonia (2004), in a water channel with $\mathrm{Re}_\lambda = 1000$.

The last column presents the predictions of the minimal model.

$\widetilde{\tau}_{ii,\infty}$, $\downarrow ii \downarrow$	DNS	LES	Water channel	Minimal Model
xx	≈ 0.53	≈ 0.46	0.50 ± 0.01	1/2
yy	≈ 0.22	≈ 0.27	0.25 ± 0.02	1/4
zz	≈ 0.27	≈ 0.27	0.25 ± 0.02	1/4

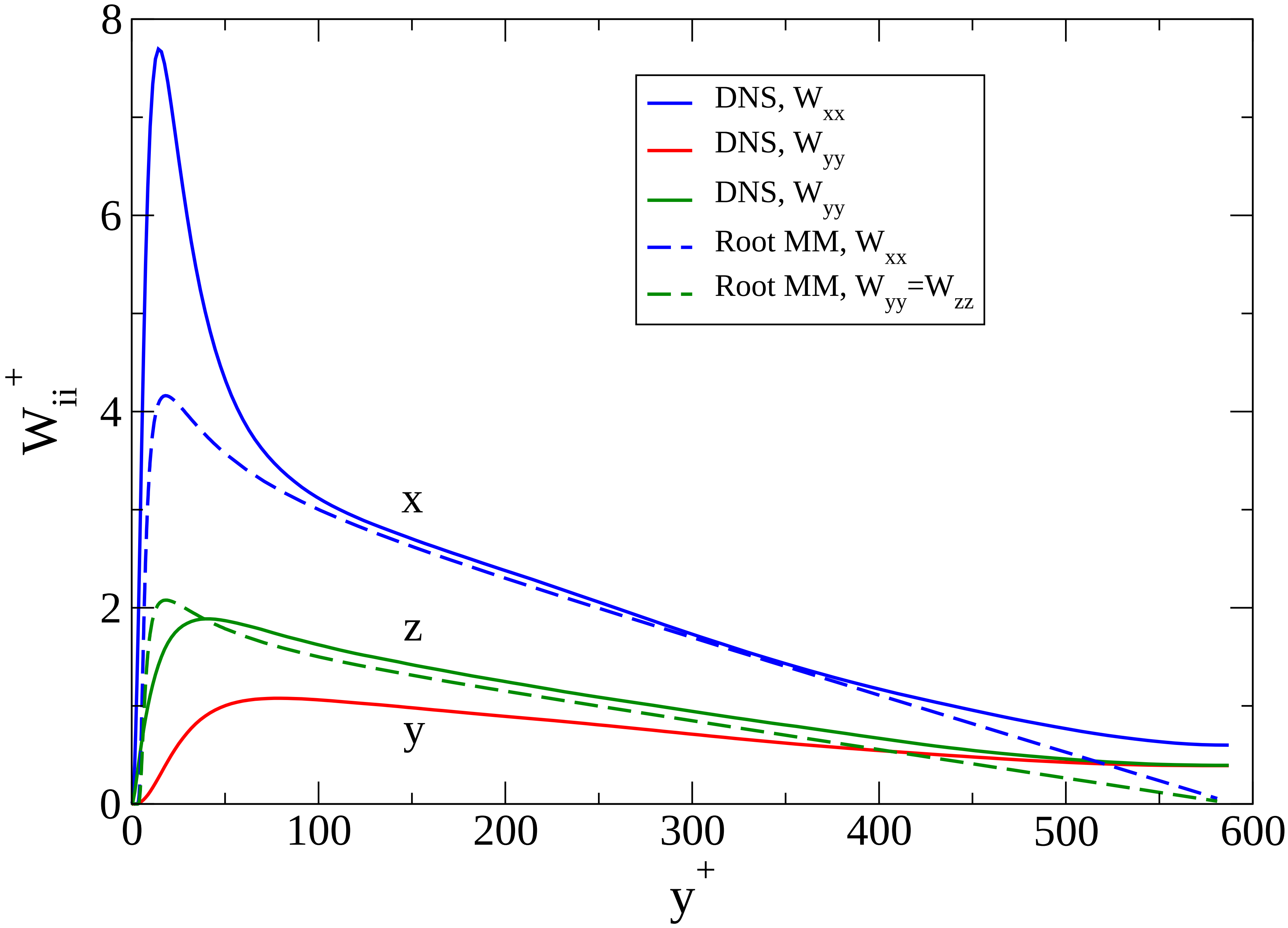


MM comparison with DNS for the total energy profile, $Re_\lambda = 590$

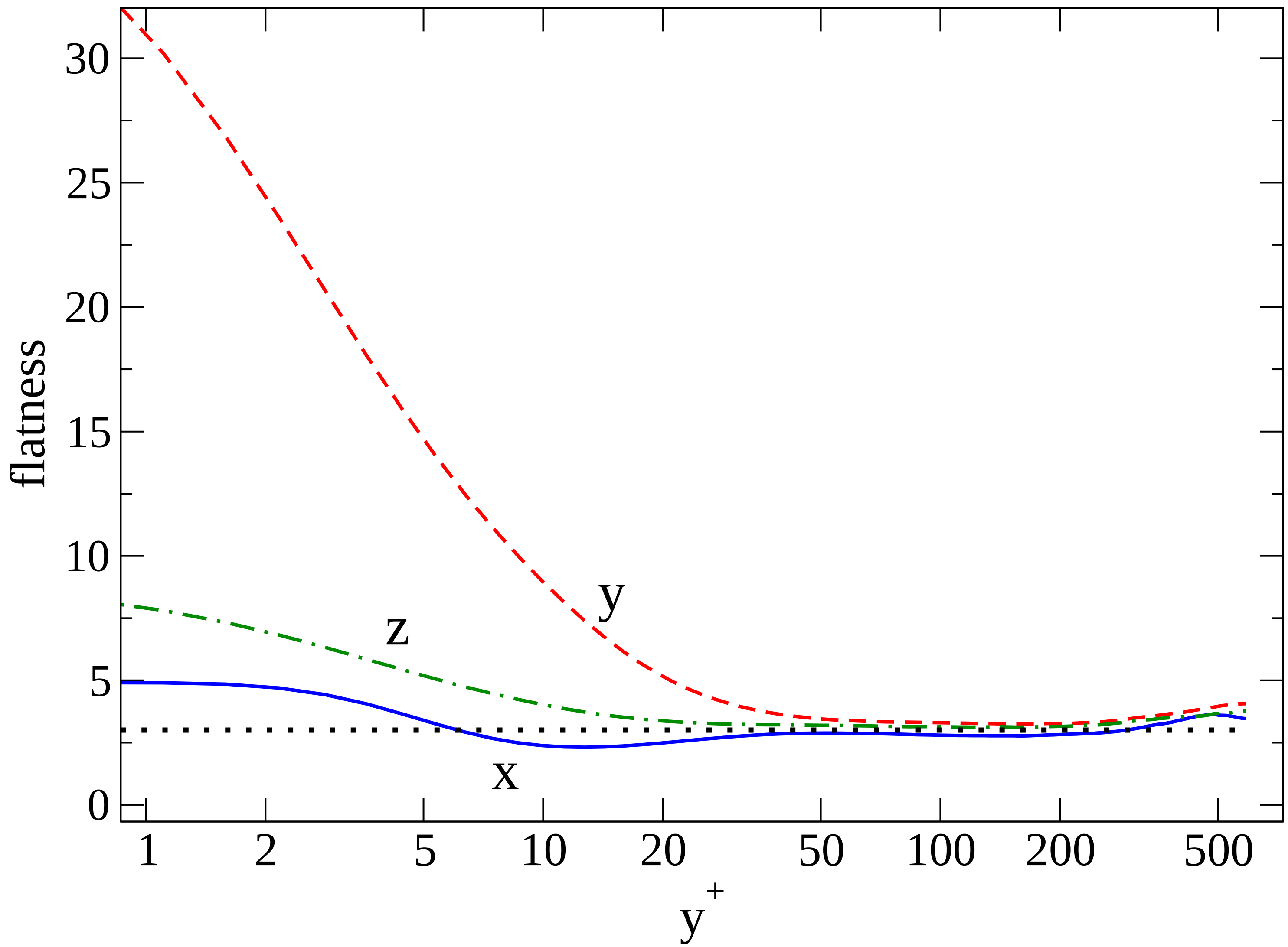


Root-MM comparison with DNS for the Reynolds-stress,

$Re_\lambda = 395,\ 590$



Root-MM comparison with DNS for the diagonal W_{ii}^{+}, $\mathrm{Re}_{\lambda} = 590$



Profiles of the flatness of the u, v and w components of the turbulent velocity fluctuations, DNS data for $\mathrm{Re}_\tau = 590$. Horizontal dotted line shows Gaussian value for the flatness, equal to three.

- **First summary: Strength & Limitations of the NS Minimal Model**

ns-MM with the given set of five parameters describes five profiles:

- $V(y)$ – with accuracy of $\simeq 1\%$ – almost throughout the channel
- the Reynolds stress $\tau_{xy}(y)$ – with accuracy of few percents
- the kinetic energy $\frac{1}{2}W(y)$ – with reasonable, semi-quantitative accuracy, including the position and width of its peak in the buffer sub-layer
- The profiles of $\tau_{xx}(y)$, $\tau_{yy}(y)$ & $\tau_{zz}(y)$, including $\frac{1}{2}$-$\frac{1}{4}$-$\frac{1}{4}$ distribution.

– NS-MM cannot pretend to describe all the aspects of the turbulent statistics: The MM ignores the quasi-two dimensional character of turbulence, coherent structures in the very vicinity of the wall, *etc.*, *etc.*

The NS MM takes into account the essential physics of wall-bounded turbulence almost throughout the channel flow. With a proper generalization, the MM will be useful in studies of more complicated turbulent flows with temperature stratifications, developing TBL, etc.

Second Story: Temperature stratified TBL

– Balance Eq. outside of the viscous & buffer layers:

(E_K – kinetic energy, $\boldsymbol{F}$ – heat flux, $E_\theta \equiv \left\langle \theta^2 \right\rangle /2$.

$$\tau^+_{xx} = E^+_K - \frac{\ell^\ddagger F^+_z}{2\,\gamma^\ddagger_{uu}}\,, \quad \tau^+_{yy} = \frac{E^+_K}{2}\,, \quad \tau^+_{zz} = E^+_K + \frac{\ell^\ddagger F^+_z}{2\,\gamma^\ddagger_{uu}}\,, \quad \ell^\ddagger \equiv \frac{\ell}{L}\,, \qquad (9)$$

$$\begin{aligned}
\gamma^\ddagger_{uu} E^+_K &= \ell^\ddagger F^+_z - \tau^+_{xz} S^\ddagger_U \Rightarrow S^\ddagger_U - \ell^\ddagger\,, & \tau^+_{xz} \Rightarrow -1\,, \quad S^\ddagger_U \equiv S^+_U \ell^+\,,\\
4\,\widetilde{C}_{RI}\,\gamma^\ddagger_{uu}\tau^+_{xz} &= \ell^\ddagger F^+_x - \tau^+_{zz}\,S^\ddagger_U\,, & F^+_z \Rightarrow -1\,, \quad S^\ddagger_\ominus \equiv S^+_\ominus \ell^+\,,\\
C_{\theta\theta}\,\gamma^\ddagger_{uu} E^+_\theta &= -F^+_z\,S^\ddagger_\ominus \Rightarrow S^\ddagger_\ominus\,, & \gamma^\ddagger_{uu} = c_{uu}\sqrt{E^+_K}\,, \qquad (10)\\
C_{u\theta}\,\gamma^\ddagger_{uu} F^+_x &= -\left(\tau^+_{xz} S^\ddagger_\ominus + C_{SU} F^+_z\,S^\ddagger_U\right) \Rightarrow S^\ddagger_\ominus + C_{SU}\,S^\ddagger_U\ . &\\
C_{u\theta}\,\gamma^\ddagger_{uu} F^+_z &= -\left(\tau^+_{zz} S^\ddagger_\ominus + 2\,C_{E\ominus}\,\ell^\ddagger E^+_\theta\right)\ . &
\end{aligned}$$

Here z-vertical and y-cross-stream directions; ℓ is the outer scale (distance to the wall for neutral stratification);

– Task: Find five functions of $\ell^\ddagger \equiv \ell/L$: $S^\ddagger_U$, $S^\ddagger_\ominus$, E^+_K, E^+_θ and F^+_x.

- **Analysis of the ℓ/L-dependence**

– Strong stratification, $\ell^\ddagger \equiv \ell/L \gg 1$:

$$
\begin{aligned}
E_K^{+\infty}(\ell^\ddagger) &= \frac{{\ell^\ddagger}^{2/3}}{c_{uu}}\left(1 - 4\frac{C_{E\ominus}}{C_{\theta\theta}}\right), \quad S_U^{\ddagger\infty}(\ell^\ddagger) = 2\,\ell^\ddagger\left(1 - \frac{2\,C_{E\ominus}}{C_{\theta\theta}}\right) \Rightarrow S_U^{+\infty} = \frac{2}{L}(...) \\
S_\ominus^{\ddagger\infty}(\ell^\ddagger) &= S_U^{\ddagger\infty}(\ell^\ddagger)\left(\frac{2\,C_{E\ominus}C_{u\theta}}{C_{\theta\theta}} + C_{SU}\right) \Rightarrow S_\ominus^{+\infty} = \frac{2}{L}(\ldots)(\ldots)\,, \qquad (11) \\
E_\theta^{+\infty}(\ell^\ddagger) &= \frac{S_\ominus^{\ddagger\infty}(\ell^\ddagger)}{{\ell^\ddagger}^{1/3} c_{uu} C_{\theta\theta}} \propto {\ell^\ddagger}^{2/3}, \quad F_x^{+\infty}(\ell^\ddagger) = -\frac{2\,C_{E\ominus}\,S_U^{\ddagger\infty}}{{\ell^\ddagger}^{1/3} c_{uu} C_{\theta\theta}} \propto {\ell^\ddagger}^{2/3}.
\end{aligned}
$$

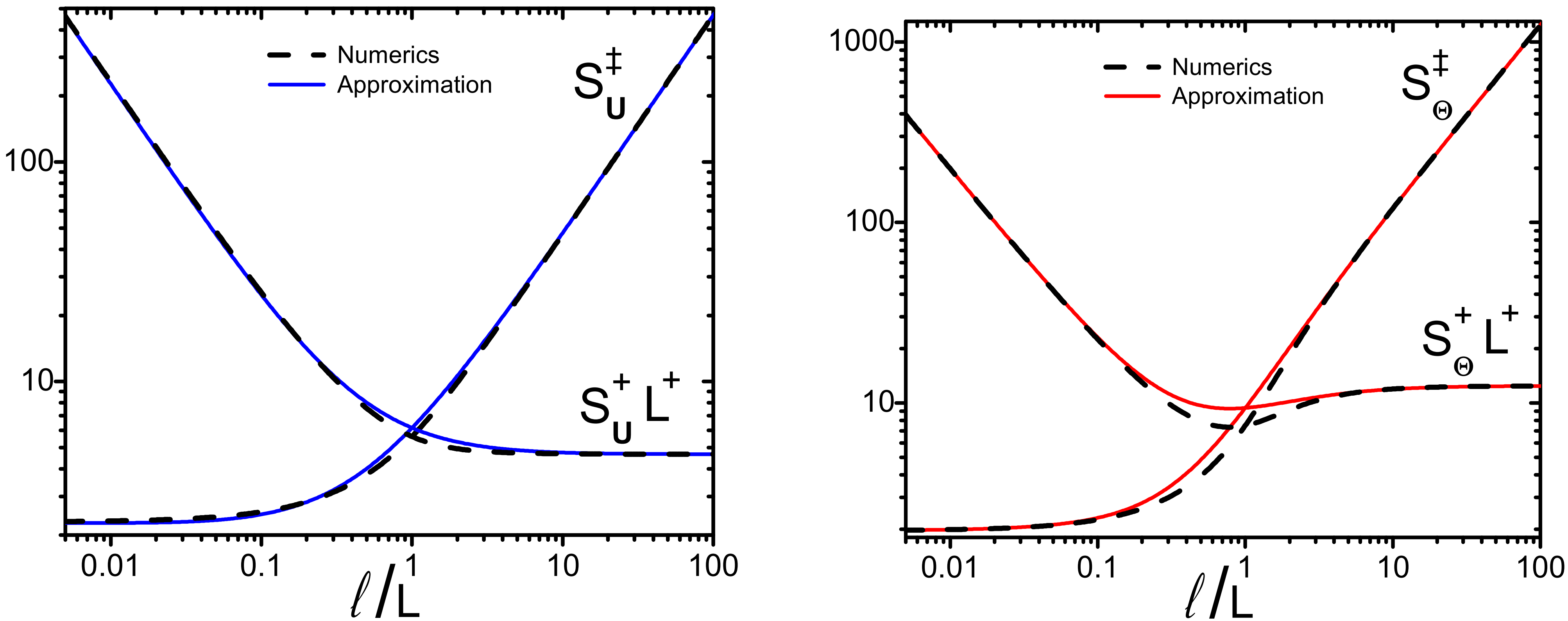
– Solution for all $\ell^\ddagger$ – interpolation formulai:

$$
E_K^{+}(\ell^\ddagger)^{3/2} \simeq E_K^{\infty}(\ell^\ddagger)^{3/2} + \frac{E_K^{+}(0)^2}{\sqrt{E_K^{\infty}(\ell^\ddagger) + E_K^{+}(0)}}, \qquad (12)
$$

$$
S_\ominus^{\ddagger}(\ell^\ddagger) \simeq S_\ominus^{\ddagger\infty}(\ell^\ddagger) + \frac{S_\ominus^{\ddagger}(0) + 6(c_{uu}\alpha)^{4/3}\beta\,\ell^\ddagger}{\left(1 + \alpha\,\ell^\ddagger\right)^{4/3}}, \qquad (13)
$$

where α and β are known combinations of constants C's.

Mean velocity shear S_U^+ and $S_U^\ddagger \equiv S_U^+ \ell^+$ (blue) – Left panel and
Mean temperature shear S_Θ^+ and $S_\Theta^\ddagger \equiv S_\Theta^+ \ell^+$ (red) – Right panel vs. ℓ/L.
Black dashed lines: - - - - - represent analytical approximate solutions



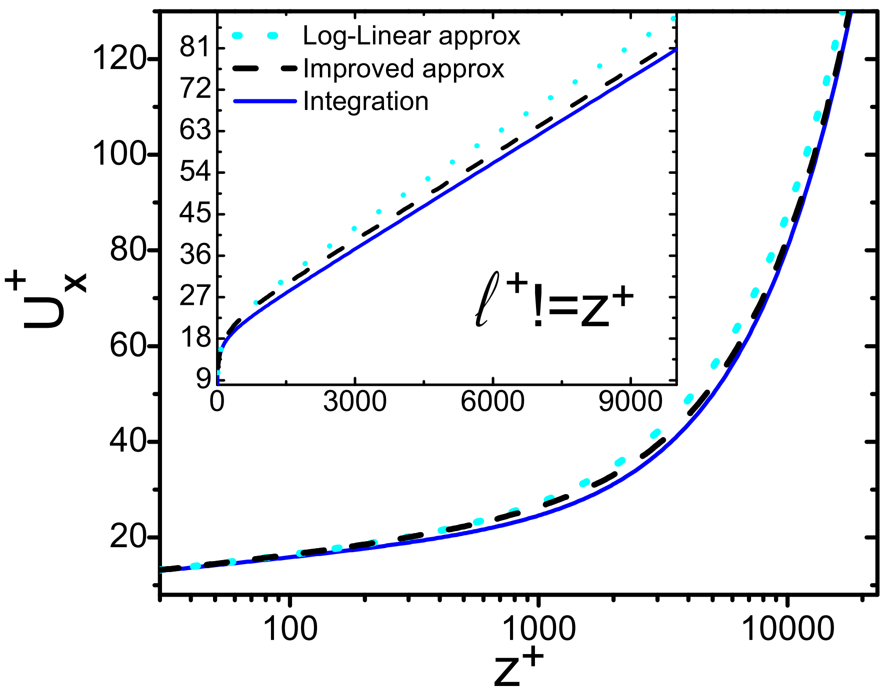
Region $\ell \geq L$ may not be realized in the Nature.

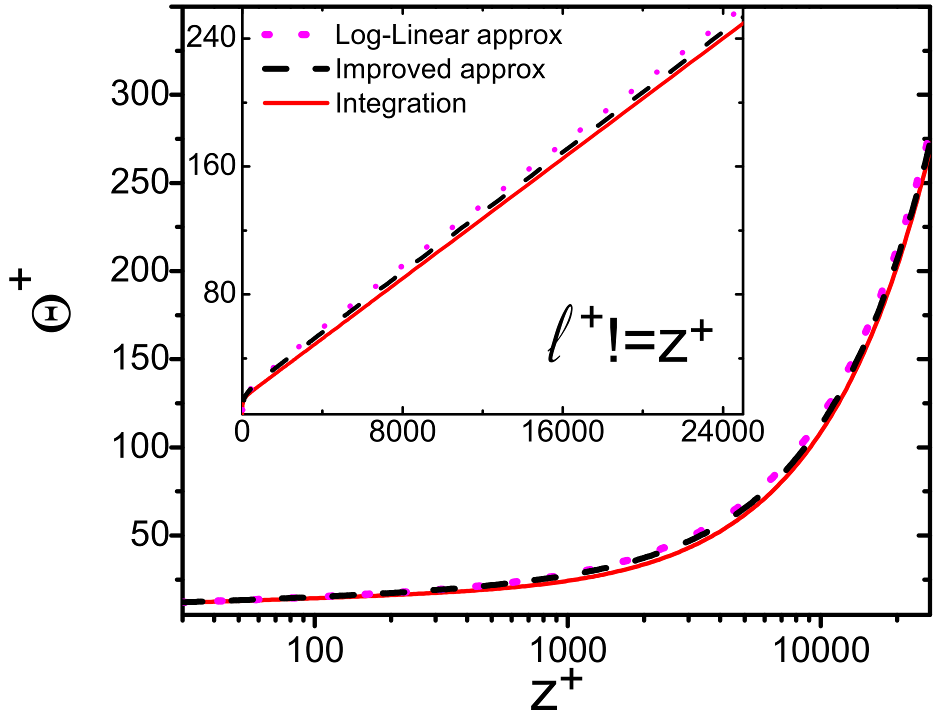
If so, it has eristic, methodological meaning.

– Mean velocity U^+ (blue) and temperature Θ^+(red) vs z^+ for $L^+ = 1000$

- - - & - - - : Log-linear appr. - - - - - - - - - - : Improved approx.

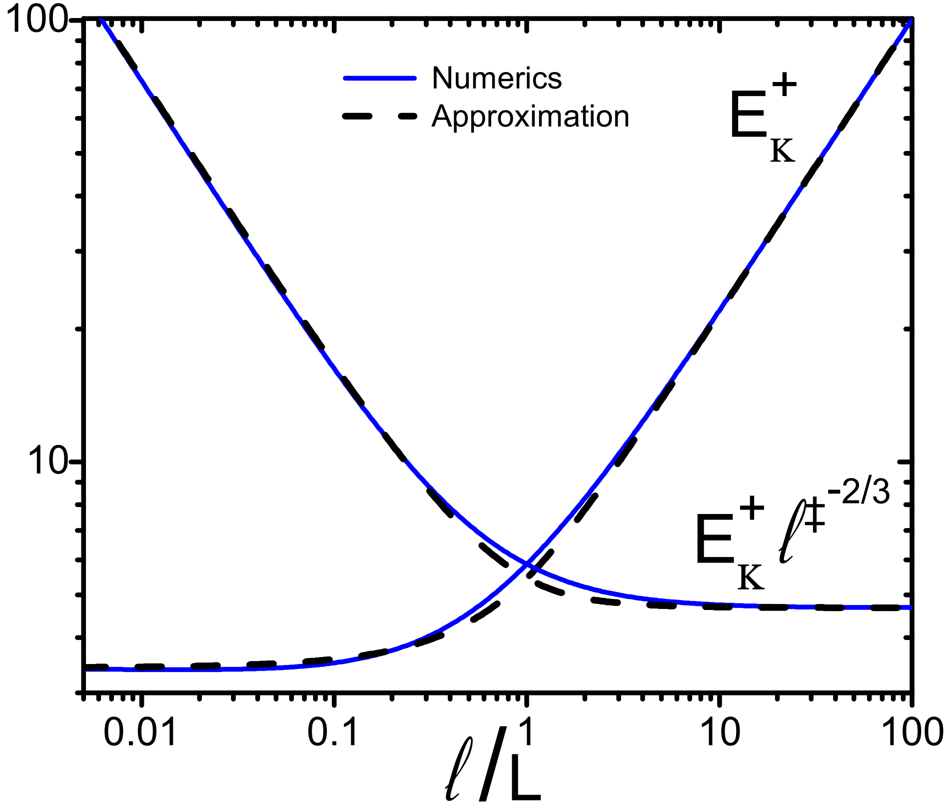
———————- : Numer. Integr. $\frac{1}{\ell} = \sqrt{\frac{1}{z^2} + \frac{1}{L^2}}$

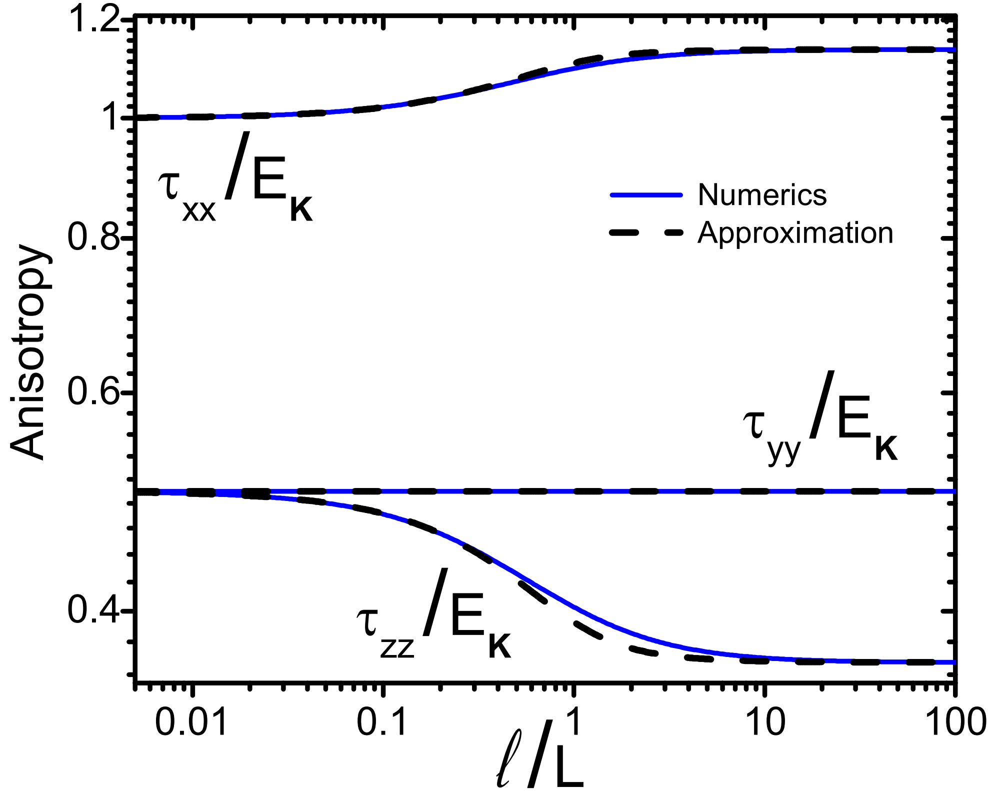




Analytical approximate solutions for Turbulent kinetic energy E_K^+ (Left) and normalized by E_K "partial kinetic energies" τ_{ii} Anisotropy – (Right)

Black dashed lines: - - - - - - represent "exact" numerical solutions





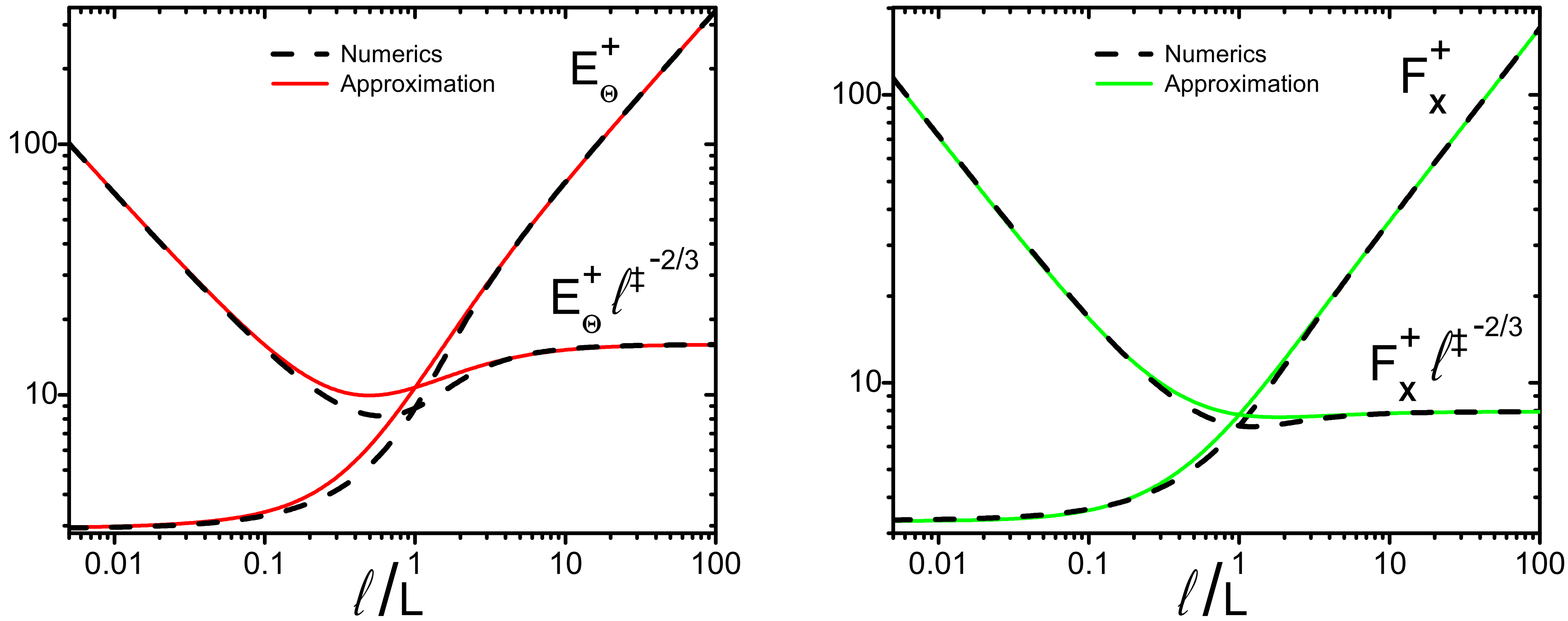
Region $\ell \geq L$ may not be realized in the Nature.

If so, it has eristic, methodological meaning.

Analytical approximate solutions for Turbulent potential energy $E_{\ominus}^{+}$ (Left)

and streamwise component of the heat flux F_x^+ – (Right)

Black dashed lines: - - - - - represent “exact” numerical solutions



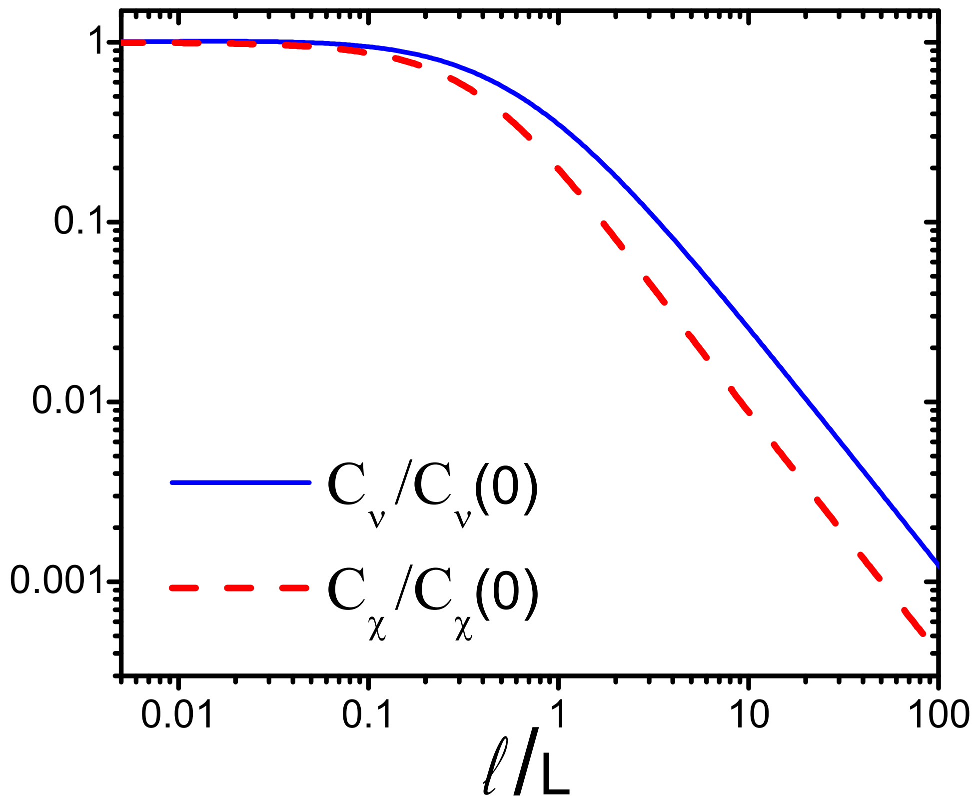
Region $\ell \geq L$ may not be realized in the Nature.

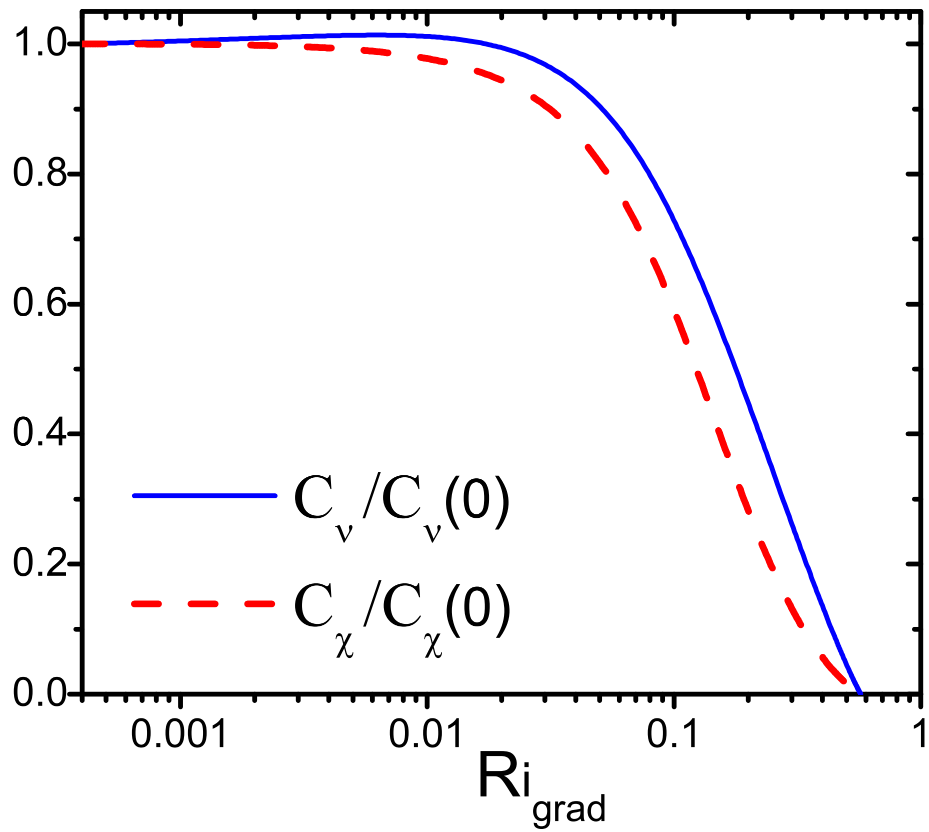
If so, it has eristic, methodological meaning.

Fail of "down-gradient" approximations for the momentum & heat fluxes

Define: $$\tau_{xz} = -\nu_{\mathsf{T}} dU_x/dz\,, \qquad \nu_{\mathsf{T}} \equiv C_\nu\, \ell_z \sqrt{\tau_{zz}}\,, \tag{14}$$

$$F_z = -\chi_{\mathsf{T}} d\Theta/dz\,, \qquad \chi_{\mathsf{T}} \equiv C_\chi\, \ell_z \sqrt{\tau_{zz}}\,, \tag{15}$$





Conclusion: C_ν & C_χ cannot be considered as constants in any reasonable approximation

• Second summary: Fail of the down-gradient approximation, internal waves and more...

Presented approach takes into full account all the second-order statistics conserving the total mechanical energy. It results in the analytic solution of the profiles of all mean quantities and all second-order correlations as a function of ℓ/L, including (ℓ/L)-dependence of effective turbulent diffusion coefficients C_ν and C_χ. This removes the unphysical predictions of previous theories, based on the assumption $C_{...}$const, and thus violating the conservation of energy.

Role of internal gravity waves

in the upper wave-dominated TBL with large $\mathrm{Ri}_{\mathrm{grad}}$

• Third Story: Time-Developing (TD) TBL

– Minimal model for TD TBL NSE $\Rightarrow$

$$\nu S(z,t) + \tau(z,t) = P(z,t)\,, \quad P(z,t) \equiv P_0(t) + \frac{\partial}{\partial t}\int_0^z dz'\, V(z',t)\ .$$

$P(z,t)$ is z,t-dependent momentum flux toward the wall. Integration constant $P_0(t)$ is the wall shear stress – the momentum flux at the wall.

– Exact Balance of the Kinetic energy: NSE $\Rightarrow$

$$\frac{\partial K}{\partial t} + \mathcal{E} + \frac{\partial T'}{\partial z} = \tau S\,,$$

– Our standard Closure for the rate of energy dissipation:

$$\mathcal{E} \equiv \nu\langle s_{ij}s_{ij}\rangle \ \Rightarrow\ \left[\nu\left(\frac{a}{z}\right)^2 + \frac{b\sqrt{K}}{z}\right]K \ \Rightarrow\ \frac{b\sqrt{K}}{z}\ .$$

New – Turbulent diffusion approximation for the spacial energy flux:

$$T'(z) \Rightarrow -D(z)\frac{\partial}{\partial z}K(z,t)\,, \quad D(z,t) = d\,z\sqrt{K(z,t)}\ .$$

New – Closure for the Reynolds stress: $\dfrac{\tau(z,t)}{K(z,t)} \equiv c^2(z) \Rightarrow c_\infty^2 \equiv c^2.$

- Self-consistent factorized asymptotical solution of the MM

– Suggested asymptotical factorization

$$K(z,t) \;=\; \frac{P_0(t)}{c^2}\,k(x)\,, \quad x \equiv \frac{z}{\mathcal{Z}(t)}\,, \quad P(x,t) = P_0(t)\,k(x)\,, \qquad \text{(Fac)}$$

$$S(z,t) \;\equiv\; \frac{\partial V(z,t)}{\partial z} = \frac{\sqrt{P_0(t)}}{\kappa\,\mathcal{Z}(t)}\,s(x)\,, \quad V(z,t) = \sqrt{P_0}\Big[B + \frac{1}{\kappa}\int\limits_{1/\mathrm{Re}}^{x=z/\mathcal{Z}} s(x')dx'\Big]\,,$$

– Factorization of the mechanical and energy balance

With the suggested factorization (Fac) partial differential balance Eqs. for $S(z,t)$ & $K(z,t)$ turn to the ordinary differential Eqs. for $k(x)$ & $s(x)$ with explicit form of $P_0(t)$ & $\mathcal{Z}(t)$:

$$\frac{d\,k(x)}{d\,x} = -\frac{\alpha}{\kappa}\,x\,s(x)\,, \;\text{(MecBal)}; \quad P_0(t) = \left[\frac{\kappa V_\infty}{\ln(t/\tau)}\right]^2, \quad \mathcal{Z}(t) = \alpha\,t\sqrt{P_0(t)}\,,$$

$$b\,k(x)s(x) = -\alpha\,c\,x\frac{dk(x)}{dx} + \frac{b[k(x)]^{3/2}}{x} - d\frac{d}{dx}x\sqrt{k(x)}\,\frac{dk(x)}{dx}\,. \quad \text{(EnBal)}$$

- Approximate Asymptotical Analytical solution for TD-TBL

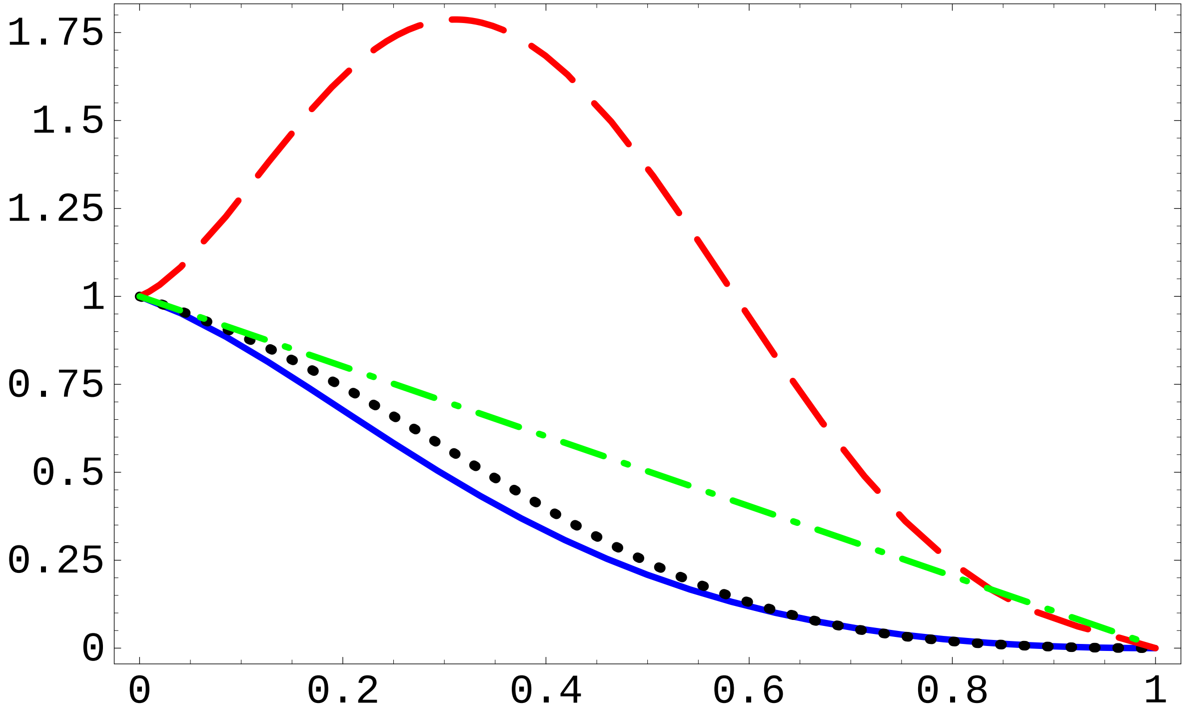
$$\boldsymbol{k(x) = (1-x)^2(1+\beta_1 x+\beta_2 x^2+\beta_3 x^3+\beta_4 x^4)^2}\,, \qquad \text{(Guess)}$$

$$\alpha = \kappa\left[\int_0^1 x s(x) dx\right]^{-1}, \ \beta_1 = 1-\frac{\beta_0}{2}, \ \beta_2 = \frac{b+\beta_0[(\beta_0-2(2+\beta_3+\beta_4)]\,d}{2\beta_0\,d},$$

$$\beta_3 = 3-\frac{\beta_0}{2}-2\beta_4-\frac{5d}{4\beta_0}, \quad \beta_4 = \frac{-b^3+b^2\beta_0^2 d+104b\beta_0^2+24(\beta_0-8)\beta_0^3 d^3}{48\beta_0^3 d^3},$$

$$\alpha \approx 0.431\,, \quad \beta_1 \approx 0.503\,, \quad \beta_2 \approx -2.17\,, \quad \beta_3 \approx 1.90\,, \quad \beta_4 \approx -0.555\,.$$

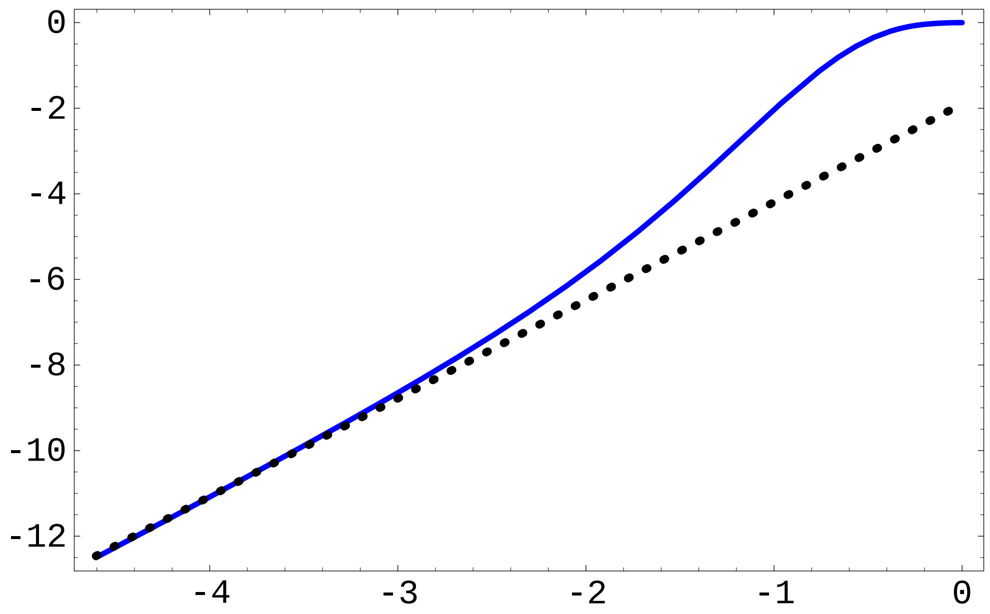
- $\boldsymbol{k(x)}$, Eq.(Guess), blue line
- $\boldsymbol{x\,s(x)}$, (EnBal) with $\boldsymbol{k}(x)$, red dashed line — — ——,
- $\boldsymbol{k(x)}$, from (MecBal) with $\boldsymbol{x\,s(x)}$, black dots $\cdots\cdot$
- small x asymptote $\boldsymbol{k}(x) = 1-x$, dash- dotted line $\cdot - \cdot - \cdot - \cdot -$

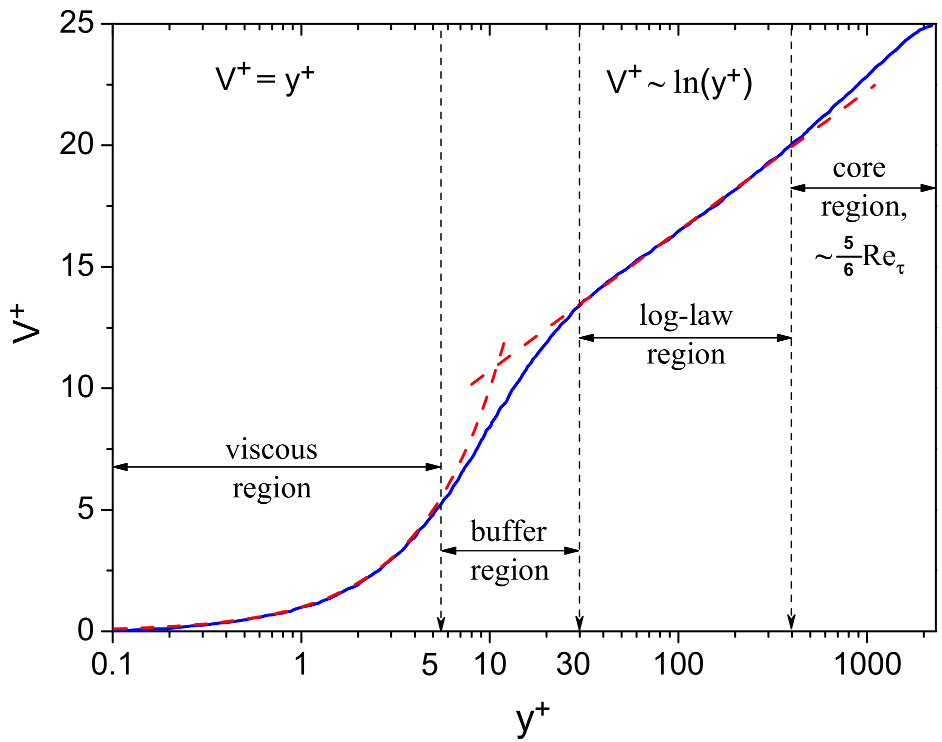


- **Discussion of the predictions**

DNS for a channel flow (Right)

— Mean velocity profile with wake $\mathbf{\Delta V^+ \approx 1.97}$ (Left)

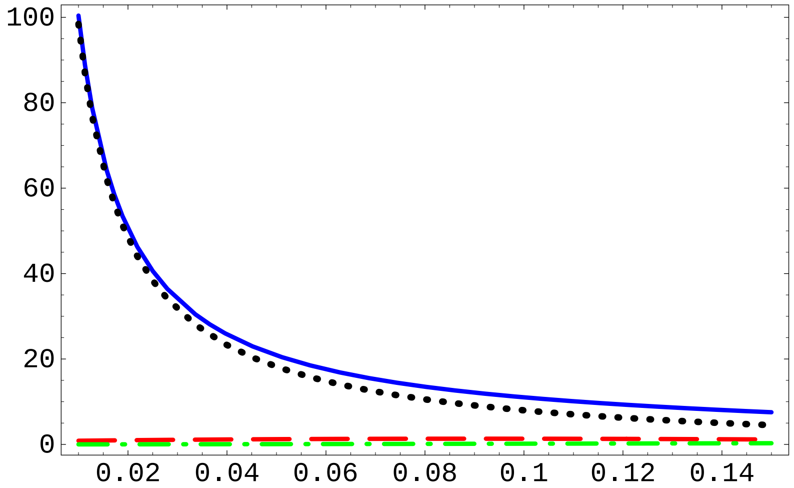


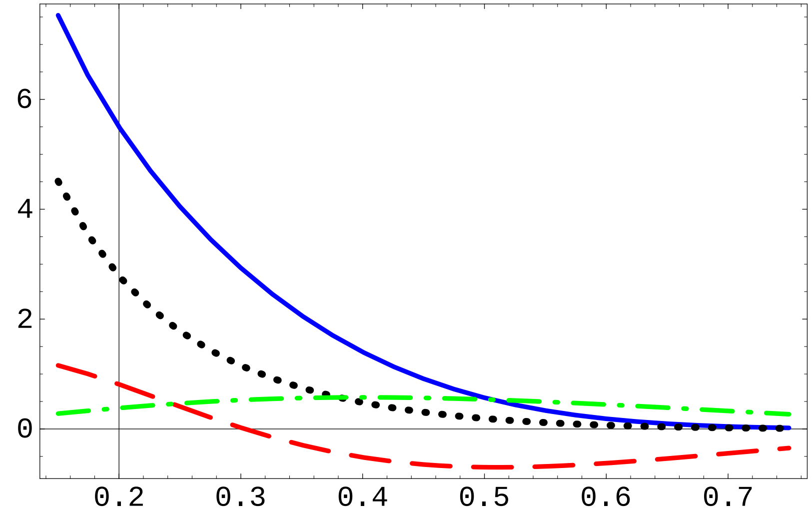


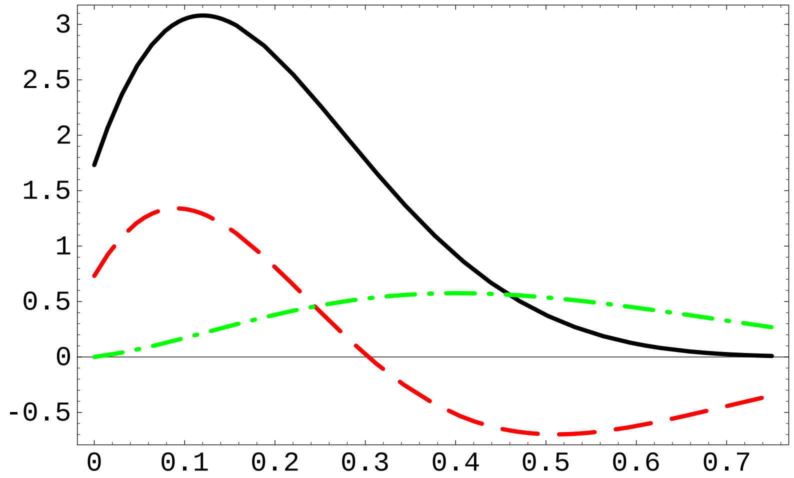
Similarity for the developing TBL and channel flow:

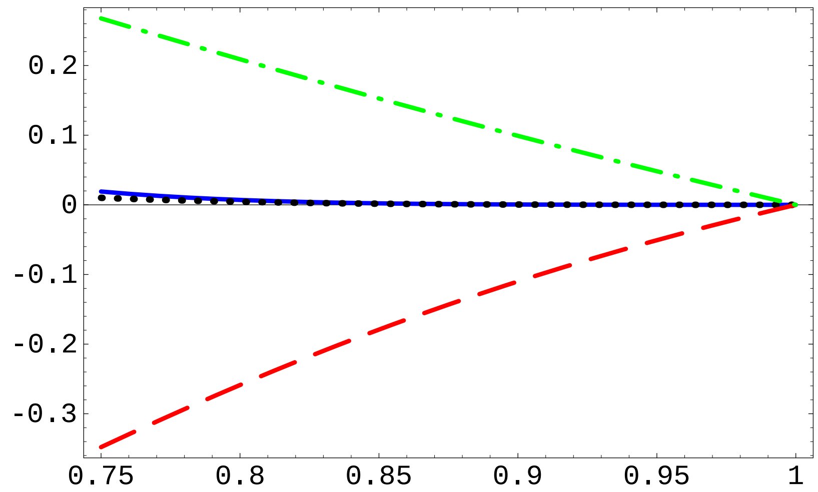
$P(x) \approx 1 - x$ in both cases

• **Energy balance:** Production term, blue solid lines, Dissipation, – black dots ($\cdots$), turbulent diffusion, dashed line ($---$), temporal term $\propto \partial K/\partial t$ – green dash-dotted lines, ($-\cdot-\cdot-$), the energy input (difference between the energy production and energy dissipation) – black solid line.









Third Summary: What do we learn

For the temporal developing TBL we have analytical predictions for

- the shear and the mean velocity profile with wake contribution;
- for the profile of the Reynolds stress and kinetic energy, which in the first half of TBL is close to that in the channel flow;
- time dependence of the friction at the wall,

$$v_\tau^2 \equiv P_0(t) \approx [\kappa V_\infty / \ln(t/T)]^2 \ ;$$

- time dependence of the turbulent front velocity

$$d\mathcal{Z}(t)/dt = \kappa^{3/2} V_\infty \Big/ \ln(t/T) \ .$$

For the spacial developing TBL one should replace t with x/V_∞ where x is the distance from the front edge.

THE END