Victor S. L'vov, Baku, January 21, 2009

Turbulent Drag Reduction by Dilute Solution of Polymers

In collaboration with

T.S. Lo (TL), A. Pomyalov (AP), I. Procaccia (IP), V. Tiberkevich (VT)

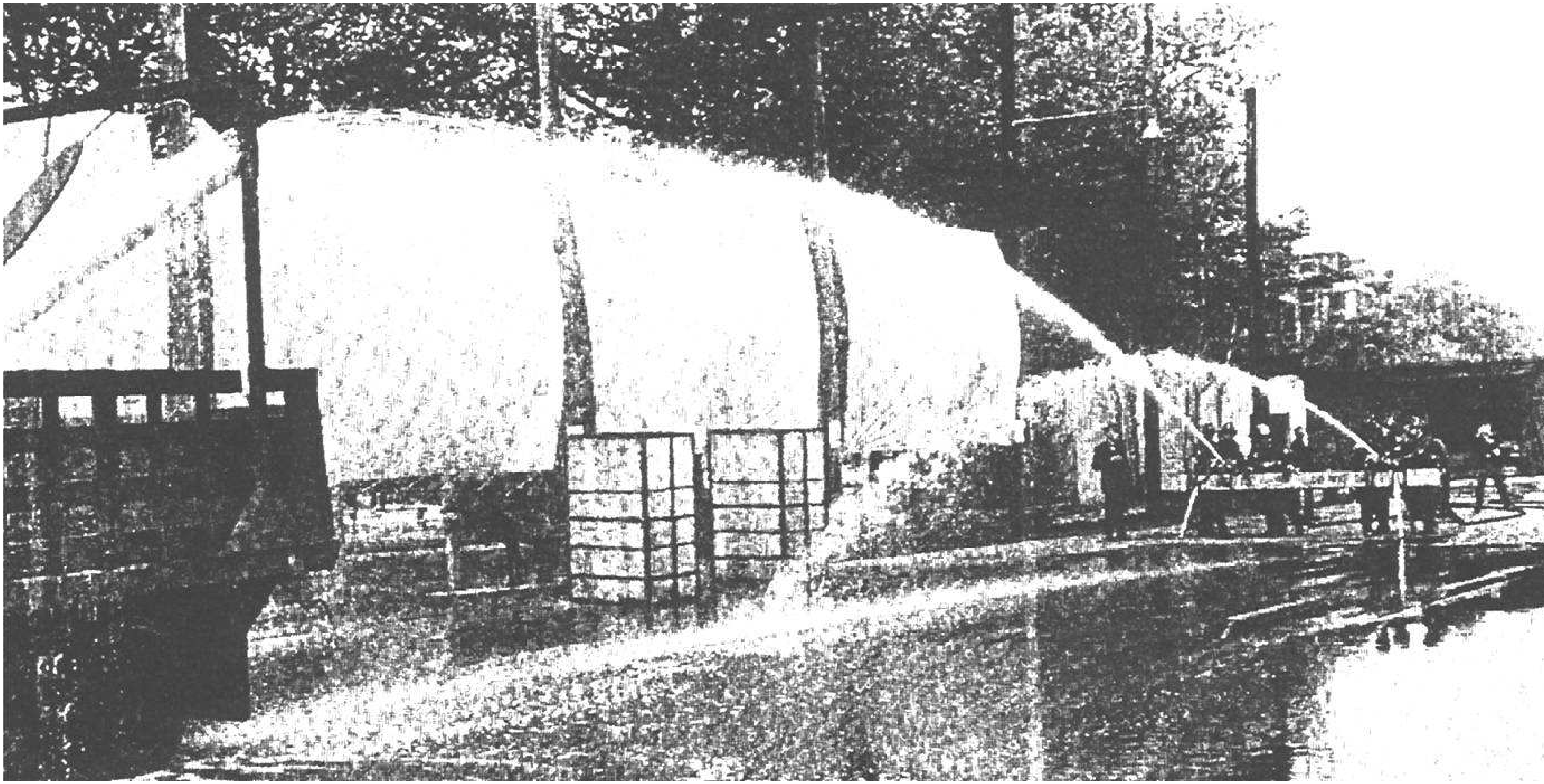
– Weizmann Inst., Rehovot, Israel

Roberto Benzi, Elizabeth de Angelis, Carlo Casciola – Rome Univ., Italy

and Emily Ching – Hong Kong Univ.

Review of our theory: IP, VL, and R. Benzi, *Theory of drag reduction by polymers in wall bounded turbulence,*, Reviews of Modern Physics, **80**, 225-247 (2008), Details in:

1. VL, AP, IP & VT, Drag Reduction by Polymers in Wall Bounded Turbulence, Phys. Rev. Lett, **92**, 244503 (2004).
2. E. de Angelis, C. Casciola, VL, AP, & VT, Drag Reduction by a linear viscosity profile, Phys. Rev. E, **70**, 055301(R) (2004).
3. R. Benzi, VL, IP & VT, Saturation of Turbulent Drag Reduction in Dilute Polymer Solutions, EuroPhys. Lett, **68** 825 (2004).
4. VL, AP, IP & VT, The polymer stress tensor in turbulent shear flow, Phys. Rev. E., **71**, 016305 (2005).
5. VL, AP & VT, Simple analytical model for entire turbulent boundary layer over flat plane, Environmental Fluid Mechanics, **5**, 373-386 (2005).
6. R. Benzi, E. de Angelis, VL, IP & VT. Maximum Drag Reduction Asymptotes and the Cross-Over to the Newtonian plug, Journ. of Fluid Mech., **551**, 185 (2006)
7. R. Benzi, E.S.C. Ching, TL, VL & IP. Additive Equivalence in Turbulent Drag Reduction by Flexible & Rodlike Polymers Phys. Rev. E **72**, 016305 (2005).
8. R. Benzi, E. de Angelis, VL & IP. Identification and Calculation of the Universal Maximum Drag Reduction Asymptote by Polymers in Wall Bounded Turbulence, Phys. Rev. Letts., **95** No 14, (2005)
9. Y. Amarouch‘ene, D. Bonn, H. Kellay, TL, VL, IP, Reynolds number dependence of drag reduction by rod-like polymers, J. Fluid Mech., submitted, Also nlin.CD/0607006



“... the increase in water flow by adding only 30 wppm of Polyethylene oxide (PEO) allows firemen to deliver the same amount of water to a higher elevation from a 1.5-inch hose which can be handled by two men, instead of four men to handle 2.5-inch hose fed only by water ...”

History: experiments, engineering developments, ideas & problems

– B.A.Toms, 1949: An addition of $\sim 10^{-4}$ weight parts of long-chain polymers can suppress the turbulent friction drag up to 80%.

– This phenomenon of "drag reduction" is intensively studied (by 1995 there were about 2500 papers, now we have many more) and reviewed by Lumley (1969), Hoyt (1972), Landhal (1973), Virk (1975), McComb (1990), de Gennes (1990), Sreenivasan & White (2000), and others.

– In spite of the extensive – and continuing – activity the fundamental mechanism has remained under debate for a long time, oscillating between Lumley's suggestion of importance of the polymeric contribution to the fluid viscosity and de Gennes's idea of importance of the polymeric elasticity. Some researches tried to satisfy simultaneously both respectable.

– Nevertheless, the phenomenon of drag reduction has various technological applications from fire engines (allowing a water jet to reach high floors) to oil pipelines, starting from its first and impressive application in the Trans-Alaska Pipeline System.

Trans-Alaska Pipeline System

$L \approx 800$ Miles, $\oslash = 48$ inches

$\Rightarrow$

TAPS was designed with 12 pump stations (PS) and a throughput capacity of 2.00 million barrels per day (BPD).

Now TAPS operates with only 10 PS (final 2 were never build) with throughput 2.1 BPD, with a total injection of ≈ 250 wppm of polymer "PEO" **drag reduction additive** $\Rightarrow$

(wppm $\equiv$ weight parts per million, 250 wppm $= 2.5 \cdot 10^{-4}$)

- **INFO PEO page:**

Typical parameters of polymeric molecules PEO – Polyethylene oxide ($N \times$ [–CH_2–CH_2–O]) and their solutions in water:

- degree of polymerization $N \approx (1.2 - 12) \times 10^4$,

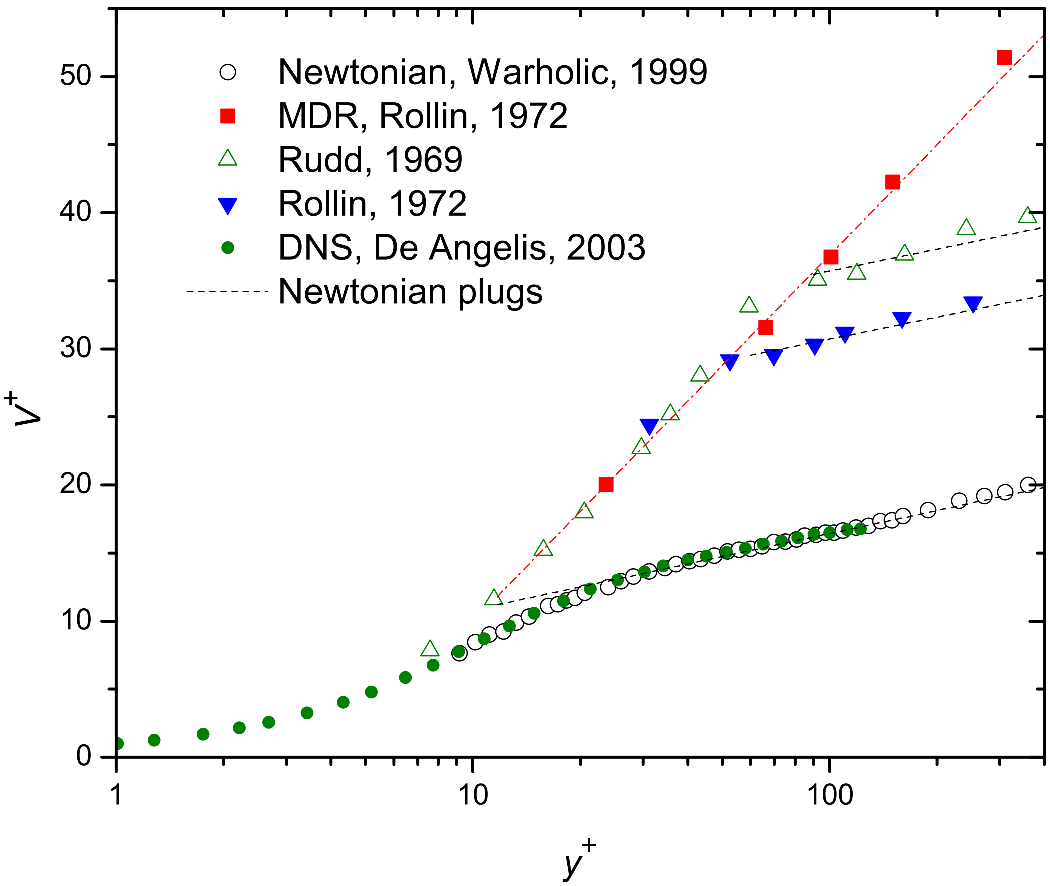
- molecular weight $M \approx (0.5 - 8) \times 10^6$,

- equilibrium end-to-end distance $R_0 \approx (7 - 20) \times 10^{-8}$ m,

- maximal end-to-end distance $R_{\text{max}} \sim \sqrt{N} R_0 \approx 6 \times 10^{-6}$ m;

- typical mass loading $\psi = 10^{-5} - 10^{-3}$. For $\psi = 2.8 \times 10^{-4}$ of PEO $\nu_{\text{pol}} \approx \nu_0$. PEO solutions is dilute up to $\psi = 5.5 \times 10^{-4}$.

• **Essentials of the phenomenon: Virk's universal MDR asymptote & cross-over to Newtonian plug**



$\Leftarrow$ **Mean normalized velocity profile as a function of the normalized distance from the wall in drag reduction**

The green circles – DNS for Newtonian channel flow, open circles – experiment. The Prandtl-Karman log-profile :

$$V^+ = 2.5 \ln y^+ + 5.5$$

The red squares – experiment, **universal** Virk's MDR asymptote

$$V^+ = 11.7 \ln y^+ - 17.0$$

The blue triangles & **green open triangles** – $\times$-over, for intermediate concentrations of the polymer, from the MDR asymptote to the Newtonian plug

Wall normalization: $\mathcal{R}e \equiv \dfrac{L\sqrt{p'L}}{\nu_0}$, $y^+ \equiv \dfrac{y\mathcal{R}e}{L}$, $V^+ \equiv \dfrac{V}{\sqrt{p'L}}$.

Asymptotical Universality of Drag Reduction by Polymers in Wall Bounded Turbulence:

Outline:

- **History of the problem**
- **Essentials of the phenomenon $\Rightarrow$ subject of the theory**

We are here

- **A theory of drug reduction and its verification:**
  - Simple theory of basic phenomena in drag reduction
  - Origin & calculation of Maximum Drag Reduction Asymptote
  - DNS verification of the the simple theory of drag reduction
- **Advanced approach**
  - Elastic stress tensor & effective viscosity
  - $\times$-over from the MDR asymptote to the Newtonian plug
- **Summary of the theory**

Simple theory of basic phenomena in drag reduction is based on:

– An approximation of effective polymeric viscosity for visco-elastic flows,

– An algebraic Reynolds-stress model for visco-elastic wall turbulence.

• **An approximation of effective polymeric viscosity** accounts for the effective polymeric viscosity (according to Lumley) and neglects elasticity

(accounting for the elasticity effects was the main point of the de Gennes' approach)

We stress: the polymeric viscosity is $\boldsymbol{r}$-dependent, Lumley's $\nu_p \Rightarrow \nu_p(\boldsymbol{r})$

• **Algebraic Reynolds-stress model** for a channel (of width $2L$):

– Exact (standard) equation for the flux of mechanical momentum:

$$\nu(y)S(y) + W(y) = p'L\,, \qquad \nu(y) \equiv \nu_0 + \nu_p(y)\ . \tag{1}$$

Hereafter: x & y streamwise & wall-normal directions, $p' \equiv -dp/dx$

mean shear: $S(y) \equiv \dfrac{d\,V_x(y)}{dy}$, Reynolds stress: $W(y) \equiv -\langle v_x v_y \rangle$.

Simple theory of basic phenomena in drag reduction:

- An approximation of effective polymeric viscosity $\nu_0 \Rightarrow \nu(y) \equiv \nu_0 + \nu_p(y)$
- **Algebraic Reynolds-stress model for viscoelastic flows**:

– Eq. for the flux of mechanical momentum: (for a channel of width $2L$)

$$\nu(y)S(y) + W(y) = p'L\,, \qquad \nu(y) \equiv \nu_0 + \nu_p(y)\ . \tag{1}$$

$$p' \equiv -dp/dx\,, \quad S(y) \equiv d\,V_x(y)/dy\,, \quad W(y) \equiv -\langle v_x v_y \rangle\ .$$

– Balance Eq. for the density of the turbulent kinetic energy $K \equiv \left\langle |\boldsymbol{v}|^2 \right\rangle /2$:

$$\left[\ \nu(y)\,(a/y)^2 + b\sqrt{K(y)}/y\ \right] K(y) = W(y)S(y)\,, \tag{2}$$

– Simple TBL closure: $$\frac{W(y)}{K(y)} = \begin{cases} c_{\mathrm{N}}^2\,, & \text{for Newtonian flow,} \\ c_{\mathrm{V}}^2\,, & \text{for viscoelastic flow.} \end{cases} \tag{3}$$

– Dynamics of polymers (in harmonic approximation, with τ_{p} – polymeric relaxation time) restricts the level of turbulent activity, (consuming kinetic energy) at the threshold level: $$1 \simeq \tau_{\mathrm{p}} \sqrt{\left\langle \frac{\partial u_i}{\partial r_j}\frac{\partial u_i}{\partial r_j} \right\rangle} \simeq \tau_{\mathrm{p}}\ \frac{\sqrt{W(y)}}{y}\ . \tag{4}$$

• Algebraic Reynolds-stress model in wall units:

$$\mathcal{R}e \equiv L\sqrt{p'L}/\nu_0\,, \quad y^+ \equiv y\mathcal{R}e/L\,, \quad V^+ \equiv V/\sqrt{p'L}\,, \quad \nu^+ = \left[1 + +\nu_{\rm p}^+\right]\,.$$

$$\text{Mechanical balance:} \qquad \nu^+ S^+ + W^+ = 1\,, \qquad (1)$$

$$\text{Energy balance:} \qquad \nu^+\left(\delta/y^+\right)^2 + \sqrt{W^+}/\kappa_{\rm K} y^+ = S^+\,, \qquad (2)$$

$$\text{Polymer dynamics:} \qquad \sqrt{W^+} = L^2\nu_0 y^+/\tau_{\rm p}{}^2\mathcal{R}e^2\,. \qquad (3)$$

• Test case: Newtonian turbulence:

Disregard polymeric terms: $\nu^+ \to 1$ & solve quadratic Eqs. (1)-(2) for $S^+(y^+)$ & integrate. **The result**:

For $\quad y^+ \le \delta$:
$$V^+ = y^+\,. \qquad (4a)$$

For $\quad y^+ \ge \delta$:

$$V^+(y^+) = \frac{1}{\kappa_{\rm K}}\ln Y(y^+) + B - \Delta(y^+)\,, \quad B = 2\delta - \frac{1}{\kappa_{\rm K}}\ln\left[\frac{e\,(1 + 2\,\kappa_{\rm K}\delta)}{4\kappa_{\rm K}}\right]\,,$$

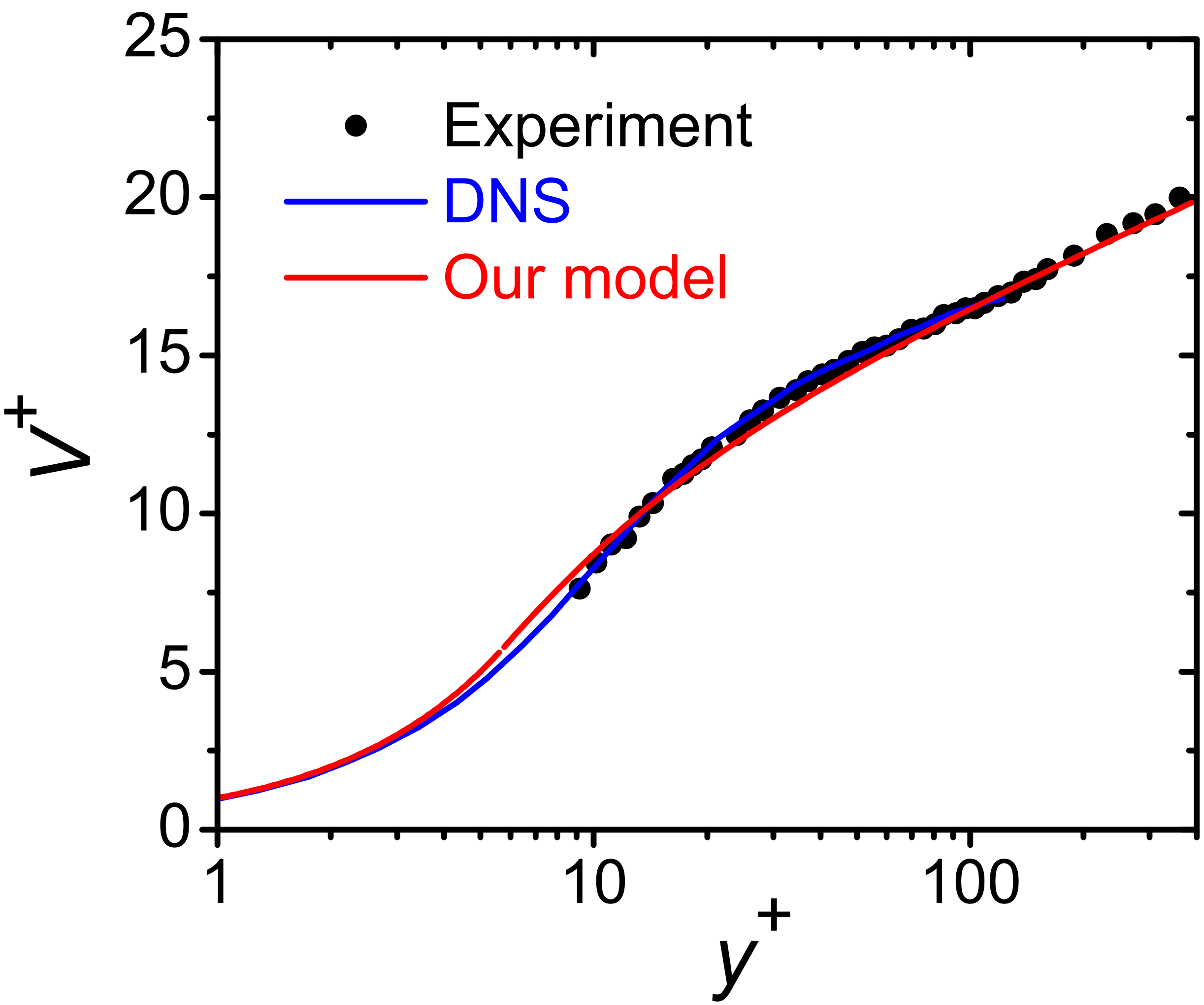
$$Y(y^+) = [y^+ + \sqrt{y^{+2} - \delta^2 + (2\kappa_{\rm K})^{-2}}]/2\,, \qquad (4b)$$

$$\Delta(y^+) = \frac{2\,\kappa_{\rm K}^2\delta^2 + 4\kappa_{\rm K}[Y(y^+) - y^+] + 1}{2\kappa_{\rm K}^2 y^+}\,.$$

Two fit parameters: $\kappa_{\rm N}$ & δ .

• **Comparison of analytical profile (4) with experiment & numerics**

using $\kappa_K^{-1} = 0.4$ and $\delta_N = 6$



Summary: Our simpl Algebraic Reynolds-stres model, based on the *exac* balance of mechanica momentum and K41 in spired *model equation* fo the local energy balance gives physically transparent analytical, semi-quantitativ description of turbulen boundary layer

- **Viscoelastic turbulent flow & Universal MDR asymptote:**

$$\text{Mechanical balance} \qquad \nu^+ S^+ + W^+ = 1\,, \tag{1}$$

$$\text{Energy balance:} \qquad \nu^+ \left(\frac{\delta_{\mathrm{N,V}}}{y^+}\right)^2 + \frac{\sqrt{W^+}}{\kappa_{\mathrm{K}} y^+} = S^+\,, \tag{2}$$

$$\text{Polymer dynamics:} \quad \sqrt{W^+} = \frac{L^2 \nu_0}{\tau_{\mathrm{p}}{}^2} \frac{y^+}{\mathcal{R}e^2} \to 0 \quad \text{at fixed } y^+ \ \&\ \mathcal{R}e \to \infty\,. \tag{3}$$

Equation (3) dictates:

Maximum Drag Reduction (**MDR**) asymptote $\Rightarrow \quad \mathcal{R}e \to \infty\,,\ W^+ = 0.$

We have learned that in the MDR regime:

— normalized (by wall units) turbulent kinetic energy [Eq.(3)]
$K^+ \propto W^+ \to 0$;

— the mechanical balance [Eq.(1)] and the balance of kinetic energy [Eq.(2)] are dominated by the polymeric contribution $\propto \nu^+$.

• **Viscoelastic turbulent flow & Universal MDR asymptote:**

Mech. & energy bal.: $\nu^+ S^+ + W^+ = 1\,,\ \nu^+ \left(\frac{\delta_{N,V}}{y^+}\right)^2 + \frac{\sqrt{W^+}}{\kappa_K y^+} = S^+\,,$ (1,2)

Polymer dynamics: $\sqrt{W^+} = \frac{L^2 \nu_0}{\tau_p{}^2}\,\frac{y^+}{\mathcal{R}e^2} \to 0$ at fixed y^+ & $\mathcal{R}e \to \infty$. (3)

Maximum Drag Reduction (**MDR**) asymptote $\Rightarrow$ $\mathcal{R}e \to \infty\,,\ W^+ = 0.$

In the MDR regime Eqs. (1,2) become $\Rightarrow$ $\nu^+ S^+ = 1\,,\ \nu^+ \delta_V^2 = S^+ {y^+}^2$

and have solution: $S^+ = \delta_V / y^+\,,\qquad \nu^+ = y^+/\delta_V$

for $y^+ \ge \delta_V$, because $\nu^+(y^+) \ge \nu_0^+ = 1.$

For $y^+ \le \delta_V\,,\ S^+ = 1\,,\ \nu^+ = 1.$ Integration $V^+(y^+) = \delta_V \int_{\delta_V}^{y^+} S^+(\xi)d\xi \ \Rightarrow$

Universal MDR asymptote: $V^+(y^+) = \delta_V \ln\left(e\, y^+/\delta_V\right)$. (4)

• Viscoelastic turbulent flow & Universal MDR asymptote:

Mech. & energy bal.: $\nu^+ S^+ + W^+ = 1\,, \ \nu^+\left(\frac{\delta_{\rm N,V}}{y^+}\right)^2 + \frac{\sqrt{W^+}}{\kappa_{\rm K} y^+} = S^+\,$, (1, 2)

Polymer dynamics: $\sqrt{W^+} = \frac{L^2\nu_0}{\tau_{\rm p}{}^2}\,\frac{y^+}{\mathcal{R}e^2} \to 0$ at fixed y^+ & $\mathcal{R}e \to \infty$. (3)

At MDR: $W^+ = 0 \ \Rightarrow$ Eqs. (1,2) $\Rightarrow \ \nu^+ S^+ = 1\,, \quad \nu^+\delta_{\rm V}^2 = S^+ {y^+}^2 \quad \Rightarrow$
$S^+ = \delta_{\rm V}/y^+\,, \quad \nu^+ = y^+/\delta_{\rm V} \qquad \Rightarrow \quad V^+(y^+) = \delta_{\rm V}\ln\left(e\, y^+/\delta_{\rm V}\right)$. (4)

Summary:

— In the MDR regime normalized (by wall units) turbulent kinetic energy $K^+ \to 0$;

— MDR regime is the edge of turbulent solution of the Navier-Stokes Eq. (NSE) with the largest possible effective viscosity $\nu(y)$, at which the turbulence still exists!

— MDR profiles of $\nu(y)$ & $S(y)$ are determined by the NSE itself and **are universal, independent of parameters of polymeric additives.**

• **Calculation of δ_{V} in the MDR asymptote:** $V^+(y^+) = \delta_{\mathrm{V}} \ln\left(e\, y^+/\delta_{\mathrm{V}}\right)$.

Consider $\nu^+ S^+ + W^+ = 1\,, \quad \nu^+ \dfrac{\delta_{\mathrm{N,V}}^2}{y^{+2}} + \dfrac{\sqrt{W^+}}{\kappa_{\mathrm{K}} y^+} = S^+$ with prescribed $\nu^+ = 1 + \alpha(y^+ - \delta_{\mathrm{N}})$ and replace flow dependent $\delta_{\mathrm{N,V}} \to \Delta(\alpha)$ with yet arbitral α:

$$[1 + \alpha(y^+ - \delta_{\mathrm{N}})]S^+ + W^+ = 1\,,\ [1 + \alpha(y^+ - \delta_{\mathrm{N}})]\frac{\Delta^2(\alpha)}{y^{+2}} + \frac{\sqrt{W^+}}{\kappa_{\mathrm{K}} y^+} = S^+ \ (*).$$

Clearly, $\delta_{\mathrm{N}} = \Delta(0)$ (Newtonian flow) and $\delta_{\mathrm{V}} = \Delta(\alpha_{\mathrm{V}})$, where $\Delta(\alpha_{\mathrm{V}})$ is the MDR solution of (*) in asymptotical region $y^+ \gg 1$ with $W = 0$:

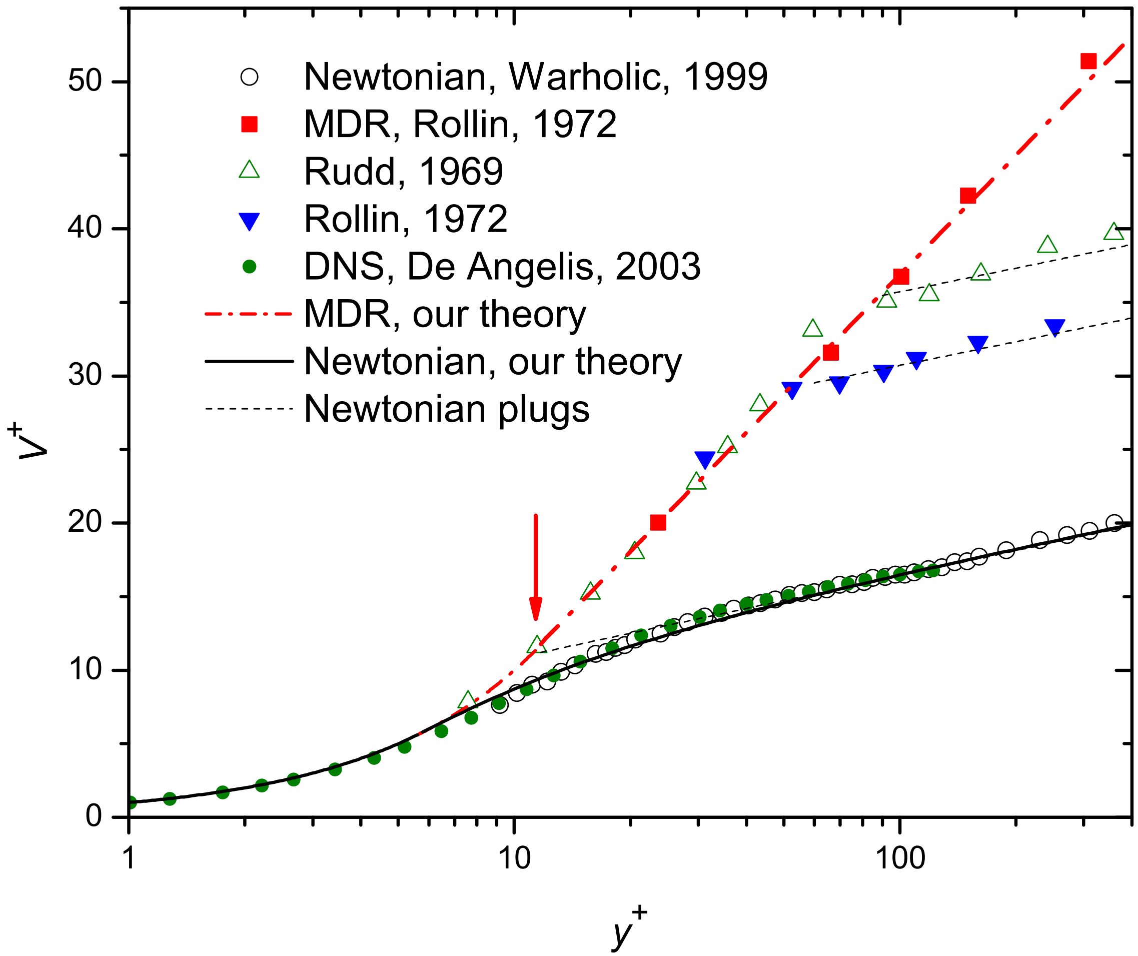
$$\alpha_{\mathrm{V}}\, \Delta(\alpha_{\mathrm{V}}) = 1\,, \quad \Delta(\alpha) = \frac{\delta_{\mathrm{N}}}{1 - \alpha\delta_{\mathrm{N}}}\,, \ \Rightarrow\ \alpha_{\mathrm{V}} = \frac{1}{2\delta_{\mathrm{N}}}\,, \ \Rightarrow\ \delta_{\mathrm{V}} = 2\,\delta_{\mathrm{N}}\ .$$

$\Delta(\alpha)$ follows from the requirement of the rescaling symmetry of Eq. (*):

$$y^+ \to y^\ddagger \equiv \frac{y^+}{g(\tilde\delta)}\,,\ g(\tilde\delta) \equiv 1 + \alpha(\tilde\delta - \delta_{\mathrm{N}})\,,\ \tilde\delta \to \delta^\ddagger \frac{\tilde\delta}{g(\tilde\delta)}\,,\ S^+ \to S^\ddagger \equiv S^+ g(\tilde\delta)\ .$$

Finally: $V^+(y^+) = 2\delta_{\mathrm{N}} \ln\left(\dfrac{e\, y^+}{2\,\delta_{\mathrm{N}}}\right)$, with Newtonian constant $\delta_{\mathrm{N}} \approx 6$ (†

• **Virk's MDR asymptote:** experiment (‡) *vs* our equation (†)



The red squares – experiment

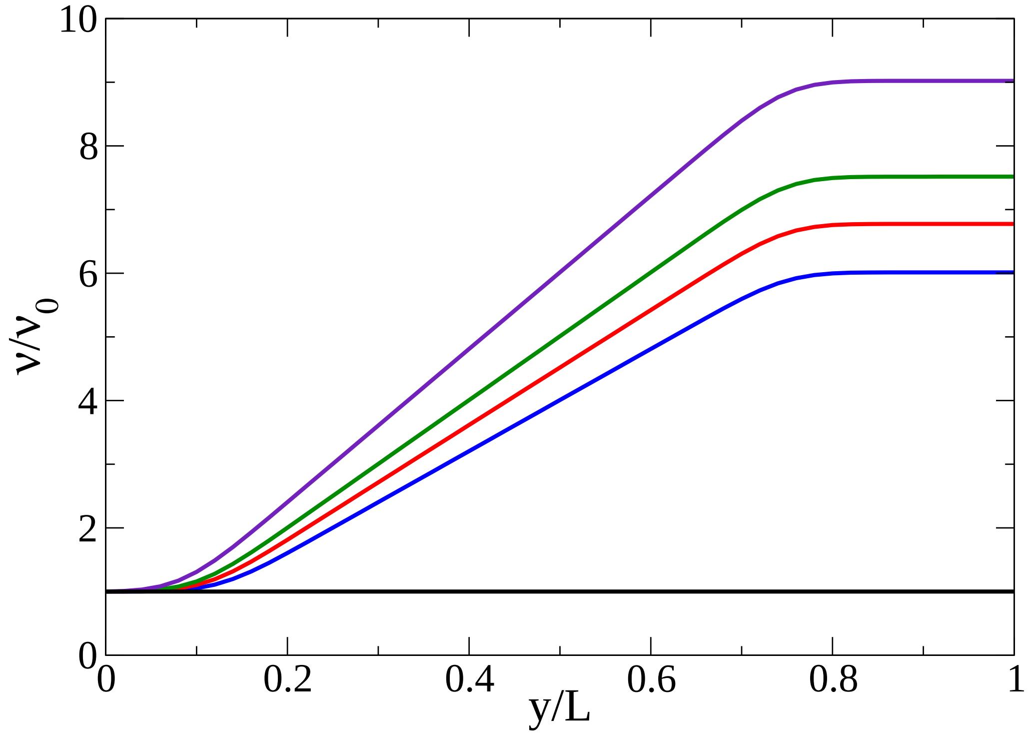
MDR: $V^+ = 11.7 \ln y^+ - 17.0$. (

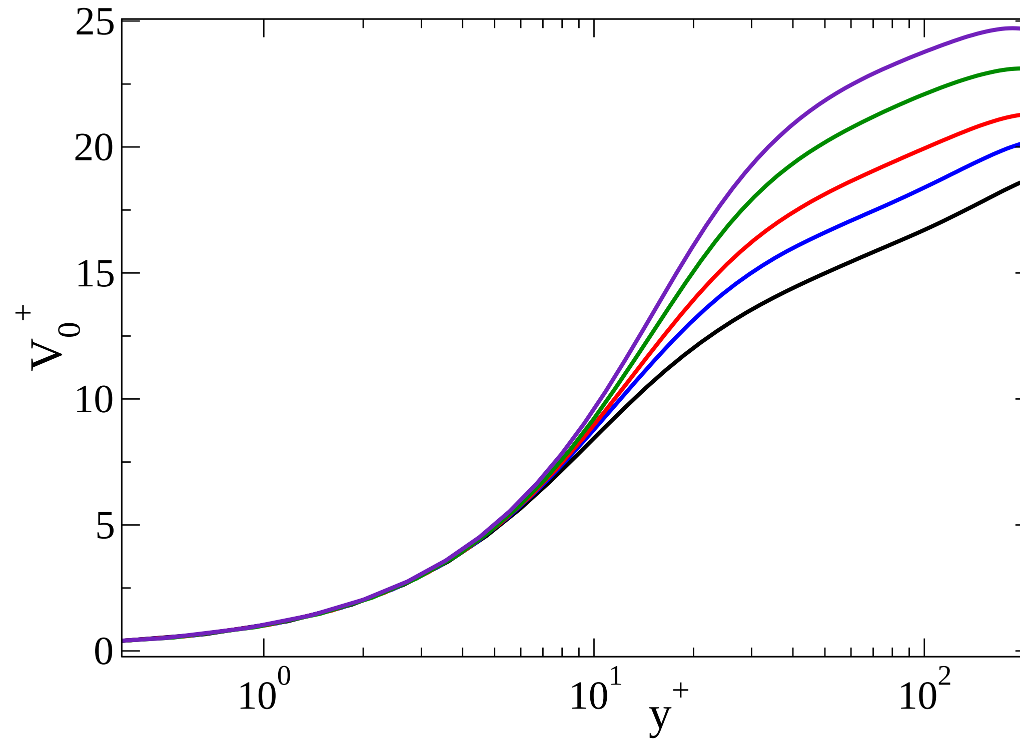
$V^+ = 2\delta_N \ln\left(e\, y^+/2\delta_N\right)$. (

Taking $\delta_N \approx 6$ from Newtoni data one has slope $2\delta_N \approx 1$ close to 11.7 in (‡) and interce $2\delta_N \ln(e/2\delta_N) \approx -17.8$, close -17.0 in (‡).

Summary: Maximum possible drag reduction (MDR asymptote) corresponds to the maximum possible viscosity profile at the edge of existence of turbulent solution and thus is universal, i.e. independent of polymer parameters, if polymers are able to provide required viscosity profile.

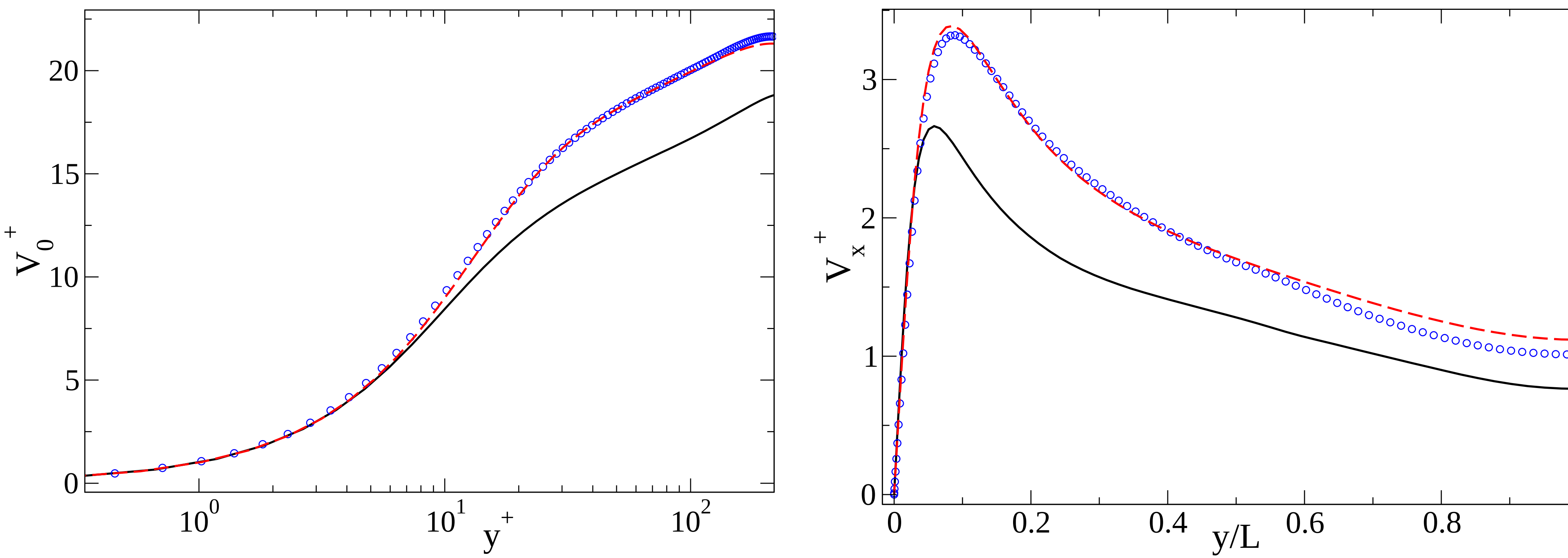
Renormalized–NSE DNS test of the scalar mean viscosity model





The viscosity (Left) & NSE-DNS mean velocity profiles (Right). $\mathcal{R}e = 6000$(with centerline velocity). Solid black line — standard Newtonian flow. One sees a drag reduction in the scalar mean viscosity model.

• Comparison of **renormalized–NSE model** (Red line : – – –) and **full FENE-P model** (Blue circles: ∘ ∘ ∘ ∘) [Newtonian flow: —-] $\mathcal{R}e = 6000$



Conclusion: Suggested simple model of polymer suspension with self-consistent viscosity profile really demonstrates the drag reduction itself and its essentials: mean velocity, kinetic energy profiles not only in the MDR regime, but also for intermediate $\mathcal{R}e$.

Riddle

Intuitively: effective polymeric viscosity $\nu_{\text{p}}(y)$ should be proportional to the (thermodynamical) mean square polymeric extension $\mathcal{R} \equiv \overline{R^2}$, averaged over turbulent assemble, $\mathcal{R}_0 = \langle \mathcal{R} \rangle$: $\nu_{\text{p}}(y) \propto \mathcal{R}_0(y)$.

However, in the MDR regime in our model $\nu_{\text{p}}(y)$ increases with the distance from the wall, $\nu_{\text{p}}(y) \propto y$, while experimentally $\mathcal{R}(y)$ decreases.

A way out

Instead of intuitive (and wrong) relationship $\nu_{\text{p}}(y) \propto \mathcal{R}_0(y)$ one needs to find correct connection between $\nu_{\text{p}}(y)$ and mean polymeric conformation tensor $\mathcal{R}_0^{ij} \equiv \left\langle \overline{R^i R^j} \right\rangle$.

This is a goal of

Advanced approach: Elastic stress tensor Π & effective viscosity

- **Advanced approach: Elastic stress tensor $\mathbf{\Pi}$ & effective viscosity**

Define: elastic stress $\Pi^{ij} \equiv \mathcal{R}^{ij}\,\nu_{0,\mathrm{p}}/\tau_{\mathrm{p}}$ & conformation $\mathcal{R}^{ij} \equiv \overline{R^i R^j}$ tensors,

polymeric "laminar" viscosity $\nu_{0,\mathrm{p}}$ & polymeric relaxation time τ_{p}.

$\boldsymbol{R} = \boldsymbol{r}/r_0$ – end-to-end distance, normalized by its equilibrium value

Write: Navier Stokes Equation (NSE) for dilute polymeric solutions:

$$\frac{\partial\,\boldsymbol{v}}{\partial t} + \big(\boldsymbol{v}\boldsymbol{\nabla}\big)\boldsymbol{v} = \nu_0 \Delta \boldsymbol{v} - \boldsymbol{\nabla} P + \boldsymbol{\nabla}\mathbf{\Pi}\,, \tag{1}$$

together with the equation for the elastic stress tensor:

$$\frac{\partial\,\mathbf{\Pi}}{\partial t} + (\boldsymbol{v}\boldsymbol{\nabla})\mathbf{\Pi} = \boldsymbol{S}\mathbf{\Pi} + \mathbf{\Pi}\boldsymbol{S}^{\dagger} - \frac{1}{\tau_{\mathrm{p}}}\big(\mathbf{\Pi} - \mathbf{\Pi}_{\mathrm{eq}}\big)\,, \qquad S^{ij} \equiv \partial v^i/\partial x^j\,, \tag{2}$$

Averaging Eq. (1) and $\displaystyle\int_0^y \ldots d\tilde{y}$ one has equation for $S(y) \equiv \langle \partial v^x/\partial y\rangle$:

$$\nu_0\, S(y) + \Pi_0^{xy}(y) + W(y) = p'\, L\,, \tag{3}$$

in which $\Pi_0^{xy}(y) \equiv \langle \Pi^{xy}(y)\rangle$ is the momentum flux, carried by polymers.

• Elastic stress tensor $\mathbf{\Pi}$ & effective viscosity in the MDR regime

In short: $\Pi^{ij} \equiv \mathcal{R}^{ij}\, \nu_{_{0,\rm p}}/\tau_{_{\rm p}}$, polymeric viscosity $\nu_{_{0,\rm p}}$ & relaxation time $\tau_{_{\rm p}}$.

$$\text{NSE for dilute polymer solutions: } \frac{\partial \boldsymbol{v}}{\partial t} + \Big(\boldsymbol{v}\boldsymbol{\nabla}\Big)\boldsymbol{v} = \nu_0 \Delta \boldsymbol{v} - \boldsymbol{\nabla} P + \boldsymbol{\nabla}\mathbf{\Pi}\,, \qquad (1)$$

$$\frac{\partial \mathbf{\Pi}}{\partial t} + (\boldsymbol{v}\boldsymbol{\nabla})\mathbf{\Pi} = \boldsymbol{S}\mathbf{\Pi} + \mathbf{\Pi}\boldsymbol{S}^\dagger - \frac{1}{\tau_{\rm p}}\Big(\mathbf{\Pi} - \mathbf{\Pi}_{\rm eq}\Big)\,, \qquad S^{ij} \equiv \partial v^i / \partial x^j\,, \qquad (2)$$

$$\text{Eq. (1) gives Eq. for the mean shear: } \nu_0\, S(y) + \Pi_0^{xy}(y) + W(y) = p'\, L\,, \qquad (3)$$

Averaging Eq. (2), and taking $D/Dt = 0$ one gets stationary Eq. for $\mathbf{\Pi}_0$:

$$\mathbf{\Pi}_0 = \tau_{\rm p}\Big(\boldsymbol{S}_0 \cdot \mathbf{\Pi}_0 + \mathbf{\Pi}_0 \cdot \boldsymbol{S}_0^\dagger + \boldsymbol{Q}\Big)\,, \qquad \boldsymbol{Q} \equiv \tau_{\rm p}^{-1}\mathbf{\Pi}_{\rm eq} + \Big\langle \boldsymbol{s}\cdot\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\boldsymbol{s}^\dagger \Big\rangle\,. \qquad (4a)$$

In the shear geometry $\boldsymbol{S}_0 \cdot \boldsymbol{S}_0 = 0$. This helps to find by the subsequent substitution of the RHS⇒RHS of Eq. (4a) its solution:

$$\mathbf{\Pi}_0 = 2\,\tau_{\rm p}^3\, \boldsymbol{S}_0 \cdot \boldsymbol{Q} \cdot \boldsymbol{S}_0^\dagger + \tau_{\rm p}^2 \Big(\boldsymbol{S}_0 \cdot \boldsymbol{Q} + \boldsymbol{Q}\cdot \boldsymbol{S}_0^\dagger\Big) + \tau_{\rm p}\boldsymbol{Q}\,. \qquad (4b)$$

• Elastic stress tensor $\mathbf{\Pi}$ & effective viscosity in the MDR regime

In short: $\Pi^{ij} \equiv \mathcal{R}^{ij}\,\nu_{_{0,p}}/\tau_{_p}$, polymeric viscosity $\nu_{_{0,p}}$ & relaxation time $\tau_{_p}$.

$$\text{NSE for dilute polymer solutions: } \frac{\partial \boldsymbol{v}}{\partial t} + \Big(\boldsymbol{v}\boldsymbol{\nabla}\Big)\boldsymbol{v} = \nu_0 \Delta \boldsymbol{v} - \boldsymbol{\nabla} P + \boldsymbol{\nabla}\mathbf{\Pi}\,, \qquad (1)$$

$$\frac{\partial \mathbf{\Pi}}{\partial t} + (\boldsymbol{v}\boldsymbol{\nabla})\mathbf{\Pi} = \boldsymbol{S}\mathbf{\Pi} + \mathbf{\Pi}\boldsymbol{S}^{\dagger} - \frac{1}{\tau_{\mathrm{p}}}\Big(\mathbf{\Pi} - \mathbf{\Pi}_{\mathrm{eq}}\Big)\,, \qquad S^{ij} \equiv \partial v^i/\partial x^j\,, \qquad (2)$$

Eq. (1) gives Eq. for the mean shear: $$\nu_0\, S(y) + \Pi_0^{xy}(y) + W(y) = p'\, L\,, \qquad (3)$$

Eq. (2) gives Eq. for $\mathbf{\Pi}_0$: $$\mathbf{\Pi}_0 = 2\,\tau_{\mathrm{p}}^3\, \boldsymbol{S}_0 \cdot \boldsymbol{Q} \cdot \boldsymbol{S}_0^{\dagger} + \tau_{\mathrm{p}}^2 \Big(\boldsymbol{S}_0 \cdot \boldsymbol{Q} + \boldsymbol{Q} \cdot \boldsymbol{S}_0^{\dagger}\Big) + \tau_{\mathrm{p}} \boldsymbol{Q}\ . \qquad (4b)$$

At the onset of drag reduction Deborah number at the wall $\mathcal{D}e_0 \simeq 1$, $\mathcal{D}e_0 = \mathcal{D}e(0)$, $\mathcal{D}e(y) \equiv \tau_{\mathrm{p}}\, S(y)$. In the MDR regime: $\mathcal{D}e(y) \gg 1$.

In the limit $\mathcal{D}e(y) \gg 1$, Eq. $(4b)$ gives:

$$\mathbf{\Pi}_0(y) = \Pi_0^{yy}(y) \begin{pmatrix} 2\,[\mathcal{D}e(y)]^2 & \mathcal{D}e(y) & 0 \\ \mathcal{D}e(y) & 1 & 0 \\ 0 & 0 & C \end{pmatrix}\,, \qquad C = \frac{Q_{zz}}{Q_{yy}} \simeq 1\ . \qquad (4c)$$

For $\mathcal{D}e(y) \gg 1$ tensorial structure of $\mathbf{\Pi}_0$ becomes universal. In particular:

$$\Pi_0^{xy}(y) = \mathcal{D}e(y)\ \Pi_0^{yy}(y)\ . \qquad (4d)$$

• Elastic stress tensor $\mathbf{\Pi}$ & effective viscosity in the MDR regime

NSE for dilute polymer solutions: $$\frac{\partial \boldsymbol{v}}{\partial t} + \left(\boldsymbol{v}\boldsymbol{\nabla}\right)\boldsymbol{v} = \nu_0 \Delta \boldsymbol{v} - \boldsymbol{\nabla} P + \boldsymbol{\nabla}\mathbf{\Pi}\,, \qquad (1)$$

$$\frac{\partial \mathbf{\Pi}}{\partial t} + (\boldsymbol{v}\boldsymbol{\nabla})\mathbf{\Pi} = \boldsymbol{S}\mathbf{\Pi} + \mathbf{\Pi}\boldsymbol{S}^{\dagger} - \frac{1}{\tau_{\mathrm{p}}}\left(\mathbf{\Pi} - \mathbf{\Pi}_{\mathrm{eq}}\right)\,, \qquad S^{ij} \equiv \partial v^i/\partial x^j\,, \qquad (2)$$

Eq. (1) gives Eq. for the mean shear: $$\nu_0\, S(y) + \Pi_0^{xy}(y) + W(y) = p'\, L\,, \qquad (3)$$

In the limit $\mathcal{D}e(y) \gg 1$, Eq. (4b) gives:

$$\mathbf{\Pi}_0(y) \;=\; \Pi_0^{yy}(y)\begin{pmatrix} 2\,[\mathcal{D}e(y)]^2 & \mathcal{D}e(y) & 0 \\ \mathcal{D}e(y) & 1 & 0 \\ 0 & 0 & C \end{pmatrix}, \quad C \simeq 1\,. \qquad (4c)$$

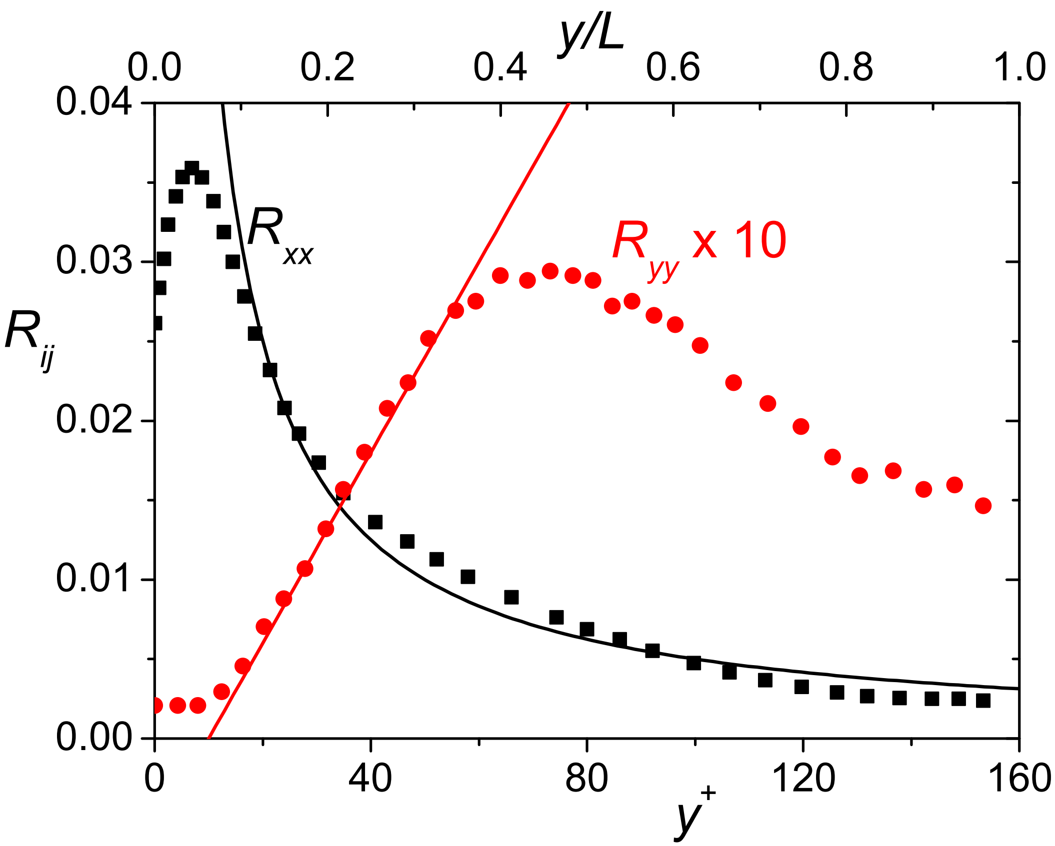
$$\Rightarrow \qquad \Pi_0^{xy}(y) = \mathcal{D}e(y)\,\Pi_0^{yy}(y) = S(y)\tau_p\,\Pi_0^{yy}(y)\,. \qquad (4d)$$

Insertion $(4d)$ into (3) $\Rightarrow$ the model eq: $\left[\nu_0 + \nu_{\mathrm{p}}(y)\right] S(y) + W(y) = p'\, L,$

in which $\nu_{\mathrm{p}}(y) \equiv \tau_p\, \Pi_0^{yy}(y) = \nu_{0,\mathrm{p}}\mathcal{R}^{yy}$ is the polymeric viscosity. (5)

Summary: Effective polymeric viscosity $\nu_{\mathrm{p}}(y)$ is proportional to yy component of the conformation (and elastic stress) tensor, not to its trace.

• **Comparison of the theory with Direct Numerical Simulation**



We found: In the MDR regime,

$$\nu_p(y) \propto \mathcal{R}_{yy}(y) \propto$$

$$\text{while} \quad S(y) \propto 1/y\,,$$

$$\mathcal{R}^{yy}(y) \propto y$$

$$\mathcal{R}^{xx}(y) = 2S^2(y)\tau_p^2\, \mathcal{R}^{yy}(y) \propto \frac{1}{y}\,,$$

DNS data for $\mathcal{R}^{yy}$ – red circles for $\mathcal{R}^{xx}$ – black squares are fitted by red y and black $1/y$ lines.

Summary: Predicted spacial profiles of effective viscosity and polymeric extension are consistent with the DNS and experimental observations

• **Cross-over from the MDR asymptote to the Newtonian plug**

Model Eqs. $$\left[\nu_0 + \nu_{\rm p}(y)\right] S(y) + W(y) = p' L\,, \qquad (1)$$
$$\left\{\left[\nu_0 + \nu_{\rm p}(y)\right](a/y)^2 + b\sqrt{K(y)}/y\right\} K(y) = W(y)S(y)\ . \qquad (2)$$
$$W(y)/K(y) = c_{\rm V}^2\,, \qquad \tau_{\rm p}^2\, W(y) \simeq y^2\ . \qquad (3,4)$$

Reminder: In the MDR regime red-marked terms in Eqs. (1,2) are small.

$\times$-over of linearly extended polymers: Eq. (1) $\Rightarrow$ $p'L \simeq W(y_\times)$ $\Rightarrow$
with Eq. (4): $p'\,L \simeq y_\times^2/\tau_{\rm p}^2$ $\Rightarrow$ $y_\times \simeq \tau_{\rm p}\sqrt{p'L}$ $\Rightarrow$ $y_\times^+ \simeq \mathcal{D}e(0)$. (5a)

$\times$-over of finite extendable polymers: $\nu_{\rm p}(y) \le \nu_{\rm p,max} \simeq \nu_0\, c_{\rm p}\,(a\,N_{\rm p})^3$.

In the MDR: $\nu_{\rm p}(y^+) \simeq \nu_0\, y^+$ $\Rightarrow$ $y_\times^+ \simeq c_{\rm p}\,(a\,N_{\rm p})^3$. (5b)

In general: $$y_\times^+ \simeq \frac{\mathcal{D}e(0)\, c_{\rm p}\,(a\,N_{\rm p})^3}{\mathcal{D}e(0) + c_{\rm p}\,(a\,N_{\rm p})^3}\ . \qquad (6)$$

Verification: $\times$-over (5a) is consistent with DNS of Yu et. al. (2001), $\times$-over (5b) is in agreement with DNA experiment of Choi et. al. (2002)

SUMMARY OF THE RESULTS

- Essentials of the drag reduction by dilute elastic polymers can be understood within the suggested Effective Viscosity Approximation.
- For the NSE with the effective viscosity we have suggested an Algebraic Reynolds-stress model that describes relevant characteristics of the Newtonian and viscoelastic turbulent flows in agreement with available DNS and experimental data.
- The model allows one to clarify the origin of the universality of the maximum possible drag reduction and to calculate universal Virk's constants, that are are a good quantitative agreement with the experiments.
- The model predicts two mechanisms of $\times$-over MDR$\Rightarrow$ Newtonian plug in a qualitative agreement with DNS and experiment.

In short: Basic physics of drag reduction in polymeric solutions is understood and has reasonable simple and transparent description in the framework of developed theory. Further developments and detailing are possible

THE END