Wave Turbulence
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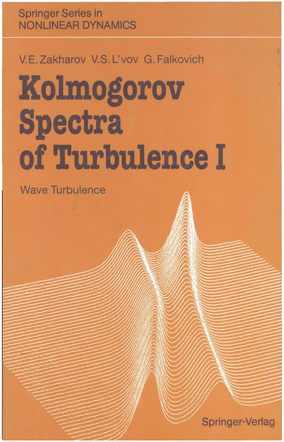

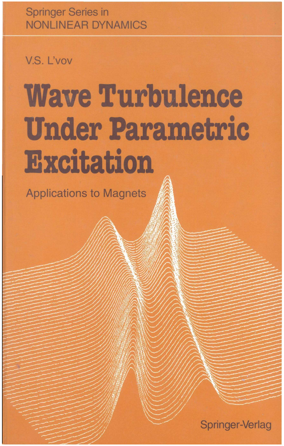

Outline

Classical Hamiltonian formalism
Examples of non-linear waves

Dispersive waves play a crucial role in a vast range of physical applications, from quantum to classical regions, from microscopic to astrophysical scales:

- Kelvin waves propagating on quantized vortex lines provide an essential mechanism of turbulent energy cascades in superfluids
- Sea waves are important for the momentum and energy transfers from wind to ocean, as well as for navigation conditions;
- Internal waves on density stratifications and inertial waves due to rotation are important in turbulence behavior and mixing in planetary atmospheres and oceans;
- Planetary Rossby waves are important for the weather and climate evolutions;
- Alfven waves are ubiquitous in turbulence of solar wind and interstellar medium.

Classical Hamiltonian formalism
Basic equations of motion

Basic equations of motion in various media are very different:

- Gravity and capillary Waves on fluid surface and in stratified fluids, including Rossby waves in rotation Atmosphere, Cyclones and Anticyclones, Intrinsic waves in the Ocean $\Rightarrow$ The Navier-Stokes Equations
- Acoustic waves, Sound in crystals, glasses (disordered media), fluids and plasmas: $\Rightarrow$ Material equations
- Electromagnetic waves: Radio-frequency, Microwaves, Light, X-rays, etc. in dielectrics; Numerous wave types in plasma $\Rightarrow$ The Maxwell + Material equations
- Spin waves in Magnetics: $\Rightarrow$ The Bloch & Landau-Lifshitz Eqs.

All these equations can be presented in a canonical form as **the Hamiltonian equations of motion** for wave amplitude $a_{\boldsymbol{k}}(t)$

$$i\,\frac{da_{\boldsymbol{k}}}{dt} = \frac{\delta\mathcal{H}\{a_{\boldsymbol{k}'}, a^*_{\boldsymbol{k}'}\}}{\delta a^*_{\boldsymbol{k}}}$$

Classical Hamiltonian formalism
Structure of the Hamiltonian at small nonlinearity: Free and three-wave interaction Hamiltonian

Step 1. Let $\boldsymbol{a}\,,\boldsymbol{a}^* = \mathbf{0}$ in the absence of a wave. Assume that $\boldsymbol{a}\,,\boldsymbol{a}^*$ are *small* in required sense, for instance, when the elevation of the surface-water waves is smaller then the wavelength.
Step 2. For small $\boldsymbol{a}\,,\boldsymbol{a}^*$ Hamiltonian $\mathcal{H}$ can be expanded over $\boldsymbol{a}$, $\boldsymbol{a}^*$:

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_{\rm int}\,, \quad \mathcal{H}_{\rm int} = \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{H}_5 + \mathcal{H}_6 + \ \dots\,, \qquad (1)$$

where free-wave Hamiltoinian $\mathcal{H}_2 = \sum_{\boldsymbol{k}} \omega(\boldsymbol{k}) a_{\boldsymbol{k}} a_{\boldsymbol{k}}^*\,,$

and $\mathcal{H}_n \propto a^{(n-m)} a^{*(m)}$ is n-wave interaction Hamiltonian:

$$\mathcal{H}_3 = \mathcal{H}_{2\leftrightarrow 1} + \mathcal{H}_{3\leftrightarrow 0} \ = \frac{1}{2} \sum_{\boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3} \Big(V_1^{2,3} b_1^* a_2 a_3 + \text{c.c.} \Big) + \frac{1}{6} \sum_{k_1 + k_2 + k_3 = 0} \Big(U_{1,2,3} a_1^* a_2^* a_3^* + \text{c.c.} \Big)\ .$$

Classical Hamiltonian formalism

Hamiltonian at small nonlinearity: Four- Five and Six-wave interaction Hamiltonian

$$\begin{aligned}\mathcal{H}_4 &= \mathcal{H}_{2\leftrightarrow 2} + \mathcal{H}_{3\leftrightarrow 1} + \mathcal{H}_{4\leftrightarrow 0}\\ &= \frac{1}{4}\sum_{k_1+k_2=k_3+k_4} W_{1,2}^{3,4}\, a_1^* a_2^* a_3 a_4\\ &\quad + \frac{1}{3!}\sum_{k_1=k_2+k_3+k_4}\left(G_1^{2,3,4}\, a_1 a_2^* a_3^* a_4^* + \mathrm{c.c.}\right)\\ &\quad + \frac{1}{4!}\sum_{k_1+k_2+k_3+k_4=0}\left(R_{1,2,3,4}^*\, a_1 a_2 a_3 a_4 + \mathrm{c.c}\right)\end{aligned}$$

$$\mathcal{H}_{2\leftrightarrow 3} = \frac{1}{12}\sum_{1+2=3+4+5}\left[V_{1,2}^{3,4,5}\, a_1 a_2 a_3^* a_4^* a_5^* + \mathrm{c.c.}\right],$$

$$\mathcal{H}_{3\leftrightarrow 3} = \frac{1}{36}\sum_{1+2+3=4+5+6} W_{1,2,3}^{4,5,6}\, a_1 a_2 a_3 a_4^* a_5^* a_6^* \,.$$

Statistical description of weakly nonlinear waves

Approximation of wave Kinetic Equation (KE)

Statistical description of weakly interacting waves can be reached in terms of the wave Kinetic Equation (KE)

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = \mathrm{St}(\boldsymbol{k},t)\,,$$

for the simultaneous pair correlation functions $n(\boldsymbol{k},t)$, defined by

$$\langle a(\boldsymbol{k},t)a^*(\boldsymbol{k}',t)\rangle = n(\boldsymbol{k},t)\delta(\boldsymbol{k}-\boldsymbol{k}')\,,$$

where $\langle \ldots \rangle$ stands for proper (ensemble, etc.) averaging.

In classical limit, when the occupation numbers of Bose particles $N(\boldsymbol{k},t) \gg 1$, $n(\boldsymbol{k},t) = \hbar N(\boldsymbol{k},t)$.

Statistical description of weakly nonlinear waves

$1 \leftrightarrow 2$ Collision term

The collision integral $\mathrm{St}(\boldsymbol{k}, t)$ can be find in various ways, including the Golden Rule of quantum mechanics.

- For the $\mathbf{1} \leftrightarrow \mathbf{2}$ process, described by the Hamiltonian $\mathcal{H}_{1\leftrightarrow 2}$:

$$
\begin{aligned}
\mathrm{St}_{1\leftrightarrow 2}(\boldsymbol{k}) &= \pi \int d\boldsymbol{k}_1 d\boldsymbol{k}_2 \Big\{ \frac{1}{2} |V_{\boldsymbol{k}}^{1,2}|^2 \, \delta_{\boldsymbol{k}}^{1,2} \, \delta\big(\Omega_{\boldsymbol{k}}^{1,2}\big) \, \mathcal{N}_{\boldsymbol{k}}^{1,2} \\
& \qquad + |V_1^{\boldsymbol{k},2}|^2 \, \delta_1^{\boldsymbol{k},2} \, \delta\big(\Omega_1^{\boldsymbol{k},2}\big) \, \mathcal{N}_1^{\boldsymbol{k},2} \Big\}, \quad \text{where} \\
\mathcal{N}_{\boldsymbol{k}}^{1,2} &\equiv n_{\boldsymbol{k}} n_1 n_2 \big(n_{\boldsymbol{k}}^{-1} - n_1^{-1} - n_2^{-1}\big), \\
\delta_{\boldsymbol{k}}^{1,2} &\equiv \delta\big(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2\big), \\
\Omega_{\boldsymbol{k}}^{1,2} &\equiv \omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}_1} - \omega_{\boldsymbol{k}_2} .
\end{aligned}
$$

Statistical description of weakly nonlinear waves

$2 \leftrightarrow 2$, $2 \leftrightarrow 3$ and $3 \leftrightarrow 3$ Collision terms

$$
\begin{aligned}
\mathrm{St}_{2\leftrightarrow 2} &= \frac{\pi}{2}\int d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \, |T^{2,3}_{\boldsymbol{k},1}|^2 \, \delta^{2,3}_{\boldsymbol{k},1} \, \delta(\omega_{\boldsymbol{k}} + \omega_1 - \omega_2 - \omega_3) \\
&\quad \times n_{\boldsymbol{k}} n_1 n_2 n_3 \left(n_{\boldsymbol{k}}^{-1} + n_1^{-1} - n_2^{-1} - n_3^{-1}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{St}_{2\leftrightarrow 3} &= \frac{\pi}{12}\int d\boldsymbol{k}_1 \ldots d\boldsymbol{k}_4 \Big\{ 2\,|V^{\,2,3,4}_{\,\boldsymbol{k},1}|^2 \, \delta^{\,2,3,4}_{\,\boldsymbol{k},1} \, \mathcal{N}^{\,2,3,4}_{\,\boldsymbol{k},1} \\
&\quad \times \delta(\omega_{\boldsymbol{k}} + \omega_1 - \omega_2 - \omega_3 - \omega_4) \\
&\quad + 3\,|V^{\,\boldsymbol{k},3,4}_{\,1,2}|^2 \, \delta^{\,\boldsymbol{k},3,4}_{\,1,2} \, \mathcal{N}^{\,\boldsymbol{k},3,4}_{\,1,2} \delta(\omega_1 + \omega_2 - \omega_{\boldsymbol{k}} - \omega_3 - \omega_4) \Big\}, \\
\mathcal{N}^{\,3,4,5}_{\,1,2} &\equiv n_1 n_2 n_3 n_4 n_5 \left(n_1^{-1} + n_2^{-1} - n_3^{-1} - n_4^{-1} - n_5^{-1}\right);
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{St}_{3\leftrightarrow 3} &= \frac{\pi}{12}\int d\boldsymbol{k}_1 \ldots d\boldsymbol{k}_5 \, |W^{\,4,5,6}_{\,\boldsymbol{k},1,3}|^2 \, \delta^{\,4,5,6}_{\,\boldsymbol{k},1,3} \\
&\quad \times \delta(\omega_{\boldsymbol{k}} + \omega_1 + \omega_2 - \omega_3 - \omega_4 - \omega_5) \, n_{\boldsymbol{k}} n_1 n_2 n_3 n_4 n_5 \\
&\quad \times \left(n_1^{-1} + n_2^{-1} + n_3^{-1} - n_4^{-1} - n_5^{-1} - n_6^{-1}\right).
\end{aligned}
$$

General properties of wave-kinetic equation
Conservation Laws

- All KEs conserve the total energy of non-interacting waves:

$$E = \int d\boldsymbol{k}\,\varepsilon_{\boldsymbol{k}}\,, \quad \text{with the energy density } \varepsilon_{\boldsymbol{k}} \equiv \omega_{\boldsymbol{k}} n_{\boldsymbol{k}} = \hbar\omega_{\boldsymbol{k}} N_{\boldsymbol{k}}\,,$$

where the quantum mechanical occupation numbers $N_{\boldsymbol{k}} \equiv n_{\boldsymbol{k}}/\hbar$. This energy does not include (small) correction to the total energy of the system of interacting waves, described by $\mathcal{H}_{\rm int}$.

Compute dE/dt using $(2 \leftrightarrow 1)$-KE to get

$$\begin{aligned}\frac{d\,E}{d\,t} &= \int d1d2d3\,\omega_k\delta(\omega_k-\omega_1-\omega_2)\dots \\ &= \frac{1}{3}\int d1d2d3\,(\omega_k-\omega_1-\omega_2)\delta(\omega_k-\omega_1-\omega_2)\dots = 0\,.\end{aligned}$$

One sees that formally the conservation of energy follows from the $\delta(\omega_k-\omega_1-\omega_2)$, that originates from the time-invariance.

General properties of wave-kinetic equation
Conservation Laws

For the $(2 \leftrightarrow 2)$-KE analogously one gets

$$\frac{dE}{dt} = \int d1d2d3d4\, \omega_k \delta(\omega_k + \omega_1 - \omega_2 - \omega_3) \dots$$
$$= \frac{1}{4} \int d1d2d3d4\, (\omega_k + \omega_1 - \omega_2 - \omega_3)\delta(\omega_k + \omega_1 - \omega_2 - \omega_3) \dots = 0\,.$$

In the same way one proves conservation of energy $E = \int \omega_k n_k d\,k$ in any high-order KE.

- All KEs conserve the total mechanical moment $\boldsymbol{P}$ of interacting waves:

$$\frac{d\boldsymbol{P}}{dt} = 0\,, \quad \text{where} \quad \boldsymbol{P} \equiv \int (\boldsymbol{k} n_{\boldsymbol{k}})\,, d\boldsymbol{k}\,.$$

- $(2 \leftrightarrow 2)$- and $(2 \leftrightarrow 2)$-KEs conserve the total number of particles N of interacting waves:

$$\frac{dN}{dt} = 0\,, \quad \text{where} \quad N \equiv \int n_{\boldsymbol{k}}\,, d\boldsymbol{k}\,.$$

General properties of wave-kinetic equation

Boltzmann's H-theorem

Introduce the entropy of the wave system $S(t) = \displaystyle\int \ln(n_k)d\,k$ and study its evolution, computing with the help of KE one gets

$$\frac{dS}{dt} = \int \frac{\partial n_k}{n_k \partial t} d\,k = \int \frac{\mathrm{St}_k}{n_k} d\,k \;\Rightarrow \text{for the } (2 \leftrightarrow 1) \text{ collision integral } \Rightarrow$$

$$= \frac{\pi}{2} \int dk \int dk_1 dk_2 \ldots \Big(\frac{1}{n_k} - \frac{1}{n_1} - \frac{1}{n_2}\Big)\Big(\frac{1}{n_k} - \frac{1}{n_1} - \frac{1}{n_2}\Big) \geq 0 \;.$$

Analogously, for the $(2 \leftrightarrow 2)$ collision integral $\quad \dfrac{dS}{dt} =:$

$$\frac{\pi}{8} \int dk \int dk_1 dk_2 dk_3 \ldots \Big(\frac{1}{n_k} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}\Big)\Big(\frac{1}{n_k} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}\Big) \geq 0 \;.$$

Similarly, one gets: for any KE wave evolution only increases entropy:

$$\frac{dS}{dt} > 0 \,, \quad \text{and in the the state of thermodynamic equilibrium} \quad \frac{dS}{dt} = 0 \,.$$

General properties of wave-kinetic equation
Thermodynamic Equilibrium

In the state of thermodynamic equilibrium $\dfrac{dS}{dt} = 0\,,\ \&\dfrac{dn_k}{dt} = 0.$

Notice:

$$
\begin{aligned}
\mathrm{St}_{2\leftrightarrow 1} &\propto \delta(\omega_k - \omega_1 - \omega_2)\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\left[\frac{1}{n_k} - \frac{1}{n_1} - \frac{1}{n_2}\right], \\
\mathrm{St}_{2\leftrightarrow 2} &\propto \delta(\omega_k + \omega_1 - \omega_2 - \omega_3)\delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\left[\frac{1}{n_k} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}\right], \\
\mathrm{St}_{3\leftrightarrow 2} &\propto \delta(\omega_k + \omega_1 + \omega_2 - \omega_3 - \omega_4)\delta(\boldsymbol{k} + \dots)\left[\frac{1}{n_k} + \frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4}\right], \\
\mathrm{St}_{3\leftrightarrow 3} &\propto \delta(\omega_k + \omega_1 + \omega_2 - \omega_3 - \omega_4 - \omega_5)\delta(\boldsymbol{k} + \boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4 - \boldsymbol{k}_5) \\
&\quad \times \left[\frac{1}{n_k} + \frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} - \frac{1}{n_5}\right].
\end{aligned}
$$

Equilibrium: Rayleigh-Jeans distribution $n_0(\boldsymbol{k}) = \dfrac{T}{\omega_{\boldsymbol{k}} - \mu - \boldsymbol{k}\cdot\boldsymbol{V}}$ with free constants: temperature T, velocity $\boldsymbol{V}$ and μ – chemical potential.

Kolmogorov spectra of wave turbulence
Turbulent spectra with constant energy flux

Consider isotropic energy spectra of turbulent waves in d-dimensional [$d = 1$ – Kelvin waves, $d = 2$ – surface-water waves, $d = 3$ – sound] scale-invariant [$\omega(\lambda k) = \lambda^\alpha \omega(k)$, i.e. $\omega_k \propto k^\alpha$, etc.] media.

Having energy pumping in the region of small k_+ and energy damping at $k_- \gg k_+$ we have "inertial interval of scales $k_+ \ll k \ll k_-$, where wave system obeys KE.
Conservation of energy implies energy-continuity equation:

$$\frac{\partial E_k}{\partial t} + \frac{\partial \varepsilon(k)}{\partial k} = 0, \quad E_{\mathbf{k}} \equiv \omega_{\mathbf{k}} n_{\mathbf{k}}\,, \text{ where energy flux } \varepsilon(k) \equiv \int_{k_+}^{k} \omega_k \mathrm{St}(k) k^{(d-1)}$$

Assuming locality of the energy transfer [integral convergence in $\mathrm{St}(k)$] one estimates in scale-invariant case $\int d\mathbf{k} \sim k^d$, $V_k^{k,k} \equiv V_3 k^{\beta_3}$

$$\varepsilon_{2\leftrightarrow1}(k) \simeq \omega_k k^d \mathrm{St}_{2\leftrightarrow1}(k) \sim k^{2d}(V_k^{k,k})^2 n_k^2 \simeq k^{2d+2\beta_3} V_3^2 n_k^2 \equiv \varepsilon_3\,,$$

$$\varepsilon_{2\leftrightarrow2}(k) \simeq k^{3d}(T_{k,k}^{k,k})^2 n_k^3 \simeq k^{3d+2\beta_4} V_4^2 n_k^3 \equiv \varepsilon_4\,, \ T_{k,k}^{k,k} \equiv V_4 k^{\beta_4}, \ \dots$$

$$\varepsilon_{3\leftrightarrow3}(k) \simeq k^{5d}(W_{k,k}^{k,k})^2 n_k^5 \simeq k^{5d+2\beta_6} V_6^2 n_k^5 \equiv \varepsilon_6\,, \ W_{k,k,k}^{k,k,k} \equiv V_6 k^{\beta_6},$$

Kolmogorov spectra of wave turbulence
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$$\begin{aligned}
\varepsilon_{2\leftrightarrow 1}(k) &\simeq \omega_k k^d \mathrm{St}_{2\leftrightarrow 1}(k) \sim k^{2d} (V_k^{k,k})^2 n_k^2 \simeq k^{2d+2\beta_3} V_3^2 n_k^2 \equiv \varepsilon_3 \,, \\
\varepsilon_{2\leftrightarrow 2}(k) &\simeq k^{3d} (T_{k,k}^{k,k})^2 n_k^3 \simeq k^{3d+2\beta_4} V_4^2 n_k^3 \equiv \varepsilon_4 \,, \quad T_{k,k}^{k,k} \equiv V_4 k^{\beta_4}, \ \dots \\
\varepsilon_{3\leftrightarrow 3}(k) &\simeq k^{5d} (W_{k,k}^{k,k})^2 n_k^5 \simeq k^{5d+2\beta_6} V_6^2 n_k^5 \equiv \varepsilon_6 \,, \quad W_{k,k,k}^{k,k,k} \equiv V_6 k^{\beta_6},
\end{aligned}$$

Kolmogorov spectra of wave turbulence

Turbulent spectra with constant energy flux: Physical examples

In general for turbulence with p-wave interactions:

$$\varepsilon_p \simeq \left(k^d n_k\right)^{p-1} + 2\,\beta_p V_p^2 \quad \Rightarrow \quad n_k \simeq \varepsilon_p^{1/(p-1)} k^{-[d+2\,\beta_p/(p-1)]} \;\Rightarrow$$

$$n_k \propto \frac{1}{k^{x_p}}\,, \qquad x_p = d + \frac{2\beta_p}{p-1}\,.$$

- Acoustic turbulence: $d = 3$, $\omega_k \propto k\,,\; p = 3\,,\; \beta_3 = 3/2 \Rightarrow$

$$x_3 = 9/2\,, \quad \text{Zakharov-Sagdeev spectrum}\,.$$

- Capillary waves on deep water: $d = 2,\; \omega_k \propto k^{3/2}$, $p = 3$,

$\beta_3 = 9/4 \quad \Rightarrow \quad x_3 = 17/4\,, \quad$ Zakharov-Filonenko spectrum .

- Gravity waves on deep water: $d = 2,\; \omega_k \propto \sqrt{k}$, $p = 4$,

$\beta_4 = 3 \Rightarrow x_4 = 4\,, \quad$ Zakharov-Filonenko spectrum .

Kolmogorov spectra of wave turbulence
Interaction locality

- Kelvin waves (KWs) in superfluids: $d = 1$, $\omega_k \propto k^2$, $p = 6$, $\beta_6 = 6 \Rightarrow$

$$x_6 = 17/5\,,\quad \text{Kozik-Svistunov-2004 (KS) spectrum ???}\,.$$

All above turbulent energy spectra are based on the *assumption* of the locality of energy transfer over scales (converges of integrals in the collision term). This is the case for all the above examples, except of the last.

Recently Laurie, L'vov, Nazarenko & Rudenko (PRB, submitted) show that KS assumption of locality the Kelvin wave interactions is happened to be wrong and thus the KS-spectrum is irrelevant.

L'vov & Nazarenko (under preparation) show that interaction of the KWs, propagated over randomly curved vortex line is dominated by the five-wave interaction with the local energy transfer. We got:

$$x_5 = 7/2\,,\quad \text{L'vov-Nazarenko-2009 (LN) local spectrum}\,.$$

Kolmogorov spectra of wave turbulence

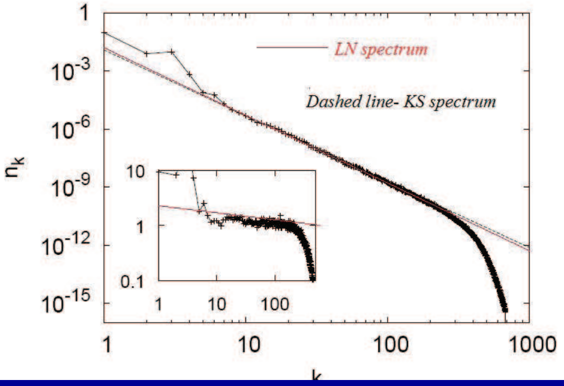
Lvov-Nazarenko vs Kozik-Svistunov spectra of Kelvin waves in superfluids

More about KW turbulence (κ – quantum of velocity circulation):

Nonlocal 3 $\leftrightarrow$ 3 KS-spectrum $\Rightarrow$ Local 3 $\leftrightarrow$ 2 LN-spectrum

$$n_{\rm KS}(k) \simeq \frac{(\varepsilon \kappa^2)^{1/5}}{k^{17/5}} \qquad \Rightarrow \qquad n_{\rm LN}(k) \simeq \frac{(\varepsilon \kappa)^{1/4}}{k^{7/2}} \,.$$

G. Boffetta, A. Celani, D. Dezzani, J. Laurie and S. Nazarenko JLTP **156**, 193 (2009)



Kolmogorov spectra of wave turbulence
Turbulent spectra with constant energy flux

Remember: $2 \leftrightarrow 2$- and $2 \leftrightarrow 2$-KEs conserves total number of particle $N \equiv \int n_k dk$. This implies particle-continuity equation:

$$\frac{\partial n_k}{\partial t} + \frac{\partial \mu(k)}{\partial k} = 0, \qquad \text{where “particle" flux } \mu(k) \equiv \int_{k_+}^{k} \mathrm{St}(k) k^{(d-1)} dk \ .$$

Assuming locality of the particle transfer one has (with $\omega_k = \omega k^\alpha$):

$$\mu_{2\leftrightarrow 2}(k) \simeq k^{3d} (T_{k,k}^{k,k})^2 n_k^3 / o_k \simeq k^{3d+2\beta_4-\alpha} V_4^2 n_k^2 / \omega \equiv \mu_4 \, ,$$

$$\mu_{3\leftrightarrow 3}(k) \simeq k^{5d} (W_{k,k}^{k,k})^2 n_k^5 / \omega_k \simeq k^{5d+2\beta_6-\alpha} V_6^2 n_k^2 / \omega \equiv \mu_6 \, ;$$

In general: $$\mu_{2p}(k) = \left(k^d n_k\right)^{2p-1} k^{2\beta_{2p}-\alpha} V_{2p}^2 / \omega \ .$$

With condition $\mu(k) = \mu =$const. the gives:

$$n_k \propto \frac{1}{k^{y_p}} \, , \qquad y_p = d + \frac{2\beta_p - \alpha}{p-1} \ .$$

Kolmogorov spectra of wave turbulence

Turbulent spectra with constant energy flux: Physical examples

- Gravity waves on deep water: $d=2,\ \omega_k \propto \sqrt{k}$, $p=4$,

$$\beta_4 = 3 \Rightarrow y_4 = 4 - \frac{1}{6} = \frac{23}{6}\,, \quad \text{Zakharov-Filonenko (local) spectrum}\,.$$

- Kelvin waves in superfluids: $d=1,\ \omega_k \propto k^2$, $p=4$,

$$\beta_6 = 6 \Rightarrow y_6 = 3\,, \quad n_k \propto \frac{1}{k^3}\,, \quad \text{Winen (weakly-nonlocal) spectrum}\,.$$

Laurie, L'vov, Nazarenko & Rudenko (PRB, submitted) log-correction:

$$n_k \propto \frac{1}{k^3\,(\ln k)^{1/5}}\,.$$

Kolmogorov spectra of wave turbulence
Direction of fluxes: Direct energy cascade

- Direct energy cascade:

From one side, we showed (for p-wave KE in d-dimensional media):

$$n_k \propto k^{-x}\,, \;\; x = d + 2\,\beta_p/p - 1\,.$$

From other side, in the thermodynamic equilibrium $n_k = T/\omega_k \propto k^{-\alpha}$. Because energy goes toward toward equilibrium distribution. , In all known examples $x > \alpha$ (e.g. for Kelvin waves $x = 7/2\,,\;\alpha = 2$), therefore energy goes toward large k. Therefore we have

“Direct energy cascade" (toward large k).

Kolmogorov spectra of wave turbulence
Direct energy & inverse particle cascades in 4- & 6-wave KE

Following Kraichnan consider energy and particle-number influxes, ε^+ and μ^+ at some $k \approx k_0$.
Denote as $\omega_\pm$, and $\gamma_\pm$ – the wave frequencies and dampings at $k_+ \gg k_0$ and $k_- \ll k_0$, $n_\pm = n(k_\pm)$ – particle numbers at $k = k_\pm$.
Then in the $k_\pm$ areas the rates of particle-number dissipations are $\mu_\pm \simeq n_\pm \gamma_\pm$, and the rates of energy dissipation: $\varepsilon_\pm \simeq \omega_\pm \mu_\pm$. Clearly the total dissipations: $\mu \equiv \mu_+ + \mu_-$, $\varepsilon \equiv \varepsilon_+ + \varepsilon_- = \omega_+ \mu_+ + \omega_- \mu_-$.
Solving these Eqs. in the limit $\omega_- \to 0$, and $\omega_+ \to \infty$ one has:

$$\mu = \mu_- + \frac{\varepsilon}{\omega_+} \to \mu^- \, , \quad \varepsilon = \varepsilon_+ + \omega_- \mu^+ \to \varepsilon_+ \, .$$

This mean that the energy mainly dissipates at large k and we have

Direct energy cascade,

whereas the particle number mainly dissipates at small k and one has

Inverse particle cascade.

• We have formulated Classical Hamiltonian formalism for nonlinear waves, considered canonical structure of the Hamiltonian at small nonlinearity;
• We presented statistical description of weakly nonlinear waves, formulated wave kinetic equations and studied their general properties;
• The main point was Kolmogorov spectra of wave turbulence with constant energy and particle fluxes. We stressed importance of the locality of the energy/particle cascades and considered direction of fluxes.

We did NOT discussed various (well studied) problems, including:
• Many-flux and anisotropic Kolmogorov spectra;
• Matching inertial-interval spectra with the pumping and dissipation regions,
• Exact flux solutions of the 3-wave & 4-wave KEs;
• Stability and evolution toward flux solutions;• Kolmogorov spectra of strong wave turbulence, etc.

$\mathcal{THE}$ $\mathcal{END}$