Quantum and Classical Turbulence @ Beijing, China, April 2012

Energy Spectra of Homogeneous Superfluid Turbulence:

from classical hydrodynamics to quantum Kelvin-wave region

Victor S. Lvov

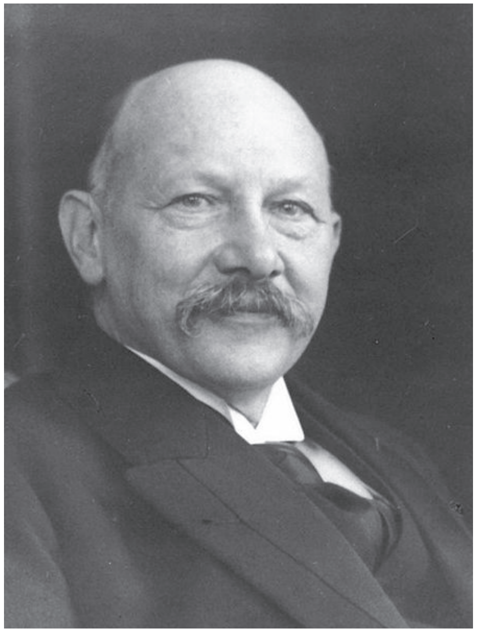
OUTLINE

- Introduction: Discovery of the superfluidity, Two-fluid equations and mutual friction.
  Where we are in the theory of superfluid vs. classical hydrodynamic turbulence
- Energy spectra at large-scales: K-41 inspired statistical description of superfluid turbulence ⇒ Energy spectra of normal and superfluid components, cross-velocity correlations.
- Weak turbulence of small-scale Kelvin waves (KWs): Hamiltonian.
  Analysis of the Interaction amplitudes. Kinetic equation for KWs, its solutions and applicability.
  Locality problem of the energy transfer over scales and KS-vs-LNR controversy
- Spectrum of superfluid turbulence at inter-vortex scales: LNR-MinModel and KS-scenario of energy transfer over inter-vortex scales. LNR vs. KS energy spectra
- Summary of the results and perspectives

Contents

0.1 Superfluids: experiments and theory

Heike Kamerlingh-Onnes



Nobel prize 1913

"for his investigations on the properties of matter at low temperatures which led, inter alia, to the production of liquid helium".

K-O discovered in 1911 superconductivity.

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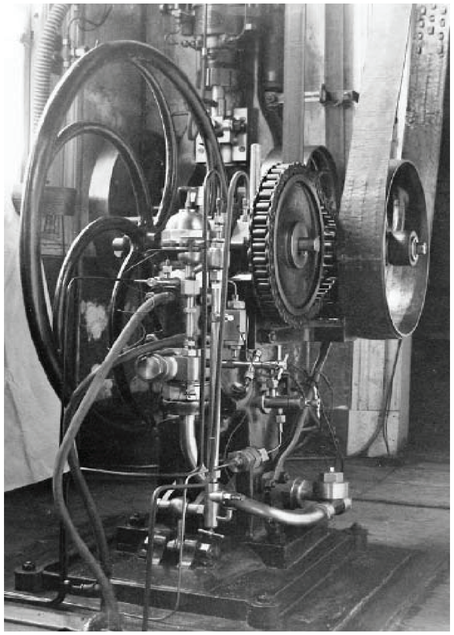


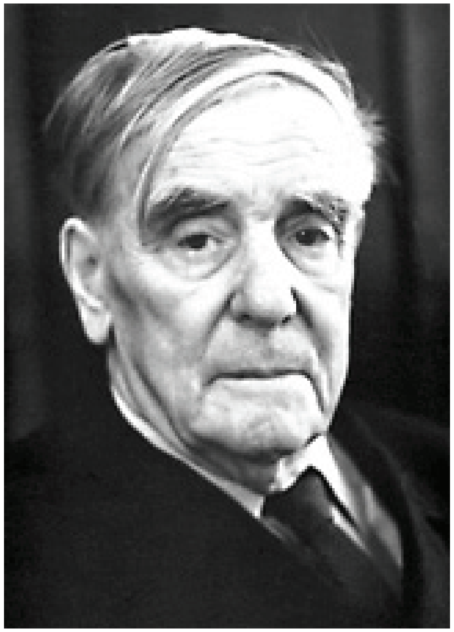
liquified He at $T = 4.2\,\mathrm{K}$ in July 10, 1908.

K-O & coworkers in 1924 discovered density change at $T = 2.18\,\mathrm{K}$.

Keesom & Wolfke, 1928: this is a phase transition

He I $\Leftrightarrow$ He II.

Piotr Leonidovich Kapitza



Nobel prize 1978

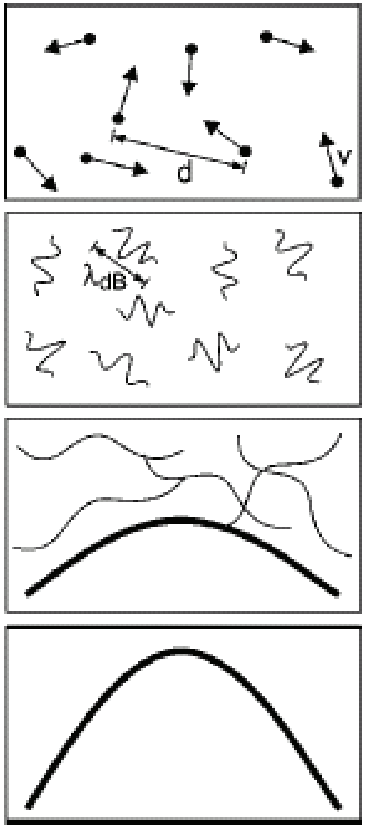
"for his basic inventions and discoveries in the area of low-temperature physics". P.L. Kapitza in Moscow discovered and named in 1937

superfluidity of ^{4}He

Jack Allen

and his student Donald Missener

independently discovered superfluidity in PLK's Cambridge lab.



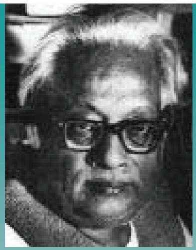
High Temperature T:
thermal velocity v
density d^{-3}
"Billiard balls"

Low Temperature T:
De Broglie wavelength
$\lambda_{dB}=h/mv \propto T^{-1/2}$
"Wave packets"

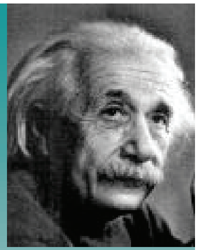
$T=T_C$: BEC
$\lambda_{dB} \approx d$
"Matter wave overlap"

T=0: Pure Bose condensate
"Giant matter wave"

Ideal Bose gas



S. Bose



A. Einstein

Bose-Einstein quantum statistics

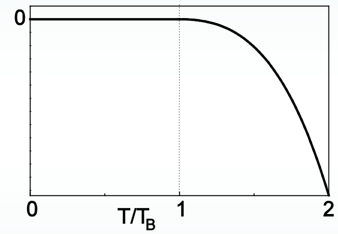
$$n_k = \frac{1}{\exp\left\{\frac{\varepsilon_k - \mu}{kT}\right\} - 1}$$

calibration

$$N = \frac{1}{V}\sum_{k=0}^{\infty} n_k$$

A. **Einstein 1924:** **(in 3D momentum space) below certain condensation temperature, macroscopically large number of particles will occupy the lowest energy state**

$$N = N_0 + \sum_{k=1}^{\infty} n_k = N_0 + \sum_{k=1}^{\infty} \frac{1}{\exp\left\{\frac{\varepsilon_k - \mu}{kT}\right\} - 1}$$



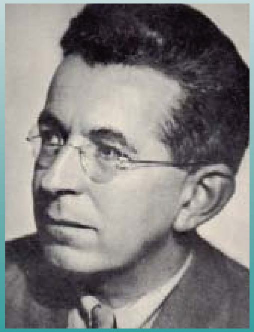
$$N = N_0 + \frac{4\pi}{(2\pi\hbar)^3}\int_0^{\infty} \frac{p^2}{\exp\left\{\frac{p^2}{2m_{He}} \Big/ kT_B\right\} - 1} dp = \frac{m_{He} kT_B}{2\pi^2\hbar^3}\sqrt{2m_{He}kT_B}\int \frac{\sqrt{z}\,dz}{e^z - 1}$$

$$= \frac{\sqrt{\pi}}{2}\xi\left(3/2\right)$$

Rieman f-n

null

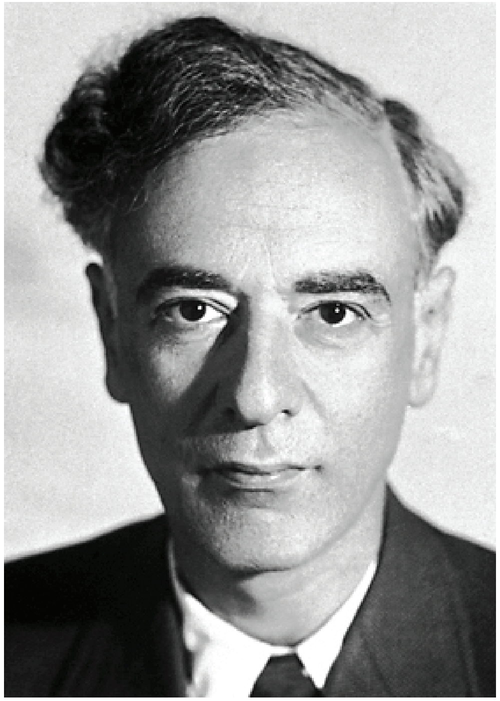
$$T_B = \frac{2\pi\hbar^2}{m_{He}k}\left(\frac{N}{\xi(3/2)}\right)^{2/3} \cong 3.15K$$



F. London

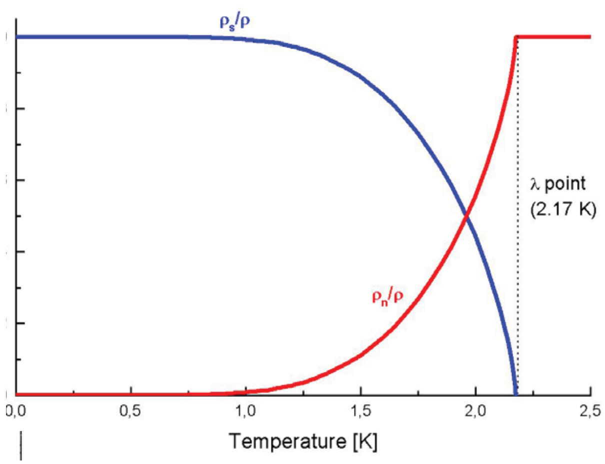
Experiment – He II

$$T_\lambda \cong 2.18K$$



Lev Davidovich Landau Nobel Prize, **1962**
"for his pioneering theories for condensed matter, especially liquid helium".
In particular, he quantized in **1941** the Tisza-**1940** two-fluid model and suggested Andronikashvilii's **1946** experiment on oscillating in He II discs.

Its period and damping measures densities of superfluid, $\boldsymbol{\rho}_{\mathrm{s}}$ and normal, $\boldsymbol{\rho}_{\mathrm{s}}$, components:



Elepter Luarsabovich Andronikashvili

Laszlo Tizsa

Landau-Tizsa two fluid model

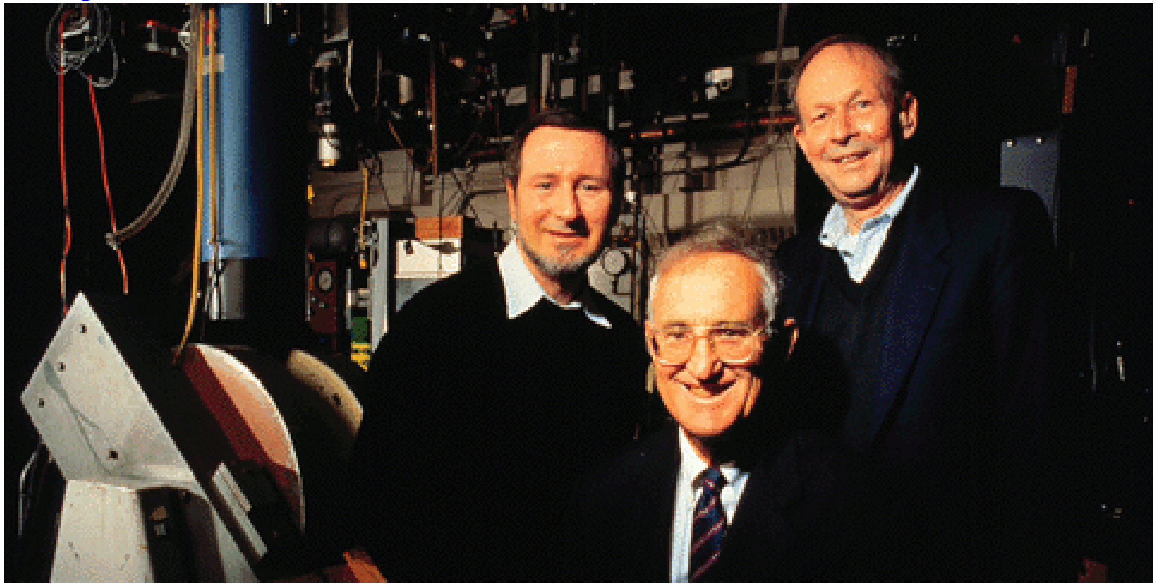
for superfluid, $\boldsymbol{V}_n$, and normal $\boldsymbol{V}_s$ velocities:

$$\rho_s \frac{D\mathbf{v}_s}{Dt} = -\frac{\rho_s}{\rho}\nabla p + \rho_s S \nabla T - \mathbf{F}_{ns}$$

$$\rho_n \frac{D\mathbf{v}_n}{Dt} = -\frac{\rho_n}{\rho}\nabla p - \rho_s S \nabla T + \eta \nabla^2 \mathbf{v}_n + \mathbf{F}_{ns}$$

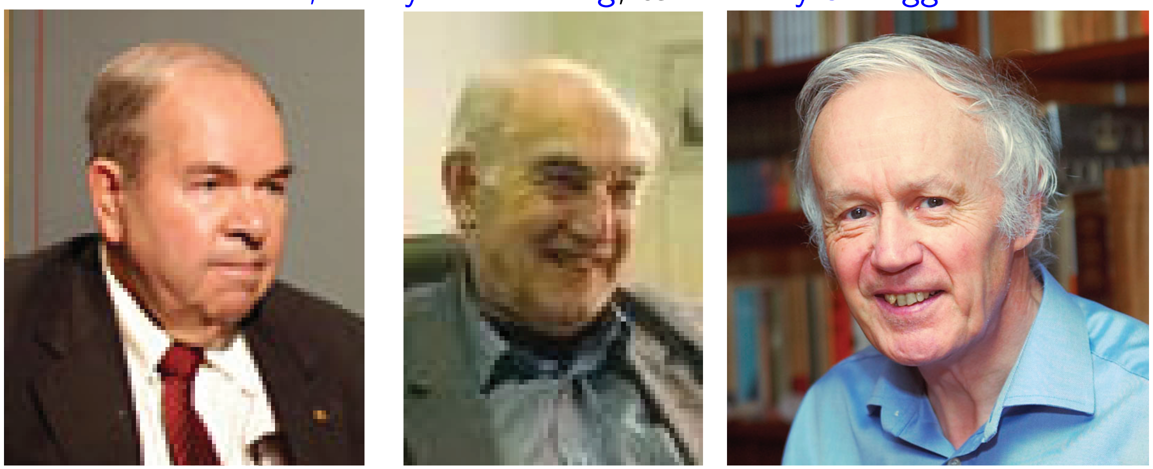
predicts "second sound", critical velocity, etc.
Here: S – entropy, T – temperature and $\boldsymbol{F}_{ns} = A\rho_n\rho_s(\boldsymbol{V}_s - \boldsymbol{V}_n)^3$ is the mutual friction between superfluid and normal components

Douglas D. Osheroff, David M. Lee & Robert C. Richardson



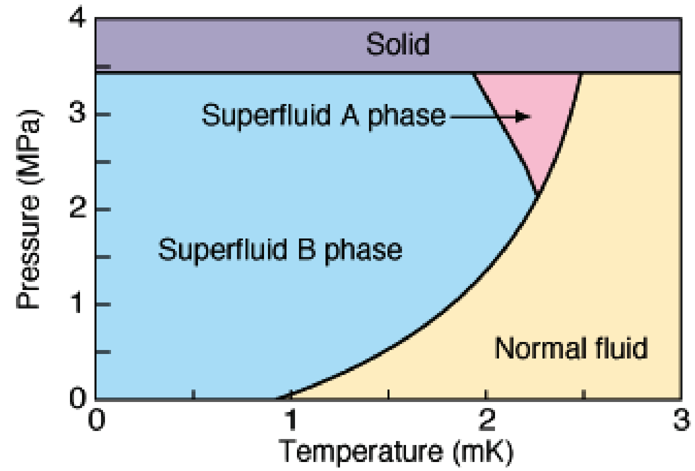
Nobel Prize 1996 "for their discovery of superfluidity in helium-3".

Alexei A. Abrikosov, Vitaly L. Ginzburg, & Anthony J. Leggett



Nobel Prize 2003 "for pioneering contributions to the theory of superconductors and superfluids"

^{3}He, the result of tririum decay, was produced (150 Kg since 1955) and liquified in LANL. Using Pomeranchuk's compressive cooling D.O, R.R&D.L discovered superfluidity of ^{3}He on April 20, 1972 at Cornell.



Knowing this before publication, J. Leggett on Sept. 5, 1972 submited to PRL explanation of their observations as Bardeen-Cooper-Schrieffer condensation of Couper pairs of ^{3}He atoms in the triplet state with the tensorial ordering parameter. The B-state has an isotropic gap.

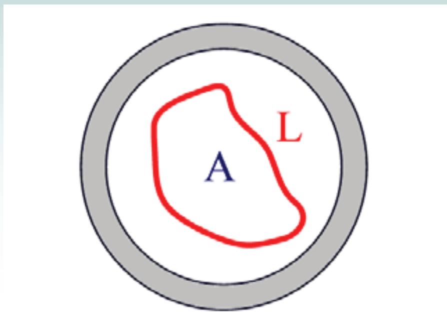
Quantum mechanical description of He II

Macroscopic wave function $\Psi = \sqrt{\rho_s}\exp\{i\varphi(r,t)\}$

$$\hat{p} = i\hbar\nabla \longrightarrow \mathbf{v}_s = \frac{\hbar}{m_4}\nabla\varphi \longrightarrow curl\mathbf{v}_s = 0$$

Circulation –singly connected region

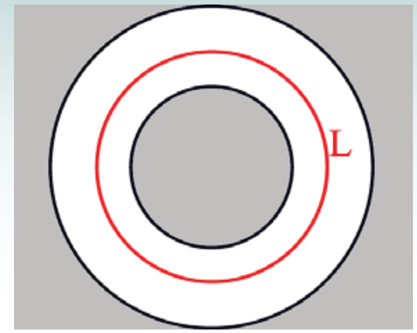
$$\Gamma = \oint_L \mathbf{v}_s d\ell = 0$$



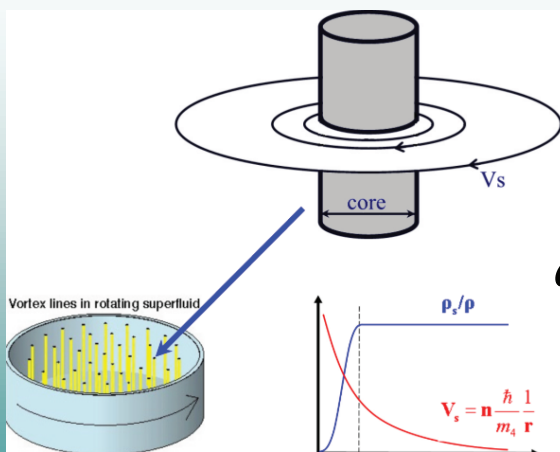
Circulation- multiply connected region

$$\Gamma = \oint_L \mathbf{v}_s d\ell = n\frac{h}{m_4} = n\kappa$$

$$\kappa \cong 10^{-7} m^2 / s$$



Quantized vortices in He II

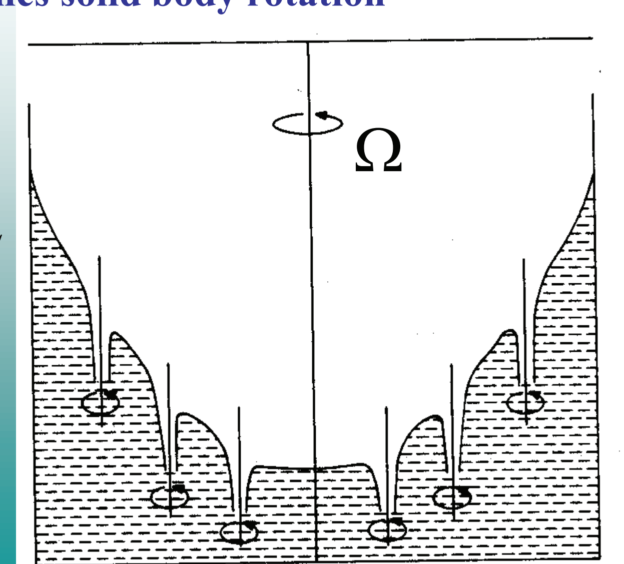


vorticity

$$\omega_N = 2\Omega \cong \langle \omega_S \rangle \cong \kappa L$$

Rotating bucket of He II

-thanks to the existence of rectilinear vortex lines He II mimics solid body rotation



Turbulence in a superfluid was predicted first by Richard Feynman in **1955** and found experimentally (in counterflow ^{4}He) by Henry Hall and Joe Vinen in **1956**. Consider

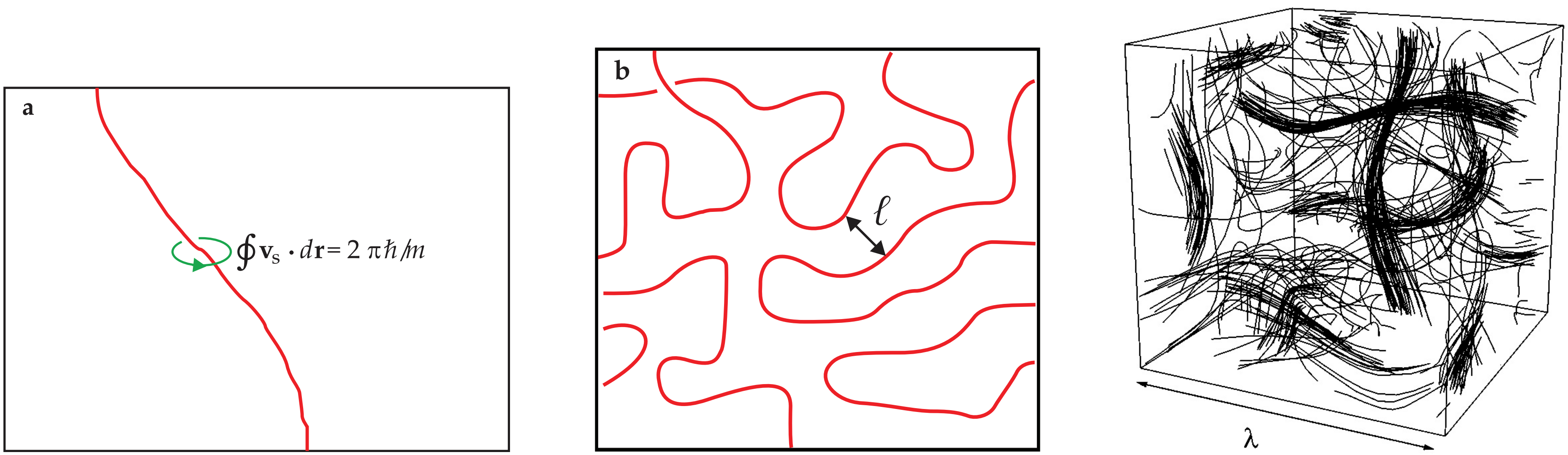
1.3.1 Normal fluid *vs.* superfluid at $T \to 0$ limit:

– Normal fluid kinematic viscosity $\nu \neq 0$ *vs.* $\nu \equiv 0$ in superfluids;

– Two scales in normal fluids: Outer scale $\mathcal{L}$ and dissipative micro-scale $\eta \ll \mathcal{L}$;

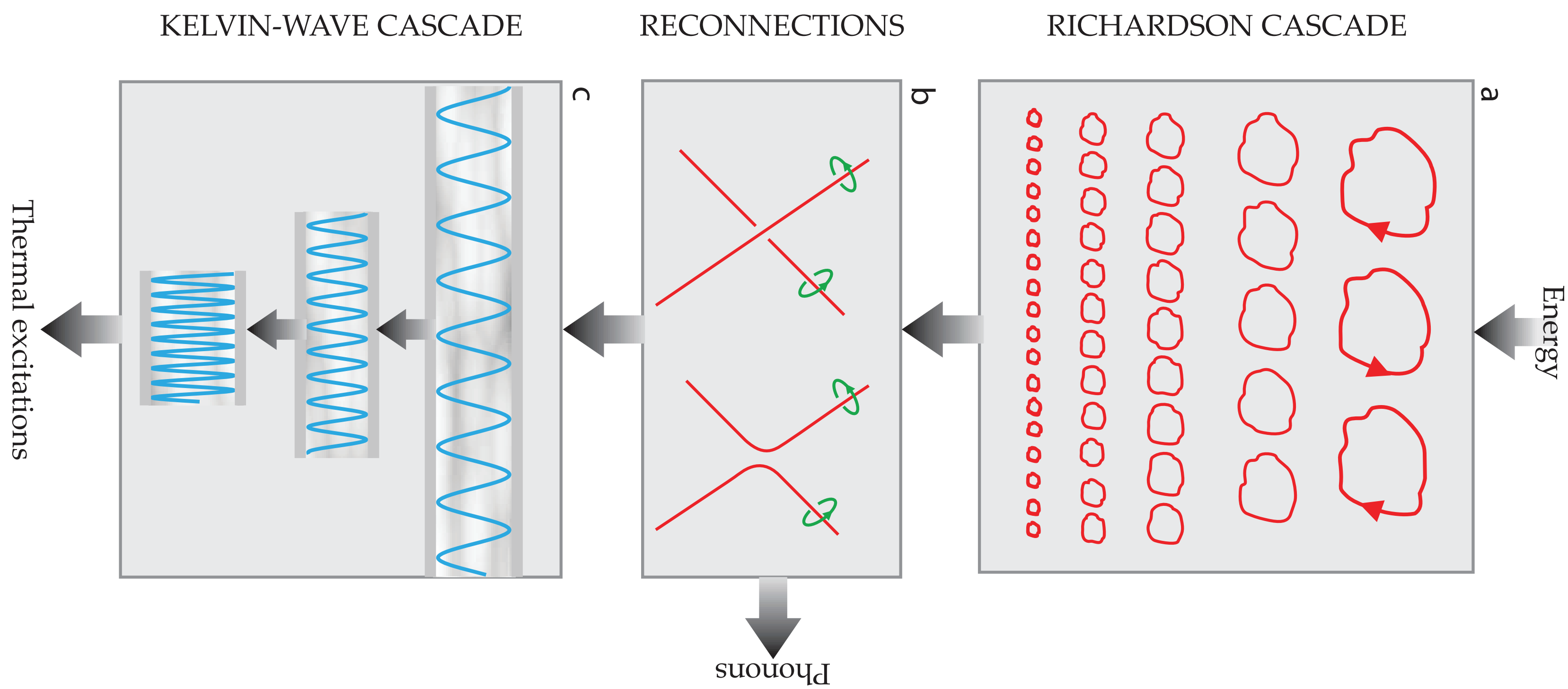
– Two additional scales in superfluids due to quantization of vortex lines:

$\Downarrow$ vortex core diameter a_0 $\Downarrow$ $\Downarrow$ mean inter-vortex distance ℓ $\Downarrow$ $\Downarrow$ Outer scale $\mathcal{L}$ $\Downarrow$



In ^{4}He $a_0 \simeq 1$ Å, in ^{3}He $a_0 \simeq 800$ Å. Experimentally, in both ^{4}He and ^{3}He, $\Lambda \equiv \ln\left(\frac{\ell}{a_0}\right) \simeq 12 \div 15$

1.2 Sketch of the quantum-turbulence cascades[1]



Very small scales: $L \ll \ell$.

Intervortex scales: $L \sim \ell$

Large scales: $L \gg \ell$

My goal here is to describe as simple as possible the homogeneous, isotropic turbulence of superfluids in zero-temperature limit with energy pumping at scales $\mathcal{L}$ much above the intervortex distance ℓ. [1]

This can help in analysis of DNS and experimental data, as well as in further development of a theory of superfluid turbulence in more realistic situations.

[1] W.F. Vinen and R. J. Donnelly, ***Quantum Turbulence***, Physics Today, v. 60, 43 (2007)

0.3 Two-fluid equations in hydrodynamic (HD) region, $R \gg \ell$

In this region one can neglect the quantization of vortex lines and make use coarse-grained two fluid equation for velocities the superfluid and normal components $\boldsymbol{u}_{\rm s}$ and $\boldsymbol{u}_{\rm n}$, with densities $\rho_{\rm s}$ and $\rho_{\rm n}$ and pressures $p_{\rm s}$ and $p_{\rm n}$

$$\rho_{\rm s}\Big[\frac{\partial\, \boldsymbol{u}_{\rm s}}{\partial t} + (\boldsymbol{u}_{\rm s}\boldsymbol{\nabla})\boldsymbol{u}_{\rm s}\Big] - \boldsymbol{\nabla} p_{\rm s} \ = \ -\boldsymbol{F}_{\rm ns}\,, \quad p_{\rm s} = \frac{\rho_{\rm s}}{\rho}[p - \rho_{\rm n}|\boldsymbol{u}_{\rm s} - \boldsymbol{u}_{\rm n}|^2]\,, \tag{1a}$$

$$\rho_{\rm n}\Big[\frac{\partial\, \boldsymbol{u}_{\rm n}}{\partial t} + (\boldsymbol{u}_{\rm n}\boldsymbol{\nabla})\boldsymbol{u}_{\rm n}\Big] - \boldsymbol{\nabla} p_{\rm n} \ = \ \rho_{\rm n}\nu\Delta\boldsymbol{u}_{\rm n} + \boldsymbol{F}_{\rm ns}\,, \quad p_{\rm n} = \frac{\rho_{\rm n}}{\rho}[p + \rho_{\rm s}|\boldsymbol{u}_{\rm s} - \boldsymbol{u}_{\rm n}|^2]\,, \tag{1b}$$

coupled by the the mutual friction between superfluid and normal components of the liquid mediated by quantized vortices which transfer momenta from the superfluid to the normal subsystem and vice versa:

$$\boldsymbol{F}_{\rm ns} = -\rho_{\rm s}\{\alpha'(\boldsymbol{u}_{\rm s} - \boldsymbol{u}_{\rm n}) \times \boldsymbol{\omega}_{\rm s} + \alpha\, \hat{\boldsymbol{\omega}}_{\rm s} \times [\boldsymbol{\omega}_{\rm s} \times (\boldsymbol{u}_{\rm s} - \boldsymbol{u}_{\rm n})]\} \approx \alpha\, \rho_{\rm s}\omega_{_{\rm T}}(\boldsymbol{u}_{\rm s} - \boldsymbol{u}_{\rm n})\,, \quad \omega_{_{\rm T}} = \sqrt{\langle|\boldsymbol{\omega}_{\rm s}|^2\rangle}\,. \tag{1c}$$

Here for $\boldsymbol{F}_{\rm ns}$ can be used simple closure (approximation) suggested by L'vov-Nazarenko-Volovik in Ref.[1].

Eqs (1) are very similar to the Navier-Stokes equation. Therefore in a theory of large-scale superfluid turbulence we can use numerous tools, developed in the theory of classical HD turbulence.

[1] V. S. L'vov, S. V. Nazarenko and G. E. Volovik, JETP Letters, v. 80, 535-539 (2004)

0.4 Classical closures for the energy flux $\varepsilon(k)$via energy spectrum $E(k)$

A simplest closure, based on dimensional reasoning was suggested in 1941 by Kolmogorov:

$$\boldsymbol{E(k)} \simeq \boldsymbol{\varepsilon^{2/3} k^{-5/3}} \qquad \Rightarrow \qquad \boldsymbol{\varepsilon(k)} \simeq \boldsymbol{E(k)^{3/2} k^{5/2}} \,. \tag{2a}$$

Numerous closures, relating energy flux with the energy density was suggested since 1960$^{\rm th}$, like Kraichnan's DIA (Direct Interaction Approximation) (DIA)[2], and his Lagrangian History DIA [3] popular Orszag's EDQNM (Eddy Damped Quasi-Normal Markovian) approximation [4] and its modern LNR-modification [5] which is simpler and better reflects interaction of very different-scale eddies:

$$\frac{d\,\varepsilon(k)}{dk} \simeq \int\limits_{-\infty}^{\infty} \frac{dq\,dp\,\delta(k+q+p)\left[k^3\,E(q)E(p)+q^3\,E(k)E(p)+p^3\,E(q)E(k)\right]}{2\pi\,k\,(k^2+q^2+p^2)(\gamma_k+\gamma_q+\gamma_p)}\,, \tag{2b}$$

Here $\gamma_k = \sqrt{|k|^3 E(k)}$ is the eddy-turnover frequency. (2c)

All Navier-Stokes based closures includes as a particular solution the thermodynamic equilibrium with equipartition of energy between all degrees of freedom, i.e. different $\vec{k}$. Namely:

$$\text{At } \ \varepsilon = 0: \quad E(k) = k^2 T/(\pi\rho)\,, \quad T - \text{Temperature} \quad \rho - \text{fluid density.} \tag{3}$$

For example, in Eq. (2b) at $E(k) \propto k^2$, $\delta(k+q+p)\left[\,\dots\right] \propto \delta(k+q+p)\,(k+q+p) = 0$

This is not the case for for much more simple algebraic K41-closure (2a), giving $E(k) \neq 0$ for $\varepsilon \neq 0$.

[2] R. H. Kraichnan, J. Fluid Mech. v. 5, 497 (1959)
[3] R. H. Kraichnan, Phys. Fluids v.7, 1030 (1964)
[4] S.A. Orszag, J. Fluid Mech. v.41, 363(1970)
[5] V. S. L'vov, S. V. Nazarenko, O. Rudenko, Phys. Rev. B v. 76, 024520 (2007)

A reasonable compromise between adequateness of the description of current state of superfluid-turbulence research and simplicity can be reached using Leith approach[6], who rewrote K41 Eq. (2a)

as $\varepsilon(k) \simeq \sqrt{k^5 E(k)}\, E(k)$ and replace $E(k)$ by $-k^3 \frac{d}{dk}\Big[\frac{E(k)}{k^2}\Big]$.

This gives differential approximation for the energy flux

$$\varepsilon(k) = -\frac{1}{8}\sqrt{k^{11}\,E(k)}\,\frac{d}{dk}\Big[\frac{E(k)}{k^2}\Big]\,, \tag{4}$$

where we introduced proper numerical factor [7]. Solving Eq. (4) one gets spectrum $E(k)$ with $\varepsilon(k)=\varepsilon$,

$$E(k) = k^2\Big[\frac{24\,\varepsilon}{11\,k^{11/2}} + \Big(\frac{T}{\pi\rho}\Big)^{3/2}\Big]^{2/3} \Rightarrow \begin{cases} (24/11)^{2/3}\ \varepsilon^{2/3}\, k^{-5/3}\,, \\ \\ k^2 T/\pi\rho\ . \end{cases} \tag{5}$$

Low k region: K41 spectrum $E(k) \propto \varepsilon^{2/3} k^{-5/3}$,

Large k region: energy equipartition with an effective temperature T.

- The differential approximation (4) for the energy flux in the hydrodynamic regime $k\ell < 1$ will be used in further analysis of the superfluid turbulence.

- More sophisticated LNR-integral closure (2b) can be used in future, when more experimental and numerical data will be available.

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[6] C. Leith, Phys. Fluids v. 10, 1409 (1967)

[7] C. Connaughton, S. Nazarenko, Phys. Rev. Lett. v. 92, 044501 (2004)

1 Superfluid Energy Spectra at Large-Scales

1.1 K-41 inspired statistical description of superfluid turbulence

In order to derive equations for the energy spectra $E_\mathrm{s}(k)$, $E_\mathrm{n}(k)$ one needs to rewrite Eqs. (1) for superfluid and normal velocity components $\boldsymbol{v}_\mathrm{s}(\boldsymbol{r},t)$ and $\boldsymbol{v}_\mathrm{n}(\boldsymbol{r},t)$ in the $\boldsymbol{k}$-representation, to multiply resulting equation for $\boldsymbol{v}_\mathrm{s,n}(\boldsymbol{k},t)$ by $\boldsymbol{v}^*_\mathrm{s,n}(\boldsymbol{k},t)$, to add complex conjugated equation and to average. Then to multiply resulting equations for the velocities (cross)-correlation functions $F_\mathrm{ss}(k,t)$ and $F_\mathrm{nn}(k,t)$ by $k^2/(4\pi^2)$. Resulting equations for $E_\mathrm{s}(k,t)$ and $E_\mathrm{n}(k,t)$ preserve total energies of each component E_s and E_n in the dissipationless limit ($\alpha_\mathrm{s}=\alpha_\mathrm{n}=\nu=0$). Therefore their nonlinear part [that originate from the nonlinear terms in Eqs. (1)] can be presented in the continuity equation as follows:

$$\frac{\partial E_\mathrm{s}(k,t)}{\partial t}+\frac{\partial \boldsymbol{\varepsilon}_\mathrm{s}(\boldsymbol{k},\boldsymbol{t})}{\partial k}=\alpha_\mathrm{s}\omega_{_\mathrm{T}}[\boldsymbol{E}_\mathrm{sn}(\boldsymbol{k},\boldsymbol{t})-E_\mathrm{s}(k,t)]\,,\quad \omega_{_\mathrm{T}}=\sqrt{\langle|\omega_\mathrm{s}|^2\rangle}\,, \tag{6a}$$

$$\frac{\partial E_\mathrm{n}(k,t)}{\partial t}+\frac{\partial \boldsymbol{\varepsilon}_\mathrm{n}(\boldsymbol{k},\boldsymbol{t})}{\partial k}=\alpha_\mathrm{n}\omega_{_\mathrm{T}}[\boldsymbol{E}_\mathrm{sn}(\boldsymbol{k},\boldsymbol{t})-E_\mathrm{n}(k,t)]-\nu k^2\,E_\mathrm{n}(k,t)\,, \tag{6b}$$

$$\boldsymbol{\varepsilon}_\mathrm{s}(\boldsymbol{k})=-\frac{1}{8}\sqrt{k^{11}\,E_\mathrm{s}(k)}\,\frac{d}{dk}\Big[\frac{E_\mathrm{s}(k)}{k^2}\Big]\,,\qquad \boldsymbol{\varepsilon}_\mathrm{n}(\boldsymbol{k})=-\frac{1}{8}\sqrt{k^{11}\,E_\mathrm{n}(k)}\,\frac{d}{dk}\Big[\frac{E_\mathrm{n}(k)}{k^2}\Big]\,. \tag{6c}$$

Here $\boldsymbol{\varepsilon}_\mathrm{s}(\boldsymbol{k},\boldsymbol{t})$ and $\boldsymbol{\varepsilon}_\mathrm{n}(\boldsymbol{k},\boldsymbol{t})$ are the turbulent energy fluxes over scales in superfluid and normal-fluid components for which we will use Leith-Nazarenko differential approximation (4) and $\boldsymbol{E}_\mathrm{sn}(\boldsymbol{k},\boldsymbol{t})$ is the normal-superfluid cross-correlation function that required separate analysis.

1.2 Cross-correlation function of the normal and superfluid velocities

Cross-correlation function of the normal and superfluid velocities [8,9] can be found using traditional in HD turbulence approximation of turbulent eddy-viscosity, replacing nonlinear terms in two-fluid equations (1) for $\boldsymbol{v}_{\rm n,s}(\boldsymbol{k},t)$ by $\gamma_{\rm n,s}(k)\boldsymbol{v}_{\rm n,s}(\boldsymbol{k},t)+\boldsymbol{f}_{\rm n,s}(\boldsymbol{k},t)$, where $\boldsymbol{f}_{\rm n,s}(\boldsymbol{k},t)$ is the Langevin random force with Gaussian statistics and simultaneous correlations $f^2(\boldsymbol{k})$ chosen such to reproduce given energy spectra $E_{\rm n}(k)$ and $E_{\rm s}(k)$. Resulting system of two coupled linear Eqs. for $\boldsymbol{v}_{\rm s}(\boldsymbol{k},t)$ and $\boldsymbol{v}_{\rm n}(\boldsymbol{k},t)$ can be analyzed by simple one-page calculations with the result:

$$\boldsymbol{E_{\rm sn}(k)} = \frac{\boldsymbol{\alpha_{\rm n}\, E_{\rm s}(k) + \alpha_{\rm s}\, E_{\rm n}(k)}}{\boldsymbol{\alpha_{\rm s} + \alpha_{\rm n} + [\nu\, k^2 + \gamma_{\rm n}(k) + \gamma_{\rm s}(k)]/\omega_{\rm T}}}\,, \quad \text{where} \quad \gamma_{\rm n,s}(k) \simeq k^{3/2}\sqrt{\mathcal{E}_{\rm n,s}(k)} \qquad (7)$$

are eddy-turnover frequencies in the normal and superfluid components. My feeling is that

Eq. (7) works better than one would expect ,

having in mind our rather crude approximations made in Refs. [9,10].

- For large mutual friction, when $\omega_{\rm T}(\alpha_{\rm n}+\alpha_{\rm s}) \gg (\gamma_{\rm n}+\gamma_{\rm s})$, it turns into a simpler form $E_{\rm sn}(k) = [\rho_{\rm s}E_{\rm s}(k)+\rho_{\rm n}E_{\rm n}(k)]/\rho$. This equation has a physically motivated solution $E_{sn}(k) = E_s(k) = E_n(k)$, that gives $\langle \boldsymbol{u}_{\rm n}(\boldsymbol{k},t)[\boldsymbol{u}_{\rm n}(\boldsymbol{k},t)-\boldsymbol{u}_{\rm s}(\boldsymbol{k},t)]\rangle = 0$ and thus requires
  $\boldsymbol{u}_{\rm n}(\boldsymbol{r},t) = \boldsymbol{u}_{\rm s}(\boldsymbol{r},t)$, – a fully coherent motion of the superfluid and the normal fluid velocities.

- For small mutual friction, $\omega_{\rm T}(\alpha_{\rm n}+\alpha_{\rm s}) \ll (\gamma_{\rm n}+\gamma_{\rm s})$, Eq. (7) gives
  decoupled motion of the superfluid and the normal fluid components, $E_{\rm sn}(k) \ll \sqrt{E_{\rm s}(k)E_{\rm sn}(k)}$.

[8] V.S. L'vov, S.V. Nazarenko, L. Skrbek, JLTP v. 145, 125 - 142 (2006)
[9] V.S. L'vov, unpublished (2010)

1.3 Energy spectra of superfluid components in ^{3}He-B

Due to large viscosity of normal fluid component in ^{3}He-B (like glycerol), as a rule it can be considered as resting: $\boldsymbol{u}_{\rm n}(\boldsymbol{r},t)=0$. This tremendously simplifiers the problem and allows it full analytical description in terms of "mutual friction frequency" $\Gamma\equiv\alpha_{\rm s}\sqrt{\langle|\omega_{\rm s}|^2\rangle}$ and its dimensionless version $\gamma\equiv\Gamma/\varepsilon_0 k_0^{3/2}$ [ε_0- energy influx, $1/k_0$-outer scale of turbulence]:

- For $\gamma=\gamma_{\rm cr}=5^{2/3}/2\approx 1.46$ there is a "critical" solution

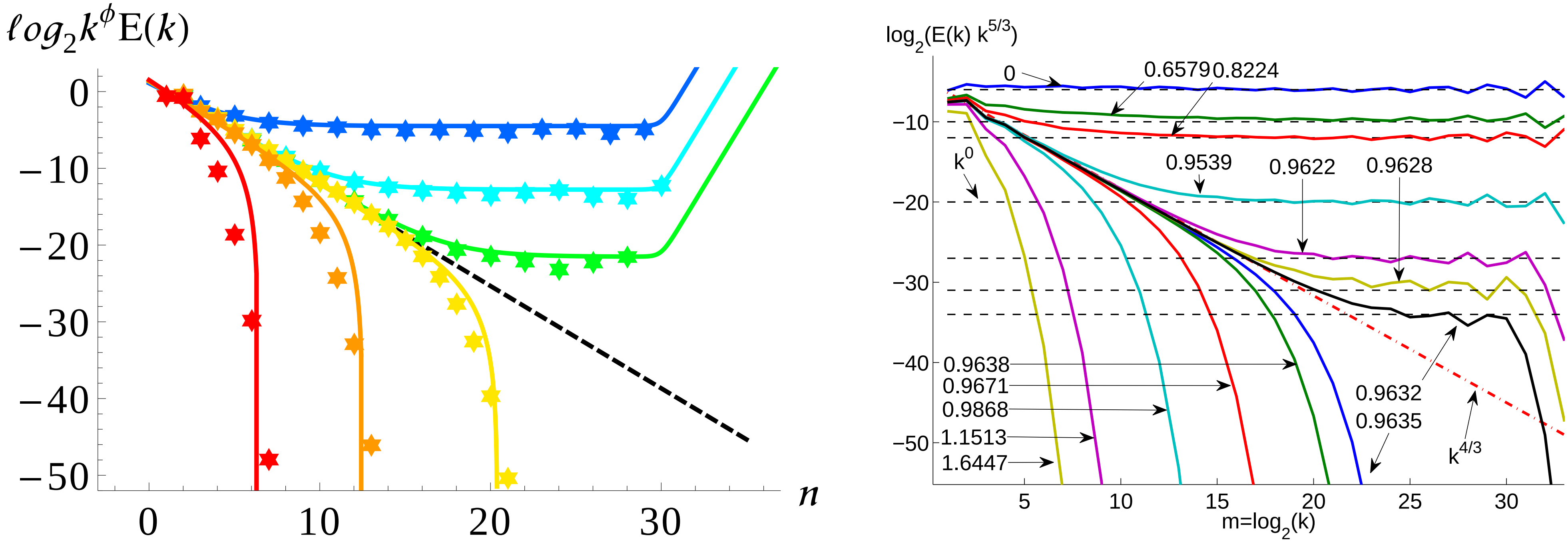
$$E_{\rm cr}(k)=\frac{16}{25}\frac{\Gamma^2}{k^3}=\Big(\frac{4\gamma}{5}\Big)^2\frac{\varepsilon_0^{2/3}k_0^{4/3}}{k^3}\,. \tag{8}$$

- For $\gamma>\gamma_{\rm cr}$ solutions become "sub-critical" $E_{\rm sb}(k)<E_{\rm cr}(k)$ and vanish at some finite $k=k_{\rm cr}$:

$$E_{\rm sb}(k)\simeq E_{\rm cr}(k)\Big(1-\frac{k^{2/3}}{k_{\rm cr}^{2/3}}\Big)^2\,,\quad k_{\rm cr}=\frac{k_0}{|\gamma-\gamma_{\rm cr}|^{2/3}} \tag{9}$$

- When $\gamma<\gamma_{\rm cr}$ the energy spectra become "super-critical" with $E_{\rm sb}(k)>E_{\rm cr}(k)$. For $k\ll k_{\rm cr}$, these spectra are close to the critical one and for $k\gg k_{\rm cr}$ they develop a K41 tail parametrized by $\varepsilon_\infty<\varepsilon_0$:

$$E_{\rm sp}(k)\simeq\Big(\frac{24}{11}\Big)^{2/3}\frac{\varepsilon_\infty^{2/3}}{k^{5/3}}\Big(1+\frac{k_{\rm cr}^{2/3}}{k^{2/3}}\Big)^2\,,\quad \varepsilon_\infty=\varepsilon_0(\gamma-\gamma_{\rm cr})^2 \tag{10}$$

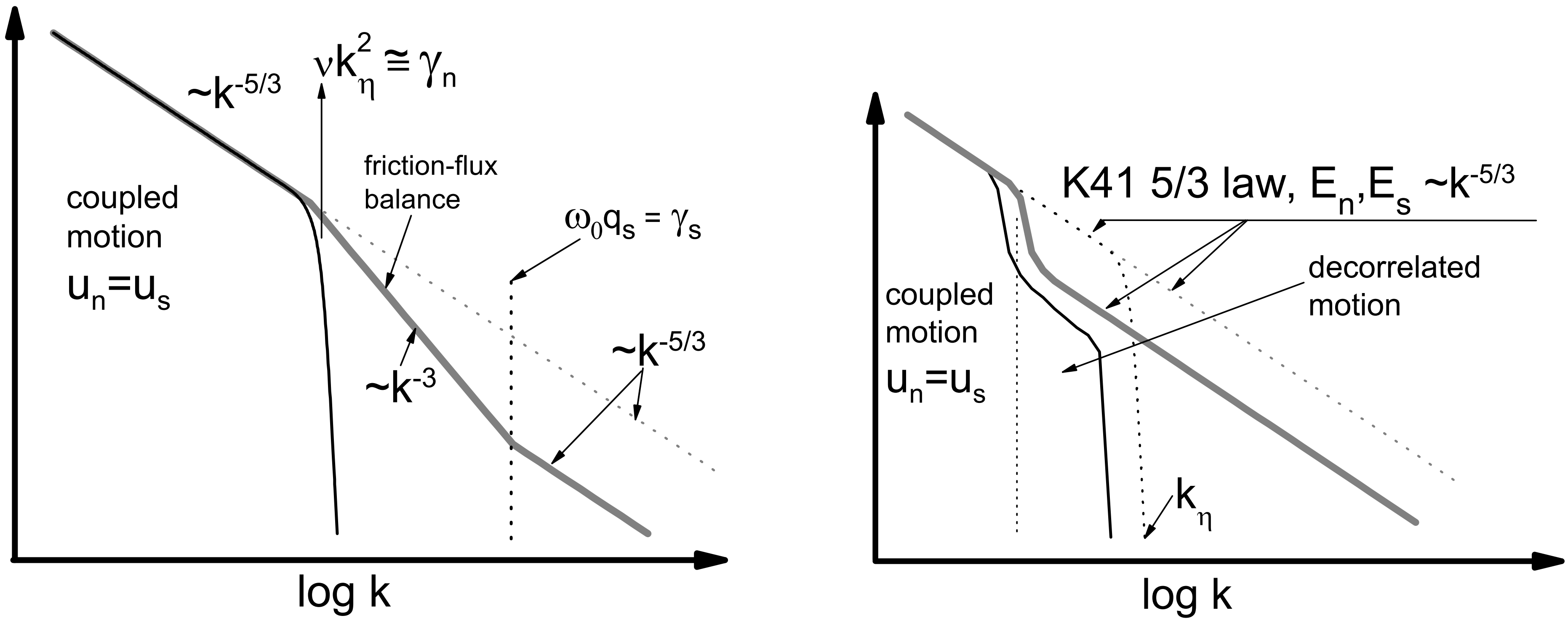


Left: Critical (black dashed line), sub-critical (red, kind of magenta, yellow) and super-critical solutions (green, cyan, blue) solution of the superfluid energy balance equation. At large k there is a bottleneck energy accumulation with energy equipartition $E(k) \propto k^2$.

Right: Sub- and super-critical solutions of the one-fluid Sabra-shell model with mutual friction (normal-fluid component is at rest). The stars on the right panel show a comparison with data from the Sabra-shell model at similar values of the ratio $\gamma/\gamma_{\rm cr}$. The bottleneck effect $E_{\rm sp}(k) \sim k^2$ at the highest wavevectors is removed from the Sabra-shell model by proper choice of parameters to simplify the simulations.

1.4 Energy spectra of normal & superfluid components in ^{4}He

Schematic log-log plots of the solutions of Eqs. (18) and (7)

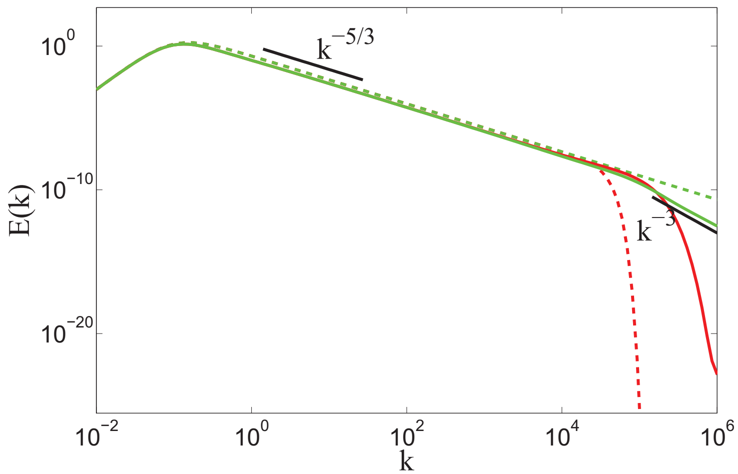


Thin black line Spectra of normal-fluid component, Thick gray line- Superfluid spectra.

Left panel: $T > 0.2T_\lambda$ – large mutual friction: $\alpha\rho/\rho_{\rm n} \simeq 1$

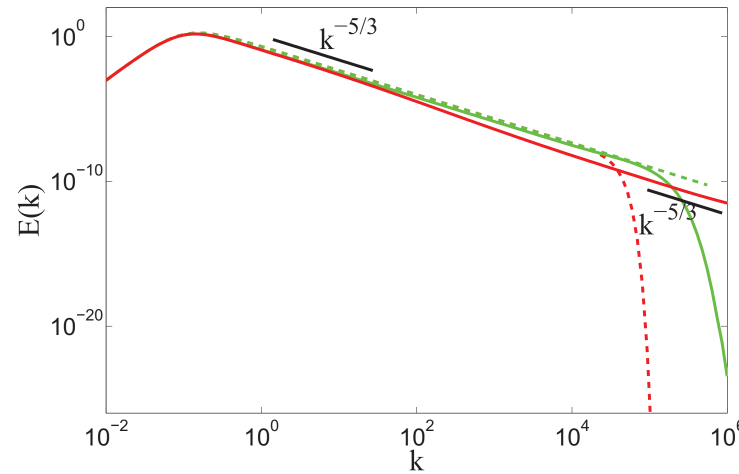
Right panel: $T < 0.1T_\lambda$ – small mutual friction: $\alpha\rho/\rho_{\rm n} \ll 1$

Recent numerical simulations[10], based on 1) 2004-LNV model (1c)[11] for the mutual friction; and 2) EDQNM closure for the energy fluxes [more realistic, than differential, Eq. (6c) and very similar to Eq. (2b)] are in "Good qualitative agreement in the small scale range with the DNS results found in Ref.[12]" and confirmed four asymptotical solutions, obtained by LNS in Ref.[13] and presented on the previous slide. See e.g,, energy spectra of normal fluid (red) and superfluid (green) components at high temperature. For each fluid, the initial (dash line) and steady states (solid line) are displayed, Ref.[11]:



Strong mutual friction:

$5/3$-K-41 and 3-LNV-04 spectra



Weal mutual friction:

$\frac{5}{3}$-K-41 and $\frac{5}{3}$-K-41 spectra

All these allows one to conclude, that

simple analytical LNS-model for the energy spectra, given be Eqs. (18) and (7), can be used for semi-qualitative analysis of superfluid turbulence for quasi-classical scales, exceeding intervortex distance.

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[10] J. Tchoufag and P. Sagaut, JFM, 2011
[11] V.S. L'vov, S.V. Nazarenko G.E. Volovik, JETP Letters, v. 80, 535-539 (2004)
[12] P.E. Roche, C.F. Barenghi, E. Leveque, Europhys. Lett., v. 87, 54006 (2009)
[13] V.S. L'vov, S.V. Nazarenko, L. Skrbek, JLTP v. 145, 125 - 142 (2006)

2 Weak turbulence of small-scale Kelvin waves

2.1 Hamiltonian formulation of Kelvin wave (KW) dynamics

2.1.1 Hamiltonian for Kelvin waves

Euler equation can be identically rewritten as the Biot-Savart Equation (BSE) for the vorticity $\boldsymbol{\omega}(\boldsymbol{r}) \equiv \boldsymbol{\nabla} \times \boldsymbol{v}$. In superfluids it should be supplemented by the condition of quantization of the vortex lines. Resulting superfluid BSE was written in the Hamiltonian form by Sonin[14], and later by Svistunov[15]:

$$i\boldsymbol{\kappa}\,\frac{\partial \boldsymbol{w}}{\partial t} = \frac{\delta \boldsymbol{H}\{\boldsymbol{w}, \boldsymbol{w}^*\}}{\delta \boldsymbol{w}^*}\,, \quad \boldsymbol{w}(\boldsymbol{z},\boldsymbol{t}) = \boldsymbol{x}(\boldsymbol{z},\boldsymbol{t}) + \boldsymbol{i}\,\boldsymbol{y}(\boldsymbol{z},\boldsymbol{t})\,, \tag{11a}$$

$$\boldsymbol{H} = \boldsymbol{H}^{\text{BSE}} = \frac{\boldsymbol{\kappa}}{4\pi}\iint \frac{\left[1 + \boldsymbol{Re}\big(\boldsymbol{w}'^*(\boldsymbol{z}_1)\boldsymbol{w}'(\boldsymbol{z}_2)\big)\right] d\boldsymbol{z}_1 d\boldsymbol{z}_2}{\sqrt{(\boldsymbol{z}_1 - \boldsymbol{z}_2)^2 + |\boldsymbol{w}(\boldsymbol{z}_1) - \boldsymbol{w}(\boldsymbol{z}_2)|^2}}\,, \tag{11b}$$

where $\boldsymbol{w}(\boldsymbol{z},\boldsymbol{t})$ is a deviation of the vortex from the straight line, $\boldsymbol{\kappa} = \boldsymbol{h}/\boldsymbol{M}$ – circulation quant, * stands for the complex conjugation, $\delta/\delta w^*$ is the functional derivative, and $w'(z) = dw/dz$.

From now on the nonlinear dynamics of KWs become part of physics of nonlinear waves in general with well described recipe of study, see e.g. my ZLF-92 book [16]:

[14] **E.B. Sonin**, **Reviews of modern physics v. 59, 87 (1987)**

[15] **B. V. Svistunov**, **Phys. Rev. B v. 52, 3647 (1995)**

[16] **ZLF-92: V.E. Zakharov, V.S. L'vov & G.E. Falkovich**, ***Kolmogorov Spectra of Turbulence***, **(Springer-Verlag, 1992)**

2.1.2 Hamiltonian description of KWs with small amplitudes

– Hamiltonian Eq. (11a) with the boundary conditions periodical on the length $\mathcal{L}$, should be written in the Fourier representation, in which $w(z,t) = \kappa^{-1/2} \sum_k a_k(t) \exp(ikz)$:

$$i\, da_k(t)/dt = \partial \mathcal{H}\{a, a^*\}/\partial a_k^*(t)\,, \tag{12}$$

New Hamiltonian $\mathcal{H}\{a, a^*\}$ is the density of the old one: $\mathcal{H}\{a, a^*\} \equiv H\{w, w^*\}/\mathcal{L}$, which has to be expanded with respect of $a_k\,, a_k^*$: $\mathcal{H} \simeq \mathcal{H}_2 + \mathcal{H}_4 + \mathcal{H}_6$. Here “Free Hamiltonian”

$$\mathcal{H}_2 = \sum_k \omega_k\, a_k\, a_k^*\,, \quad \omega_k \simeq {}^\Lambda\omega_k + {}^1\omega_k\,, \quad {}^\Lambda\omega_k = \kappa\, \Lambda\, k^2/4\pi\,, \quad {}^1\omega_k = -\kappa\, k^2 \ln(k\ell)/4\pi\,, \tag{13a}$$

describes free propagation of KWs with the frequency ω_k, expanded in powers of parameter $\Lambda = \ln(\ell/a) \simeq 10 \div 15$ in ^{3}He, ^{4}He. Interaction Hamiltonians $\mathcal{H}_4$ and $\mathcal{H}_6$ describe $2 \leftrightarrow 2$ and $3 \leftrightarrow 3$ scattering:

$$\mathcal{H}_4 = \frac{1}{4} \sum_{1+2=3+4} T_{1,2}^{3,4}\, a_1 a_2 a_3^* a_4^*\,, \quad T_{1,2}^{3,4} \equiv T(k_1, k_2 | k_3, k_4)\,, \tag{13b}$$

$$\mathcal{H}_6 = \frac{1}{36} \sum_{1+2+3=4+5+6} W_{1,2,3}^{4,5,6}\, a_1 a_2 a_3 a_4^* a_5^* a_6^*\,, \quad a_j \equiv a(k_j, t)\,, \tag{13c}$$

$$\text{where } T_{1,2}^{3,4} = {}^\Lambda T_{1,2}^{3,4} + {}^1 T_{1,2}^{3,4}\,, \quad {}^\Lambda T_{1,2}^{3,4} = -\frac{\Lambda k_1 k_2 k_3 k_4}{4\pi}\,, \quad {}^1 T_{1,2}^{3,4} \sim k_j^4\,; \tag{13d}$$

$$W_{\dots}^{\dots} = {}^\Lambda W_{\dots}^{\dots} + {}^1 W_{\dots}^{\dots}\,, \quad {}^\Lambda W_{\dots}^{\dots} = -\frac{9\Lambda k_1 k_2 k_3 k_5 k_6}{8\pi\kappa}\,, \quad {}^1 W_{\dots}^{\dots} \sim k_j^6\,. \tag{13e}$$

are also expanded in powers of Λ. Eqs. for ${}^1T_{1,2}^{3,4}$ and ${}^1W_{1,2,3}^{4,5,6}$ were found in KS-04 PRL [17] and later confirmed in LLNR-10 PRB [18].

[17] KS-04: E. Kozik & B. Svistunov, Phys. Rev. Lett. v. 92, 035301 (2004)
[18] LLNR-10: J. Laurie, V. S. Lvov, S. Nazarenko & O. Rudenko, Phys. Rev. B., v. 81, 104526 (2010)

We will see below that the leading in $\mathbf{\Lambda}$ approximation gives no energy exchange between KWs and therefore one has to account in $\mathcal{H}$ for subleading terms, zero order in $\mathbf{\Lambda}$, denoted by superscript ^{1}.

2.1.3 Effective Hamiltonian of "six-KWs" dynamics

Prehistory: A general classical perturbation formalism in Hamiltonian dynamics with weak interaction is "well know to those who know this well" and is presented, e.g. in my ZLF-92 bookZLF-92. This approach for the case of KWs was first used in KS-04 PRL, Ref. [17]. Later, in LLNR-10 PRB, Ref. [18] we found and corrected two mistakes in the KS-2004-PRL by adding two sets of contributions to the interaction Hamiltonian (of the same order of magnitude, as already accounted for) overlooked by KS.

Overview of the approach: The specifics KWs, propagating along vortex line is absence of 4-wave dynamics is absent because the conservation laws of energy and momentum in 1D media allow only trivial processes with $\boldsymbol{k}_1 = \boldsymbol{k}_3$, $\boldsymbol{k}_2 = \boldsymbol{k}_4$, or $\boldsymbol{k}_1 = \boldsymbol{k}_4$, $\boldsymbol{k}_2 = \boldsymbol{k}_3$.

Therefore by a proper non-linear canonical transformation $\{a, a^*\} \Rightarrow \{b, b^*\}$, $\mathcal{H}_4$ can be eliminated from the problem. This comes at a price of appearance of additional terms in the full interaction

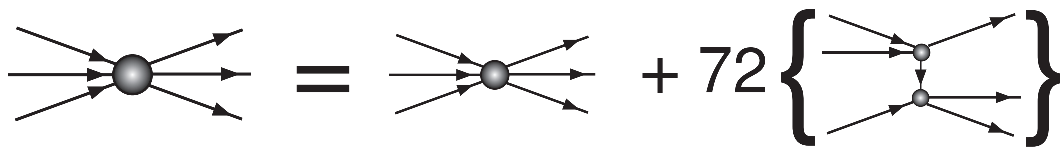
Hamiltonian $\widetilde{\mathcal{H}}_6$:

$$\widetilde{\mathcal{H}}_4 = 0\,, \quad \widetilde{\mathcal{H}}_6 = \frac{1}{36} \sum_{1+2+3=4+5+6} \widetilde{W}^{4,5,6}_{1,2,3}\, b_1 b_2 b_3 b_4^* b_5^* b_6^*\,, \tag{14a}$$

$$\widetilde{W}^{4,5,6}_{1,2,3} = W^{4,5,6}_{1,2,3} + Q^{4,5,6}_{1,2,3}\,, \quad Q^{4,5,6}_{1,2,3} = \frac{1}{8} \sum_{\substack{i,j,m=1\\ i\neq j\neq m}}^{3} \sum_{\substack{p,q,r=4\\ p\neq q\neq r}}^{6} q^{\,p,q,r}_{\,i,j,m}\,, \tag{14b}$$

$$q^{\,p,q,r}_{\,i,j,m} \equiv \frac{T^{j,\,m}_{r,\,j+m-r}\, T^{q,\,p}_{i,\,p+q-i}}{\Omega^{\,r,\,j+m-r}_{\,j,\,m}} + \frac{T^{q,\,r}_{m,\,q+r-m}\, T^{i,\,j}_{p,\,i+j-p}}{\Omega^{\,m,\,q+r-m}_{\,q,\,r}}\,,$$

$$\Omega^{3,4}_{1,2} \equiv \omega_1 + \omega_2 - \omega_3 - \omega_4 = {}^{\Lambda}\Omega^{3,4}_{1,2} + {}^{1}\Omega^{3,4}_{1,2} + \mathcal{O}(\Lambda^{-1})\,. \tag{14c}$$



$$\widetilde{W}^{4,5,6}_{1,2,3} = {}^{\Lambda}W^{4,5,6}_{1,2,3} + {}^{1}W^{4,5,6}_{1,2,3} + 72\, \frac{({}^{\Lambda}T + {}^{1}T)^{2}}{({}^{\Lambda}\Omega_k + {}^{1}\Omega_k)}\,. \tag{14d}$$

The 72 Q-terms in the *full 6-wave interaction coefficient* $\widetilde{W}^{4,5,6}_{1,2,3}$ can be understood as contributions of two 4-wave scattering into resulting 6-wave process via a virtual KW with $\boldsymbol{k} = \boldsymbol{k}_j + \boldsymbol{k}_m - \boldsymbol{k}_r$ in the 1-st term in Q and via a KWs with $\boldsymbol{k} = \boldsymbol{k}_q + \boldsymbol{k}_r - \boldsymbol{k}_m$ in the 2-nd term.

– Resulting Eq. (14b) for $\widetilde{W}^{4,5,6}_{1,2,3}$ involve about $\mathbf{20,000}$ contribution. Careful analysis of all these contributions, using `Mathematica` was done in LLNR-10 paper[18]. It was shown that:

1. The leading, linear $\boldsymbol{\Lambda}$ terms in $\widetilde{W}^{4,5,6}_{1,2,3}$, are exactly canceled at the LIA-manifold ${}^{\Lambda}\boldsymbol{\Omega}^{3,4}_{1,2} = \mathbf{0}$ as it is expected from the full integrability of the Line-Induced-Approximation.

2. All next-order terms, of the order of unity, thus giving main contribution to $\widetilde{W}^{4,5,6}_{1,2,3}$, in physically relevant region, where $\min\{k, k_1, k_2, k_3, k_4, k_5\} \ll \max\{k, k_1, k_2, k_3, k_4, k_5\}$ can be presented in the remarkably simple and physically transparent form, see left frame

$$\widetilde{\mathcal{W}}^{3,4,5}_{k,1,2} \simeq -\frac{3kk_1k_2k_3k_4k_5}{4\pi\kappa}, \tag{15a}$$

3. Resulting 6-KW Hamiltonian does not include irrelevant for KW dynamics 4-wave interaction:

$$\mathcal{H}_{\rm eff} = \sum_k \omega(k) b_k b_k^* + \frac{1}{36} \sum_{1+2+3=4+5+6} \widetilde{W}^{4,5,6}_{1,2,3}\, b_1 b_2 b_3 b_4^* b_5^* b_6^* \,. \tag{15b}$$

Resulting Hamiltonian $\mathcal{H}_{\rm eff}$ of interacting KWs served as a starting point in our next step:

2.2 Statistical description of weak turbulence of KWs

Approximation of the Kinetic Equation (KE) is valid for weakly nonlinear KWs and governs evolution of "occupation numbers" $\boldsymbol{n(k,t)}$ defined by $\langle b(\boldsymbol{k},t)b^*(\boldsymbol{k}',t)\rangle = (2\pi/\mathcal{L})^2\,\delta(\boldsymbol{k}-\boldsymbol{k}')\,\boldsymbol{n(k,t)}$,

according to

$$\boldsymbol{\partial n(k,t)/\partial t} = \mathrm{St}(\boldsymbol{k},\boldsymbol{t})\,, \tag{16a}$$

KE can be derived in various ways, including the Golden Rule of quantum mechanics. The collision integral $\mathrm{St}(\boldsymbol{k},\boldsymbol{t})$ for the $3\leftrightarrow 3$ process of KW scattering is:

$$\mathrm{St}_{3\leftrightarrow3}(\boldsymbol{k}) = \frac{\pi}{12}\iiiint\!\!\int \left|\mathcal{W}^{\,3,4,5}_{\,k,1,2}\right|^2 \delta^{\,3,4,5}_{\,k,1,2}\,\delta\Big({}^{\Lambda}\Omega^{\,3,4,5}_{\,k,1,2}\Big)\, n_k n_1 n_2 n_3 n_4 n_5 \tag{16b}$$
$$\times\big(n_k^{-1}+n_1^{-1}+n_2^{-1}-n_3^{-1}-n_4^{-1}-n_5^{-1}\big)\,d\boldsymbol{k}_1\,d\boldsymbol{k}_2\,d\boldsymbol{k}_3\,d\boldsymbol{k}_4\,d\boldsymbol{k}_5\;.$$

Detailed analysis of KE (16), sketched below, leads to LN-energy spectrum of KWs, given below.

KEq. (16) conserves the total number of (quasi)-particles $\boldsymbol{N}$ and the total (bare) energy of the system $\boldsymbol{E}$ and and has a Rayleigh-Jeans solution $\boldsymbol{n}_{\mathrm{T}}(\boldsymbol{k})$:

$$\boldsymbol{N} \equiv \int n_k\,d\boldsymbol{k}\,,\quad \boldsymbol{E} \equiv \int {}^{\Lambda}\omega_k n_k\,d\boldsymbol{k}\,,\quad \boldsymbol{n}_{\mathrm{T}}(\boldsymbol{k}) = \frac{\boldsymbol{T}}{\hbar\,{}^{\Lambda}\omega_k + \boldsymbol{\mu}}\,, \tag{17}$$

corresponding to thermodynamic equilibrium of KWs with the temperature $\boldsymbol{T}$ and chemical potential $\boldsymbol{\mu}$. In addition, KEq. (16b), has also flux-equilibrium solutions, $n_{\mathrm{E}}(k)$ and $n_{\mathrm{N}}(k)$.

Phenomenology of solution with constant energy flux and dimensional reasoning

Conservation laws (17) allow one to introduce the continuity equations for the energy spectrum of KWs: ${}^{\Lambda}\!E_k \equiv {}^{\Lambda}\omega_k n_k$ and energy flux in the k-space, ϵ_k:

$$\frac{\partial\, {}^{\Lambda}\!E_k}{\partial t} + \frac{\partial \epsilon_k}{\partial k} = 0\,, \quad \epsilon_k \equiv -\int_0^k {}^{\Lambda}\omega_k\, \mathrm{St}_{3\leftrightarrow 3}(k)\, dk\ . \tag{18a}$$

In scale-invariant systems one guesses the flux equilibrium solutions of KE (16):

$$n_{\mathrm{E}}(k) = A_{\mathrm{E}} k^{-x_{\mathrm{E}}}\,, \qquad A_{\mathrm{E}} - \text{dimensional constant.} \tag{19}$$

Scaling exponent x_{E} can be found in the case of *locality* of the energy flux, i.e. when the integrals over $k_1, \dots k_5$ in Eq. (18) and (16b) *converge*. In this case, the leading contribution to these integrals originates from regions $k_1 \sim k_2 \sim k_3 \sim k_4 \sim k_5 \sim k$ and fluxes (18) can be estimated as follows:

$$\epsilon_k \simeq k^5 [\mathcal{W}(k,k,k|k,k,k)]^2 n^5_{\mathrm{N}}(k)\ . \tag{20}$$

Stationarity of solutions of Eq. (18) require k-independence of the flux $\epsilon_k = \epsilon$. Together with Eq. (20) this gives the KS-04, Ref.[17] scaling exponent $x_{\mathrm{E}} = 17/5$.

However, as shown in LLNR-10-PRB, Ref.[18], this locality assumption is wrong and therefore the KS spectrum of KWs is irrelevant.

Local $\frac{5}{3}$ - energy spectrum of Kelvin waves

To find physical solution of KE (16), L'vov and Nazarenko [19] (LN) accounted for the leading divergent contribution in Eq. (16b), and found that $\mathbf{St}_{3\leftrightarrow 3}(\boldsymbol{k})$ takes the form:

$$|\Psi|^2\widetilde{\mathbf{St}}_{3\leftrightarrow 1}(\boldsymbol{k}) = \mathrm{St}_{1\leftrightarrow 3} = \frac{\pi}{12}\int d\boldsymbol{k}_1 \dots d\boldsymbol{k}_3 \Big\{ |\boldsymbol{V}_{\boldsymbol{k}}^{1,2,3}|^2 \, \delta_{\boldsymbol{k}}^{1,2,3} \boldsymbol{\mathcal{N}}_{\boldsymbol{k}}^{1,2,3} \tag{21}$$

$$\times \delta(\omega_{\boldsymbol{k}} - \omega_1 - \omega_2 - \omega_3) + 3\,|\boldsymbol{V}_{\mathbf{1}}^{\boldsymbol{k},2,3}|^2 \, \delta_1^{\boldsymbol{k},2,3} \boldsymbol{\mathcal{N}}_{\mathbf{1}}^{\boldsymbol{k},2,3} \delta(\omega_1 - \omega_{\boldsymbol{k}} - \omega_2 - \omega_3) \Big\} ,$$

$$\boldsymbol{\mathcal{N}}_{\mathbf{1}}^{2,3,4} \equiv n_1 n_2 n_3 n_4 \big(n_1^{-1} - n_2^{-1} - n_3^{-1} - n_4^{-1} \big) \, ; \quad \boldsymbol{V}_{\mathbf{1}}^{2,3,4} = -\frac{3\boldsymbol{\Psi}\, \boldsymbol{k}_1 \boldsymbol{k}_2 \boldsymbol{k}_3 \boldsymbol{k}_4}{4\pi\sqrt{2}} \, ,$$

where $\Psi = (2/\kappa)\int k^2 n(k) dk$ is mean square deviation of vortex from straight line. $\mathrm{St}_{1\leftrightarrow 3}$ corresponds to usual $1 \leftrightarrow 3$ KE with the effective interaction amplitude $\boldsymbol{V}_{\mathbf{1}}^{2,3,4}$ that describes interacting quartets of KWs propagating along randomly banded vortex lines with a large-scale curvature $R \simeq \ell/\sqrt{\Psi} \geq \ell$.

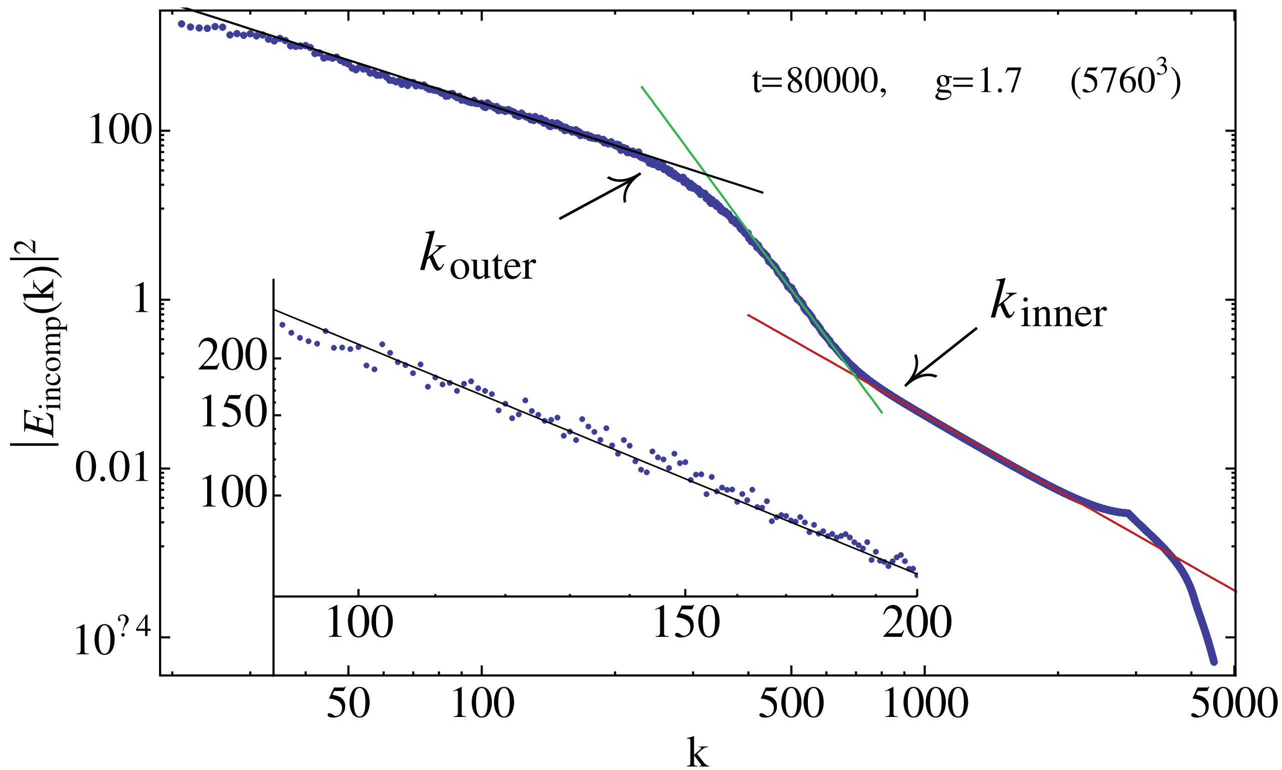
The proof of convergence in Eq. (21) is a delicate issue and cannot be done only by power counting because the latter would give a divergent answer. Only a quadruple cancelation of the largest, next to the largest and the two further sub-leading contributions appear to result in the final, convergent result for the collision term (21), as was shown by LN. Based on the locality of 4-KW interactions LN found:

$$\boldsymbol{E}(\boldsymbol{k}) = \frac{\boldsymbol{C}_{\mathrm{LN}} \boldsymbol{\Lambda} \boldsymbol{\kappa}\, \boldsymbol{\varepsilon}^{1/3}}{\boldsymbol{\Psi}^{2/3} \boldsymbol{\ell}^{4/3} \boldsymbol{k}^{5/3}} , \quad \boldsymbol{C}_{\mathrm{LN}} \simeq \mathbf{3.98} \, , \quad \text{Ref.[25]} \tag{22}$$

[19] **LN: V. S. L'vov & S. Nazarenko, Pis'ma v ZhETF , v. 91, 464 (2010).**
[20] **L. Boue, R. Dasgupta, J. Laurie, V.S. Lvov, S. Nazarenko I. Procaccia, PRB 84 064516 (2011).**

– DNS evidence for the LN $\frac{5}{3}$-KW spectrum

Gross-Pitaevskii 5760^3 " DNS by J. Yepez, G. Vahala, L. Vahala & M Soe[21] demonstrate energy spectrum (with exponents between 1.84 and 1.68, interpreted as an evidence of the K41 law.



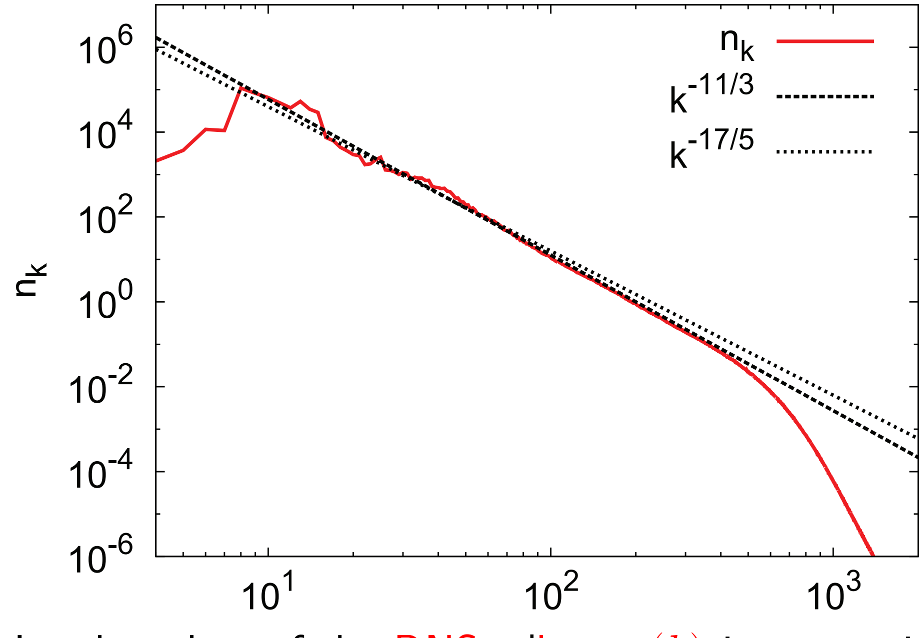
However, absence of bungles in their vortex configurations shows no evidence for large scale HD motions with $r \gg \ell$. Just the opposite, this motions can be considered as KWs with $\ell > r > a$. Thus we relate the observed scaling to LN $\frac{5}{3}$ - spectrum.

[21] **J. Yepez, G. Vahala, L. Vahala & M Soe, PRL v. 103, 084501 (2009)**

To preserve all the necessary physics of the full BSE, without its numerical complexity we derived from BSE in LLNR-10, Ref.[18] differential equation for small-amplitude, long KWs:

$$i\frac{\partial w}{\partial t} = -\frac{\kappa}{4\pi}\frac{\partial}{\partial z}\left[\left(\Lambda - \frac{1}{4}\left|\frac{\partial w}{\partial z}\right|^4\right)\frac{\partial w}{\partial z}\right]. \tag{23}$$

where $w(z,t)$ is the complex variable describing a weak perturbation of an unperturbed vortex line along z of the form $w(z,t) = x(z,t) + iy(z,t)$. In Ref. 21 we perform a DNS of Eq. (23) to check the power law scaling of the energy spectrum for KW turbulence in a statistically non-equilibrium stationary state – see plots on the Right ⇒



Log-log plots of the DNS values $n(k)$ in comparison against both the KS-$\frac{17}{5}$ and LN-$\frac{11}{3}$ predictions.

We observe a remarkable agreement with the LN-spectrum for about a decade in the k-space.

3 Bottleneck energy accumulation at cross-over scales

3.1 Differential model for small-scale KW turbulence

is based on LN spectrum and also has the solution with constant energy flux $\varepsilon_k = \varepsilon$, (Right):

$$\varepsilon(k) = -\frac{3}{5}\frac{\left[\Psi k^3 \ell^2 E_{\rm KW}(k)\right]^2}{\left(C_{\rm LN}\Lambda\,\kappa\right)^3}\frac{dE(k)^{\rm KW}}{dk}\,,\ E_k^{\rm KW} = \left[\frac{\left(C_{\rm LN}\Lambda\,\kappa\right)^3\varepsilon}{\Psi^2\,\ell^4 k^5} + \left(\frac{T}{2\pi\rho}\right)^3\right]^{1/3} \Rightarrow \begin{cases} \dfrac{C\Lambda\,\kappa\,\varepsilon^{1/3}}{\left(\Psi\,\ell^2\right)^{2/3} k^{5/3}}\,, \\ T\big/(2\pi\rho)\,. \end{cases}$$

Again: Low k region: LN spectrum $E^{\rm KW} \propto \varepsilon^{1/3} k^{-5/3}$,

Large k region: energy equipartition with an effective temperature T.

3.2 Superposition Model of turbulent motions in superfluids

suggested in LNR-08 paper [22], approximates superfluid turbulence as a **superposition** of "pure" rotational HD motions with the

$$\text{HD-spectrum:} \quad E^{\rm HD}(k) \equiv g(k\ell)E(k)\,, \tag{24a}$$

and KW motions with the

$$\text{KW-spectrum:} \quad E^{\rm KW}(k) \equiv [1 - g(k\ell)]E(k)\,. \tag{24b}$$

To find "blending" function $g(k\ell)$ we present $E(k)$ as

$$E(k) = \frac{1}{2}\int d\boldsymbol{r}\left\langle \Big|\sum_j \boldsymbol{v}_{j,k}(\boldsymbol{r})\Big|^2\right\rangle = E^{\rm KW}(k) + E^{\rm HD}(k)\,, \tag{25a}$$

[22] LNR-08: V. S. L'vov, S. V. Nazarenko & O. Rudenko, J. of Low Temp. Phys. v. 153, 140 (2008).

where $\boldsymbol{v}_{j,k}(\boldsymbol{r})$ is a velocity field, produced by a particular (with label j) vortex, bent by a KW with wave-vector k. At very small scales, $k\ell \gg 1$, the bending motions of individual vortexes are uncorrelated and total energy spectrum $\boldsymbol{E(k)}$ is dominated by KWs, which can be computed as follows

$$\boldsymbol{E}^{\rm KW}(\boldsymbol{k}) = \frac{1}{2}\int d\boldsymbol{r} \sum_j \langle |\boldsymbol{v}_{j,k}(\boldsymbol{r})|^2 \rangle = [\boldsymbol{1} - \boldsymbol{g(k\ell)}]\boldsymbol{E(k)} \ . \tag{25b}$$

After some calculations under controlled approximations the blending function $\boldsymbol{g}$ was found in LNR-08 by comparison of spectra Eq. (25) at different $x \equiv k\ell$:

$$\boldsymbol{g(x)} = g_0[0.32 \ln(\Lambda + 7.5)x] \,, \ g_0(x) = \left[1 + \frac{x^2 \exp(x)}{4\pi(1+x)}\right]^{-1} \ . \tag{25c}$$

3.3 Differential Min-Model for the energy flux in superfluids

suggested in LNR-08 paper [10], also approximates $\boldsymbol{\varepsilon(k)}$ as a sum of HD and KW contributions:

$$\begin{aligned} \boldsymbol{\varepsilon(k)} &= \widetilde{\boldsymbol{\varepsilon}}^{\rm HD}(\boldsymbol{k}) + \widetilde{\boldsymbol{\varepsilon}}^{\rm KW}(\boldsymbol{k}) \,, \qquad \text{where} && (26a)\\ \widetilde{\boldsymbol{\varepsilon}}^{\rm HD}(\boldsymbol{k}) &= \boldsymbol{\varepsilon}^{\rm HD}(\boldsymbol{k}) + \boldsymbol{\varepsilon}^{\rm HD}_{\rm KW}(\boldsymbol{k}) \,, \qquad \text{and} && (26b)\\ \widetilde{\boldsymbol{\varepsilon}}^{\rm KW}(\boldsymbol{k}) &= \boldsymbol{\varepsilon}^{\rm KW}(\boldsymbol{k}) + \boldsymbol{\varepsilon}^{\rm KW}_{\rm HD}(\boldsymbol{k}) \,, && (26c) \end{aligned}$$

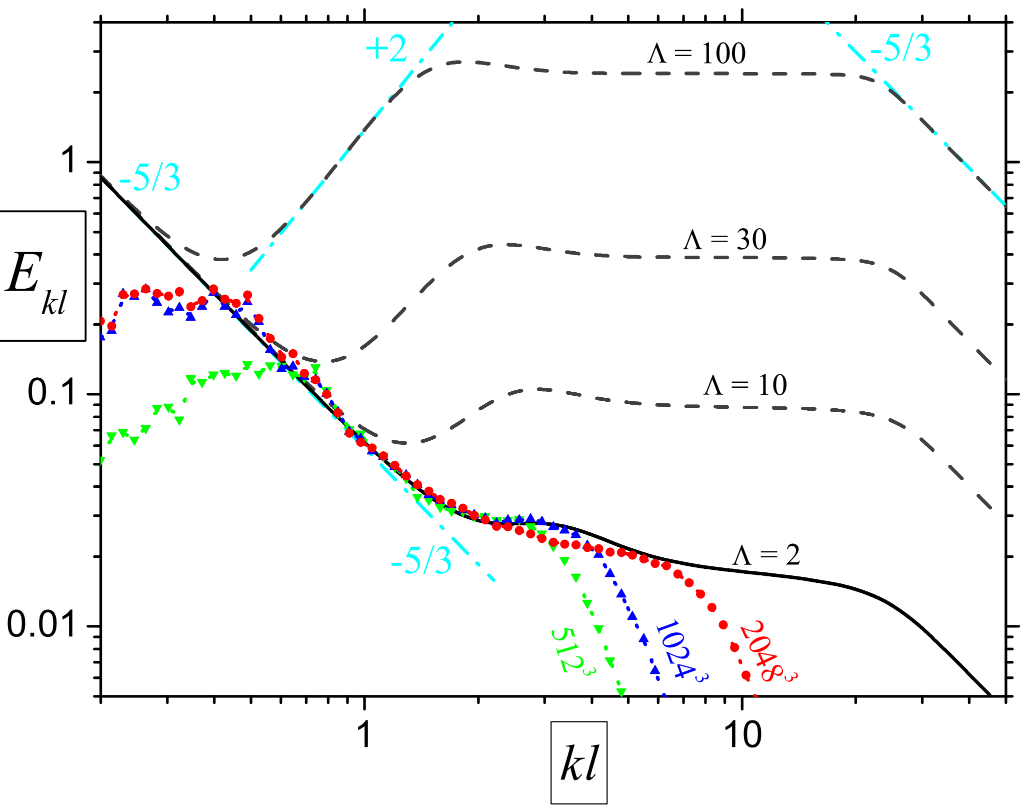
where $\boldsymbol{\varepsilon}^{\rm HD}(\boldsymbol{k})$ and $\boldsymbol{\varepsilon}^{\rm KW}(\boldsymbol{k})$ are the energy fluxes in "pure" HD, Eq. (4) and KW systems. Additional contributions $\boldsymbol{\varepsilon}^{\rm HD}_{\rm KW}(\boldsymbol{k})$ and $\boldsymbol{\varepsilon}^{\rm KW}_{\rm HD}(\boldsymbol{k})$, originate from influence of KW on the HD-energy flux and vise versa, was found in LNR-08 paper [10] by some additional arguments, such as form of thermodynamical equilibrium.

All above reasoning finally give

$$\varepsilon(k) = -\Big\{\frac{1}{8}\sqrt{k^{11} g(k\ell) E(k)} + \frac{3}{5}\frac{\big\{\Psi k^3 k_*\, \ell^2 [1-g(k\ell)] E(k)]\big\}^2}{\big(C\Lambda\, \kappa\big)^3} \times \frac{d}{dk}\Big\{E(k)\Big[\frac{g(k\ell)}{k^2} + \frac{[1-g(k\ell)]}{k_*^2}\Big]\Big\}, \quad \text{Minimal Model.} \tag{27}$$

where $k_*\ell = 6.64/\ln(\Lambda + 7.5)$. With the constant flux requirement $\varepsilon(k) = \varepsilon =$const. this is an ordinary differential equation, that allows to find energy spectrum $E(k)$ in the entire region of scales: classical HD, quantum KW and crossover scales $k\ell \sim 1$ for different $\Lambda = \ln(\ell/a_0)$:

3.4 Comparison of Min-Model with Tokio 2048^3-DNS & Manchester ^{4}He experiment



I. For $k\ell \ll 1$ $E(k)$ and $\varepsilon(k)$ are dominated by HD components and one sees K41 law (2a), $E^{\rm HD}(k) \propto k^{-5/3}$, with constant HD energy flux.

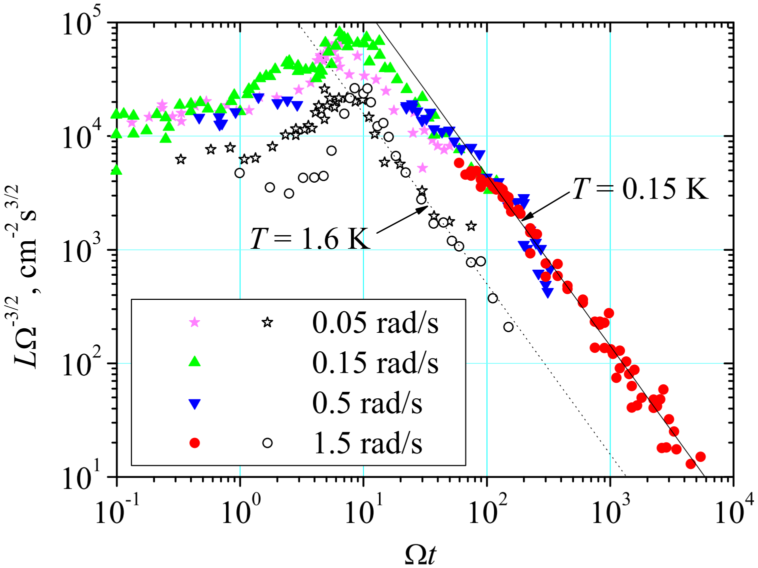
II. At $k\ell \lesssim 1$ and for $\Lambda \gg 1$ one sees the bottleneck with thermodynamic equilibrium: equipartition between HD degrees of freedom, $E^{\rm HD}(k) \propto k^2$.

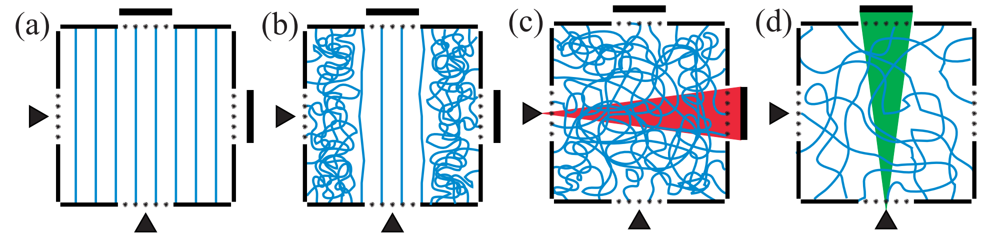
III. At $k\ell \gtrsim 1$ the energy flux is still carried by HD motions, $\varepsilon(k) \simeq \varepsilon^{\rm HD}(k)$ while energy is already dominated by KWs, $E(k) \simeq E^{\rm KW}(k)$. In the flux-free system of KWs one again sees thermodynamic equilibrium: but with equipartition between KW degrees of freedom, $E^{\rm KW}(k)$ =const

IV. For $k\ell \gg 1$ $E(k) \simeq E^{\rm KW}(k)$ and $\varepsilon(k) \simeq \varepsilon^{\rm KW}(k)$ i.e. are dominated by KWs. In this pure KW region, as expected, one observes the LN-spectrum of KWs, $E^{\rm KW}(k) \propto k^{-5/3}$ with constant KW-energy flux.

• As Λ decreases, the bottleneck effect becomes less pronounced. The equilibrium HD region II practically disappears for $\Lambda \simeq 2$. However the equilibrium KW region III is still well featured, being less sensitive to Λ. This agrees with the Tokio-DNS results for 2048^3, 1024^3, and 512^3, shown by dots.

– Comparison with the Manchester ^{4}He spin-down experiment[23] at $\boldsymbol{t} \to 0$





⇑ Cartoon of the vortex configurations ⇑

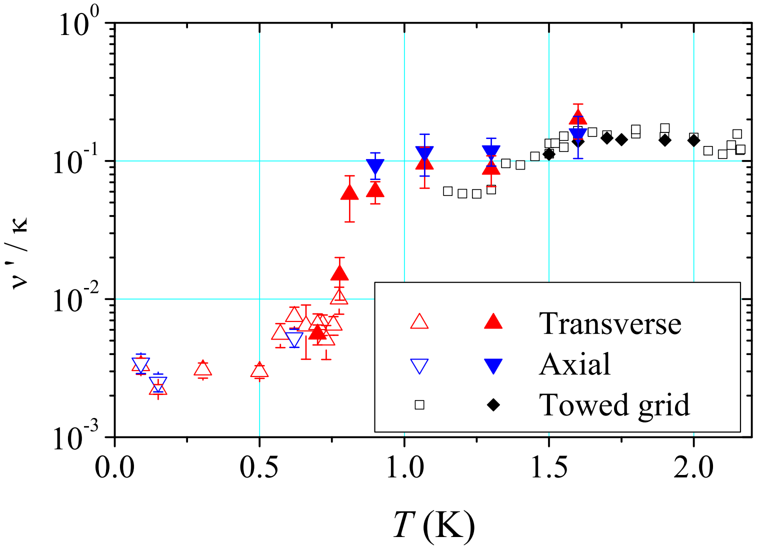
⇐ Vortex line density $(L\,\Omega^{-3/2})$ *vs.* $(\Omega\, t)$.

Measuring the time-decay of the vortex line density by negative-ion scattering, they found the temperature dependence of the effective viscosity $\boldsymbol{\nu'}$, defined via rate of energy dissipation $\boldsymbol{\varepsilon}$ and mean square vorticity:

$$\frac{dE(t)}{dt} = \boldsymbol{\varepsilon}(t) = \boldsymbol{\nu'}\langle|\boldsymbol{\omega}|^2\rangle\ ,\ \langle|\omega|^2\rangle = (\kappa L)^2\,,$$

$$\text{Turb. Energy } E \propto \varepsilon^{2/3} \Rightarrow E(t) \propto (t - t_*)^2$$

$$\Rightarrow L(t) \propto 1/[\kappa\sqrt{\boldsymbol{\nu'}(t-t_*)^3}]$$



- Effective viscosity $\boldsymbol{\nu'}$ at $T \to 0$, that follows from Min-Model: $\boldsymbol{\nu'} = \dfrac{\boldsymbol{\varepsilon}}{\langle|\boldsymbol{\omega}|^2\rangle} \approx \dfrac{8.65\,\kappa}{10^3 + 45.8\Lambda + 1.98\Lambda^2}$.

For $\Lambda \simeq 15$ we got $\nu'_{\rm theor} \approx 0.004\,\kappa$, which is close to the Manchester ^{4}He spin-down result $\nu'_{\rm exp} \approx 0.003\,\kappa$. We consider this agreement (achieved without fit parameters) as satisfactory.

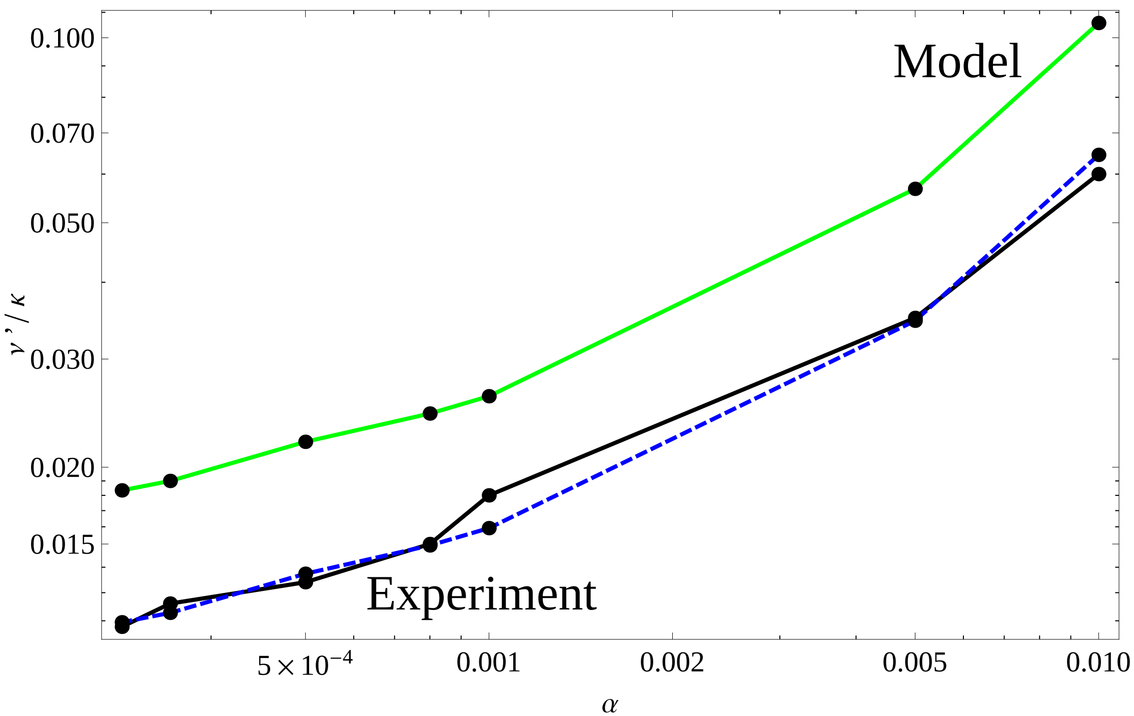
[23] **Walmsley, Golov, Hall, Levchenko and Vinen**, PRL, v. 99, 265302 (2007)

– Comparison with the Manchester ^{4}He spin-down experiment [[11]] at finite temperature

At finite temperature one has to account for the mutual friction between normal and superfluid components, proportional to the dimensionless mutual-friction coefficient $\alpha(T)$. This leads to appearance of two additional terms in the RHS of the balance equation for the superfluid energy $E(k)$ describing:

1. Large-scale dissipation of the eddy-dominated energy and
2. Small-scale dissipation of the KW energy:

$$\frac{\partial E(k,t)}{\partial t} + \frac{\partial \varepsilon(k)}{\partial k} = -\alpha(T)\left\{\omega_{\rm T}\, g(k\ell) + \frac{\kappa \Lambda k^2}{4\pi[1-g(k\ell)]}\right\} E(k,t)\,, \tag{28}$$



Solution of Eq. (28) allows one to find dependence of the effective viscosity on mutual friction parameter $\alpha(T)$. For $\Lambda = 15$ this gives *green line* without fitting constants, *black line* – experimental Manchester results and *blue dashed line* – the same as green, but shifted by numerical prefactor in order to overlay theoretical and experimental lines at the first point.

Having in mind that the Manchester experiment does not control [up to $\simeq (\frac{1}{2} \div 2)$] temperature independent prefactor in ν' we consider this agreement as unexpectedly good.

4 Summary and perspectives

- I presented a simple, physically transparent and analytically tractable "differential" model of energy spectra of homogeneous, isotropic turbulence in superfluids with energy pumping at scales $\mathcal{L}$ much above the intervortex distance ℓ.

- The detailed comparison of the model with energy spectra for the entire interval of scales belongs to future. Partial comparison with currently available DNS and experimental data demonstrate a reasonable and even unexpectedly good agreement . The possible reason is that in the most questionable for description crossover region, the model predicts a thermodynamical equilibrium in which the energy spectra are universal and non-sensitive for details of microscopic mechanisms of interactions (like vortex-reconnections, etc.).

- Nevertheless, a lot of questions remains unsolved:
  detailed description of the vortex-reconnection dynamics, may be in the line of KS scenario, and its effect on the temporal and spatial evolution of superfluid turbulence, microscopic description of the counterflow turbulence, etc., etc.

- Much more experimental, analytical and numerical studies are required
  to achieve desired level of understanding in superfluid-turbulence research.