Home Work 3

1. (Ch. 5, ex. 5.1) Consider a particle in a harmonic well, starting at $t = 0$ with $q = -q_0$ and $p = 0$. Sketch the location of the particle in phase space at $t = 0$, $1/4\tau$, $1/2\tau$, $3/4\tau$, andτ, where τ is a period of motion. Connect the points and show that the orbit is an ellipse. Which way does the phase space orbit circulate, clockwise or counterclockwise?

2. (Ch. 5, ex. 5.2) Consider a pendulum, starting at $t = 0$ with $q = 0$ and $p = p_0$. Sketch the location of the particle in phase space at $t = 0$, $1/4\tau$, $1/2\tau$, $3/4\tau$, andτ, where τ is a period of motion. You should get discover three different regions, two corresponding to rotation and one to libration, depending on the value of p_0. Show that the direction of the rotational orbits in phase space correlate smoothly with those of the libration at separatrix.

3. (Ch. 5, ex. 5.3) To understand why the Fourier transform variable p in the definition of the Wigner function has the physical meaning of a momentum, examine the structure of the quantity $<x'|\Psi><\Psi|x>$ for the case where $\Psi(x) = Ne^{-\alpha(x-x_0)^2+ip_0(x-x_0)}$. This product is a function of the variables x and x' and is complex. Plot the real part of this product and discuss the wavelength of oscillations along the $s = x - x'$ coordinate. If the object is Fourier transformed along the $s = x - x'$ coordinate, where will its peak be in the Fourier variable, p?

4. (Ch. 5, ex. 5.4) The Wigner distribution has several important properties which suggest its interpretation as a probability distribution:

$$\int_{-\infty}^{\infty} f_W(p,q)\, dp = <q|\Psi><\Psi|q> = |\Psi(q)|^2 \tag{4.1}$$

$$\int_{-\infty}^{\infty} f_W(p,q)\, dq = <p|\Psi><\Psi|p> = |\tilde{\Psi}(p)|^2 \tag{4.2}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_W(p,q)\, dpdq = 1 \tag{4.3}$$

where $\tilde{\Psi}(p)$ is the momentum representation of the wavefunction $|\Psi>$. Derive these relations.

5. (Ch. 5, ex. 5.5) (divided here into a preliminary and a full calculation)) Consider the normalized wavepacket

$$\Psi(q,t) = \left(\frac{2Re\alpha_t}{\pi}\right)^{1/4} e^{-\alpha_t(q-q_t)^2+\frac{i}{\hbar}p_t(q-q_t)+\frac{i}{\hbar}\gamma_t}, \tag{5.1}$$

and assume that γ_t is real.

a. Taking $\alpha_t = \frac{m\omega}{2\hbar}$, show that

$$f_W(q,p) = \frac{1}{\pi\hbar} e^{-\frac{m\omega}{\hbar}(q-q_t)^2} e^{-\frac{1}{m\omega\hbar}(p-p_t)^2}. \tag{5.2}$$

b. Show that in the more general case, where

$$\alpha_t = a\left(\frac{\alpha_0 \cos\omega t + ia\sin\omega t}{i\alpha_0 \sin\omega t + a\cos\omega t}\right) \tag{5.3}$$

($a = \frac{m\omega}{2\hbar}$), the Wigner distribution takes the form:

$$f_W(q,p) = \frac{1}{\pi\hbar} e^{-\frac{2|\alpha_t|^2}{\mathrm{Re}\,\alpha_t}(q-q_t)^2} e^{-\frac{1}{2\hbar^2\mathrm{Re}\,\alpha_t}(p-p_t)^2} e^{-\frac{2\mathrm{Im}\,\alpha_t}{\hbar\mathrm{Re}\,\alpha_t}(q-q_t)(p-p_t)}. \tag{5.4}$$

6. (Ch. 5, EOC ex. 1) Calculate $\mathrm{Tr}(\rho)$ and $\mathrm{Tr}(\rho^2)$ for the generic 2x2 density matrix:

$$\rho_{mn} = \begin{pmatrix} |a|^2 & a^*b \\ ab^* & |b|^2 \end{pmatrix} \tag{6.1}$$

Then repeat, this time with the off-diagonal elements set equal to 0. Show that in the first case $Tr(\rho^2) = 1$ (pure state) and in the second case $Tr(\rho^2) \leq 1$ (mixed state).

7. (Ch. 5, EOC ex. 3) Verify that

$$Tr(\rho^2) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \rho_W^2(p,q)$$

for the harmonic oscillator potential. First calculate the LHS using both the energy representation for ρ:

$$< \psi_n|\rho|\psi_m >= e^{-\beta E_n}/Q\delta_{nm} \qquad Q = \sum_n e^{-\beta E_n}. \tag{7.1}$$

Then repeat the calculation using the x representation for ρ:

$$\frac{< x'|e^{-\beta H}|x >}{Q} = (\frac{m\omega}{\pi\hbar} \tanh(f/2))^{1/2} exp\{\frac{-m\omega}{2\hbar \sinh(f)}[(x^2 + x'^2)\cosh(f) - 2xx']\} \tag{7.2}$$

where $f = \hbar\omega/kT$ and $Q = \int_{-\infty}^{\infty} < x|e^{-\beta H}|x > dx$. Note that the higher the temperature, the farther ρ gets from a pure state, i.e. $Tr(\rho^2)$ decreases with temperature. Plot $Tr(\rho^2)$ as a function of T.