Introduction to Fourier optics

Textbook: Goodman (chapters 4-5)

Overview:

Fourier transform Properties of lenses.
Imaging properties of lenses.
Frequency analysis of imaging systems
Coherent and incoherent imaging

Lenses as phase transformations

Let us now introduce what is perhaps the most important optical element into the framework of Fourier optics – the thin lens. For simplicity, let's just consider a plano-concave lens with radius of curvature R. The thickness of the lens is then:

$$\Delta(x,y) = \Delta_0 - R\left(1 - \sqrt{1 - \frac{x^2+y^2}{R^2}}\right) \approx \Delta_0 - \frac{x^2+y^2}{2R}$$

Where the last term is in the paraxial (small angle) approximation. Since the optical pathlength difference between passing through the lens and passing in the vacuum outside is proportional to (n-1), we immediately get that the phase transformation imposed by the lens is:

$$t_l(x,y) = \exp\left[-i\frac{k}{2f}\left(x^2+y^2\right)\right]$$

Where f is defined as the focal length of the lens by the "lensmaker's formula":

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Let us now consider the combination of a lens with free-space propagation, having an object illuminated by a plane wave, followed immediately by a lens of focal length f.

Let us derive again the Fresnel approximation. We start with:

$$U(x,y) = \frac{1}{i\lambda}\iint_{aperture} U(\xi,\eta)\frac{e^{ikr_{01}}}{r_{01}}\cos\theta\, d\xi\, d\eta$$

Which we approximate as:

$$U(x,y) = \frac{e^{ikz}}{i\lambda z}\iint_{aperture} U(\xi,\eta) e^{\frac{ik}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]} d\xi\, d\eta$$

Taking the x^2,y^2 phase terms out of the expression, we get:

$$U(x,y) = \frac{e^{ikz}}{i\lambda z} e^{\frac{ik}{2z}\left(x^2+y^2\right)} \iint_{aperture} U(\xi,\eta) e^{\frac{ik}{2z}\left(\xi^2+\eta^2\right)} e^{-\frac{ik}{2z}(\xi x+\eta y)} d\xi\, d\eta$$

Where we have approximated r_{01} as z in the denominator, neglected cosθ, and approximated the phase term to lowest order.

Clearly, upon addition of the "correct" amount of quadratic phase at the aperture plane, the Fresnel formula would be transformed into something equivalent to the Fraunhoffer one.

Now, let us consider a lens of focal length f with an input placed immediately before it.

After the lens, we get:

$$U_l(\xi,\eta) = At_a(\xi,\eta)P(\xi,\eta)e^{-\frac{ik}{2f}\left(\xi^2+\eta^2\right)}$$

where P is the pupil aperture of the lens (a circ function).

Inserting this to the Fresnel formula we get:

$$U(x,y) = \frac{Ae^{ikz}}{i\lambda z}e^{\frac{ik}{2z}\left(x^2+y^2\right)}\iint t_a(\xi,\eta)P(\xi,\eta)e^{\left(\frac{ik}{2z}-\frac{ik}{2f}\right)\left(\xi^2+\eta^2\right)}e^{-\frac{ik}{2z}(\xi x+\eta y)}d\xi d\eta$$

which at z=f is just:

$$U(x,y) = \frac{Ae^{ikf}}{i\lambda f}e^{\frac{ik}{2f}\left(x^2+y^2\right)}\iint t_a(\xi,\eta)P(\xi,\eta)e^{-\frac{ik}{2f}(\xi x+\eta y)}d\xi d\eta$$

which is just the Fraunhoffer diffraction pattern – a Fourier transform multiplied by a phase prefactor.

The integral in the above expression is simply, from the ASPW approach, the ASPW of the object t_a in these parameters: $F_0\left(\frac{x}{\lambda f},\frac{y}{\lambda f}\right)$

Let us now consider the case where the object of interest is placed a distance d before the lens. In this case we only have to propagate the object diffraction pattern a distance z before applying the previous result.

The ASPW of the input, assuming a plane wave of amplitude A is:

$$F_0\left(f_{x,}f_y\right) = F\{At_a\}$$

Which after propagating a distance d to the lens becomes:

$$F_l\left(f_{x,}f_y\right) = F_0\left(f_{x,}f_y\right)\exp\left[-i\pi\lambda d\left(f_x^2+f_y^2\right)\right]$$

From the previous result we now get:

$$U(x,y) = \frac{Ae^{ikf}}{i\lambda f}e^{\frac{ik}{2f}\left(x^2+y^2\right)}F_l\left(\frac{x}{\lambda f},\frac{y}{\lambda f}\right)$$

but substituting the two above equations we get:

$$U(x,y)=\frac{Ae^{ikf}}{i\lambda f}e^{\frac{ik}{2f}\left(1-\frac{d}{f}\right)\left(x^2+y^2\right)}F_o\left(\frac{x}{\lambda f},\frac{y}{\lambda f}\right)$$

Choosing d=f completely eliminates also the phase prefactor, leaving us with a perfect Fourier transform.

This implies also that the following 4f system is a perfect 1:1 imager (also in the phase sense – not just in intensity sense). This system will be used for a variety of applications when we discuss image processing.

<u>Relation to Basics of spectroscopy</u>

In a previous talk we write the Fraunhofer solution for diffraction by an amplitude grating. There, we say that the first order diffraction pattern is directed at an angle λf_0 to the incoming beam. Let us now position a grating in the back focal plane and illuminate it with polychromatic light such that its central frequency is directed towards the lens. For other values of l, this will (for a "narrow" enough spectrum) fan out such that $\Delta\theta=\Delta\lambda f_0$. Thus the frequency spectrum is directly mapped onto the angular spectrum, which is mapped onto the focal plane intensity distribution. Thus, the Fourier transformation in space is mapped into a frequency domain spectrum. This will be further discussed in the next tutorial. This is also of relevance to Fourier domain pulse shaping which will be discussed in relation to all optical processing.

<u>Magnification in Fourier transformation</u>

More flexibility in the Fourier transform can be achieved if the object in placed between the lens and the focal plane. In this case (see full derivation in Goodman) it can be shown that the Fourier transform is magnified by a factor f/d.

<u>Image formation by a lens</u>

Let us write the electric field at the output of a lens using an impulse response:

$$U_i(u,v)=\iint h(u,v;\xi,\eta)U_o(\xi,\eta)d\xi d\eta$$

For this to be an image we must require that the image of a point be a point, and that distances be preserved up to magnifications:

$$h(u,v;\xi,\eta)\approx K\delta(u\pm M\xi,v\pm M\eta)$$

To find the impulse response let us use Fresnel formulae for propagation up to the lens, include the quadratic phase of the lens, and propagate from the lens.

For a pointlike object:

$$U_l(x,y)=\frac{1}{i\lambda z_o}e^{\frac{ik}{2z_{oi}}\left[(x-\xi)^2+(y-\eta)^2\right]}$$

To which we have to add the quadratic phase of the lens:

$$U_l'(x,y)=U_l'(x,y)P(x,y)e^{-\frac{ik}{2f}\left(x^2+y^2\right)}$$

Then we propagate a distance zi from the lens:

$$h(u,v;\xi,\eta)=\frac{1}{i\lambda z_i}\iint U_l'(x,y)e^{\frac{ik}{2z_i}\left[(u-x)^2+(v-y)^2\right]}dxdy$$

Overall this gives us:

$$h(u,v;\xi,\eta)=\frac{1}{\lambda^2 z_i z_o}e^{\frac{ik}{2z_i}\left(u^2+v^2\right)}e^{\frac{ik}{2z_{oi}}\left(\xi^2+\eta^2\right)}\iint P(x,y)e^{\frac{ik}{2}\left(\frac{1}{z_o}+\frac{1}{z_i}-\frac{1}{f}\right)\left(x^2+y^2\right)}e^{-ik\left[\left(\frac{\xi}{z_o}+\frac{u}{z_i}\right)x+\left(\frac{\eta}{z_o}+\frac{v}{z_i}\right)y\right]}dxdy$$

There are a lot of phase terms in this expression. Let us now see what is required in order to have approximately a delta function in this expression.

The term outside the integration with the u,v coordinates is just a phase term at the output, and vanishes if we look at the output intensity.

The quadratic term inside the integration is only eliminated if the prefactor is zero, that is:

$$\frac{1}{z_o}+\frac{1}{z_i}-\frac{1}{f}=0$$

Which is just the lens law as we derived in the ray optics picture.

Now we are left with a quadratic phase term dependent on the object coordinates. This term is not easily eliminated. For the special case of an object illuminated by a sonverging spherical wave, this can be exactly compensated for. Otherwise, the requirement is such that the object dimensions are small enough so that this phase term is small and can be neglected. In practice, this requires that the object be no larger than ~1/4 of the size of the lens.

Assuming all these are satisfied, we are now left with the following expression for the impulse response:

$$h(u,v;\xi,\eta)=\frac{1}{\lambda^2 z_i z_o}\iint P(x,y)e^{-ik\left[\left(\frac{\xi}{z_o}+\frac{u}{z_i}\right)x+\left(\frac{\eta}{z_o}+\frac{v}{z_i}\right)y\right]}dxdy$$

Defining the magnification as:

$$M = -\frac{z_i}{z_o}$$

we finally get:

$$h(u,v;\xi,\eta) = \frac{1}{\lambda^2 z_i z_o} \int\int P(x,y) e^{-\frac{ik}{z_i}[(u-M\xi)x+(v-M\eta)y]} dxdy$$

which is the result we actually wanted. Clearly the expression on the right is not a delta function but is rather convolved with the pupil of the lens. This sets the basis for the diffraction limit of an optical system.

For the rest of this discussion, it may be convenient to redefine the impulse response in such a manner so as to remove the effect of magnification. To do so, we can redefine the coordinates in the object space such that:

$$\bar{\xi} = M\xi; \bar{\eta} = M\eta$$

Thus, we have:

$$h(u,v;\bar{\xi},\bar{\eta}) = h(u-\bar{\xi},v-\bar{\eta}) = \frac{A}{\lambda^2 z_i z_o} \int\int P(x,y) e^{-\frac{ik}{z_i}[(u-\bar{\xi})x+(v-\bar{\eta})y]} dxdy$$

If we define the perfect image:

$$U_g(\bar{\xi},\bar{\eta}) = \frac{1}{|M|} U_o\left(\frac{\bar{\xi}}{M},\frac{\bar{\eta}}{M}\right)$$

We then get:

$$U_i(u,v) = \int\int h(u-\bar{\xi},v-\bar{\eta}) U_g(\bar{\xi},\bar{\eta})$$

with:

$$h(u,v) = \frac{A}{\lambda^2 z_i z_o} \int\int P(x,y) e^{-\frac{ik}{z_i}[ux+vy]} dxdy$$

So that for a diffraction limited optical system the image is a convolution of the Gaussian (perfect) image predicted by geometrical optics with an impulse response which is the Fraunhoffer diffraction pattern of the pupil.

Coherent and incoherent illumination

When analyzing the frequency response of an optical system we have to also consider the illumination conditions. Until now we have generally assumed plane wave illumination of the object, i.e. that the object is illuminated by a coherent wave. In most cases, however, the object of interest is illuminated by spatially incoherent light. In the context of optical microscopy this is also the case when observing incoherent scattering excited by a coherent source (such as fluorescence). The basic result from that is that scatterings from two different points in the object do not interfere with each other, but only with themselves. We will now proceed to analyze the properties of imaging systems through the use of transfer functions (applicable for the case of **invariant** linear systems, which according to the above derivation applies for imaging scaled up to magnification)

Coherent case

We have just seen, for the coherent case, that

$$h(u,v;\xi,\eta)=\frac{1}{\lambda z_i}\iint P(x,y)e^{-\frac{ik}{z_i}[(u-M\xi)x+(v-M\eta)y]}dxdy$$

When considering a small object, on the lens axis, we can assign a very clear meaning to the integration coordinates x and y – relating them to spatial frequencies of the object and of the image, such that:

$$\theta_o=\frac{x}{z_o};\theta_i=\frac{x}{z_i}$$

In this description it is also clear why the pupil function of the lens serves as a cutoff in the spatial frequency space, with

$$f_c=\frac{d}{2\lambda z_i}$$

for typical parameters (d=2cm,z=10cm, λ=0.5μm the cutoff frequency is about 200/mm).

The transformation to the magnified coordinates is necessary to make the transformation from object to image spatially invariant. Under these conditions, we can write:

$$U_i(u,v)=\iint h(u-\bar{\xi},v-\bar{\eta})U_g(\bar{\xi},\bar{\eta})$$

Corresponding, in Fourier space, to:

$$U_i(f_x,f_y)=H(f_x,f_y)U_o(f_x,f_y)$$

Where the transfer function is simply the pupil in the normalized coordinates:

$$h(u,v)=\int\int P(\lambda z_i f_x,\lambda z_i f_y)e^{-\frac{ik}{z_i}[f_x u+f_y v]}df_x df_y = F\{P(\lambda z_i f_x,\lambda z_i f_y)\}$$

Which due to the symmetry of P can be written as (rather than the minus signs on the argument):

$$H(f_x,f_y)=F\{F\{P(\lambda z_i f_x,\lambda z_i f_y)\}\}=P(\lambda z_i f_x,\lambda z_i f_y)$$

And has only values of either unity or zero.

<u>Incoherent case</u>

The incoherent case is a bit more complicated. We are dealing only with intensity patterns, and thus we have to introduce an intensity transfer function.

We get:

$$I_i(u,v)=\int\int\left|h(u-\bar{\xi},v-\bar{\eta})\right|^2 I_g(\bar{\xi},\bar{\eta})d\bar{\xi}d\bar{\eta}$$

For intensity distributions, it is convenient to describe the Fourier conjugate in normalized coordinates. This is since that for positive definite real functions, the maximal value of the FT is at (0,0).

In Fourier domain, this corresponds to:

$$F\{I_i\}=F\left\{|h|^2\right\}F\{I_o\}$$

rather than the coherent version:

$$F\{I_i\}=\left|F\{A_i\}\right|^2=\left|F\{h\}F\{A_o\}\right|^2$$

Again, working in normalized coordinates, and using the identities of FT we can write:

$$\breve{H}_{inc}=\frac{F\left\{|h|^2\right\}}{\int\int|h(u,v)|^2 dudv}=\frac{\int\int H(p,q)H^*(p-f_x,q-f_y)dpdq}{\int\int|H(p,q)|^2 dpdq}$$

Thus, the incoherent OTF is the normalized autocorrelation of the amplitude transfer function.

Some properties immediately arise from the above definition:

$$\breve{H}_{inc}(0,0)=1$$

$$\breve{H}_{inc}\left(-f_x,-f_y\right)=\breve{H}_{inc}^{\ *}\left(f_x,f_y\right)$$
$$\left|\breve{H}_{inc}\left(f_x,f_y\right)\right|\le\left|\breve{H}_{inc}(0,0)\right|$$

The last property follows from Schwartz inequality with H,H*

$$\left|\int\int XYdpdq^2\right|\le\int\int|X|^2 dpdq^2\int\int|Y|^2 dpdq^2$$

In particular, since we saw that H, the (coherent) amplitude transfer function is just the pupil function:

$$H\left(f_x,f_y\right)=P\left(\lambda z_i f_x,\lambda z_i f_y\right)$$

We get that (since P is real):

$$\breve{H}_{inc}=\frac{\int\int P\left(x+\frac{\lambda z_i f_x}{2},y+\frac{\lambda z_i f_y}{2}\right)P\left(x-\frac{\lambda z_i f_x}{2},y-\frac{\lambda z_i f_y}{2}\right)dxdy}{\int P^2(x,y)dxdy}$$

Geometrically, this means that we have to calculate the area of overlap of two shifted pupils relative to the pupil area. This is very simple to calculate for a square aperture, where the OTF has a triangular cross section.

From the above definition it also follows that for an aberration free system, the OTF is positive definite too.

<u>Aberrations and apodization</u>

The above consideration was done for an aberration-free system, and assuming a unity value of the transmission at the pupil. Both of these do not necessarily have to be fulfilled.

Let us consider first apodization, which is a simpler effect. We saw that, for example, using a circular aperture we get an Airy pattern, which has significant sidelobes. If the pupil is amplitude modulated, we can trade off the bandwidth of the central lobe with the intensity of the sidelobes. Let us compare two cases: a circular aperture, and one apodized (multiplied) by a Gaussian. The latter has a broader FWHM but smaller sidelobes. In the limit where the Gaussian width is much smaller than the circular aperture, there are, in fact, no sidelobes, since the FT of a Gaussian is a Gaussian.

Alternatively, we can inverse apodize to enhance the higher frequency components, by allowing more light transmission at the edges

The discussion of aberrations closely follows the one we used in the geometrical optics case. There, we took the derivative of the phase shift at the output pupil in order to find the geometrical shift of the imaged point as a function of the position it passed the pupil. Here, we simply have to introduce this phase term into the frequency response H to get the exact effect of aberrations. Thus, the pupil function P has now an addition space-dependent phase term W(x,y)

$$H(f_x, f_y) = P(\lambda z_i f_x, \lambda z_i f_y) e^{ikW(\lambda z_i f_x, \lambda z_i f_y)}$$

Let us consider now the effects of aberrations on the OTF. For unapodized pupils, we simply have, for eaxh frequency component, to integrate over the area of overlap of the two shifted pupils. In this notation, for an aberrated image, we get:

$$\breve{H}_{inc}(f_x, f_y) = \frac{\iint\limits_{A(f_x, f_y)} e^{ik\left[W\left(x+\frac{\lambda z_i f_x}{2}, y+\frac{\lambda z_i f_y}{2}\right) - W\left(x-\frac{\lambda z_i f_x}{2}, y-\frac{\lambda z_i f_y}{2}\right)\right]} dxdy}{\iint\limits_{A(0,0)} dxdy}$$

From the Scwartz inequality, using the two exponents, it is clear that this reduces the absolute value of the OTF. In particular, the OTF can become negative with aberrations.

One can immediately note that:
1. aberrations do not affect the (0,0) component
2. Near the cutoff (where A is amall) aberrations do not affect the OTF
3. The effect of aberrations is negligible for W<<λ. In practice, the system is nearly aberration free for W<~λ/4.

A simple example is a focusing error, which corresponds to a quadratic phase shift. For a square aperture of width w, we can write this as:

$$W(x,y) \approx \frac{1}{2}\frac{\Delta z}{z_i^2}\left(x^2+y^2\right) = \left[\frac{1}{2}\frac{w^2}{z_i^2}\Delta z\right]\frac{\left(x^2+y^2\right)}{w^2} = W_m \frac{\left(x^2+y^2\right)}{w^2}$$

This has a solution of the form:

$$\breve{H}_{inc}(f_x, f_y) = \breve{H}_{unaberrated}(f_x, f_y) \sin c\left[\frac{8W_m}{\lambda}\left(\frac{f_x}{2f_o}\right)\left(1-\left|\frac{f_x}{2f_o}\right|\right)\right] \sin c\left[\frac{8W_m}{\lambda}\left(\frac{f_y}{2f_o}\right)\left(1-\left|\frac{f_y}{2f_o}\right|\right)\right]$$

From this, we can readily see the above qualities:

Fow Wm<<l, the OTF is unchanged. The cutoff and the (0,0) values are unchanged. In practice, however, the OTF has a low absolute value beyond the first zero of the sinc, which occurs for f_x such that:

$$\frac{8W_m f_x}{2\lambda f_o} = \pi \Rightarrow f_x = \frac{\lambda/4}{W_m} f_o$$

Alternatively, we could say that the image is undistorted as long as:

$$\left[\frac{1}{2}\frac{w^2}{z_i^2}\Delta z\right] < \frac{\lambda}{4} \Rightarrow \Delta z < \frac{\lambda}{2}\frac{z_i^2}{w^2}$$

For a focused Gaussian beam, this corresponds to the Rayleigh range

$$R.r. = \lambda \frac{f^2}{d^2}$$

It should be noted that from this analysis it is clear that the high frequency components are always depth resolved. We will get back to this point when we discuss optical microscopy, in particular confocal microscopy.

Comparison of coherent and incoherent transfer functions

Frequency response:

As we saw, in the incoherent case the cutoff frequency was twice that of the coherent case. The amplitude of the high frequency components, however, is smaller for the incoherent case. Thus, the resolving power of either of these two cases can differ for different samples. Let us consider, for example, two objects with the same amplitude transmittance, but with a different phase transmittance:

$$t_A(\xi,\eta) = \cos 2\pi\tilde{f}\xi$$
$$t_A(\xi,\eta) = \left|\cos 2\pi\tilde{f}\xi\right|$$

The frequency spectrum of the first is two lobes at $\pm\tilde{f}/2$, while for the second the frequency spectrum it corresponds to a central lobe at zero frequency with two lobes at $\pm\tilde{f}$. The first is transmitted in full by a coherent system, but we get reduced contrast for an incoherent system. For the second, we get no contrast for the coherent case, but some contrast for the incoherent one.

Resolution:

Assume the Rayleigh criterion (two points separated by 1.22λ). For the incoherent case we get a small dip in the intensity response. For the coherent case this depends on the phase of the two scatterings, and can go from full constructive interference, where the peak appears at the center, to full destructive interference, where we get a zero at the center. Clearly, the very definition of resolution in the coherent case is more complicated.