Polarization

Textbook: Born and Wolf (chapters 1)

Overview:

Fresnel formulae ; Derivation of Brewster's angle
Walkoff ; The beam displacer
Polarizers and waveplates
Nematic liquid crystal as polarization elements

1. Fresnel formulae

Until now we have considered the scalar for of maxwells equations, where ray propagation depended only on the refractive index. Historically, light was thought to be a longitudinal wave, and it was only the discovery of polarization phenomena that led to the conclusion that it is a transverse wave. In this case, a vector treatment is necessary to derive optical properties. The simplest case is that of the reflection and transmission of a polarized electromagnetic wave from a plane interface between two media with different refractive indices. Let's solve this using the continuity conditions of Maxwell's equations in source-free media:

The incident wave is, using $\tau_i = \omega\left(t - \frac{r \cdot s}{v_1}\right) = \omega\left(t - \frac{x \sin\theta_i + z\cos\theta_i}{v_1}\right)$:

$$E_x = -A_\parallel \cos\theta \exp(i\tau_i)\,;\; E_y = -A_\perp \exp(i\tau_i)\,;\; E_z = A_\parallel \sin\theta \exp(i\tau_i)$$

From which we can get the magnetic field too ($H = \sqrt{\varepsilon} s \times E$):

$$H_x = -\sqrt{\varepsilon_1} A_\perp \cos\theta \exp(-i\tau_i)\,;\; H_y = -\sqrt{\varepsilon_1} A_\parallel \exp(-i\tau_i)\,;\; H_z = \sqrt{\varepsilon_1} A_\perp \sin\theta \exp(-i\tau_i)$$

Writing similar equations for the transmitted field (with v2) and for the reflected field, we get the following set of equations for the 4 continuity conditions in the tangential direction ($E_i^j + E_r^j = E_t^j$, $j = x, y$, and similarly for H):

$$\begin{aligned}
&\cos\theta_i\left(A_\parallel - R_\parallel\right) = \cos\theta_t T_\parallel \\
&A_\perp + R_\perp = T_\perp \\
&\sqrt{\varepsilon_1}\cos\theta_i\left(A_\perp - R_\perp\right) = \sqrt{\varepsilon_2}\cos\theta_t T_\perp \\
&\sqrt{\varepsilon_1}\left(A_\parallel + R_\parallel\right) = \sqrt{\varepsilon_2} T_\parallel
\end{aligned}$$

We get two sets of two coupled equations for the parallel and tangential components, respectively, which are not mixed with one another. The solution is thus simple:

$$\begin{aligned}
T_\parallel &= \frac{2n_1\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t} A_\parallel \\
T_\perp &= \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t} A_\perp \\
R_\parallel &= \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t} A_\parallel \\
R_\perp &= \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t} A_\perp
\end{aligned}$$

Rewriting for the reflection (multiplying by $\sin\theta_i$ and using Snell's law) we get:

$$R_\parallel = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} A_\parallel$$
$$R_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} A_\perp$$

The expression for $R_\perp$ is always regular. For $R_\parallel$ the denominator can become infinite, such that $R_\parallel = 0$. This condition is called Brewster's angle and is fulfilled for the case of propagation from air when:

$$\tan\theta_i = n$$

Also discuss solution for normal incidence

Let's now consider the case of total internal reflection. In this case we have to substitute τ_t with a complex number in order to describe the evanescent wave in the outer medium:

$$\sin\theta_t = \frac{\sin\theta_i}{n} \text{ ; } \cos\theta_t = \pm i\sqrt{\frac{\sin^2\theta_i}{n^2} - 1}$$

The Fresnel formulae can be written in this case as:

$$R_\parallel = \frac{n^2\cos\theta_i - i\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i + i\sqrt{\sin^2\theta_i - n^2}} A_\parallel$$
$$R_\perp = \frac{\cos\theta_i - i\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i + i\sqrt{\sin^2\theta_i - n^2}} A_\perp$$

While the absolute value of both is unity, we can see that there is a relative phase shift δ between the two upon reflection, such that:

$$\tan\frac{\delta}{2} = \frac{\cos\theta_i\sqrt{\sin^2\theta_i - n^2}}{\sin^2\theta_i}$$

Thus, total internal reflection in a prism can be used to change the polarization state of light. (Describe the Fresnel Rhomb)

What are the properties of the evanescent wave?

The wavevector k of the transmitted field has to fulfill $k_t = 2\pi n/\lambda$.
Along the surface $k_{tx} = k_t \sin\theta_t$ and perpendicular to it $k_{ty} = k_t \cos\theta_t$
Clearly this means that:

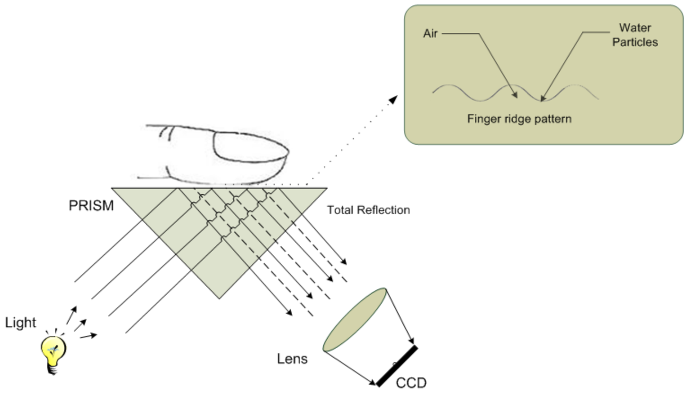
1. The field decays exponentially toward the y axis.

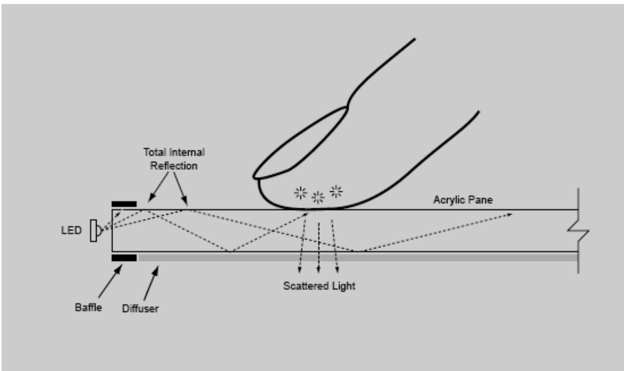
2. In the evanescent field there is a 90 degree phase shift between the y-polarized component and the x-polarized component (relevant only for parallel polarization) due to the imaginary component in the amplitude.

$$E_x = -A_{\parallel} \cos\theta \exp(i\tau_i)\,;\; E_z = A_{\parallel} \sin\theta \exp(i\tau_i)$$

Hence, the evanescent field is elliptically (circularly) polarized with a polarization that depends on the direction of propagation of light! This enables unidirectional coupling of fluorescence (if it is circularly polarized).

Another interesting phenomenon is frustrated total internal reflection. This occurs when there are two interfaces spaced very close such that the evanescent wave does not completely decay. In this case, some light will propagate to the second medium (same as "tunneling"). One interesting application is in fingerprint detection.





Poincare representation of polarization and Jones matrices

In free space, polarized light can be represented using three generalized parameters (equivalent to two amplitudes and a relative phase in a fixed basis). There are several useful representations for this. One uses the Stokes parameters and a geometrical representation on a sphere (Poincare sphere).

In this case:

$$
\begin{aligned}
S_1 &= |E_x|^2 - |E_y|^2 \\
S_2 &= 2\,\mathrm{Re}\left(E_x E_y^*\right) \\
S_3 &= 2\,\mathrm{Im}\left(E_x E_y^*\right)
\end{aligned}
$$

Where S0 is just the normalization of the total intensity. The ratios Si/S0 describe points on a sphere where the poles (|S3|=1 describe circular polarization and the equator describes linear polarizations at various angles)

An alternative representation is Jones calculus, where the polarization is described by a vector of two complex quantities, and polarizing elements are described by 2x2 matrices.

Examples:

$$
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}
$$

describe horizontal, vertical, 45 degrees linear and left circular polarizations, respectively.

Simple optical elements such as the half wave plate, quarter wave plate or polarizer , or their cascadings, can be described by 2x2 matrices:

$$
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
$$

Propagation in anisotropic crystals

In a general anisotropic crystal E and D are not parallel to one another, but rather related through the dielectric tensor, which is a symmetric 3x3 matrix.

$$\vec{D} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{pmatrix} \vec{E}$$

Therefore, the poynting vector (propagation of energy), which is perpendicular to E, is not parallel to the phase front, which is perpendicular to D. This leads to the phenomenon of walkoff.

To model this, we have to revisit Maxwell's equations, retaining the fact that the dielectric constant is a tensor.

From:

$$\nabla x \left(\frac{1}{\mu} \nabla x E \right) - \frac{\varepsilon}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$$

and

$$\nabla \cdot D = \nabla \cdot \vec{\varepsilon} \vec{E} = 0$$

We get (for a plane wave $E(r,t) = E_0 e^{i(kr - \omega t)}$):

$$|k|^2 E_0 - (k \cdot E_0) k = \frac{\omega^2}{c^2} \varepsilon E_0$$
$$k \cdot \vec{\varepsilon} \vec{E} = 0$$

This is a set of 3 linear equations for E which have a nontrivial solution only if the determinant is zero.

The dielectric tensor can be represented as a diagonal matrix in a system of coordinates determining the principal axes. Effectively, this means that the index of refraction along each of these is different, leading also to the phenomenon of double refraction (different Snell angles for different polarizations).

Most crystals we deal with are uniaxial, where two principal axes are identical. These are used to generate the most common optical elements such as polarizers and waveplates. The most commonly used ones are crystal quartz (ne=1.55, no=1.54) and calcite (ne=1.49, no=1.65), and more recently yvo4 (ne=2.22, no=1.99).

In the frame determined by these axes, the dielectric tensor can be represented as:

$$\vec{D} = \begin{pmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{pmatrix} \vec{E}$$

Under these conditions the equations can be written as:

$$\left(-k_y^2 - k_z^2 + \frac{\omega^2 n_0^2}{c^2} \right) E_x + k_x k_y E_y + k_x k_z E_z = 0$$

with the last equation having ne instead of no. With some algebra this can be rewritten as:

$$\left(\frac{k_x^2 + k_y^2 + k_z^2}{n_o^2} - \frac{\omega^2}{c^2} \right) \left(\frac{k_x^2 + k_y^2}{n_e^2} + \frac{k_z^2}{n_o^2} - \frac{\omega^2}{c^2} \right) = 0$$

Thus, for each propagation direction there are two possible solutions, one with refractive index no, and the other with refractive index:

$$\frac{1}{n^2} = \frac{\cos^2 \theta}{n_e^2} + \frac{\sin^2 \theta}{n_o^2}$$

where θ is the angle between the polarization axis and the extraordinary axis.

Let us now consider the effect of walkoff in such a crystal.

We want to calculate the angle r between D and E. Let's assume that we have in the plane

$$E = (\cos \theta, 0, \sin \theta)$$

Then

$$D = (\varepsilon_o \cos \theta, \varepsilon_e \sin \theta)$$

Thus:

$$\cos(\rho) = \frac{D \cdot E}{|D||E|} = \frac{\varepsilon_o \cos^2 \theta + \varepsilon_e \sin^2 \theta}{\sqrt{\varepsilon_o^2 \cos^2 \theta + \varepsilon_e^2 \sin^2 \theta}}$$

$$\sin(\rho) = \frac{|D \times E|}{|D||E|} = \frac{(\varepsilon_o - \varepsilon_e) \sin \theta \cos \theta}{\sqrt{\varepsilon_o^2 \cos^2 \theta + \varepsilon_e^2 \sin^2 \theta}}$$

$$\tan(\rho) = \frac{(n_o^2 - n_e^2) \tan \theta}{n_e^2 + n_o^2 \tan^2 \theta}$$

It can be easily shown that the walkoff is maximal at $\theta = \arctan\left(\frac{n_e}{n_o}\right) \approx 45^o$.

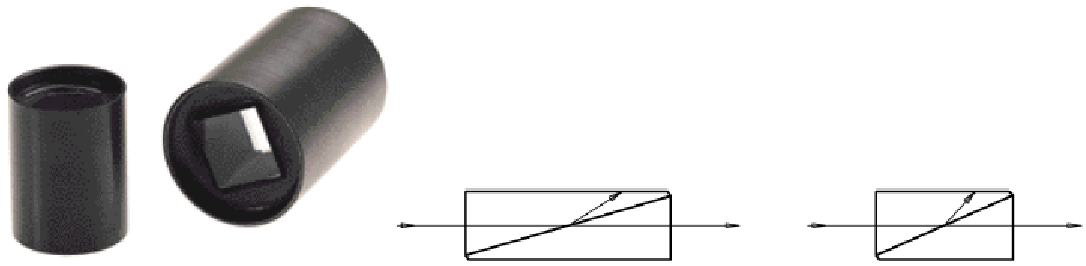
For this value:

$$\tan\rho \approx \rho \approx \frac{2n\Delta n}{2n^2} \approx \frac{\Delta n}{n}$$

Description of the Glan polarizer ; waveplate (multiple order, zero order, achromatic) ; Beam displacer and analogs (wollaston, Rochon etc,).

Waveplates (and the use as a variable attenuator)

<u>Glan polarizer</u>

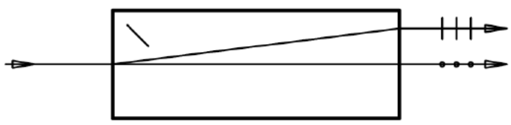


Cemented prism with propagation along the ordinary axis, based on total internal reflection (and a different refractive index for both polarizations). Air spaced prism gives a larger angular acceptance.

For air: critical angle is: 42 degrees for e ray and 37 degrees for o ray. With the cement (n~1.4) the angular separation is greater, enabling a larger angular acceptance.

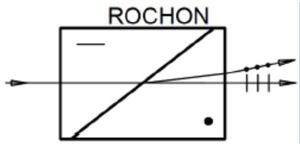
Typical extinction ratios are 10^5 – 10^7

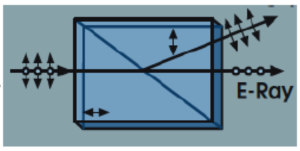
<u>Beam displacer</u>



Typically a slab of uniaxial material at 45 degrees. Lateral shift is tan(ρ)*d. Some astigmatism for e-ray. For calcite ρ is about 6 degrees, so that the displacement is about 10% of the length.

Rochon





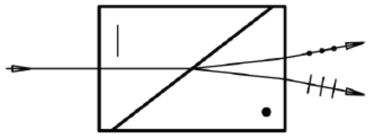
Cemented uniaxial prism with propagation along e direction, (essentially isotropic) with prism where propagation is along the o direction, so that one polarization component is refracted.

Snell's law: for angle θ of the prism:

$$n_o \sin\theta = n_e \sin\theta'$$
$$\Delta\theta = \arcsin\left(\frac{n_o}{n_e}\sin\theta\right) - \theta$$

Outside the crystal we usually have n=1, so the overall deviation angle is larger (for small angles by ne).

Wollaston



Cemented uniaxial prism with propagation along o direction, with prism where propagation is along the o direction but with the other two axes rotated, so that both polarization components are refracted, almost symmetrically.

Snell's law: for angle θ of the prism:

$$n_e \sin\theta = n_o \sin\theta'$$
$$\Delta\theta' = \arcsin\left(\frac{n_e}{n_o}\sin\theta\right) - \theta$$

In the small angle regime (sinθ~θ) the two are symmetric in the crystal but become somewhat asymmetric outside.

Other polarizers:

Polarcor polarizers (metallic lines with distance < λ/2).

Plastic sheet polarizers.

Polarizing cubes (dielectric layers)

Discussion on birefringence phase matching in nonlinear media (second harmonic generation)

Liquid crystal elements

It is sometimes desirable to have a variable wave plate with an external control or a spatially varying waveplate. For this the most commonly used elements are liquid crystals.

Describe the simplest elements:

Nematic liquid crystal. Voltage and calibration.

Twisted nematic liquid crystal.

A twisted nematic liquid crystal as an array of waveplates:

A retarder at an angle α is:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \exp(i\varphi) \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

This is now multiplied in sequence with a retarder at an angle Δα with this one, so that overall each element is just a multiplication of a series of matrices of the form:

$$\begin{pmatrix} \cos\alpha_f & -\sin\alpha_f \\ \sin\alpha_f & \cos\alpha_f \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & \exp(i\Phi/N) \end{pmatrix} \begin{pmatrix} 1 & \alpha_f/N \\ -\alpha_f/N & 1 \end{pmatrix} \right]^N$$

or

$$\begin{pmatrix} \cos\alpha_f & -\sin\alpha_f \\ \sin\alpha_f & \cos\alpha_f \end{pmatrix} \left[\begin{pmatrix} 1 & \alpha_f/N \\ -\alpha_f(1+i\Phi)/N^2 & (1+i\Phi)/N \end{pmatrix} \right]^N$$

at the limit N-> Infinity. For a large enough value of the total retardance across the device (in practice Φ>~4π) the limit goes to the unity matrix, leaving only a rotation at the final angle. In practice, this means that the polarization of light follows the director in a LC device.

<u>Light propagation in media</u>

Consider a wavepacket (necessary so that it has finite spatial extent and finite duration since a plane wave is infinite). For simplicity we assume that the range of k vectors (or of frequencies) is much smaller than the carrier

$$a(x,t) = \int_{-\infty}^{\infty} dk\, A(k) e^{i(kx-\omega t)}$$

Expand the frequency to first order around the carrier

$$\omega(k) = \omega_0 + (k - k_0)\left.\frac{d\omega}{dk}\right|_{k_0}$$

With some algebra this can be easily shown to equal

$$a(x,t) = e^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} dk\, A(k) e^{i(k-k_0)\left(x - \frac{d\omega}{dk}t\right)}$$

This defines two velocities. The phase velocity (propagation of the carrier) is $\frac{\omega}{k}$ whereas the envelope propagates at $\frac{d\omega}{dk}$, termed the group velocity.

The next derivative, $\frac{d^2\omega}{dk^2}$ (or GVD, group velocity dispersion) represents the distortion of the envelope upon propagation in the medium and is relevant for ultrashort pulses.

<u>Form birefringence</u>

The last kind of birefringent medium I wish to describe is one in which the birefringence is of geometrical origin. That is – this material is composed of isotropic media which are distributed in an anisotropic manner in a subwavelength grating.

Considering the case of such a subwavelength grating, calculation of the effective dielectric constant for TE and for TM polarizations, is equivalent to the problem of connecting capacitors in parallel to one another or in a row. Thus, for one direction of the polarization (assuming the fill factor of material 1 is f):

$$\varepsilon_{eff} = f\varepsilon_1 + (1 - f)\varepsilon_2$$

while for the other:

$$1/\varepsilon_{eff} = f/\varepsilon_1 + (1 - f)/\varepsilon_2$$

which corresponds to refractive indices

$$n_{TE} = \sqrt{n_1^2 f + n_2^2 (1 - f)}$$

$$n_{TM} = \frac{n_1 n_2}{\sqrt{n_1^2 (1 - f) + n_2^2 f}}$$

This is a particularly useful tool for relatively long (NIR) wavelengths, where fabrication of subwavelength gratings is easy, and where a very high index contrast (almost 1:4 in air:Si) can be achieved. For visible frequencies, this is more esoteric.

<u>A few words on polarization by scattering (due to the dipole pattern emission).</u>

A few words on optical activity (different refractive indices for left and right circularly polarized light) and on Faraday rotation

A few words on the pockels effect