Exam 2014

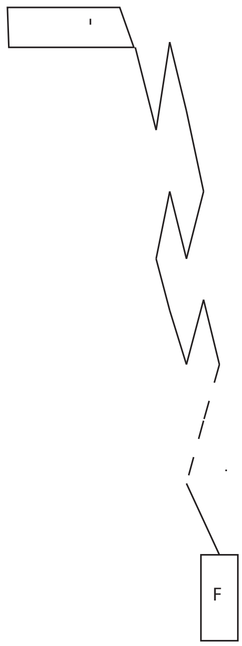
Problem 1:
Two different phases of the same solid have respectively the specific heats $c_1 = aT^3$ and $c_2 = bT^2$.

a) Assuming that they both satisfy the third law of thermodynamics, find the entropies of the phases.

b) Assuming that their internal energies (per particle) at zero temperature are the same and equal to e_0, find how their energies depend on the temperature.

c) Assuming that the densities are the same, find the temperature of the phase transition and determine which is the low-temperature phase.

Problem 2.
As a simplest model for rubber, consider a chain consisting of $N \gg 1$ segments, each of the length a. Every segment can rotate freely and be oriented either up or down. An upper end of the chain is fixed, at the lower end we have weight F.



i) Determine the dependence of the mean chain length l on the temperature T.

How the length changes (increases or decreases) upon heating?

ii) How the temperature changes (increases or decreases) upon adiabatic stretching?

To answer that, use

$$\left(\frac{\partial T}{\partial l}\right)_S = -\left(\frac{\partial l}{\partial T}\right)_F \left(\frac{\partial F}{\partial l}\right)_T \left(\frac{\partial T}{\partial S}\right)_l = -\left(\frac{\partial l}{\partial T}\right)_F \left(\frac{\partial F}{\partial l}\right)_T \frac{T}{C_l} . \qquad (*)$$

iii) Bonus question: prove (*).

Problem 3. Find the specific heat of N fermions at (low) temperature T in the three-dimensional potential $U = m\omega^2 r^2/2$.

Problem 4. Consider the 1d spin chain where spins can have values $1, 2, \ldots q$. The Hamiltonian is determined by the interaction of the nearest neighbors: $\beta\mathcal{H} = -K\sum_i \delta_{\sigma_i,\sigma_{i+1}}$. Here $\delta_{a,b} = 1$ when $a = b$ and zero otherwise.

a) Do Renormalization Group decimation of every second site ($k = 2$) and find the RG recursion relations $g(K)$ and $K'(K)$.

b) Find the fix points and describe their stability.

c) Find the correlation radius for $q = 2$ and for arbitrary q. One can find the correlation radius either from RG or from transfer matrix using $r_c = \ln^{-1}(\lambda_1/\lambda_2)$.

Solutions

Solution 1:
a) Entropies are obtained by integrating dQ/T starting from $s(0) = 0$:

$$s_1(T) = \int_0^T \frac{c_1 dT}{T} = \frac{aT^3}{3}\,, \quad s_2(T) = \frac{bT^2}{2}\,.$$

b) Since $de = Tds + \mu dN$ then for $dN = 0$ we have $de = Tds(T) = c(T)dT$ and

$$e_1 = e_0 + \frac{aT^4}{4}\,, \quad e_2 = e_0 + \frac{bT^3}{3}\,.$$

c) Since the densities are the same, only chemical work needs to be considered so that $\mu = e - Ts$. Phase transition happens when $\mu_1 = aT^4(1/4 - 1/3) = \mu_2 = bT^3(1/3 - 1/2)$ which gives

$$T = \frac{2b}{a}\,.$$

At low temperatures, $\mu_1 > \mu_2$ so the second phase is realized there.

Solution 2: The problem is equivalent to that of a two-level system, non-interacting spins in an external field etc. The number of up/down segments respectively is $N_+ = N/2 + l/2a$, $N_- = N/2 - l/2a$. The energy of the system is the potential energy of the weight $E = -Fl$. The entropy of the system is $S = \ln(N!/N_+!N_-!)$. We now write the free energy $\mathcal{F}(l) = E(l) - TS(l)$ and requiring $\partial\mathcal{F}/\partial l = 0$ we find $l = Na\tanh(Fa/T)$. The length *decreases* with the temperature since

$$\left(\frac{\partial l}{\partial T}\right)_F < 0\,. \tag{1}$$

The change of temperature under adiabatic stretching is determined by $(\partial T/\partial l)_S$. It can be found using the identity $\partial(l,F)/\partial(T,S) = 1$, which is the version of $\partial(V,P)/\partial(T,S) = 1$ for our system:

$$\left(\frac{\partial T}{\partial l}\right)_S = \frac{\partial(T,S)}{\partial(l,S)} = \frac{\partial(T,S)}{\partial(l,S)}\frac{\partial(l,F)}{\partial(T,S)} = -\frac{\partial(F,l)}{\partial(F,T)}\frac{\partial(F,T)}{\partial(l,T)}\frac{\partial(l,T)}{\partial(l,S)} = -\left(\frac{\partial l}{\partial T}\right)_F\left(\frac{\partial F}{\partial l}\right)_T\frac{T}{C_l} > 0\,. \tag{2}$$

The last inequality follows from (1) and from the stability conditions $C_l > 0$ and $(\partial l/\partial F)_T > 0$.

Solution 3. For 3d harmonic oscillator, the energy levels are $\epsilon_n = \hbar\omega(n + 3/2)$ and their degeneracy $g_n = 2C^2_{n+1} = n(n+1) \approx n^2$ or equivalently the density of states $g(\epsilon) = \epsilon^2$ (compare with home exercise 2.1). Therefore, the number of particles and the total energy are respectively

$$N = \sum_0^\infty g_n f(\epsilon_n) \approx (\hbar\omega)^{-3}\int_0^\infty \epsilon^2 f(\epsilon)\,d\epsilon \approx (1/3)(\mu/\hbar\omega)^3\left[1 + (\pi T/\mu)^2\right]\,, \tag{3}$$

$$E = \sum_0^\infty \epsilon_n g_n f(\epsilon_n) \approx (\hbar\omega)^{-3}\int_0^\infty \epsilon^3 f(\epsilon)\,d\epsilon \approx (\mu^4/4)(\hbar\omega)^{-3}\left[1 + 2(\pi T/\mu)^2\right]\,. \tag{4}$$

Here we used the Fermi-Dirac distribution $f(\epsilon) = [1 + e^{\beta(\epsilon-\mu)}]^{-1}$ and the low-temperature approximation $\int_0^\infty F(\epsilon)f(\epsilon)\,d\epsilon \approx \int_0^\mu F(\epsilon)\,d\epsilon + F'(\mu)\pi^2T^2/6$. From (3) we find $\mu_0 = \mu(T=0) = \hbar\omega(3N)^{1/3}$ and $\mu(T) \approx \mu_0(1 - \pi^2T^2/3\mu_0^2)$. We substitute it into (4) and obtain

$$E(T) = E_0 + N\pi^2T^2/2\mu_0 \quad C = \frac{\pi^2 T}{3^{1/3}\hbar\omega}N^{2/3}\,. \tag{5}$$

Solution 4: a) Consider three spins, $\sigma_1, \sigma_2, \sigma_3$ and sum over the values of the spin σ_2:

$$\sum_{\sigma_2=1}^q e^{K(\delta_{\sigma_1\sigma_2} + \delta_{\sigma_2\sigma_3})} = \begin{cases} q - 1 + e^{2K} & \text{if } \sigma_1 = \sigma_3, \\ q - 2 + 2e^K & \text{if } \sigma_1 \neq \sigma_3. \end{cases} \tag{6}$$

Now require that this is equal to $e^{g+K'\delta_{\sigma_1\sigma_3}}$ and obtain

$$e^g = q - 2 + 2e^K \, , \quad e^{K'} = \frac{q - 1 + e^{2K}}{q - 2 + 2e^K} \, . \tag{7}$$

b) The fix points correspond to $K' = K$. $K = 0$ is a stable point and $K = \infty$ is an unstable one.

c) From RG one writes $r_c(K') = r_c(K)/2$. To solve this equation we need to find such function $x(K)$ so that $x(K') = x^2(K)$, then $r_c \propto \ln^{-1}(x)$. That function is $x = 1 + q/(e^K - 1)$ - one way to find it is to introduce first $e^K - 1 = z$ so that $z' = z^2/(2z + q)$. Alternatively, one can use the transfer matrix, which in this case has eigenvalues $\lambda_1 = e^K + q - 1$ and $\lambda_2 = e^K - 1$. Correlations decay with the distance r as $(\lambda_2/\lambda_1)^r$ so that the correlation length is

$$r_c = \ln^{-1}(\lambda_1/\lambda_2) = \ln^{-1}[1 + q/(e^K - 1)] \, .$$

One can check that for $q = 2$ everything coincides with that of 1d Ising model.

It is called Potts model, see Kardar, Fields, Problems 6.2a, 6.3c.