Exam 2013 -1

1. To see how much of the exam grade spread is pure luck, consider the exam with ten identically difficult problems of ten points each. The exam is multiple-choice that is the correct choice of the answer brings 10 points and a wrong choice brings zero. Assume that all students are identical and they choose answers at random with the probability $p = 0.8$ to get it right. What are the expected mean grade $\bar{g}$ and the standard deviation $\sqrt{\langle (g-\bar{g})^2 \rangle}$?

Solution. For one question the mean grade is $\bar{g}_1 = 10p = 8$ and the standard deviation is $\Delta_1 = \sqrt{\langle g_1^2 \rangle - \bar{g}_1^2} = 10\sqrt{p(1-p)} = \sqrt{0.8 \times 100 - 64} = \sqrt{16}$. Since the problems are independent they can be treated as a random walk with ten steps. For a random walk, $\Delta_N = \Delta_1 \sqrt{N} = \sqrt{100Np(1-p)}$ and $\Delta_{10} = \sqrt{160} \approx 13$. Alternatively, you can notice that the probability to have $k = g/10$ right answers is $P(k, N) = C_N^k p^k (1-p)^{N-k}$, called binomial distribution. For binomial distribution, the mean value $\langle k \rangle = p$ and the standard deviation is $\sqrt{Np(1-p)}$. One can also treat N as a large number and obtain the standard deviation by expanding $\ln P(k, N)$ around its maximum $k_* \approx pN$ where the binomial distribution is close to Gaussian.

2. Consider a cubic sample of solid dielectric with the volume $1\,cm^3$. The speed of sound in this material is $u = 1$ km/sec. Planck constant and Boltzmann constants are respectively $\hbar = 10^{-34}$ Jsec and $k = 1.4 \cdot 10^{-23}$ J/K. Estimate at what temperature the relative fluctuation of the thermal energy (temperature-dependent part of the energy) is of order unity.

Solution: The squared energy fluctuation $\langle (\Delta E)^2 \rangle = kT^2 C_v$ is expressed via the specific heat. We expect the relative fluctuation to be of order unity at low temperatures where we can use the Debye formula $C_v \simeq kN(kT/\theta)^3$ see (86) from Lecture Notes. The thermal energy is $E \simeq NkT(kT/\theta)^3$ so that $\langle (\Delta E)^2 \rangle / E^2 \simeq (\theta/kT)^3/N \simeq (hu/kT)^3 V^{-1}$. Therefore, the relative fluctuations are of order unity when $kT \simeq hu/L$ which corresponds to $T \simeq 10^{-6}$K.

3. Gas molecules interact with the wall. The energy of interaction $U(x)$ depends on the distance to the wall x and changes from ∞ at $x = 0$ to 0 at $x \to -\infty$. Find the pressure on the wall if the temperature is T and the concentration of molecules far from the wall is n_0.

Solution: The molecules at x are under the action of the force $-dU/dx$ and act on the force with the opposite force. The total force acting on the unit area of the wall is

$$\int_{-\infty}^{0} \frac{dU}{dx} n(x) dx = \int_{-\infty}^{0} \frac{dU}{dx} n_0 e^{-U(x)/T} dx = n_0 \int_0^{\infty} e^{-U(x)/T} dU = n_0 T \,,$$

i.e. ideal gas pressure (it is important that there is no interaction between molecules themselves).

If one is lazy but smart, one can obtain the answer without any computation: imagine a wall at $x = -\infty$, where the pressure must be $n_0 T$ - the same pressure must be on the wall at $x = 0$.

4. Renormalization group and central limit theorem. Consider the space of random variables x having the probability distribution $\rho(x)$ with zero mean and variance $\sigma = \int x^2 \rho(x) dx$. Consider the renormalization-group transformation which consists of two steps:

1) Take two random variables and add them. The new distribution of sums is $\rho'(z) = \int \rho(x)\rho(y)\delta(x + y - z) dx dy$.

2) Since the step 1) increases variance, make the re-scaling such that returns the variance back to σ but keeps the distribution normalized: $\rho''(z) = \lambda \rho'(\lambda z)$.

a) Determine λ.
b) Which equation $\rho(x)$ must satisfy to be a fixed point of the procedure 1) + 2)?
c) Find the solution of this equation (hint: use Fourier representation).

Solution: The step 1) increases the variance by the factor 2, so we need $\lambda = \sqrt{2}$. The equation for the fixed point,

$$\rho(z) = \sqrt{2} \int \rho(\sqrt{2}z - y)\rho(y)dy,$$

is a convolution equation, so it is easiest to solve it in Fourier representation, where $\rho(k) = \int \rho(z)e^{ikz}dz$ satisfies the equation $\rho(\sqrt{2}k) = \rho^2(k)$. The solution of this equation is Gaussian whose Fourier transform is Gaussian too: $\rho(z) \propto \exp(-z^2/2\sigma^2)$. The hint was given in the title of the problem by words "central limit theorem", which tells us that the Gaussian distribution appears after adding random variables.

Bonus question:

B. A raw chicken egg, when put into a large pot with boiling water usually cooks in about five minutes. An ostrich egg has about the same shape as a chicken egg, but its linear size is three times larger. Approximately how long does it take to cook an ostrich egg?

B. Solution: Let us recall what happens when we put an egg into boiling water. We assume that initially the entire egg is at some uniform room temperature. Once in the boiling water, the surface layer of the egg quickly heats up to the temperature of the water (we assume rapid heat exchange in the water outside the egg). Then, due to the temperature gradient, heat flows from the outside of the egg towards the yoke, eventually raising its temperature to the point when it coagulates, i.e., becomes solid. At this point, we proclaim the egg cooked. The heat conduction is governed by the diffusion equation, which suggests the scaling "time=distance squared". Therefore, cooking an ostrich egg must take about 45 minutes.

Exam 2013-2

Problem 1. A d-dimensional container is divided into two regions A and B by a fixed wall. The two regions contain identical Fermi gases of spin 1=2 particles which have a magnetic moment τ. In the region A there is a magnetic field H, no field in the region B. Initially, the entire system is at zero temperature, and the numbers of particles per unit volume are the same in both regions. If the wall is now removed, particles may flow from one region to the other. Determine the direction in which particles begin to flow, and how the answer depends on the space dimensionality d.

Solution: In general, particles flow from a region of higher chemical potential to a region of lower chemical potential. We therefore need to find out in which region the chemical potential is higher, and we do this by considering the grand canonical expression for the number of particles per unit volume. In the presence of a magnetic field, the single- particle energy is $\epsilon \pm \tau H$, where ϵ is the kinetic energy. The sign depends on whether the magnetic moment is parallel or antiparallel to the field. The total number of particles is then given by

$$N = \int_0^\infty d\epsilon g(\epsilon) \frac{1}{\exp[\beta(\epsilon - \eta - \tau H)] + 1} + \int_0^\infty d\epsilon g(\epsilon) \frac{1}{\exp[\beta(\epsilon - \eta + \tau H)] + 1} \, . \tag{1}$$

For non-relativistic particles in a $d-$dimensional volume V, the density of states is $g(\epsilon) = \gamma V \epsilon^{d/2-1}$, where γ is a constant. At $T = 0$, the Fermi distribution function is

$$\lim_{\beta\to\infty} \left(\frac{1}{e^{\beta(\epsilon-\mu\pm\tau H)} + 1} \right) = \theta(\mu \mp \tau H - \epsilon) \tag{2}$$

where $\theta(\cdot)$ is the step function, so the integrals are easily evaluated with the result

$$\frac{N}{V} = \frac{2\gamma}{d} \left[(\mu + \tau H)^{d/2} + (\mu - \tau H)^{d/2} \right] \, . \tag{3}$$

At the moment that the wall is removed, N/V is the same in regions A and B; so (with $H = 0$ is the region B) we have

$$(\mu_A + \tau H)^{d/2} + (\mu_A - \tau H)^{d/2} = 2\mu_B^{d/2} \, . \tag{4}$$

For small fields, we can make use of the Taylor expansions

$$(1 \pm x)^{d/2} = 1 \pm \frac{d}{2} x + \frac{d}{4} \left(\frac{d}{2} - 1 \right) x^2 + ... \tag{5}$$

to obtain

$$\left(\frac{\mu_B}{\mu_A} \right)^{d/2} = 1 + \frac{d(d-2)}{8} \left(\frac{\tau H}{\mu_A} \right)^2 + ... \tag{6}$$

We see that, for $d = 2$, the chemical potentials are equal, so there is no flow of particles. For $d > 2$, we have $\mu_B > \mu_A$ so particles flow towards the magnetic field in region A while, for $d < 2$, the opposite is true. We can prove that the same result holds for any magnetic field strength as follows. For compactness, we write $\lambda = \tau H$. Since our basic equation $(\mu_A + \lambda)^{d/2} + (\mu_A - \lambda)^{d/2} = 2\mu_B^{d/2}$ is unchanged if we change λ

to $-\lambda$, we can take $\lambda > 0$ without loss of generality. Bearing in mind the μ_B is fixed, we calculate $d\mu_A/d\lambda$ as

$$\frac{d\mu_A}{d\lambda} = \frac{(\mu_A - \lambda)^{d/2-1} - (\mu_A + \lambda)^{d/2-1}}{(\mu_A - \lambda)^{d/2-1} + (\mu_A + \lambda)^{d/2-1}} \,. \tag{7}$$

Since $\mu_A + \lambda > \mu_A - \lambda$, we have $(\mu_A + \lambda)^{d/2-1} > (\mu_A - \lambda)^{d/2-1}$ if $d > 2$ and vice versa. Therefore, if $d > 2$, then $d\mu_A/d\lambda$ is negative and, as the field is increased, μ_A decreased from its zero-field value μ_B and is always smaller than μ_B. Conversely, if $d < 2$, then μ_A is always greater than μ_B. For $d = 2$, we have $\mu_A = \mu_B$ independent of the field.

Problem 2. One way to create Bose-Einstein condensation is to put atoms in a three-dimensional optical lattice which is an atom trap created by standing light waves. There is no more than one atom per cite cooled to the lowest vibrational state. The ratio between the number of atoms and number of cites is $\kappa \leq 1$. The lattice is then adiabatically removed (the intensity of light is lowered) so that the entropy of the gas is preserved. If $\kappa = 1$ then after lattice removal the atomic wavefunctions delocalize and overlap so that the atoms become a zero-temperature Bose-Einstein condensate (BEC).

Now consider $\kappa < 1$.

i) Calculate the entropy of a partially filled lattice and compare it with the entropy of the Bose gas.

ii) Find the condition that determines the smallest κ for which the formation of BEC is still possible.

iii) Use the approximate value $5g_{5/2}/2g_{3/2} \approx 1.3$ and estimate the smallest κ numerically.

Solution 2: Suppose we have $P \gg 1$ sites of which κP sites are occupied. The entropy is

$$S = \ln[P!/(\kappa P)!(P - \kappa P)!] \approx P[(\kappa - 1)\ln(1 - \kappa) - \kappa \ln \kappa] \,. \tag{8}$$

The entropy of the κP atoms of a Bose gas below T_c is given by the formula (93) from the lecture notes

$$S = \frac{5g_{5/2}}{2g_{3/2}} \kappa P \left(\frac{T}{T_c}\right)^{3/2} \approx 1.3 \cdot \kappa P \left(\frac{T}{T_c}\right)^{3/2} \,. \tag{9}$$

The condition that the entropies (8) and (9) are equal gives the temperature of the Bose gas which appears after the trap is adiabatically removed. Requiring that this temperature is equal to $T = T_c$ gives the $\kappa \approx 0.5$.

Problem 3. Consider the over-damped Brownian particle in the potential $V(q)$ so that the respective equation of motion is

$$\dot{q} = -\frac{dV}{dq} + \eta \,. \tag{10}$$

Here the noise is white Gaussian with $\langle \eta(0)\eta(t)\rangle = 2\delta(t)$. The potential has the asymptotic $V(q) \to q^3$ at $q \to \pm\infty$, that is $V(q) \to +\infty$ at $q \to +\infty$ and $V(q) \to -\infty$ at $q \to -\infty$. The space q has the topology of a circle i.e. $q = \infty$ and $q = -\infty$ is the same point.

i) Find the stationary probability distribution $\rho(q)$. Does this distribution have a Boltzmann-Gibbs form with a zero probability current?

ii) Describe the form of the asymptotic of $\rho(q)$ at large $|q|$.

Solution 3: It is straightforward to check that the Boltzmann-Gibbs distribution $e^{-V(q)}$ is non-normalizable, i.e. its integral over q diverges. Yet the Fokker-Planck equation is of the second order, so it must have two solutions. Therefore, we need another solution:

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial q}\left(\frac{\partial\rho}{\partial q} + \frac{dV}{dq}\rho\right) = -\frac{\partial J}{\partial q}\ . \tag{11}$$

Apart from the Boltzmann-Gibbs distribution, which has zero current, a solution with a constant current is steady as well. Solving equation

$$\frac{\partial\rho}{\partial q} + \frac{dV}{dq}\rho = -J = \text{const}\ ,$$

we find the true steady-state solution,

$$\rho(q) = -Je^{-V(q)}\int_{-\infty}^{q} e^{V(q')}dq'\ ,$$

which is normalizable. Note that the current must be negative i.e. directed towards $-\infty$. For this solution to have a physical meaning, the phase space must be a circle i.e. the current that flows to $-\infty$ returns from $+\infty$. Asymptotic at large q is $\rho(q) \propto q^{-2}$, which corresponds to a fast (finite-time) escape to infinity according to the law $\dot{q} = -q^2$.

Problem 4

Consider particles having coordinates x on a line: $-\infty < x < \infty$. Find the probability distribution $p(x)$ in two cases:

case 1) The only information established by measurement is that the mean distance from zero is $\langle|x|\rangle = X$.

case 2) The only information established by measurement is that the variance is given by $\langle x^2\rangle = X^2$.

Which measurement provided more information on the coordinate distribution? Quantify the difference in bits.

Solution 4, $p_1 = (2X)^{-1}\exp(-|x|/X)$, $p_2 = (2\pi X^2)^{-1/2}\exp(-x^2/2X^2)$. $S_1 = \ln(2X) + 1$, $S_2 = \ln X + 1/2[1 + \ln(2\pi)]$. $I_2 - I_1 = (S_1 - S_2)/\ln 2 = (1 + \ln 2 - \ln\pi)/2\ln 2 \approx 0.4$ bits. See Kardar, Particles, Problem 2.6