Introduction to Turbulence for Physicists

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Abstract

One often asks what is "the great problem of turbulence" so one can solve it and get a Nobel prize. These lecture notes for an 8-hour course do just that adding as a bonus the formulation of a related Milennium problem and one for Fields medal (for mathematically oriented and deprived of the Nobel prize). No prior knowlwdge of fluids is required.

1 Fluids and Flows

We deal with continuous media where matter may be treated as homogeneous in structure. The term fluid means that resistance cannot prevent deformation from happening because the resisting force vanishes with the rate of deformation. With patience, anything can be deformed. Therefore, whether one treats the matter as a fluid or a solid depends on the time available for observation. As the prophetess Deborah sang, 'The mountains flowed before the Lord' (Judges 5:5). The ratio of the relaxation time to the observation time is called the Deborah number. The smaller the number the more fluid the material.

Fluid mechanics is the macroscopic study of two conservation laws, mass and momentum. Mass conservation is expressed as a continuity equation

$$\frac{\partial \rho}{\partial t} = -\partial_k \left(\rho v_k\right) \ . \tag{1}$$

Here $\mathbf{v}$ is the fluid velocity, ρ is density Translation invariance brings momentum conservation. The time derivative of the momentum density is the divergence of the momentum flux:

$$\frac{\partial \rho v_i}{\partial t} = -\partial_k \left(\rho v_i v_k - P\delta_{ik} - \nu\rho\partial_k v_i\right) \ . \tag{2}$$

This is the Navier-Stokes equation named after the first and last persons who derived it. The first term describes the momentum transport by the flow. The two other terms in the right-hand side describe stresses. The pressure P plays the role of the potential energy of interaction between fluid particles, so its gradient is the normal force per unit area. The last term describes the tangential force due to viscous friction; ν is the kinematic viscosity which is the diffusivity of momentum (estimated for gases as the molecular velocity times the mean free path). The diffusive flux of momentum is not proportional to the gradient of momentum (as the flux of any other substance) but to the gradient of velocity only. In other words, density gradient does not cause any friction and does not bring momentum diffusion in a uniform flow; that is the medium is in thermal equilibrium (say, in a gravity field) absent any velocity gradient.

Both (2) and (1) contain velocity gradients which are thus assumed finite. That corresponds to a continuous flow where the trajectories of fluid particles fo not intersect. Indeed, the equation for the distance vector between two fluid particles, $dR_i/dt = \delta v_i(\mathbf{R})$ has a unique solution if the vector field is Lipshits, that is $\delta v_i(\mathbf{R})$ goes to zero with R not slower than linearly. One of the Milennium problems in Mathematics is to establish whether finite velocity gradient could turn infinite in a finite time. As we shall see below, turbuence statistics looks like fluid trajectories could stick and split and the velocity field is statistically non-Lipshits. That by itself does not guarantee finite-time singularity, since the probability of $\nabla\mathbf{v}$ could go to zero when $|\nabla\mathbf{v}| \to \infty$, just too slow so that some moments are infinite.

In d dimensions, there are $d+2$ variables but only $d+1$ equations in (1,2). One needs to supplement it by a medium-specific equation of state relating P and ρ. We start from the simplest case of an incompressible fluid of a uniform density and temperature. Does that mean that we need to consider the pressure uniform too? That would be true only in thermal equilibrium that is with no flow or uniform flow. Indeed, the continuity equation in this case is reduced to $div\,\mathbf{v} = 0$. Applying div to (2), we obtain $div\,(\mathbf{v}\nabla)\mathbf{v} = \Delta P$. Pressure inhomogeneeity is due to deviations from thermal equilibrium caused by the velocity gradients. The nonlocality of the inverse Laplace operator means that the pressure in the whole space is instantenously adjusted to any local velocity change. This is because incompressibility presumes that we took the speed of sound (wave of density and pressure changes) to infinity or assumed $v \ll c$. Now we need to remember that this is a singular limit which diminished the order of time derivative in our system from second to first. This is also clear from writing the operator of the wave equation as $c^{-2}\partial_t^2 - \Delta$.

For incompressible fluid, we write

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{\nabla P}{\rho} + \nu\Delta\mathbf{v} + \mathbf{f}\,. \tag{3}$$

We use both Eulerian descriprion based on fields (like electromagnetic or field theory) and Lagrangian description based on considering fluid particles.

Apparently, d/dt is a Lagrangian derivative, while $\partial/\partial t$ is an Eulerian one.

Note that there is no energy conservation because the friction terms in (2,3) break time reversibility and provide for energy dissipation. For example, (3) gives

$$\frac{d}{dt}\int v^2\, d\mathbf{r} = -\nu \int \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)^2 d\mathbf{r}\,. \tag{4}$$

Simplest flow. Let us see how well (3) describes reality. Taking for water $\nu = 0.01\, cm^2/sec$ and the external force per unit mass to be gravity, $\mathbf{f} = \mathbf{g}$, let us apply (3) to the simplest stationary flow of a fluid sliding along the plane inclined with the angle α. Directing x-axis along the flow and the plane, and z-axis perpendicular to them, we write two components of (3):

$$\nu\partial_z^2 v = -g\sin\alpha\,, \quad \partial_z P = -g\cos\alpha\,.$$

Imposing zero velocity at the bottom $z = 0$ and zero friction at the surface $z = h$ we obtain

$$v = z(2h - z)\frac{g\sin\alpha}{2\nu}\,. \tag{5}$$

With $g \approx 10^3 cm/sec^2$, this is expected to describe puddles, creeks and rivers, as well as horizontal channel and pipe flows driven by a pressure gradient (replacing $g\sin\alpha$ by $\nabla P/\rho$). Let's see how well it does the job. Taking for a puddle $h = 0.1\, cm$ and $\alpha = 10^{-2}$ we obtain on the surface reasonable $v = 5\, cm/sec$. Taking conservately for a river on a large plane (like Missisipi or Volga) $\alpha \simeq 100\, m/1000\, km \simeq 10^{-4}$ and $h = 10\, m$ we obtain $v = 5 \cdot 10^8 cm/sec = 5 \cdot 10^6 m/sec$. When we are wrong by a factor 10^6, it is a chance to get one million times smarter.

The huge diference between the case where we succeeded and the one where we failed so miserably must be characterized by a dimensionless parameter. As always, it must be the ratio of the terms (and forces) in our equation. Indeed, fluid mechanics is essentially a story of the struggle between the inertia that tries to keep the flow and the friction that tries to stop it. The inertia is characterized by the nonlinear term $(\mathbf{v}\nabla)\mathbf{v} \simeq v^2/h$ and the friction by $\nu\Delta v \simeq \nu v/h^2$. Their ratio is called the Reynolds number

$$Re = \frac{vh}{\nu}\,. \tag{6}$$

Wait, but the nonlinear term is identically zero for our solution (5) because $\mathbf{v} \perp \nabla\mathbf{v}$. Why then the solution is realized for a puddle but is not realized for a river? We suspect that in the second case it must be unstable, but with respect to what type of perturbation? Let us step closer to the real world and take into account that the bottom is not an ideal smooth plane. Any misalignment by a small angle β makes $Re = \beta vh/\nu$ which is 50β for a puddle and $5 \cdot 10^{11}\beta$ for a river. It is then clear that even microscopic bottom inhomogeneities must change the river flow everywhere. Indeed, everyday experience suggests that the river flow must be turbulent. But how it affects the resistance which must balance $g\alpha$ drive?

2 Guessing the right resistance force

What mean flow velocity we expect for a turbulent river? We expect that turbulence transports momentum to the bottom much faster than molecular diffusion. Replacing (after Prandtl) ν by the "turbulent viscosity" $\nu_T \simeq vh$ we obtain a force balance $g\alpha \simeq v^2/h$ which gives a reasonable estimate:

$$v \simeq \sqrt{g\alpha h} \quad \text{or} \quad v \simeq \sqrt{h\nabla P/\rho}\,. \tag{7}$$

For a river it gives $v \simeq 10\,cm/sec$. By a similar argument Newton estimated the resistance (drag) force experienced by a body of a size h moving with the velocity u. That force must be the momentum impacted to the fluid per unit time. The volume of the fluid we put in motion in one second is h^2v and the momentum it gets per unit volume is ρv so that the force is as follows:

$$F = C\rho u^2 h^2\,. \tag{8}$$

Newton assumed that the dimensionless (so-called friction) factor C is determined solely by the body shape.

Viscosity enters neither (7) nor (8) even though it is clear that there is no resistance without friction. Indeed, the resistance force is supposed to change sign with v while our "resistance" is proportional to v^2. It is clear that viscosity must dominate in the limit $Re \to 0$ when the force is $F \simeq \rho\nu hu$ so that the friction factor must depend on Re, in particular, $C(Re) \propto 1/Re$ at $Re \to 0$. What Newton wants us to believe is that $C(Re)$ saturates to a constant at $Re \to \infty$.

From the engineering viewpoint, the resistance is the central problem of turbulence. From the physics perspective, the most interesting part of it is what we call anomaly: when the symmetry-breaking factor goes to zero, the effect of symmetry-breaking has a finite limit. In this case (of the so-called dissipative anomaly), the symmetry is time-reversibility, the symmetry-breaking factor is viscosity and the effect is finite resistance in the inviscid limit. How can we understand its mechanism? That was done by Prandtl (1905) who discovered the phenomenon of separation and wake creation. Consider, for instance, the flow around a cylinder or sphere. The flow must have up-down symmetry, so that the points on the axis (forward D and backward C) are stagnation points. Denote A the point of the sphere farthest from the symmetry axis. On the upstream half DA, the fluid particles accelerate and the pressure decreases. Indeed, the energy of a fluid particle (that is per unit mass), $v^2/2 + P/\rho$ must be conserved without a friction. On the downstream part AC, the reverse happens, that is every particle moves against the pressure gradient. A small viscosity changes pressure only slightly across the boundary layer. Indeed, if the viscosity is small, the boundary layer is thin and can be considered locally flat. Denote u the velocity right outside the boundary layer. In the boundary layer, at $z < \nu/u$, no-slip condition prescribes $v_x \simeq u^2 z/\nu$ and $\partial v_x/\partial x \simeq u^2 z/\nu R \simeq \partial v_z/\partial z$. The normal velocity is then $v_z \simeq u^2 z^2/\nu R$, which gives the pressure gradient, $\partial p/\partial z = -\rho(v\nabla)v - \eta\Delta v_z \simeq \rho u^2/R$, so that the pressure change across the layer is $\rho u^2/Re$ that is small when Re is large. In other words, the pressure inside the boundary layer is almost equal to that in the main stream, which is the pressure of the ideal fluid flow. But the velocities of the fluid particles that reach the point A are lower in a viscous fluid than in an ideal fluid because of viscous friction in the boundary layer. Then those particles have insufficient energy to overcome the pressure gradient downstream. The particle motion in the boundary layer is stopped by the pressure gradient before the point C is reached. The pressure gradient then becomes the force that accelerates

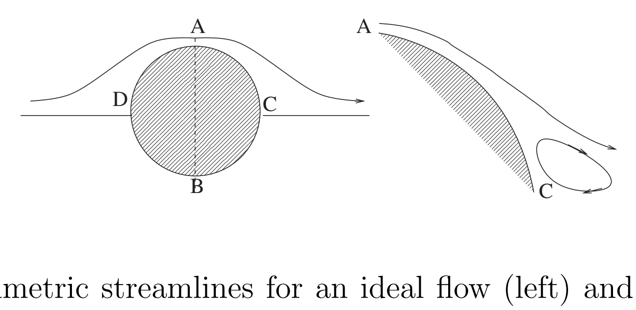

Figure 1: Symmetric streamlines for an ideal flow (left) and appearance of separation and a recirculating vortex in a viscous fluid (right).

the particles from the point C upwards, producing separation. See more in Section 1.5.2 in Fluid Mechanics by Falkovich.

2.1 Drag with a wake

We can now describe the way Nature resolves reversibility paradox. In the reference frame of the body, far from it we have a uniform flow (with $\mathbf{u}, p_0$), the body adds $\mathbf{v}, p'$. Let us relate the momentum flux through a closed distant surface to the force acting on the body assuming the existence of the wake. The total momentum flux transported by the fluid through any closed surface is equal to the rate of momentum change, which is equal to the force acting on the body:

$$F_i = \oint \Pi_{ik} df_k = \oint (p_0 + p')\delta_{ik} + \rho(u_i + v_i)(u_k + v_k)\, df_k \tag{9}$$
$$= (p_0\delta_{ik} + \rho u_i u_k) \oint df_k + \rho u_i \oint v_k\, df_k + \oint [p' + \rho(u_k v_i + v_i v_k)\, df_k .$$

Here $d\mathbf{f}$ is the vector normal to the surface and equal to an area element. In the last line, the first integral vanishes because the surface is closed and the second one because of mass conservation: $\rho \oint v_k\, \partial f_k = 0$. Far from the body $v \ll u$ and we neglect terms quadratic in v:

$$F_i \approx \left(\iint_{X_0} - \iint_{X} \right) (p'\delta_{ix} + \rho u v_i)\, \partial y \partial z . \tag{10}$$

Assuming that $\mathbf{u}$ is along x, the drag is the x component of (10):

$$F_x = \left(\iint_{X_0} - \iint_{X} \right) (p' + \rho u v_x)\, \partial y \partial z .$$

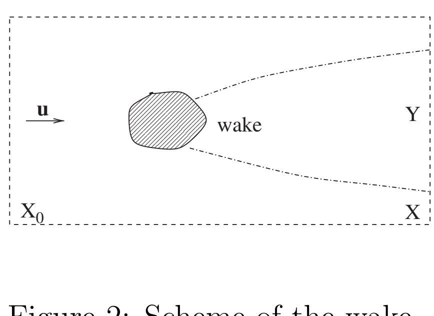

Figure 2: Scheme of the wake.

Outside the wake we have potential flow where the Bernoulli relation, $p + \rho|\mathbf{u}+\mathbf{v}|^2/2 = p_0 + \rho u^2/2$, gives $p' \approx -\rho u v_x$ so that the integral outside the wake vanishes. Inside the wake, the pressure is about the same (since it does not change across the almost straight streamlines, as we argued above but the velocity perturbation v_x is much larger than outside, so that

$$F_x = -\rho u \iint_{\text{wake}} v_x \, \partial y \partial z. \tag{11}$$

Force is positive (directed to the right) since v_x is negative. The integral in (11) is equal to the deficit of fluid flux Q through the wake area (i.e. the difference between the flux with and without the body). The wake breaks the fore-and-aft symmetry and thus resolves the paradoxes, providing for a non-zero drag in the limit of vanishing viscosity. This justifies Newton's intuition about impacting momentum to the fluid. Indeed , the integral (which is independent on the distance X) can be estimated closer to the body as uh^2 which gives (8).

2.2 Resistance of a pipe flow

A straightforward application of the above logic to a mean unidirectional flow of rivers and pipes is impossible because now we must describe the z-dependence of the mean flow which must carry the momentum injected by gravity or pressure gradient towards the bottom or walls to be absorbed there. Let us write the momentum conservation without assuming the flow unidirectional. Denote the velocity x-component as $U(z) + u(z, y, z, t)$ and z-component as $v(x, y, z, t)$, where u, v describe turbulent fluctuations. Then the continuity equation for the x-component of the mean momentum states that the divergence of the momentum flux τ is equal to the force:

$$\frac{d}{dz}\left(\nu \frac{dU}{dz} + \langle uv \rangle\right) \equiv \frac{d\tau(z)}{dz} = -\alpha g. \tag{12}$$

Integrating we get $\tau(z) = \tau(0) - \alpha g z$. The flux is zero on the river surface or at the center of a pipe, which gives $\tau(0) = \alpha g h$. Let us now consider the flow close to the solid surface, that is at $z \ll h$, where the momentum flux can be considered independent of z, and denote $v_*^2 \equiv \tau(0) = \alpha g h$. In this region the mean velocity is independent of h and must depend only on ν, z, v_*. By dimensional reasoning the dependence must have a form $U = v_* f(z v_*/\nu)$.

The dimensionless parameter zv_*/ν is the Reynolds number with the scale set by the distance to the solid boundary. Near the boundary, viscosity absorbs the flux: $\nu dU/dz = \alpha gh$ and $U(z) = \alpha ghz/\nu$. The width of that viscous boundary layer can be estimated requiring the Reynolds number to be of order unity: $l = \nu/v_*$. Outside of this layer, for $z \gg l$, one may expect viscosity to be unimportant and the flux carried by turbulence. As we cannot yet develop a consistent theory of such inhomogeneous turbulence (see more in Section 7 below), let us use plausible arguments. Since there is no momentum flux in a uniform flow, then it is natural to relate the mean flux to the flow nonuniformity whose simplest characteristics is the mean velocity gradient, dU/dz. We now assume that the flow must be determined solely by v_* and z at $l \ll z \ll h$. The only dimensionally possible relation is $dU/dz \simeq v_*/z$, which gives a logarithmic velocity profile for turbulent boundary layer (Karman 1930, Prandtl 1932):

$$U(z) \simeq v_* \log(z/l) = \sqrt{\alpha gh}\,\log[z(\alpha gh)^{1/2}/\nu]. \tag{13}$$

It is equivalent to the turbulent viscosity argument: $\tau = v_*^2 = \nu_T dU/dz$ with $\nu_T(z) \simeq v_* z$. We used l to make the argument of the logarithm dimensionless since for $z \simeq l$ one must have $U(l) \simeq v_*$. One can further illuminate the hypothesis underlying the log law (13) using so-called overlap argument. The dimensionless quantity $U(z)/v_*$ must be a function of two dimensionless arguments, $\ell = z/h$ and $Re = v_*h/\nu$. Near the wall we expect h to disappear: $U(z)/v_* \to f(\ell Re)$. Near the center, we expect ν to disappear from the law of the velocity change: $U(h) - U(z) = v_* f_1(\ell)$. Denote $U(h)/v_* = f_2(Re)$. We now make an assumption that the two asymptotic regions overlap. In this overlap region we have $f(\ell Re) = f_2(Re) - f_1(\ell)$, which requires all the functions to be logarithmic. Logarithmic turbulent profile is more flat than parabolic laminar profile, which is natural since turbulence better mixes momentum. The overlap argument and claim that the momentum flux completely determines the mean flow in a turbulent boundary layer are curiously similar to assuming inertial interval with the energy flux determining everything in the cascade picture of homogenous turbulence. We shall see in the next Section that the cascade picture correctly describes only the third moment of the velocity statistics, while other moments depend on the large scale. It is not yet clear whether the Prandtl-Karman theory must be modified in a similar way. Experiments support logarithmic mean flow profile but show that turbulence statistics depends on h even at $z \ll h$.

We see that (13) corrects (7) by a viscosity-dependent logarithmic factor. That makes velocity everywhere, even outside of the viscous layer, dependent on viscosity. While this dependence is very slow and for most cases negligible, conceptually it has dramatic consequences. It tells us that when viscosity goes to zero, the width l of the viscous layer shrinks to zero but $U(l) \simeq v_*$ i.e. stays finite. That means that we have an effective slip on the solid boundary. At any finite z, the velocity $U(z)$ goes to infinity, so that the friction factor goes to zero at $\nu \to 0$ as $\log^{-2}(hv_*/\nu)$. All this is because we consider the boundary straight and smooth, which explains the dramatic difference from the flow past a body, where curved surface provides for separation of the boundary layer and resulting wake provides for a finite drag coefficient. It is then reasonable to assume that if the logarithmic decrease of the friction factor with the Reynolds number takes place, it stops when l is getting comparable to the size r of the boundary inhomogeneities (experiments support that). When $\nu < rv_*$ one cannot assume the mean flow to be parallel to the solid boundary. Every inhomogeneity then provides its own wake with a finite drag so that $U(r) \simeq v_*$, the logarithms saturates at $\log(h/r)$, and the friction factor is getting independent of Re.

We thus see that large-Re wake flow is insensitive while the pipe flow is sensitive to the surface smoothness. The unsolved question is to what extend the turbulence in the wake is similar to the turbulence in pipes and channels.

3 Energy Cascade

We can look at this anomaly not from the viewpoint of momentum loss but from the viewpoint of energy dissipation. That will allow us to see fluxes in Fourier space rather than in a real space. Consider a fan circulating air in the room. The power per unit mass can be estimated as the force (8) times velocity divided by the mass:

$$\epsilon \simeq \frac{Fu}{\rho h^3} \simeq \frac{u^3}{h} \,. \tag{14}$$

This power goes into heat with the rate independent of viscosity. This is less bizarre than it looks since any shock wave does that. That can be demonstrated using the poor relative of (3), one-dimensional Burgers equation which describes one-dimensional compressible flow $w(x,t)$ in the reference frame moving with the sound velocity,

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + ww_x = \nu w_{xx} + f(x,t)\,. \tag{15}$$

This equation with $f = 0$ has a shock-wave solution

$$w(x,t) = \frac{2u}{1 + \exp[u(x - ut)/\nu]}\,.$$

At the front moving with velocity u, fluid particles moving from the left with the velocity $2u$ hit standing particles, stick to them and continue with half-speed due to momentum conservation. Assuming that such shocks are spaced by the distance L, the mean energy dissipation rate (per unit mass) is $\epsilon = L^{-1}\nu \int w_x^2 = 2u^3/3L$. As is often in anomalies, saving one conservation law (momentum) means sacrificing another conservation law (energy).

But how such viscosity-independent dissipation could happen in an incompressible flow? One understand the nature of this anomaly using the concept of cascade. This understanding came from an unexpected perspective: Releasing baloons in a turbulent air, Richardson (in 1926) discovered that the squared distance between two trajectories, R^2, grows with time not linearly (as was expected in that diffusion-dominated period) but as t^3. The ratio R^2/t^3 has the dimensionality of ϵ. If $R(t) \simeq (\epsilon t^3)^{1/2}$, then $dR/dt \simeq (\epsilon t)^{1/2} \simeq (\epsilon R)^{1/3}$. Since $dR/dt = \delta v(R)$, one can make sense of the Richardson data by assuming that the average velocity difference grows with the distance by the law

$$\delta v(R) \simeq (\epsilon R)^{1/3}\,. \tag{16}$$

That suggests that one can define the energy transfer rate through a given scale R as squared energy per unit mass, $[\delta v(R)]^2$, divided by the typical time $R/\delta v(R)$. Such a transfer rate is independent of R and equal to the energy dissipation rate, which is equal to the input power (assuming a steady state):

$$\epsilon \simeq \frac{(\delta v)^3}{R}\,. \tag{17}$$

This corresponds to the energy cascade picture: all the kinetic energy we generate at largest scale (of the moving body) is transferred without loss through the intermediate scales until it is dissipated into heat by viscosity. That makes the dissipative anomaly less mysterious: Cascade acts as a pipe in the Fourier space; when viscosity goes down, the pipe is getting longer but it still carries the same energy flux. Proving that a steady state exists at the limit $\nu \to 0$ is another nontrivial problem.

Similarly, we can compute the mean cube of the velocity difference per unit length from the shock wave:

$$S_3(x) = L^{-1} \int_{-L/2}^{L/2} [w(x+x') - w(x)]^3 dx' = -8\frac{u^3 x}{L} = -12\epsilon x\,. \tag{18}$$

4 Velocity statistics and anomalous scaling

In our attempts to understand the resistance of fluids, we postulated some form of the third moment of the velocity difference, (17) and (18), which is related to the energy flux through scales. The latter was even derived for a particular case of a single shock. Could we derive these in a more general setting of force-generated turbulence? This can be done both for (3) and (15). Let us derive (18) for a generic acoustic turbulence generated by a large scale random force, whose variance $\langle f(x)f(0)\rangle$ decays with x on a scale L, which is much larger than the viscous scale $\eta \equiv \nu^{3/4}\epsilon^{-1/4}$. Denote $\epsilon = \langle fw\rangle$, $w_1 = w(x_1)$, $f_1 = f(x_1)$, $\partial_1 = \partial/\partial x_1$, $w_{1xx} = \partial_1^2 w_1$, etc. In a steady state, all time derivatives must be zero including that of the second moment:

$$\frac{\partial}{\partial t}\langle w_1 w_2\rangle = \big\langle w_1\big(f_2 + \nu w_{2xx} - w_2 w_{2x}\big) + w_2\big(f_1 + \nu w_{1xx} - w_1 w_{1x}\big)\big\rangle\,.$$

For $|x_1 - x_2| \ll L$ we can put $\langle f_1 w_2\rangle = \langle f_2 w_1\rangle \approx \langle fw\rangle = \epsilon$. Since the derivatives are large on the scale η, they are uncorrelated at the distances $|x_1 - x_2| \gg \eta$, so we can neglect $\langle w_1 w_{2xx}\rangle = \partial_2\langle w_1 w_{2x}\rangle = -\partial_1\langle w_1 w_{2x}\rangle = -\langle w_{1x} w_{2x}\rangle$. We consider the force statistics to be uniform in space, then any moment like $\langle w_1 w_2^2\rangle$ is a function of $x_1 - x_2$ so that $\partial_2\langle w_1 w_2^2\rangle = -\partial_1\langle w_1 w_2^2\rangle$. Adding (zero) term $\partial_1\langle w_1^3 - w_2^3\rangle$ we derive $2\epsilon = -\partial_1\langle (w_1 - w_2)^3\rangle/6$ which gives (18) for $\eta \ll |x_1 - x_2| \ll L$.

Respective derivation for an incompressible d-dimensional turbulence gives for the cube of the longitudinal velocity difference

$$S_3(r) = -\frac{12\epsilon r}{d(d+2)}\,. \tag{19}$$

Alternatively, both can be written for the time derivative of the squared velocity difference, $\delta\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$, along the flow $(d/dt = \partial_t + \mathbf{v}\cdot\nabla)$:

$$\langle d|\delta\mathbf{v}|^2/dt\rangle = -2\,\langle(\mathbf{v}_1\cdot d\mathbf{v}_2/dt + \mathbf{v}_2\cdot d\mathbf{v}_1/dt)\rangle$$
$$= -2\,\langle(v_{1i}v_{2j}\nabla_{2j}v_{2i} + v_{2i}v_{1j}\nabla_{1j}v_{1i})\rangle = \langle\delta\mathbf{v}\delta\mathbf{f} + \nu(\delta\mathbf{v}\cdot\Delta\delta\mathbf{v})\rangle = -4\epsilon\,. \quad (20)$$

Nonzero third moment means that the statistics is time irreversible. If someone screens the movie of turbulence backwards, we now can tell the difference.

Note that the velocity field giving (16) is non-Lipshits. Solving $dR_{12}/dt = \delta v(R_{12}) = (\epsilon R_{12})^{1/3}$, we obtain

$$R_{12}^{2/3}(t) = R_{12}^{2/3}(0) + \epsilon t\,. \quad (21)$$

Compare it with an exponential separation for a smooth (Lipshits) velocity $v_1 - v_2 = \delta v(R_{12}) \simeq \lambda R_{12}$, which gives $R_{12}(t) = R_{12}(0)\exp(\lambda t)$. For the smooth case (subject of the dynamical chaos theory) we have $R_{12}(t) \to 0$ when $R_{12}(0) \to 0$, which corresponds to uniqueness of trajectories. On the contrary, (21) shows that the distance between trajectories $R_{12}(t)$ may stay finite when the initial distance $R_{12}(0) \to 0$, which would mean splitting of trajectories. Indeed, non-Lipshits equation $dx/dt = x^{1-\gamma}$ with the initial condition $x(0) = 0$ has two solutions: $x(t) \equiv 0$ and $x(t) = [\gamma t]^{1/\gamma}$.

Onsager conjectured that the power of the velocity non-smoothness must be at least 1/3 to provide an inviscid dissipation in incompressible flows, which is now proved by the mathematicians. Non-smoothness of the velocity field gives trajectories splitting or sticking which necessarily violates conservation of any non-additive quantity.

We thus see two types of non-uniqueness: trajectories sticking for compressible flows and splitting for incompressible ones. Yet the cube of the velocity differences (the third structure function S_3) scales the same with the distance and the dissipation rate. That shows that the cascade idea captures only the energetic side of turbulence and fixes the third moment of the velocity difference. What about the whole probability distribution of the velocity difference at a given scale, $P(\delta v, r)$ and the other moments $S_n(r) = \langle(\delta v)^n\rangle = \int P(\delta v, r)(\delta v)^n d\delta v$? The cascade idea is of little help here. Indeed, early proponents of the cascade assumed the distribution to be self-similar, $P(\delta v, r) = (\delta v)^{-1}F(\delta v/\left(\epsilon r\right)^{1/3})$, assuming that the cascade determines at least the scaling of all the moments. Experimens show that this is not the case: $S_n(r) \propto r^{\zeta_n}$, where ζ_n is some convex function of n satisfying $\zeta_0 = 0$ and $\zeta_3 = 1$.

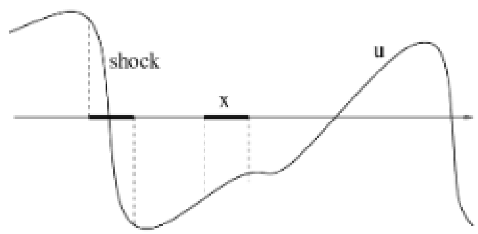

Figure 3: Typical velocity profile in Burgers turbulence

4.1 Bi-fractality of Burgers turbulence

The simplest is to find this function for Burgers turbulence. That would be wrong to assume $S_n = \langle [u(x) - u(0)]^n \rangle \simeq (\epsilon x)^{n/3}$, since shocks give a much larger contribution for $n > 1$: $S_n \simeq w^n x/L$, here x/L is the probability of finding a shock in the interval x. In terms of Fourier harmonics, every shock contributes $u_k \propto 1/k$, which indeed gives $S_2(x) = \langle [u(x) - u(0)]^2 \rangle = \int |u_k|^2 (1 - e^{ikx})\, dk \propto \int^{1/x} |u_k|^2\, dk \propto x$.

Generally, $S_n(x) \sim C_n |x|^n + C'_n |x|$, where the first term comes from the smooth parts of the velocity while the second comes from $O(x)$ probability having a shock in the interval x. The scaling exponents, $\xi_n = \partial \ln S_n / \partial \ln x$, thus behave as follows: $\xi_n = n$ for $0 \le n \le 1$ and $\xi_n = 1$ for $n > 1$. This means that the probability distribution of the velocity difference $P(\delta u, x)$ is not scale-invariant in the inertial interval, that is one cannot find such a that makes the function of the re-scaled velocity difference $\delta u / x^a$ scale-independent. The simple bi-modal nature of Burgers turbulence (shocks and smooth parts) means that the PDF is actually determined by two (non-universal) functions, each depending on a single argument: $P(\delta u, x) = \delta u^{-1} f_1(\delta u / x) + (x / L u_{\rm rms}) f_2(\delta u / u_{\rm rms})$. The breakdown of scale invariance means that the low-order moments decrease faster than the high-order ones as one goes to smaller scales. That means that the level of fluctuations increases with the resolution: the smaller the scale the more probable are large fluctuations. When the scaling exponents ξ_n do not lie on a straight line, this is called an anomalous scaling since it is related again to the symmetry (scale invariance) of the PDF broken by pumping and not restored even when $x/L \to 0$.

4.2 Multi-fractality of scalar and velocity statistics

Burgers bi-fractality is the consequence of only two possible flow configurations in 1d, smooth profile and shock. Generally, $S_n(r) \propto r^{\zeta_n}$ with the exponents lying on some smooth convex curve signalling multi-fractality. The meaning of the numbers ζ_n has been quite remarkably understood in terms of the geometric statistical conservation laws of Lagrangian evolution of n trajectories of the fluid particles.

Such conservation laws exist already for the usual diffusion. Consider particles with trajectories $\mathbf{R_i}(t)$ undergoing random walks. The squared distance between any two of them, $R_{ij}^2 = |\mathbf{R_i} - \mathbf{R_j}|^2$, grows on average as $\langle R_{ij}^2(t)\rangle = R_{ij}^2(0) + 2\kappa t$. However, the combinations (called martingales) like $f_2 = \langle R_{ij}^2(t) - R_{kl}^2(t)\rangle$ and $f_4 = \langle 2(d+2)R_{ij}^2 R_{kl}^2 - d(R_{ij}^4 + R_{kl}^4\rangle$ do not grow at all - all powers of t cancel. The joint probability $P(\mathbf{R_1}\dots\mathbf{R_n}) = P_n$ of n random walkers in d-dimensional space satisfies the nd-dimensional diffusion equation:

$$\partial_t P_n = \kappa \Delta P_n = \kappa \sum_{i=1}^{n} \nabla_i^2 P_n\,. \tag{22}$$

For the above two pairs, one can write the respective part of the Laplacian as follows: $\Delta = R^{1-2d}\partial_R R^{2d-1}\partial_R + \Delta_\theta$, where $R^2 = R_{ij}^2 + R_{kl}^2$ and Δ_θ is the angular Laplacian on $2d-1$-dimensional unit sphere. Introducing the angle, $\theta = \arcsin(R_{ij}/R)$, we see that the conservation of both $f_2 = \langle R^2\cos 2\theta\rangle$ and $f_4 = \langle R^4[(d+1)\cos^2 2\theta - 1]\rangle$ can be described as due to cancellation between the growth of the radial part (as powers of t) and the decay of the angular part (as inverse powers of t). In other words, the above two martingales are the harmonic polynomials, which are zero models of the respective Laplacian in the $2d$-dimensional space of $\mathbf{R}_{ij}$, $\mathbf{R}_{kl}$: the radial part is cancelled by an angular part, that is the growth of the distances between particles is compensated by the decay of angular correlations. For n particles, the polynomial that involves all distances is proportional to R^{2n} (i.e. $\zeta_n = n$) and the respective shape fluctuations decay as t^{-n}.

The scaling exponents of the martingales of the usual diffusion are integers proportional to n. One can model turbulent (Richardson) dispersion (21), by a diffusion with the scale-dependent diffusivity. Schematically, $R_{ij}^{2/3} \simeq \epsilon t$ can be written as $R_{ij}^2 = \kappa(R_{ij})t = R_{ij}^{4/3}\epsilon t$. We then replace (22) by

$$\partial_t P_n = \sum_{i,j} \kappa_{ab}(\mathbf{R_i} - \mathbf{R_j})\nabla_i^a \nabla_j^b P_n\,. \tag{23}$$

This so-called Kraichnan model presumes fluid velocities δ-correlated in time but preserve their power-law space correlations: $\kappa_{ab}(\lambda\mathbf{R_i}-\lambda\mathbf{R_j}) = \lambda^{4/3}\kappa_{ab}(\mathbf{R_i}-\mathbf{R_j})$. In this case, one can also build geometry-related martingales, but their respective exponents depend nonlinearly on n because the relative diffusion is now dependent on the distances to other particles. Moreover, how fast a polygone made out of particles forgets its shape depends also on the the velocity non-smoothness and space dimensionality.

The scaling exponents of the martingales made out of interparticle distances determine the structure functions of a scalar field $\theta(\mathbf{r},t)$ mixed by a turbulent flow. The scalar field is passive that is does not affect velocity. The transport is described by the equation

$$\frac{d\theta}{dt} = [\partial_t + (\mathbf{v}\cdot\nabla)]\theta = \varphi(\mathbf{r},t)\,. \tag{24}$$

Here the last term describes pumping, whose correlation function

$$\langle\varphi(\mathbf{r},t)\varphi(0,0)\rangle = \delta(t)\chi(r)$$

is assumed nonzero for $r < L$. The scalar field at every point is given by the integral of pumping integrated over the fluid trajectory that comes to this point: $\theta(\mathbf{r},t) = \int_{-\infty}^{t} dt'\varphi(\mathbf{R}(t'),t')$, where $\mathbf{R}(t) = \mathbf{r}$. The single-time two-point moment is then proportional to the time it takes for two partices to separate from r_{12} to L:

$$\langle\theta(\mathbf{r}_1)\theta_2)\rangle = \int dt'\chi\left(R_{12}(t')\right) \simeq \chi(0)t(r_{12}\to L)\,. \tag{25}$$

For a spatially smooth (Lipshits) velocity like one considers in dynamical chaos theory, $\delta v(R) \simeq \lambda R$, one has exponential separation of trajectories, $R(t) = R(0)e^{\lambda t}$, so that the time is logarithmic:

$$\langle\theta(\mathbf{r}_1)\theta_2)\rangle \simeq \frac{\chi(0)}{\lambda}\ln\frac{L}{r_{12}}\,. \tag{26}$$

For the velocity field from the energy cascade where one has Richardson diffusion, one substitutes the time from (21):

$$\begin{aligned}\langle\theta(\mathbf{r}_1)\theta_2)\rangle = \int dt'\chi\left(R_{12}(t')\right) \simeq \frac{\chi(0)}{\epsilon}\left(L^{2/3} - r^{2/3}\right)\,,\\ S_2(r_{12}) = \langle[\theta(\mathbf{r}_1) - \theta(\mathbf{r}_2)]^2\rangle \simeq \chi(0)r_{12}^{2/3}/\epsilon \propto r_{12}^{\sigma_2}\,.\end{aligned} \tag{27}$$

Note that $\chi(0)$ is the pumping rate of θ^2 so that (27) is the flux relation analogous to (17) and (18):

$$\frac{d}{dt}\left\langle(\theta_1-\theta_2)^2\right\rangle = \langle(v_1\cdot\nabla_1+v_2\cdot\nabla_2)\theta_1\theta_2\rangle = 4\chi(0), . \qquad (28)$$

Applying the same consideration, we see that the third structure function,

$$S_3 = \langle(\theta_1-\theta_2)^3\rangle = 3\langle(\theta_1^2\theta_2 - 3\theta_2^2\theta_1)\rangle \,,$$

is proportional to the time during which one can distinguish two triangles that started from two very different configurations with large aspect ratios: one with two particles near 1, another with two particles near 2. With time, triangles forget their inital shape evolving towards symmetrical configurations with aspect ratios of order unity. That time grows with r_{12} (as it takes longer to forget more elongated triangle) by the law $S_3(r_{12}) \propto r_{12}^{\sigma_3}$ can be inferred from the law of the decrease of the shape fluctuations of a triangle. For usual diffusion, geometrical shapes decay as inverse integer powers of time (equivalently, powers of R^2) as determined by the Laplacian. For turbulent diffusion, the powers σ_n are not equal to $\sigma_2 n/2$ but are determined by some convex function of n dependent on the velocity scaling and space dimensionality. This function was computed in the limiting cases of very rough (almost Brownian) velocity field and $d\to\infty$ (see Rev Mod Phys 73, 913, 2001). This shows that the probability distribution $P(\delta\theta, r)$ is not self-similar function of $\delta\theta/r^{1/3}$.

Let us see how the scaling of the Lagrangian conservation law could determine the scaling of the velocity structure functions. It involves not only the trajectories but also geometries of the velocity vectors. Let us consider two fluid trajectories $\mathbf{R_1}(t)$ and $\mathbf{R_2}(t)$. Denote the time-dependent distance between them $R(t) = |\mathbf{R_1}-\mathbf{R_2}|$ and the velocity difference $\delta\mathbf{v}\,(R(t))$. Let us assume that there exists a martingale, $\langle|\delta\mathbf{v}(R)|^2 g\big(R(t)/R(0)\big)\rangle$, built out of the squared velocity difference and some yet unknown function of the distance $R(t)$ normalized by $R(0)=r$. Assuming without loss of generality that $g(1)=1$, the initial value $\langle|\delta\mathbf{v}(r)|^2\rangle \propto r^{\zeta_2}$ is just the Eulerian moment whose scaling is given by ζ_2. Let us now consider the limit $t\to\infty$, when it is natural to assume that neither $R(t)$ nor $\delta v(R)$ remember their initial distance. Since the martingale must still be proportional to r^{ζ_2}, that requires that the function $g(x)$ is asymptotically a power law: $g(x)\propto x^{-\zeta_2}$. In other words, the asymptotic large-time Lagrangian conservation law for squared belocity difference, $\langle \big(\delta v(R)\big)^2 R^{-\zeta_2}\rangle$, determines the scaling exponent of the second structure function. The exponent ζ_2 quantifies how fast the directions of two velocity vectors decorrelate with the distance and thus determines which power of the distance to take for compensating the growth of the squared velocity difference with time. Let us stress that $g(x)$ is not a pure power law for all x: Similarly to (20), it is straightforward to compute the time derivatives at $t = 0$: $d\langle|\delta v|^2 R^b\rangle/dt = -4\epsilon r^{b-1}(1 + b/d)$, so that $g(x) \to x^{-d}$ at $x \to 1$ (see PRL 110, 214502, 2013). To conclude: while we can compute neither $g(x)$ nor its asymptotics exponent ζ_2 for a real turbulent flow, we can relate the existence of this Lagrangian conservation to the breakdown of scale invariance of the Eulerian single-time statistics.

Intermittency of turbulence. Let us appreciate the dramatic consequence of the breakdown of scale invariance in the energy cascades, expressed in $\zeta_n \neq n/3$. Both for Burgers and Navier-Stokes, the exponents $\zeta_n > n/3$ for $n < 3$ and $\zeta_n < n/3$ for $n > 3$. Writing the structure functions as

$$S_n = C_n(\epsilon r)^{n/3}\left(\frac{L}{r}\right)^{n/3-\zeta_n}, \qquad (29)$$

we see that lower moments go to zero when we increase the forcing scale keeping ϵ and r. This is true in particular for the second moment whose Fourier transform is the energy spectral density $E(k) = \int S_2(r)e^{ikr}dr$. If (as Kolmogorov and Obukhov initially assumed) we had $S_2 \simeq (\epsilon r)^{2/3}$, then we would have $E(k) = \epsilon^{2/3}k^{-5/3}$ - this 5/3 is probably the best known incorrect result of turbulence. Engineers still use 5/3 for estimates because the difference $\zeta_2 - 2/3 \simeq 0.02$ is small, but its positivity carries a remarkable message for a physicist: when the number of cascade steps L/r increases unbounded, one needs vanishingly small level of turbulent energy at a given flux. What carries the flux then? The moments with $n > 3$ grow unbounded with L/r, which means an increase of probability of very strong fluctuations. Most of the energy dissipation takes place on rare strong fluctuations. Not only turbulence does not forget about the pumping scale after many steps of the cascade, turbulence statistics changes with every step. Maybe then the law of change is scale invariant? Data support another Kolmogorov assumption: that the statistics of the ratios (called Kolmogorov multipliers) $\delta v(r)/\delta v(r/a)$ is scale invariant, that is independent of r for fixed a, but it has not been explicitly demonstrated.

The dimensionless constants C_n in (29) are determined by the pumping statistics. The existence of statistical conserved quantities breaks the scale invariance of statistics in the inertial interval and explains why turbulence in the inertial interval knows about pumping "more" than just the value of the flux. Note that both symmetries, one broken by pumping (scale invariance) and another by damping (time reversibility) are not restored even when $r/L \to 0$ and $\eta/r \to 0$.

4.3 Strongest fluctuation and instanton.

Could we also connect the incompressible velocity exponents ζ_n with some flow configurations like we did for Burgers turbulence where $\zeta_n = 1$ is equal to the spatial codimension of shocks (which are the most singular objects of compressible flows). Present-day experiments on incompressible turbulence measure ζ_n up to $n \simeq 10 - 12$. For transverse structure functions, when one take the difference of the velocity components perpendicular to the line connecting the points, $S_{n\perp} = \langle [\mathbf{v} \times \mathbf{r}/r]^n \rangle \propto r^{\zeta_{n\perp}}$, one starts to see signs of saturation, $\zeta_{n\perp} \to 2$ (see Phys. Rev. Fluids 5, 054605, 2020). If this is indeed true, that would mean that the most singular flow configurations in incompressible flows are lines, most likely the vortex lines. One may try to show this theoretically using the so-called instanton formalism where one considers $n \to \infty$ and uses the small parameter $1/n$ to justify the saddle-point approximation in the path integral that determines the respective moment (see Phys Rev E 54, 4896, 1996). This method works for any equation, $\partial_t + \mathcal{L}(u) = f$ driven by a random force f with the Gaussian probability distribution $P(f)$ defined by the variance $\langle f(0,0) f(\mathbf{r},t) \rangle = \delta(t) D(\mathbf{r})$. The averages are given by the path integral over different force histories:

$$\langle F\{u(0,0)\}\rangle = \int DuDf\, F(u)\delta\big(\partial_t u + \mathcal{L} - f\big)P(f) = \int DuDp\, F(u)e^{\imath I} ,$$

$$I = \imath \int dt d\mathbf{r}\, p(\mathbf{r},t)\big[\partial_t u + \mathcal{L}(u)\big] - \frac{1}{2}\int dt d\mathbf{r} d\mathbf{r}'\, D(\mathbf{r}-\mathbf{r}')p(\mathbf{r},t)p(\mathbf{r}',t). \quad (30)$$

We have presented the delta function as an integral over an extra field $\mathbf{p}$ and explicitly made Gaussian integration over the force.

For the Navier–Stokes equation, the most natural is to consider the high moment of the vorticity $F = \omega^n = |\nabla \times \mathbf{v}|^n$ as a continuous limit of transverse structure functions. The saddle-point approximation is based on the assumption that the main contribution into the large-n moment is made by some optimal fluctuation; it then reduces computing the path integral to finding this optimal flow history $u(\mathbf{r},t)$ and $p(\mathbf{r},t)$ by solving two nonlinear partial differential equations which follow from the action extremum:

$$\partial_t u + \mathcal{L}(u) = \int dt' d\mathbf{r}'\, D(\mathbf{r}-\mathbf{r}')p(\mathbf{r}',t')\,,$$
$$\partial_t p = \frac{\delta \mathcal{L}}{\delta u} p = \delta(t)\frac{\delta \ln F}{\delta u}\,. \qquad (31)$$

The equations are solved at $t<0$ since $p(t)\equiv 0$ for $t>0$ due to causality. The right-hand side of the second equation then provides the final condition on p, and one assumes that $u(\mathbf{r},t)\to 0$ at $t\to-\infty$. Such (so-called instanton) solutions were found for the field structure functions and spatial derivatives for the passive scalar and Burgers equation, for the vorticity moments in 2d direct cascade (see below). For the vorticity instanton giving $\langle \omega_z^n\rangle$, the second equation then has the right-hand side $\delta(t)n/\omega_z(0)$. The first equation of (31) for the z-component takes the form

$$\frac{d\omega_z}{dt} = \omega_z \frac{dv_z}{dz} + \int dt' d\mathbf{r}'\, D_{zk}(\mathbf{r}-\mathbf{r}')p_k(\mathbf{r}',t')\,. \qquad (32)$$

The challenge is to find the global finite-action flow configuration that starting from zero vorticity at the distant past first generates it by the last (force) term and then amplifies it by stretching (the first term in the rhs).

5 Two-dimensional turbulence

The two-dimensional incompressible inviscid flow described by the Euler equation $d\mathbf{v}/dt = -\nabla P/\rho$ is a special case. First, mathematicians proved that there is no finite-time singularity, so that a smooth flow stays smooth forever. Second, taking the curl we can write it for the vorticity $\omega = \nabla\times\mathbf{v}$:

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + (\mathbf{v}\nabla)\omega = \frac{\partial \omega}{\partial t} + \{\psi,\omega\} = 0\,. \qquad (33)$$

Here we introduced the streamfunction such that $\omega = \Delta\psi$. It is clear from (33) that ω is a Lagrangian invariant transported by the flow just like θ in the unforced version of (24). That means that the integral over space of any function of ω is conserved. Of particular importance for turbulence is the qudratic invariant (called enstrophy) whose spectral density is $\omega_k^2 = |\mathbf{k}\times\mathbf{v}_k|^2$.

Let us assume that an external force acts on some wavenumbers of order k and generates per unit time (per unit mass) the energy ϵ and the enstrophy $k^2\epsilon$. Let us assume that dissipation happens at $k_1 < k$ and $k_2 > k$ and denote the respective energy dissipation rates ϵ_1 and ϵ_2. The enstrophy dissipation rates are then $k_1^2\epsilon_1$ and $k_2^2\epsilon_2$. Let us show that such turbulence must contain two cascades. Indeed, the two conservation laws, $\epsilon_1+\epsilon_2 = \epsilon$ and $k_1^2\epsilon_1+k_2^2\epsilon_2 = k^2\epsilon$ give

$$\epsilon_1 = \epsilon\frac{k_2^2 - k^2}{k_2^2 - k_1^2}\,, \quad \epsilon_2 = \epsilon\frac{k^2 - k^1}{k_2^2 - k_1^2}\,.$$

The dissipation at large k_2 is usually provided by viscosity and at small k_1 by the bottom or wall friction. We see that when $k_2 \gg k > k_1$ (large Reynolds number), then $\epsilon_1 \approx \epsilon$, that is all the energy goes towards small k. This phenomenon is called an inverse cascade and it is quite counter-intuitive: one expects from a random turbulent flow fragmentation, not creation of large entitites out of a small-scale noise. Yet Nature consistently demonstrates such self-organization by inverse cascades both in creating large vortices and system-size coherent flows out of small-scale quasi-2d turbulence in oceans and atmospheres and in creating long waves out of storms on the ocean surfaces. It is the enstrophy (i.e. vorticity) which goes via a direct cascade to small scales: $k_2^2\epsilon_2 \approx k^2\epsilon$ when $k_1 \ll k < k_2$. One can guess the scaling of vorticity correlation function from the flux relation similar to (28):

$$\langle (v_1 \cdot \nabla_1 + v_2 \cdot \nabla_2)\omega_1\omega_2\rangle = P_2,\,. \tag{34}$$

Here P_2 is the pumping rate of squared vorticity. The power counting suggests that the velocity scales as the first power and vorticity as the zeroth power of the distance. Plausible arguments by analogy with the passive scalar relation (26) suggest

$$\langle \omega_1\omega_2\rangle \simeq \frac{P_2}{\lambda(r_{12})}\ln\frac{L}{r_{12}} \simeq [P\ln(L/r_{12})]^{2/3}\,. \tag{35}$$

Here we estimated the scale-dependent stretching rate as the vorticity coarse-grained on the scale: $\lambda(r) \simeq \omega_r \simeq [P\ln(L/r)]^{1/3}$. A consistent theory is still ahead of us, see Problem 1 in Section 8.

The 2d vorticity cascade is the only direct cascade where we do not find an anomalous scaling, probably because of its logarithmic nature. Let us now discuss the general features of an inverse cascades. As we coarse-grain over larger and larger scales in the inverse cascade, it is natural to expect that the properties of the pumping will be forgotten except the energy flux. One way to interpret this profound difference between direct and inverse cascades is to argue that fluid motions are slower when scales are larger (because the velocity is non-Lipshits, the turnover time $R/\delta v(R)$ grows with R). As an inverse cascade proceeds upscale, it has ample time to be effectively averaged over small-scale fluctuations including those of the pumping, whose only memory left is the flux value it generates. On the contrary, small-scale fast fluctuations in a direct cascade stay sensitive to the statistics of slow fluctuations at large scales; nonlinearity enhances variability down the cascade so that small-scale statistics is dominated by rare strong fluctuations. One can also explain the difference between direct and inverse cascades using the Lagrangian language. Correlation functions are accumulated along the Lagrangian trajectories. Inverse cascades are related to trajectories approaching each other back in time, then two-particle behavior effectively determines the evolution of multiparticle configurations and the second moment determines the scaling of higher moments. On the contrary, direct cascades correspond to trajectories separating back in time, one then relates the breakdown of scale invariance at vanishing viscosity to nonuniqueness of explosively separating trajectories in a nonsmooth velocity field; exponents of higher moments are then related to the laws of decay of the fluctuations of the shapes of multiparticle configurations. These laws depend on the number of particles so that an infinite number of forcing-related parameters is needed to predict the statistics at small scale.

Another perspective can be learned from (20), where for the 2D inverse energy cascade, there is no energy dissipative anomaly and the right-hand side in the inertial range is determined by the injection term $4\langle \mathbf{fv} \rangle = -4\epsilon$ so that the energy flux now must be considered negative (directed upscale). That means that the mean squared velocity difference has positive time derivative already at $t = 0$ and grows monotonically. On the contrary, in $3d$ the time derivative is negative at the beginning. Of course, any couple of Lagrangian trajectories eventually separates and their velocity difference increases. That shows nontrivial Lagrangian evolution in 3d energy cascade (possibly related to an anomalous scaling of the Lagrangian conservation laws): squared velocity difference between two trajectories generally behaves in a nonmonotonic way in 3d: the transverse contraction of a fluid element makes initially the difference between the two velocities decrease, while eventually the stretching along the trajectories takes over.

Another crucial difference between 2d and 3d follows from the equation for the separation

$$\frac{d\delta\mathbf{v}}{dt} = \frac{d^2\mathbf{R}}{dt^2} = \delta\mathbf{f} - \nabla\delta P/\rho\,. \tag{36}$$

The corresponding forcing term $\delta\mathbf{f} = \mathbf{f}(\mathbf{r}+\mathbf{R}) - \mathbf{f}(\mathbf{r})$ has completely different properties for an inverse energy cascade in 2D and for a direct energy cascade in 3D. For the direct cascade the large-scale forcing corresponds in the inertial range to $\delta\mathbf{f} \propto R$, which is negligible and even the scaling behavior comes from the pressure term accounting for interaction between infinitely many fluid trajectories. For 2d, R in the inertial range is much larger than the forcing correlation length. The forcing can therefore be considered short-correlated both in time and in space. That gives the diffusive growth of the squared velocity and cubic growth of the squared distance like in the Richardson law. Was the pressure term absent, one would get $R^2 = 4\epsilon t^3/3$. The experimental data give a smaller numerical factor (0.5 instead of 4/3) which is natural since the incompressibility constrains the motion. What is, however, important is that already the forcing term prescribes the law of separation consistent with the scaling of the energy cascade.

Despite all these hints, nobody has yet proved that the flow statistics must be scale-invariant in 2d inverse cascade, see Problem 2 in Section 8. Experiments indeed show scale-invariant statistics in the 2d inverse energy cascade. Moreover, the data suggest that some part of the statistics has even higher symmetry - conformal invariance (which can be thought of as local scale invariance — conformal transformations can expand here and compress there but go from here to there smoothly preserving the angles).

6 Conformal invariance in turbulence

It is likely that the existence of the third moments (fluxes) prevents the whole turbulence statistics to be ocnformally invariant, even though that was not properly demonstrated. However, there is a subset of turbulent statistics (present for different inverse cascades) which is empirically found to be conformally invariant. This subset is related to the isolines of some turbulent field. Let us first introduce the general notion of a conformally invariant random curve.

6.1 Schramm-Löwner Evolution (SLE)

Non-self-intersecting curve growing from the domain boundary can be described by a conformal map of the domain with the curve inside into a domain without the curve. For example, in the simplest case the curve $\gamma(t)$ starts at the real axis of the half-plane H. Here t parameterizes the curve, it should not be confused with the time in hydrodynamic equations The map $g_t : H\backslash\gamma(t) \to H$ is fixed by the asymptotics $g_t(z) \sim z + 2t/z + \mathrm{O}(1/z^2)$ at infinity. If the curve touches itself, one must define the domain $K(t)$ as the union of the curve and all points that cannot be reached from infinity and consider $g_t : H\backslash K(t) \to H$. The growing tip of the curve is mapped into a real point $\xi(t)$. Loewner found in 1923 that the conformal map $g_t(z)$ and the curve $\gamma(t)$ are fully parameterized by tip image $\xi(t)$ called the driving function. For that one needs to solve the remarkably simple Loewner equation $dg_t(z)/dt = 2[g_t(z) - \xi(t)]^{-1}$. Almost eighty years later, Schramm considered random curves in planar domains and showed (first, in a particular case) that the measure on the curves is conformal invariant if and only if $\xi(t) = \sqrt{\kappa}B_t$, where B_t is a standard one-dimensional Brownian walk. In addition, the measure $\mu_H(\gamma; z_1, z_2)$ on the curves γ connecting z_1 and z_2 is Markovian: if to divide γ into two pieces γ_1 from the boundary z_1 to z and γ_2 from z to z_2, then the conditional measure is as follows: $\mu_H(\gamma_2|\gamma_1; z_1, z_2) = \mu_{H\backslash\gamma_1}(\gamma_2; z, z_2)$. Diffusivity κ allows one to calssify the classes of conformal invariance random curves called SLE_κ. Such curves have been encountered in physics before as the boundaries of clusters of 2d critical phenomena described by conformal field theories. The language and formalism of SLE is a new natural communication tool for physicists and mathematicians.

Let us list here few basic facts about SLE curves. When $\kappa = 0$, γ is a vertical straight line. The larger the κ, the more curve wiggles. The curve is simple (i.e. with probability 1 does not touch nether itself nor real axis) when $0 \le \kappa < 4$. For SLE_κ with $4 \le \kappa < 8$, the curve touches itself but does not fill the space. In this case, one can define an external perimeter (as a part one can reach from infinity) which belongs to a dual class SLE_{κ_*} with $\kappa_* = 16/\kappa$. The fractal dimension of SLE_κ curves is $D_\kappa = 1 + \kappa/8$ for $\kappa < 8$.

Among the dual pairs, κ and κ_*, one is special from the viewpoint of locality. The curves from SLE_6 do not feel the boundary until they touch it (property called SLE locality).The dual curve $\mathrm{SLE}_{8/3}$ have the "restriction property": the statistics of the curves conditioned not to visit some region is the same as in the domain without this region. Intuitively, one can appreciate these properties by considering lattice (discrete) models which turn into the respective SLE in the continuous limit. For example, consider a honeycomb lattice. A random walk along the bonds starts from the boundary point that has all black hexagons to the left and white to the right and keeps that property as it moves turning right/left as it meets black/white hexagon. SLE_6 is obtained from the classical model of critical percolation when hexagons get their colors independently with the probability 1/2. $\mathrm{SLE}_{8/3}$ corresponds to a self-avoiding random walk when every bond is visited only once. Also the value $\kappa = 4$ is special because it is self-dual it corresponds to the so-called harmonic navigator. In this case, the probability of the color for the hexagon encountered is determined by the harmonic function defined in the domain with the boundary that includes the hexagons colored before; in other words, a new random walk starts from the hexagon and colors it by the color of the boundary the walk hits. Both SLE_6 and SLE_4 appear as isolines of Gaussian random fields. If one considers the surface of a random function of two variables, $a(x, y)$, as a landscape during a great flood then at some water level the probability to sail across is equal to probability to walk. At this level, the shoreline belongs to SLE_6 (critical percolation) if the correlation functions of $a(x, y)$ decay sufficiently fast. In particular, non-rigorous but plausible Harris criterium claims that if $\langle a(\mathbf{r})a(0)\rangle \sim r^{-2h}$ and $h < 3/4$, then isolines of the Gaussian field a are not equivalent to critical percolation i.e. do not belong to SLE_6. As far as SLE_4 is concerned, this class contain isolines of Gaussian (free) fields with $\langle a(\mathbf{r})a(0)\rangle \sim \ln r$. How all that is related to turbulence where the only thing we are sure about its being non-Gaussian (because the flux makes the third moment nonzero)?

6.2 Isolines in turbulence

Figure 4 shows a nodal line of vorticity obtained by a numerical solution of (**??**) with $m = 2$ on a torus (that is 2d Navier-Stokes equation with periodic boundary conditions and added external force and uniform friction). Force scale is $l_f = 2\pi/k_f = 0.05$. The curve looks fractal at the scales exceeding l_f, i.e. in the interval of an inverse cascade. Indeed, the length P grows nonlinearly with the end-to-end distance L. Power-law exponents of this grows for the curve and its external perimeter are found to be close within the resolution to the dimensionalities $7/4, 4/3$ of the dual pair SLE_6 and $\mathrm{SLE}_{8/3}$. Let us briefly describe how we identified possible curves from an SLE class and determined the driving function $\xi(t)$. We drew quite arbitrarily

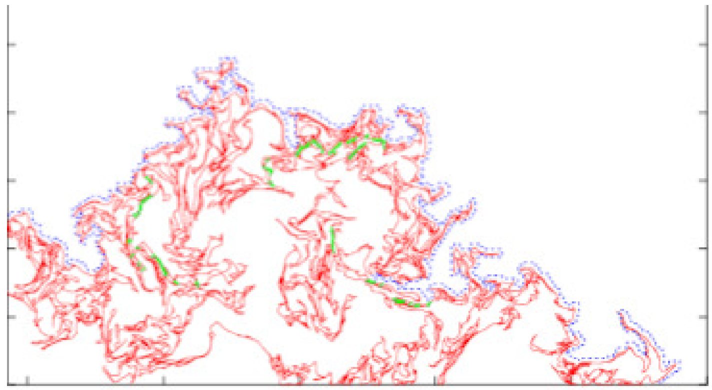

Figure 4: A portion of a candidate SLE trace obtained from the vorticity field.

a straight line to be a real axis and at the end checked that translations and rotations of the axis did not change the results. We then start from the intersection of a zero isoline and the axis and move along the curve or along the axis (when return to it) preserving orientation i.e. keeping positive vorticity always to the right. Such a procedure faithfully reproduces the statistics only in the local case, indeed we expected (and found!) $\kappa \approx 6$. We then divided our curve into small straight segments and approximated the family of conformal maps $g_t(z)$ by a discrete set of standard conformal maps absorbing one segment one by one). The resulting set of "times" t_i and values ξ_i defines the driving function $\xi(t)$. The only thing left is to run the Schramm test i.e. to check how well this function corresponds to a Brownian walk. The data presented by upward oriented triangles in Figure 5 show that the ensemble average $\langle \xi(t)^2 \rangle$ indeed grows linearly in time: the diffusion coefficient κ is very close to the value 6, with an accuracy of 5% (lower inset). The probability distribution functions of $\xi(t)/\sqrt{\kappa t}$ collapse onto a standard Gaussian distribution at all times t (upper inset). Therefore, we expect that in the limit of vanishingly small L_f the driving $\xi(t)$ tends to a true Brownian motion and zero-vorticity lines become SLE_κ traces with κ very close to 6. Note that the vorticity field has $h = 2/3 < 3/4$, that is the Harris criterium is violated. However, our field is non-Gaussian - while the probability distribution looks like Gaussian, the deviations are measurable including the third moment. Triangles pointing down on the lower are obtained for the isolines of a Gaussian field having the same Fourier

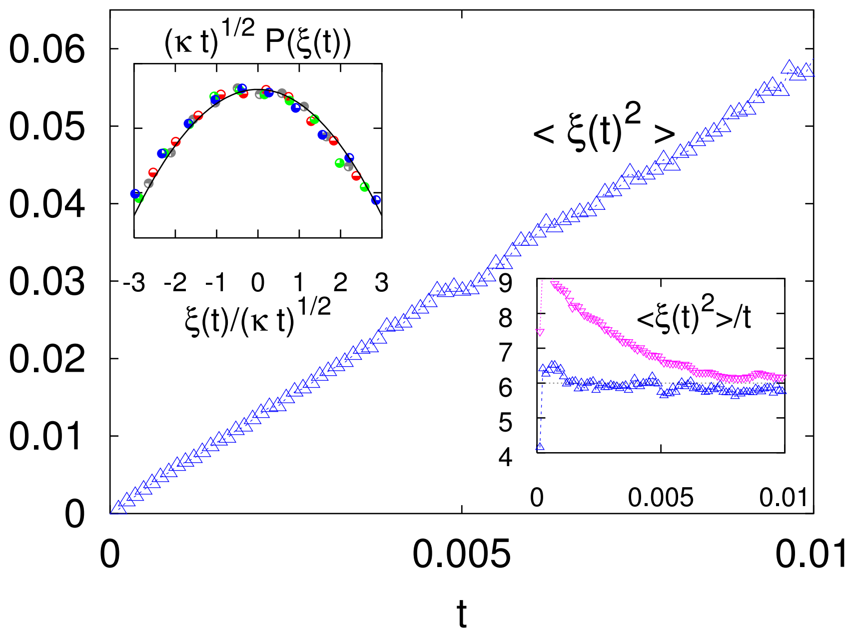

Figure 5: Demonstration of conformal invariance of the isolines of vorticity in the Euler equation. The driving function is an effective diffusion process with diffusion coefficient $\kappa = 6 \pm 0.3$. Right (lower) inset: triangles pointing up correspond to the vorticity, triangles pointing down to the Gaussian field with the same second moment. Left (upper) insets: the probability density function of the re-scaled driving function $\xi(t)/\sqrt{\kappa t}$ at four different times $t = 0.0012, 0.003, 0.006, 0.009$ (left) and $t = 0.02, 0.04, 0.08$ (right); the solid lines are the Gaussian distribution $g(x) = (2\pi)^{-1/2}\exp(-x^2/2)$.

spectrum as vorticity but randomized phases. Apparently, our accuracy is sufficient to make sure that it does not correspond to any SLE including SLE_6. Indeed, $E[\xi^2]/t \equiv \langle \xi^2 \rangle /t$ is not constant and approaches the limiting value $\kappa = 6$ only at the scales exceeding $2\pi/k_\alpha$ where the power-law correlation is already cut-off by friction and the field becomes truly uncorrelated.

6.3 Family of hydrodynamic models

The Euler equation is not unique in describing a 2d inverse cascade with an emergent conformal invariance. There exists the whole family of reality-based models, which describe the evolution of a scalar field θ transported by an incompressible two-dimensional velocity $\mathbf{u} = (\partial_y\psi, -\partial_x\psi)$, expressed via the stream function ψ. The scalar field θ is "active" because it is linearly related to ψ and $\mathbf{u}$. In Fourier space the relation reads: $\theta(\mathbf{k}) = |\mathbf{k}|^m\psi(\mathbf{k})$. The system is thus governed by the equation

$$\partial_t\theta + (\mathbf{u}\nabla)\theta = \partial_t\theta + \{\theta, \psi\} = F + D \;, \tag{37}$$

where $\{\theta, \psi\} = \theta_x\psi_y - \theta_y\psi_x$, F and D are external forcing and dissipation respectively. Different values of m give different well-known hydrodynamic equations. For $m = 2$ one obtains two-dimensional Navier-Stokes (NS) equation, θ being the vorticity. For $m = 1$ the field θ represents the temperature in the so-called Surface Quasi Geostrophic turbulence. Finally, for $m = -2$ the model corresponds to that derived by Charney and Oboukhov for waves in rotating fluids and by Hasegawa and Mima for drift waves in magnetized plasma in the limit of vanishing Rossby radius (ion Larmor radius for plasma physics).

At all values of m equation (37) possesses two positive-definite invariants for $F = D = 0$, namely $Z = \int \theta^2 d\mathbf{x}$ and $E = \int \theta\psi d\mathbf{x}/2$. When the system is forced by an external source of scalar fluctuations F, with a correlation length $\ell_f \sim 1/k_f$, the existence of two conserved quantities causes double turbulent cascade. The sign of m determines the direction of the cascades. For $m > 0$ the "energy" E is transferred toward large scales $\ell > \ell_f$ giving rise to an inverse cascade, and the "enstrophy" Z flows toward small scales. The cascades are reversed for $m < 0$.

For all the inverse cascades checked ($m = 2, 1, 1/2, -2$), one finds that the zero isolines belong to SLE with κ varying from 6 to 4 (arXiv:1012.3868, 2010).

7 Turbulence-flow interaction

What happens when an inverse energy cascade reaches the box size? It creates a system-size flow. Because this flow create system-size correlations, it is sometimes called condensate in analogy with the Bose-Einstein condensation. In a square box, such flow is a big vortex in the center (accompanied by four recirculating vortices in the corners); on a torus, the flow is the vortex dipole. One can derive analytically the radial profile of the vortex flow and the turbulence feeding it. Let us assume that polar and radial velocities are respectively $U(r) + u(\mathbf{r}, t)$ and $v(\mathbf{r}, t)$, that is u, v describe turbulence. Viscosity is irrelevant for the inverse cascade. The mean radial component of the Euler equation can be written as the continuity equation for the mean angular momentum

$$r\frac{\partial U}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}r^2\langle uv\rangle\,. \tag{38}$$

When the mean angular momentum flux $r\langle uv\rangle$ is nonzero, the flow is irreversible, i.e. the sign of $\langle uv\rangle$ does not change upon the transformation $t \to -t$ while the sign of U does. Opposite signs of U and $\langle uv\rangle$ imply that the momentum flows towards the vortex center (turbulence feeds the vortex).

Let us write now the energy balance assuming that small-scale forcing pumps ϵ:

$$\frac{1}{2}\frac{\partial}{\partial t}\left(U^2 + \langle u^2 + v^2\rangle\right) = \epsilon - \frac{1}{r}\frac{\partial}{\partial r}r\left[U\langle uv\rangle + \langle v(p + (u^2 + v^2)/2\rangle\right]\,. \tag{39}$$

In treating the vortex interior, we assume that the main contribution in the energy density is given by the mean flow and neglect $u^2 + v^2$ on the left and the last term on the right. We also neglect $\langle vp\rangle$ term assuming that it does not participate in the energy fow from turbulence to the vortex. That gives the energy balance,

$$U\frac{\partial U}{\partial t} = \epsilon - \frac{1}{r}\frac{\partial}{\partial r}rU\langle uv\rangle,, \tag{40}$$

which comprises a closed system together with (38). Solving it gives the r-independent mean vortex flow with the energy growing as $U^2/2 = 3\epsilon t$, that is three times faster than without turbulence feeding it. Note that the turbulent momentum flux is proportional not to the velocity gradient (which is zero) but to the angular momentum gradient. Turbulent viscosity notion has little value here.

Back to the pipe fow. Note that (38) is a direct analog of (12). Why don't we add the energy balance and describe at least the section of the channel or pipe flow where the flow dominate turbulence? Of course, we can do that writing for a statistically steady flow

$$\nu\frac{dU}{dz} + \langle uv\rangle = \alpha g(h - z)\,, \quad \frac{dU}{dz}\langle uv\rangle = -\epsilon(z)\,. \tag{41}$$

Unfortunately, here $\epsilon(z)$ is an unknown rate of the energy dissipation by an inhomogeneous turbulence inside the z-dependent mean flow. To find how the energy dissipation rate depends on the mean flow profile, one needs to develop a theory of the direct energy cascade inside a mean flow. This is the great problem of turbulence worthy of a Nobel prize.

8 The Seven Problems of Turbulence Theory

I list the problems according to my subjective feeling of ascending difficulty.

Direct (vorticity) cascade in 2d. There are plausible arguments based on the similarity to the passive scalar turbulence in an (almost) spatialy smooth flow, as sketched in (35). These arguments go back to Kraichnan (Phys Fluids 10, 1417, 1967) and were later developed by others. Yet nothing was really computed by a consistent theory.

1. Compute the vorticity correlation functions in 2d direct cascade and compare with the prediction (Phys Rev E 50, 3883, 1994):

$$\langle \omega^n(\mathbf{r_1})\omega^n(\mathbf{r_2})\rangle \simeq [P_2 \ln(L/r_{12})]^{2n/3}\,, \tag{42}$$

where P_2 is the input rate of the squared vorticity and L is the pumping scale. This formula may be true only or $n < \ln(L/r)$.

2. For very large n, consider the moments of the coarse-grained vorticity ω_r (by a saddle-point instanton formalism or other methods). For the instanton, the axial symmetry of the problem turns 2d nonlinearity term $(\mathbf{v}\nabla)\omega$ into zero, that is the stretching term in (32) is absent. There is neither stretching nor contraction for axially symmetric flows in 2d so that the force can pump the vorticity forever. The optimal flow realizations that determine the vorticity moments in 2d must have their axial symmetry broken. Since the stretching in the 2d vorticity cascade is exponential, it was argued that the angle-dependent part of the vorticity remains much smaller than the isotropic part during most of the (slow) evolution, which allowed to integrate over the angle-dependent degrees of freedom (in the Gaussian approximation) and obtain a renormalized action for the angle-averaged vorticity. That would be great to verify the prediction is that the asymptotic of the probability density function (PDF) is exponential: $\ln P(\omega_r) \propto -\omega_r/\overline{\omega_r}$ where the rms value is $\overline{\omega_r} \simeq (P_2 \ln(L/r)^{1/3}$, and that the whole PDF is self-similar that is depends on the single argument $\omega_r/\overline{\omega_r}$ (Phys Rev E83, 045301, 2011).

Inverse (energy) cascade in 2d. Here the first challenge is to establish self-similarity of the whole velocity statistics (or find an anomalous scaling as in 3d). Related or separate is the second challenge to prove conformal invariance of isolines; particularly puzzling is where in the hydrodynamic equations is encoded the central change of the conformal field theory which describes the isolines.

3. Establish self-similarity of the probability distribution of the distance $R(t)$ between two fluid trajectories at large times and large distances:

$$\lim_{t\to\infty, R\to\infty} P(R,t) = R^{-1} f(R^{2/3}/\epsilon t)\,. \tag{43}$$

One needs the large-time limit, since the initial stage of separation is expected to proceed by a ballistic law $R^2 = r^2 + S_2(r)t^2$. Using $\langle \delta |\mathbf{v}|^2/dt\rangle = 4|\epsilon|$ at $t = 0$, one can estimate the transition time as $\tau(r) = S_2(r)/4\epsilon \simeq r^{2/3}\epsilon^{-4/3}$. One way to proceed is to use (36). The difficulty is the pressure term, which either can be treated perturbatively or one can impose some inequilities on its contribution. I have a hunch that the function f may be close to exponential, that is the statistics of $R^{1/3}$ is close to Gaussian.

4. The velocity field is non-smooth (1/3-Holder continuous) so that all the arguments from Section 4.2 about relative diffusion dependent on the number of particles are valid. In particular, the statistics is not self-similar for the passive scalar advected by the velocity field from 2d inverse cascade. Yet, experimental and numerical data do not show any signs of an anomalous scaling of the velocity and vorticity fields. One way to prove that is to use the instanton formalism described in Section 4.3 to show that $S_n(r) \simeq (\epsilon r)^{n/3}$ for $n \to \infty$. The difference from the 3d case is that there is no vorticity stretching term, which must have this moment much smaller in 2d. Together with $S_3 = 3\epsilon r/2$ and convexity, that would mean normal scaling for all n. If self-similarity of the Eulerian velocity statistics is true, it must be related to the small-scale random force which decorrelates velocities of different fluid particles independently on each other. The normal scaling $S_n \simeq (\epsilon r)^{n/3}$ means, in particular, that the Lagrangian velocity PDF is also-self-similar in the long-time limit: $P(\delta v, t) = (\delta v)^{-1} g(\delta v/\sqrt{\epsilon t})$. It must be possible to prove that the velocity difference cannot grow slower than diffusively. Note that in 3d, the anomalous scaling means that the velocities of two fluid particles decorrelate faster than diffusively, and of more than three particles - slower than diffusively. Apart from velocity diffusion, another possible factor to exploit is two-dimenisonality which severely restricts possible laws of velocity decorrelation.

5. Consider the inverse cascades in the family of hydrodynamic equations from Section 6.3. Show that the zero-level isolines correspond to SLE and find the relation between κ and m.

6. Anomalous scaling in direct cascades. Derive the transversal velocity structure function for large order using the vorticity instanton.

7. Wall-bounded flow. Derive the profile of the mean flow $U(z)$ along the wall and the correlation functions of turbulence, starting from $\langle uv \rangle = \tau(z)$. Derive the resistance of the pipe at large Re.

to be continued...

Notations: $\mathbf{v}$ - velocity, $\mathbf{R}(t)$ - distance between two fluid particles, $\delta\mathbf{v}(R)$ - velocity difference between two fluid particles, h - river depth, pipe radius, body size, U - mean flow velocity, u - typical velocity or body velocity, w - one-dimensional velocity in the Burgers equation.