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New relations for correlation functions in Navier–Stokes turbulence

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We consider the steady-state statistics of turbulence in the inertial interval. The Kolmogorov flux relation (4/5-law) is shown to be a particular case of the general relation on the current–density correlation function. Using that, we derive an analogous flux relation for compressible turbulence and a new exact relation for incompressible turbulence.

1. Introduction

The Kolmogorov flux relation is a rare exact analytic result for the correlation functions of the velocity $\boldsymbol{v}$ in the Navier–Stokes equation. It is traditionally derived by considering quadratic invariants like kinetic energy in an incompressible flow (or squared vorticity in two dimensions). Conservation of the kinetic energy, $\int v^2/2$, by an unforced Euler equation means that one can define the energy flux in k space and write the continuity equation for the energy spectral density (see e.g. Frisch 1995):

$$\left.\begin{aligned} \Pi_k &= -\frac{1}{8\pi^2}\int d^3r\,\frac{\sin(\boldsymbol{k}\cdot\boldsymbol{r})}{r}\frac{\partial}{\partial r_i}\left[\frac{r_i}{r^2}\frac{\partial}{\partial r_j}\langle u_j u^2\rangle\right],\\ &\frac{\partial\langle|v_k|^2\rangle}{2\partial t}+\operatorname{div}\Pi_k=0,\ \ \boldsymbol{u}(\boldsymbol{r})\equiv\boldsymbol{v}(\boldsymbol{r})-\boldsymbol{v}(0). \end{aligned}\right\} \tag{1.1}$$

Stationarity under the action of a large-scale force and a small-scale viscosity means the constancy of the energy flux Π_k over intermediate wavenumbers (Kolmogorov 1941; Frisch 1995; Gawedzki 1999):

$$\nabla_i\langle u_i u^2\rangle = -4\nabla_i\langle v_i(\boldsymbol{r})v_j(\boldsymbol{r})v_j(0)\rangle = -4\bar{\epsilon}. \tag{1.2}$$

That can also be written for the third moment of the longitudinal velocity difference, $u_l=(\boldsymbol{u}\cdot\boldsymbol{r}/r)$:

$$\langle u_l^3\rangle = -12\bar{\epsilon}r/d(d+2). \tag{1.3}$$

Here, d is the space dimensionality. For $d=3$, one obtains $\langle u_l^3\rangle=-4\bar{\epsilon}r/5$, which is why the flux relation is sometimes called 4/5-law. Analogous relations are derived for the passive scalar turbulence, magnetized and helical flows (Yaglom 1949; Chandrasekhar 1951; Chkhetiani 1996; L'vov, Podivilov & Procaccia 1997; Politano & Pouquet 1998; Gomez, Politano & Pouquet 2000; Podesta, Forman & Smith 2007; Galtier 2008; Podesta 2008). If, however, the respective conservation law is non-quadratic (as the

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energy in compressible turbulence, for instance), then it is not clear that there is an analogue of (1.2). In addition, there are no known analogues of this relation for velocity moments of the orders n different from three. Experiments demonstrate that $\langle u_l^n \rangle \propto r^{\zeta_n}$ with the scaling exponents ζ_n, which are larger than one but smaller than $n/3$ for $n > 3$ (see e.g. Frisch 1995; Falkovich & Sreenivasan 2006).

In this letter, we reinterpret the Kolmogorov relation in terms of currents and densities of the conserved quantities, which allows us to derive an analogue for a compressible case and a new fundamental relation for an incompressible case.

2. General relation

Consider a general class of classical field dynamics,

$$\partial_t q^a + \nabla \cdot \boldsymbol{j}^a = f^a, \tag{2.1}$$

where $q^a, a = 1, \ldots, N$ are densities, $\boldsymbol{j}^a$ are currents and f^a are the external random source fields. From a microscopic point of view, these equations describe local conservation laws and provide a canonical form for the effective hydrodynamic description of the slow evolution at large scales. Since the zero wavenumber component of the density is conserved, low wavenumber components evolve slowly by continuity (see e.g. Forster 1975). The equations are closed via a constitutive relation that expresses currents in terms of the densities as a series in gradients:

$$j_i^a = F_i^a(\{\rho\}) + \sum_{jb} G_{i,jb}^a(\{\rho\}) \nabla_j \rho^b + \cdots, \tag{2.2}$$

where dots stand for higher-order terms involving more derivatives. The zeroth-order, reactive, term leads to a conservative dynamics while the first-order term describes dissipation. For fluid mechanics, the consideration of higher-order terms in (2.2) is usually unnecessary, so we limit ourselves to the following general class, which also contains the Navier–Stokes equation:

$$\partial_t q^a + \frac{\partial F_i^a}{\partial r_i} = f^a - \frac{\partial}{\partial r_i}\left(\sum_{jb} G_{i,jb}^a(\{\rho\}) \nabla_j \rho^b\right). \tag{2.3}$$

We assume the standard mathematical formulation of the problem of turbulence where the forcing term f^a is random and its statistics is stationary, spatially homogeneous and isotropic (Frisch 1995). This implies that the same properties hold for the steady-state statistics of q^a. The correlation length of the force will be denoted below by L.

The derivation of the Kolmogorov-type relation for (2.3) proceeds as follows. Consider the steady-state condition $\partial_t \langle q^a(0,t) q^a(\boldsymbol{r},t)\rangle = 0$ (no summation over a). Employing the dynamical equation (2.3) and using statistical symmetries, one finds

$$\begin{aligned}
0 &= \partial_t \langle q^a(0,t) q^a(\boldsymbol{r},t)\rangle \\
&= -2\frac{\partial}{\partial r_i}\langle q^a(0,t) F_i^a(\boldsymbol{r},t)\rangle + 2\langle q^a(0,t) f^a(\boldsymbol{r},t)\rangle \\
&\quad -\frac{\partial}{\partial r_i}\left\langle q^a(0,t)\left(\sum_{jb} G_{i,jb}^a(\{\rho\}(\boldsymbol{r},t))\nabla_j \rho^b(\boldsymbol{r},t)\right)\right\rangle.
\end{aligned}$$

We consider the limit of large correlation length L of the forcing, which allows one to consider r much smaller than L yet still large enough so that the last term becomes negligible as containing higher-order spatial derivatives. Because $r \ll L$, we have $f^a(\boldsymbol{r}, t) \approx f^a(0, t)$ and $\langle q^a(0, t) f^a(\boldsymbol{r}, t)\rangle \approx \langle q^a(0, t) f^a(0, t)\rangle \equiv \bar{\epsilon}_a$, where the constant $\bar{\epsilon}_a$ is the mean input rate of $(q^a)^2$. Hence we obtain for the single-time correlation function,

$$\nabla_i \langle q^a(0) F_i^a(\boldsymbol{r})\rangle = \bar{\epsilon}_a. \tag{2.4}$$

Assuming, in addition, isotropy one finds

$$\langle q^a(0) F_i^a(\boldsymbol{r})\rangle = \frac{\bar{\epsilon}_a r_i}{d}. \tag{2.5}$$

3. Particular cases

A simple example is the passive scalar turbulence where some substance with the density q and diffusivity κ is carried by a flow with the velocity $\boldsymbol{v}$. The current is $\boldsymbol{j} = q\boldsymbol{v} - \kappa \nabla q$. The flux (Yaglom) relation is as follows: $\langle q(0) q(\boldsymbol{r}) \boldsymbol{v}(\boldsymbol{r})\rangle = \bar{\epsilon} \boldsymbol{r}/d$.

Another example is the turbulence of a barotropic fluid where the pressure $p(\rho)$ is a function of the fluid density ρ only. In this case, $\boldsymbol{q} = (\rho \boldsymbol{v}, \rho)$, $F_j^i = \rho v_i v_j + p(\rho)\delta_{ij}$ and $\boldsymbol{j}^4 = \rho \boldsymbol{v}$. The equations have the form

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{3.1}$$

$$\partial_t(\rho v_i) + \partial_j(\rho v_i v_j + p\delta_{ij}) = -\partial_j \left[G^i_{j,kb}(\{\rho\}) \nabla_k \rho^b\right] + f^i, \tag{3.2}$$

where we took into account that $\boldsymbol{j}^4 = \rho \boldsymbol{v}$ is exact to all orders. The source often has the form $f^i = \rho \nabla_i \Phi$ that corresponds to an external potential Φ (the analysis below can be easily generalized to other types of forcing as well). The exact form of $G^i_{j,kb}$ is not important but its presence is necessary for a steady state. Indeed, the right-hand side of (3.2) breaks the energy conservation; the steady state holds due to the balance of the forcing that pumps fluctuations into the system and the dissipation that erases them. Now, we use the general relation (2.5) to derive the new relation for the compressible turbulence described by (3.1)–(3.2). The application of the relation to $q^4 = \rho$ with $\bar{\epsilon}_4 = 0$ gives $\langle \rho(0, t) \rho(\boldsymbol{r}) v_i(\boldsymbol{r})\rangle = 0$. In fact, this result holds for any relation between r and L. Indeed, the steady-state condition $\partial_t \langle \rho(0, t) \rho(\boldsymbol{r}, t)\rangle = 0 = \partial_i \langle \rho(0) \rho(\boldsymbol{r}) v_i(\boldsymbol{r})\rangle$ applied to the general (isotropic) form $\langle \rho(0) \rho(\boldsymbol{r}) v_i(\boldsymbol{r})\rangle = A(r) r_i$, gives $A = C r^{-d}$, so that regularity at the origin requires $C = 0$.

Application of (2.5) to q^j with $j = 1, 2, 3$ gives a non-trivial relation

$$\sum_j \left\langle \rho(0) v_j(0) \left[\rho(\boldsymbol{r}) v_j(\boldsymbol{r}) v_i(\boldsymbol{r}) + p(\boldsymbol{r})\delta_{ij}\right]\right\rangle = \frac{\bar{\epsilon} r_i}{d}, \tag{3.3}$$

where $\bar{\epsilon}$ is defined in this case as $\langle \rho(0) \boldsymbol{v}(0) \cdot \boldsymbol{f}(0)\rangle = \bar{\epsilon}$ (we summed over j to get a more symmetric result). To the best of our knowledge, the formula (3.3) is new and presents a desired analogue of Kolmogorov relation for the compressible turbulence (see e.g. Kritsuk *et al.* 2007). Probably, this relation was not derived before because it demands considering the steady-state condition for the fourth-order correlation function $\langle \rho(0, t) v_j(0, t) \rho(\boldsymbol{r}, t) v_j(\boldsymbol{r}, t)\rangle$, while usually in trying to find Kolmogorov-type relations one considers steady-state conditions for the second moment, like in magnetohydrodynamics by Politano & Pouquet (1998). Note, in passing, that one can choose q as the energy density and obtain yet another relation analogous to (4.3). Similar relations can be derived for other systems with fluxes of conserved quantities, particularly in wave turbulence (see e.g. Zakharov, L'vov & Falkovich 1992) and in stochastic coagulation (see Connaughton, Rajesh & Zaboronski 2008).

In the incompressible limit, $\rho = \text{const}$ and $\nabla \cdot \boldsymbol{v} = 0$, the pressure term is zero; again, since it is a divergence-free vector that must be regular at the origin (see Landau & Lifshits 1987). In this case, (3.3) is reduced to (1.2),

$$\langle v_i(\boldsymbol{r}) v_j(\boldsymbol{r}) v_j(0)\rangle = \bar{\epsilon} r_i/d\,, \tag{3.4}$$

and (1.3) is implied. Hence, (3.3) is indeed a general form of the Kolmogorov relation for an arbitrary Mach number. As we see, from a general viewpoint, the relation follows from the stationarity of the pair correlation function of the momentum density rather than from the energy spectral density. Indeed, $\bar{\epsilon}$ in (3.3) is the input rate of the squared momentum and not that of the energy, which coincide (up to the factor 1/2) only in the incompressible case.

4. New relation for incompressible turbulence

We have shown how the Kolmogorov relation exploits the momentum conservation. Now in the same way, we shall exploit the energy conservation and derive a new fundamental relation for incompressible turbulence. Energy conservation in the incompressible case means that the Navier–Stokes equation can be written as a continuity equation for the kinetic energy:

$$\frac{\partial}{\partial t}\frac{v^2}{2} = -\frac{\partial}{\partial r_i}\left[v_i\left(\frac{v^2}{2} + p\right)\right] + \boldsymbol{f}\cdot\boldsymbol{v} + \nu v_i \nabla^2 v_i. \tag{4.1}$$

That means one can choose $q = v^2$, which turns the general form (2.5) into a fifth-order relation. The only difference from the third-order relation (3.4) is that the pressure term is now non-zero. Note, first, that the condition $\partial\langle v^4\rangle/\partial t = 0$ gives the single-point pressure–velocity correlation function:

$$\langle v^2 v_i \nabla_i p\rangle = \langle \boldsymbol{f}\cdot\boldsymbol{v} v^2\rangle + \nu\left\langle v^2 v_i \nabla^2 v_i\right\rangle. \tag{4.2}$$

For the different-point moment, (2.5) gives

$$\frac{\partial}{\partial r_i}\left\langle v_i(\boldsymbol{r})\left[\frac{v^2(\boldsymbol{r})}{2} + p(\boldsymbol{r})\right] v^2(0)\right\rangle = \langle \boldsymbol{f}(\boldsymbol{r})\cdot\boldsymbol{v}(\boldsymbol{r}) v^2(0)\rangle + \nu\left\langle v^2(0) v_i(\boldsymbol{r})\nabla^2 v_i(\boldsymbol{r})\right\rangle. \tag{4.3}$$

This relation can be further simplified by decoupling small-scale and large-scale fields. Force and velocity are large-scale fields that change over the scale L so that one can put $\langle \boldsymbol{f}(\boldsymbol{r})\cdot\boldsymbol{v}(\boldsymbol{r}) v^2(0)\rangle = \langle \boldsymbol{f}\cdot\boldsymbol{v} v^2\rangle$ at $r \ll L$ as we did in deriving (2.5). Velocity differences and derivatives are small-scale fields that change respectively over the scale r and the viscous length η (see e.g. Frisch 1995). In particular, the local energy dissipation, $\epsilon \equiv -\nu v_i(\boldsymbol{r})\nabla^2 v_i(\boldsymbol{r})$, is a small-scale field with the correlation radius η. When the distance r is in the inertial interval, $r \gg \eta$, one can decouple large-scale and small-scale fields. Averaging over the scales larger than r and η but smaller than L, we obtain the Kolmogorov relation (1.2) in its local form (see e.g. Eyink 2003). It means that in averaging with v^2, one can replace $u_i u^2$ by $-4\epsilon r/d$ and express everything via the local energy dissipation rate. We denote $\boldsymbol{V} = \boldsymbol{v}(\boldsymbol{r}) + \boldsymbol{v}(0)$ and present

$$\begin{aligned} 4\left\langle v_i(\boldsymbol{r}) v^2(\boldsymbol{r}) v^2(0)\right\rangle &= -\left\langle u_i\left[v^2(\boldsymbol{r}) - v^2(0)\right]^2\right\rangle = -\left\langle u_i(\boldsymbol{u}\cdot\boldsymbol{V})^2\right\rangle \\ &\approx -\left\langle u_i u^2 V^2\right\rangle\left\langle \cos^2\alpha\right\rangle = -4\langle v^2 u_i u^2\rangle/d = 16\langle \epsilon v^2\rangle \boldsymbol{r}/d^2. \end{aligned} \tag{4.4}$$

Here, α is the angle between $\boldsymbol{u}$ and $\boldsymbol{V}$ and we substituted $\langle\cos^2\alpha\rangle = 1/d$. In the last equality, we used the Kolmogorov relation (1.2) in its local form (averaged over the scales intermediate between r and L). We thus find that in the inertial range, $\eta \ll r \ll L$, (4.3) is reduced to

$$\frac{\partial}{\partial r_i} \left\langle v_i(\boldsymbol{r}) p(\boldsymbol{r}) v^2(0) \right\rangle = \langle \boldsymbol{f} \cdot \boldsymbol{v} v^2 \rangle - (1 + 2/d) \langle \epsilon v^2 \rangle \equiv D. \tag{4.5}$$

We thus obtain the fundamental relation on the pressure–velocity correlation function, which is the counterpart of the Kolmogorov relation (1.3):

$$\left\langle \boldsymbol{v}(\boldsymbol{r}) p(\boldsymbol{r}) v^2(0) \right\rangle = D\boldsymbol{r}/d\,. \tag{4.6}$$

This relation can be tested experimentally and numerically. The input rate of the squared energy in (4.5) cannot be expressed via the energy flux for a general force statistics. In the particular case of a white Gaussian force, such expression is possible: $\langle f_i v_i v^2 \rangle = (1 + 2/d)\bar{\epsilon}\langle v^2 \rangle$, and D is proportional to the cumulant:

$$D = \frac{d+2}{d} \left(\bar{\epsilon}\langle v^2 \rangle - \langle \epsilon v^2 \rangle \right) = -\frac{d+2}{d} \langle\langle \epsilon v^2 \rangle\rangle. \tag{4.7}$$

Let us stress that for a Gaussian force, the single-point velocity statistics is non-Gaussian; its tails decay faster than that of Gaussian as shown by Falkovich & Lebedev (1997).

From a formal viewpoint, (3.3) and (4.5) are particular cases of the Hopf equations that express the stationarity of the correlation functions. Hopf equations are generally not closed; they impose some relations between different structure functions but do not allow to find them (see e.g. Yakhot 2001). On the contrary, our relations (3.3) and (4.5) allow one to find the correlation functions that are fluxes. Writing (4.5) as $\langle v^2(0) v_i(\boldsymbol{r}) \nabla_i p(\boldsymbol{r}) \rangle = D$ and comparing it with the single-point expression (4.2), we see that the difference is independent of the distance r when it is in the inertial interval:

$$\langle v^2(0)[v_i(0)\nabla_i p(0) - v_i(\boldsymbol{r})\nabla_i p(\boldsymbol{r})] \rangle = 2\langle \epsilon v^2 \rangle / d. \tag{4.8}$$

Inverting the incompressibility condition, $\Delta p = -\nabla_i v_j \nabla_j v_i$, one expresses pressure:

$$p(\boldsymbol{r}) = \frac{1}{(d-2)\sigma_d} \int \frac{\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^{d-2}} \frac{\partial^2 u_i(\boldsymbol{r}') u_j(\boldsymbol{r}')}{\partial r_i' \partial r_j'}, \quad \sigma_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \tag{4.9}$$

where $u_i(\boldsymbol{r}') = v_i(\boldsymbol{r}') - v_i(0)$. This allows one to present (4.6) as an integral relation on the fifth-order three-point moment of the velocity field (we denote $\hat{r}' = \boldsymbol{r}'/r'$):

$$\frac{1}{\sigma_d} \int \frac{\mathrm{d}\boldsymbol{r}'}{r'^d} \left\langle v^2(\boldsymbol{r}) \boldsymbol{v}(0) \left[u^2(\boldsymbol{r}') - d \left[\boldsymbol{u}(\boldsymbol{r}') \cdot \hat{r}' \right]^2 \right] \right\rangle = D\boldsymbol{r}/d. \tag{4.10}$$

We expressed it via the differences in the square brackets to make it explicit that the integrand is regular at the origin.

Note that the direct energy cascade, which is studied here, is absent for $d = 2$. For two-dimensional inverse energy cascade, one derives in a similar way: $\left\langle \boldsymbol{v}(\boldsymbol{r}) p(\boldsymbol{r}) v^2(0) \right\rangle = \epsilon_4 \boldsymbol{r}/2$, where ϵ_4 is the dissipation rate of the squared energy due to a large-scale sink. For the direct cascade of vorticity $\omega = \nabla \cdot \boldsymbol{v}$ in two-dimensional, analogous flux relations on the correlation functions $\langle \boldsymbol{v}(\boldsymbol{r}) \omega^n(\boldsymbol{r}) \omega^n(0) \rangle$ do not contain pressure and were derived and analysed by Falkovich & Lebedev (1994).

Let us discuss the validity limits of (4.4) where we neglected cumulants like the structure function $\langle\langle \boldsymbol{u} u^4 \rangle\rangle$. Such structure function scales as r^{ζ_5} and is subleading at sufficiently small r since $\zeta_5 > 1$. In other words, there exists a scale ℓ, below which the decoupling is possible. We ask now, how ℓ and ζ_5 may depend on the only parameter that enters Navier–Stokes equation, the space dimensionality d. For

$d = 3$, we expect $\ell \simeq L$. Usually, in statistical physics (Stanley 1968) and passive scalar theory (Kraichnan 1974; Chertkov *et al.* 1995; Falkovich, Kazakov & Lebedev 1998; Falkovich, Gawedzki & Vergassola 2001), the statistics are getting closer to Gaussian and decoupling improves when d grows. Somewhat counter-intuitively, one may expect a different behaviour in incompressible turbulence because of the diminishing role of pressure. To understand how the pressure depends on d, consider how the identity $\Delta p = -\nabla_i v_j \nabla_j v_i$ behaves when d increases while the velocity components are kept fixed. The Laplacian has d terms of different signs. Assuming that in the limit $d \to \infty$ those terms can be considered independent, their sum grows like $\sqrt{d}$. The right-hand side contains d^2 terms and grows like d so that the pressure is expected to grow as $p \propto \sqrt{d}$. This is slower than $v^2 \propto d$ so that assuming d large one may neglect the pressure contribution into the correlation functions. Consider, for instance, the tensor of the fourth moment of the velocity difference:

$$\begin{aligned}\langle u_i u_j u_k u_l \rangle &= A(r)\hat{r}_i\hat{r}_j\hat{r}_k\hat{r}_l + C(r)\left[\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right] \\ &\quad + B(r)\left[\delta_{ij}\hat{r}_k\hat{r}_l + \delta_{ik}\hat{r}_j\hat{r}_l + \delta_{il}\hat{r}_k\hat{r}_j + \delta_{jl}\hat{r}_k\hat{r}_i + \delta_{jk}\hat{r}_i\hat{r}_l + \delta_{kl}\hat{r}_i\hat{r}_j\right].\end{aligned}$$

At $d \to \infty$, the difference between longitudinal and transversal velocity components is expected to decrease, as follows from the analysis of the second moment where incompressibility and isotropy requires $\langle u_\perp^2 - u_l^2 \rangle = r\langle u_l^2 \rangle'/(d-1)$. Comparing $\langle u_l^4 \rangle = A + 6B + 3C$ with $\langle u_\perp^4 \rangle = 3C$, we see that $A, B \ll C$ at $d \to \infty$. Now, we generalize for d dimensions the relation between the velocity and pressure correlation functions, derived by Hill & Wilczak (1995) for three dimensions and remarkably find out that C-terms cancel out:

$$\langle p^2 \rangle = \frac{d^2-1}{12}\int_0^\infty A(r')\frac{\mathrm{d}r'}{r'} - \frac{d-1}{3}\int_0^\infty \frac{\mathrm{d}r'}{r'}\left[A(r') + 3B(r')\right]. \tag{4.11}$$

It means that indeed $\langle p^2 \rangle$ grows slower than $\langle u^4 \rangle$ as $d \to \infty$. Note that generally the correlation functions containing p and u may scale differently, but considering the limit $d \to \infty$ *at fixed* r, one may neglect the pressure term in (4.3). That would give

$$\nabla_i \left\langle v_i(\boldsymbol{r}) v^2(\boldsymbol{r}) v^2(0) \right\rangle = 2\langle \boldsymbol{f} \cdot \boldsymbol{v} v^2 \rangle - 2\langle \epsilon v^2 \rangle. \tag{4.12}$$

It is much different from (4.4), that is at $d \to \infty$, there may exist an interval of scales, $L \gg r \gg \ell$, where cumulants are comparable to the reducible part so that ℓ is a crossover scale between (4.5) and (4.12). It would mean that ℓ decreases with d while $\zeta_5 \to 1$ as $d \to \infty$. Moreover, at $d \to \infty$, the same analysis can be done for all odd moments: $\zeta_n = 1$ for $n \geqslant 1$. We thus come to the conclusion that the scaling of incompressible turbulence in the limit $d \to \infty$ may be the same as the scaling of Burgers turbulence. This would not be surprising since a single incompressibility condition imposed on d velocity components is expected to be less restrictive as d grows, so that flow configurations close to shocks give the main contribution to the moments. Technically, if one assumes incompressibility, yet considers the pressure terms in the equations for $\partial v^{2n}/\partial t$ small, then $\int v^{2n}\,\mathrm{d}\boldsymbol{r}$ are integrals of motion of the Euler equation for all n, so that the linear scaling of all odd velocity moments expresses the constancy of fluxes of these integrals of motion, similar to Burgers in scaling but with different factors (see e.g. Cardy, Falkovich & Gawedzki 2008). Holder inequality then requires the linear scaling for even moments too, which corresponds to an extreme non-Gaussianity of the small-scale velocity statistics. Physically, pressure is a non-local field that couples different regions in space and is expected to act like 'an intermittency killer' as remarked by Kraichnan (1991) (see also Nelkin 1975;

Gotoh & Nakano 2003). One way to find the exponents at finite d may be a large-d expansion, which must thus be very different from that used by Stanley (1968), Chertkov *et al.* (1995) and Falkovich *et al.* (1998): one needs to start here from a Burgers-like limit rather than from a Gaussian statistics. Large-d limit was considered by Fournier, Frisch & Rose (1978) for the non-steady turbulence with an initial data close to Gaussian; it was found that the pressure terms are substantial in driving the statistics away from Gaussian. This does not contradict our statement that if the pressure terms are negligible then the steady-state statistics must be very far from Gaussian.

Needless to say, the tendency of the exponents to approach unity with d growing remains purely hypothetical at that level of analysis. However, if true, it means that the degree of non-Gaussianity of the statistics of a single velocity component grows with d, which agrees with the numerical comparison between three and four dimensions made in the remarkable work by Gotoh *et al.* (2007). Scalar quantities made out of vector products contain the sum of d terms; one cannot generally conclude whether their statistics gets closer to Gaussian as d increases because of the competition between increasing non-Gaussianity of a single term and the averaging over d terms.

To conclude, (3.3), (4.5) and (4.6) are the main results of this work.

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