Physica 10D (1984) 379–386
North-Holland, Amsterdam

ON THE STABILITY OF NONLINEAR WAVES IN INTEGRABLE MODELS

E.A. KUZNETSOV, M.D. SPECTOR and G.E. FAL'KOVICH
Institute of Automation and Electrometry, Sib. Br. USSR Ac. Sci., Novosibirsk 90, 630090, USSR

Received 16 October 1982

A new method of stability investigation is presented for solutions of nonlinear equations integrable with the help of the inverse scattering transform (IST). The stability problem for periodic nonlinear waves in weakly dispersive media is solved with respect to transverse perturbations. It is shown that for positive dispersion media one-dimensional waves are unstable, and for negative dispersion such waves are stable.

1. Introduction

Recently, the progress in the investigation of nonlinear waves dynamics has been associated to considerable extent with the development of a new method of mathematical physics, the inverse scattering transform (see, e.g. [1]). This approach reduces the Cauchy problem for a given nonlinear equation to a set of linear problems. Constructively, however, this method gives an answer to the question on the asymptotic behaviour for the given nonlinear system. In particular, with the help of such a procedure the stability problem has been proved for the solitons, which are the most important solutions from the physical point of view.

On the other hand it is more or less evident that the IST method should provide considerable advantages in the case of direct study of stability of an arbitrary solution.

In the present paper we show that if the nonlinear equation $u_t = s(u)$ admits the Zakharov–Shabat representation [2]

$$\frac{\partial L}{\partial t} - \beta \frac{\partial A}{\partial y} + [L, A] = 0\,, \tag{1}$$

where L and A are matrix differential operators depending on $u(x, t)$ and its derivatives, then the problem of stability with respect to small perturbations reduces to solution of two joint equations for the matrix function $F(x, z, y, t)$,

$$\begin{aligned} &\beta \frac{\partial F}{\partial y} - \vec{L}(x)F + F\overleftarrow{L}(z) = 0\,, \\ &\frac{\partial F}{\partial t} - \vec{A}(x)F + F\overleftarrow{A}(z) = 0. \end{aligned} \tag{2}$$

Here perturbation $\delta u(x, y, t)$ can be expressed in terms of $F(x, z)$ and its derivatives on the characteristic $x = z$ and the arrow shows the direction of differentiation.

The system of equations (2) posseses an important feature. In fact, eqs. (2) admit separation of variables, and as a result the resulting spectral problems are of a lower order than the initial linearized equation. It is just the effective reduction of the order of differential operator that yields the advantage of the IST method application to the stability problem.

We apply this method to the stability problem for stationary periodic waves in a weakly dispersive medium described by the Kakomtsev–Petviashvili (KP) equation. Until now the stability problem of such waves has been examined in the limits of small wave amplitudes when the waves are supposed to be sinusoidal. In particular, it has been shown for the positive wave dispersion that small-amplitude waves are unstable with respect to decay into two other waves [3]. For negative dispersion waves (ion-sound waves in plasma, long surface gravitational waves) decay processes are forbidden and so instability connected with the higher order process should be expected.

However, as it was shown recently in [4] within the Kadomtsev–Petviashvili approximation, the matrix element for four-wave interactions turns into zero on the resonant surface. Thus, the solution of the stability problem is not evident at all, even in the limit of small amplitudes. In this paper we show that the stationary waves with arbitrary amplitudes in the case of negative dispersion are stable with respect to transverse perturbations. For the positive wave dispersion we obtain an exact solution of the stability problem from which in both limits small amplitude waves and solitons—well-known expressions for increments—follow [3, 5].

The order of this paper is the following. The first section is devoted to the consideration of our method. The cnoidal wave stability is considered in the next sections.

2. Operator dressing method and stability problem

Let us consider the stability problem relative to small perturbations for integrable nonlinear equations for which representation (1) is valid.

We show that the operator dressing method developed by Zakharov and Shabat [2] and the stability problem are closely connected. It should be noted that in fact the indication to such connection is contained in [6].

Let L_0 and A_0 be matrix differential operators,

$$L_0(x) = \sum_{k=0}^{n} u_k \frac{\partial^{n-k}}{\partial x^{n-k}}, \quad A_0(x) = \sum_{k=0}^{m} w_k \frac{\partial^{m-k}}{\partial x^{m-k}},$$

corresponding to some solution $u(x, t)$ of the given nonlinear equation $u_t = s(u)$. Let us apply the dressing procedure to the operators $\beta(\partial/\partial y) - L_0$ and $\partial/\partial t - A_0$. For this purpose we consider according to [2] the reversible integral operator

$$\hat{F}\psi = \int_{-\infty}^{\infty} F(x, z, y, t)\psi(z)\,\mathrm{d}z$$

which commutates with $\beta(\partial/\partial y) - L_0$ and $\partial/\partial t - A_0$:

$$\beta\frac{\partial F}{\partial y} - \vec{L}_0(x)F + F\overleftarrow{L}_0(z) = 0\,,$$

$$\frac{\partial F}{\partial t} - \vec{A}_0(x)F + F\overleftarrow{A}_0(z) = 0$$

and is expressed through two Volterra operators $K^{\pm}$,

$$1 + F = (1 + K^+)(1 + K^-)\,. \tag{3}$$

Then the kernel $K(x, z, y, t)$ of the integral operator K^+ obeys two ajoint equations

$$\begin{gathered} \beta\frac{\partial K}{\partial y} - \vec{L}(x)K + K\overleftarrow{L}_0(z) = 0\,, \\ \frac{\partial K}{\partial t} - \vec{A}(x)K + K\overleftarrow{A}_0(z) = 0\,, \end{gathered} \tag{4}$$

where the dressed operators L and A have the same structures as L_0 and A_0,

$$L(x) = \sum_{k=0}^{n} \tilde{u}_k \frac{\partial^{n-k}}{\partial x^{n-k}}, \quad A(x) = \sum_{k=0}^{m} \tilde{w}_k \frac{\partial^{m-k}}{\partial x^{m-k}}:$$

Here functions $\tilde{u}_k$ and $\tilde{w}_k$ are defined from the recurrent formulae (see [2])

$$\delta u_0 = \tilde{u}_0 - u_0 = 0\,,$$

$$\delta u_1 = \tilde{u}_1 - u_1 = [u_0, \xi_0]\,,$$

$$\delta u_2 = \tilde{u}_2 - u_2$$
$$= (n-1)u_0 \frac{d\xi_0}{dx} + \frac{1}{2}\left\{\frac{d\xi_0}{dx}, u_0\right\} + \frac{1}{2}[u_0, \xi_1] + [u_1, \xi_0] + \xi_0 \frac{du_0}{dx} \dots, \tag{5}$$

where

$$\xi_i = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right)^i K(x,z)|_{x=z}.$$

Analogous formulae take place for δw_k. As result of the dressing procedure for the functions $\tilde{u}_i$ we obtain the same equations as the previous ones. So the equation for δu_i describes propagation of finite perturbations on the background of the wave u (see, [6]). For the study of stability relative to small perturbations in accordance with (3) and (5) it is sufficient to take K with a norm much less than 1, i.e. to put

$$K(x,z) = -F(x,z).$$

Thus to solve the stability problem with respect to small perturbations one needs to seek a joint solution of system (2) for the kernel F while the perturbation δu is constructed from F with the aid of formulae (5). It should be noted also that in contrast to (3) system (2) is local so there are no restrictions of F except for the requirement to δu being bounded for all x.

New let us consider the stability problem when nonlinear equations are given in the form of a commutator of two operator bundles L and A:

$$\frac{\partial L}{\partial t} - \frac{\partial A}{\partial x} + [L, A] = 0. \tag{6}$$

The bundles L and A are assumed to be the rational functions of spectral parameter λ, for example,

$$L = \sum_{k=0}^{n} u_k \lambda^{n-k} + \sum_{i=1}^{N}\sum_{k=1}^{n_i} \frac{P_{ik}}{(\lambda-\lambda_i)^k}.$$

As is known [7, 8] in this case the dressing procedure can be developed as well. For this purpose consider the matrix function $\psi_0(x,t)$ defined from equations

$$\psi_{0x} = L_0\psi_0 - \psi_0 L_0,$$
$$\psi_{0t} = A_0\psi_0 - \psi_0 A_0, \tag{7}$$

being joint to (6) where L_0 and A_0 are designated as the values of L and A on some solution $u(x,t)$ of the nonlinear equation (6). According to [7, 8] to dress bundles L_0 and A_0 let us introduce a new function ψ connected with ψ_0 by the singular integral equation

$$\psi(\lambda) = \psi_0(\lambda) + \int_\Gamma \frac{\psi(\lambda')\,d\lambda'}{\lambda - \lambda' + i0}\psi_0(\lambda), \tag{8}$$

where contour Γ should be chosen from the solvability condition of (8). Then due to (7) the function ψ is the solution of two joint equations:

$$\psi_x = L\psi - \psi L_0,$$
$$\psi_t = A\psi - \psi A_0.$$

Here the operators L and A are the results of operator dressing of L_0 and A_0 and have the same structure as the initial operators, for example,

$$L = \sum_{k=0}^{n} \tilde{u}_k \lambda^{n-k} + \sum_{i=1}^{N}\sum_{k=0}^{k_i} \frac{\tilde{P}_{ik}}{(\lambda-\lambda_i)^k},$$

where (see [8])

$$\delta u_0 = 0,$$
$$\delta u_1 = -[J_0, u_0],$$
$$\delta u_2 = J_0 u_1 - \tilde{u}_1 J_0 - [u_0, J_1],$$
$$\dots,$$
$$\delta P_{ik_i} = [1 - I_{1i}]P_{ik_i}(1 - I_{1i})^{-1} - P_{ik_i},$$
$$\delta P_{ik_i-1} = (1 - I_{1i})P_{ik_i-1}(1 - I_{1i})^{-1} + (\tilde{P}_{ik_i} I_{2i} - I_{2i} P_{ik_i})(1 - I_{1i})^{-1} - P_{ik_i-1}, \dots .$$

Here

$$J_n = \int_\Gamma \lambda^n \psi(\lambda)\,\mathrm{d}\lambda, \quad I_{ni} = \int_\Gamma \frac{\psi(\lambda)\,\mathrm{d}\lambda}{(\lambda - \lambda_i)^n}.$$

Thus, as a result of the dressing procedure we obtain a new solution of the same equations (6). So while linearizing after solution $u(x, t)$ we have to consider the functions ψ and ψ_0 as small values. According to (6) one should put

$$\psi = \psi_0$$

and keep only the linear terms in the expressions for perturbations δu:

$$\begin{aligned}
&\delta u_1 = -[J_0, u_0],\\
&\delta u_2 = [J_0, u_1] - [u_0, J_1],\\
&\ldots,\\
&\delta P_{in_i} = [P_{in_i}, I_{1i}],\\
&\delta P_{in_i-1} = [P_{in_i-1}, I_{1i}] + [P_{in_i}, I_{2i}],
\end{aligned} \tag{9}$$

and so on.

Thus, as formerly, the stability problem reduces to a solution of the joint linear equations (7). In this case the perturbations are given with the help of formulae (9).

3. Stability of cnoidal waves in weak dispersion media

Let us apply our method to stability investigation of cnoidal waves in weak dispersion media relative to small non-one-dimensional perturbations. Allowance for the weak transverse modulation of such waves and for the weak nonlinearity leads to the Kadomtsev–Petviashvili equation [9]

$$\frac{\partial}{\partial x}(u_t + 6uu_x + u_{xxx}) = -3\beta^2 \frac{\partial^2 u}{\partial y^2},$$

which generalizes a well-known RDV equation over a two-dimensional case. In this equation the condition $\beta^2 \gtrless 0$ corresponds to negative or positive dispersion of sound waves.

It is well known that IST in formulation (1) is valid for this equation:

$$\begin{aligned}
L &= -\frac{\partial^2}{\partial x^2} - u_0(x, y, t),\\
A &= -4\frac{\partial^3}{\partial x^3} - 6u_0\frac{\partial}{\partial x} - 3u_{0x} + 3\beta w_0,
\end{aligned} \tag{10}$$

where

$$w_x = u_y.$$

Thus, in stability investigation we can use the method outlined in section 2, accordingly the function F being determined from equations

$$\begin{aligned}
&\left[\beta\frac{\partial}{\partial y} - L_0(x) + L_0^\dagger(z)\right]F(x, z, y, t) = 0,\\
&\left[\frac{\partial}{\partial t} - A_0(x) + A_0^\dagger(z)\right]F(x, z, y, t) = 0
\end{aligned} \tag{11}$$

(here the dagger denotes a conjugative operator) and the perturbation δu is given by formula

$$\delta u(x, y, t) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)F(x, z, y, t)\Big|_{x=z}. \tag{12}$$

The system of equations (11), (12) is equivalent to the linearized KP equation that in particular can be checked by direct calculations. For that is is necessary to apply the operator $\partial/\partial x + \partial/\partial z$ to the first equation of system (11) and the operator $\partial^2/\partial x^2 - \partial^2/\partial z^2$ to the second one and then consider the results on characteristic $x = z$.

Let us choose u_0 in the form of a cnoidal wave

$$u_0(x - Vt) = -2\wp(x + \mathrm{i}\omega' - Vt) + V/6,$$

where $\wp(z)$ is the Weierstrass elliptic function with the periods 2ω and $2\mathrm{i}\omega'$.

Proceed now to the solution of equation system (11). These equations admit the separation of variables, so the partial solution of (11) has the following form:

$$F(x, z, y, t) = c(t)\, e^{iky} \psi(x) \varphi(z),$$

where $\psi(x)$ and $\varphi(z)$ are determined from the Lamé equations [9]:

$$\begin{aligned} &\left[\frac{\partial^2}{\partial x^2} - 2\wp(x + i\omega') + i\beta k\right]\psi(x) = -E\psi(x), \\ &\left[\frac{\partial^2}{\partial z^2} - 2\wp(z + i\omega')\right]\varphi(z) = -E\varphi(z). \end{aligned} \tag{14}$$

(Here we put without any restrictions the velocity V to be equal to zero.)

The solution of equations (14) can be expressed through the Weierstrass functions $\zeta(x)$ and $\sigma(x)$,

$$\begin{aligned} \psi(x) &= e^{-\zeta(a)x} \frac{\sigma(x + i\omega' + a)}{\sigma(x + i\omega')}, \\ \varphi(z) &= e^{\zeta(b)z} \frac{\sigma(z + i\omega' - b)}{\sigma(z + i\omega')}, \end{aligned} \tag{15}$$

where the parameters a and b give the "energy" of the Schrödinger equations:

$$E = -\wp(a) + i\beta k = -\wp(b)$$

or, equivalently,

$$\wp(a) - \wp(b) = i\beta k.$$

The dependence $c(t)$ is found after substitution (23) into the second equation (11),

$$c(t) = c(0) \exp\{-2[\wp'(a) - \wp'(b)]t\}.$$

The perturbation δu is defined from (12), (13),

$$\begin{aligned} \delta u(x, t) &= c(t)\, e^{iky} \frac{d}{dx} [\psi(x)\psi(x)] \\ &= c(t)\, e^{iky}\, e^{ip(a,b)x} \chi(x) \end{aligned}$$

and has the Bloch form with quasi-momentum

$$p(a, b) = p(a) - p(b),$$

where $p(a) = i[\xi(a) - \zeta(\omega)a/\omega]$ is a quasi-momentum of the Schrödinger equations (14). In the fundamental rectangle with sides ω and $i\omega'$ the function $p(a)$ has real values on two segments $(\omega, \omega + i\omega')$ and $(i\omega', 0)$ which correspond to two gaps. On the other two segments $(0, \omega)$ and $(i\omega', \omega + i\omega')p(a)$ is a purely imaginary function.

On account of perturbations δu being bounded for all x the quasi-momentum should be restricted by the following natural condition

$$\operatorname{Im} p(a, b) = 0.$$

Thus the solution of the stability problem for cnoidal waves reduces to the analysis of the algebraic expression

$$\Gamma(a, b) = -2[\wp'(a) - \wp'(b)], \tag{16}$$

with two additional conditions

$$\wp(a) - \wp(b) = i\beta k, \tag{17}$$

and

$$\operatorname{Re}\left[\zeta(a) - \zeta(b) - \frac{\zeta(\omega)}{\omega}(a - b)\right] = 0. \tag{18}$$

4. Analysis of the dispersion equation

Conditions (17), (18) impose the restrictions on two complex parameters a and b, these conditions yield in the a-(b-) plane the curve on which the increment Γ is determined. Due to periodic dependence of the functions of (17), (18) one need seek the curve only in the fundamental period rectangle with sides 2ω and $2i\omega'$.

First, let us consider the case of negative dispersion medium $\operatorname{Im}\beta = 0$. Notice that the elliptic function $\wp(a)$ in the fundamental period rectangle takes up to same value at two symmetric points a and $-a$. It means that eq. (17) relative to parameter a has only two solutions. Besides we should remind the following properties of the elliptic Weierstrass functions:

$$\wp'^*(a^*) = \wp'(a) = -\wp'(-a),$$

$$\wp^*(a^*) = \wp(a) = \wp(-a),$$

$$\zeta^*(a^*) = \zeta(a) = -\zeta(-a).$$

Then it is evident that eq. (18) is solvable for $a = b^*$. This solution satisfies also eq. (17),

$$2\,\mathrm{Im}\,\wp(a) = \beta k.$$

This relationship gives the curve in the plane of parameter a. The increment in this case appears to be purely imaginary,

$$\Gamma = -2\mathrm{i}\,\mathrm{Im}\,\wp'(a).$$

It should be added that due to enumerated properties of elliptic functions the given solution is unique.

Thus, the periodic stationary waves in media with negative dispersion turn out to be stable against transverse perturbations.

Another situation takes place for waves with positive dispersion. Here the perturbations being neutrally stable in the one-dimensional case arise at the presence of transverse modulations. If $k = 0$ the perturbations are neutrally stable for $a = b$. For this case $\Gamma(a, b) = 0$ and the perturbation

$$\delta u(x) \approx \frac{\mathrm{d}}{\mathrm{d}x}[\psi_a(x)\psi_{-a}(x)]$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\wp(x + \mathrm{i}\omega') = -\frac{1}{2}\frac{\partial u_0}{\partial x}$$

corresponds to a small shift.

Supposing $\beta k \ll 1$ let us find the boundary of gap defined from the condition $p(a, b) = 0$. It is clear that in this case an appropriate point b is close to a: $b = a + \epsilon$. So expanding (16)–(18) on ϵ we obtain in the first order of perturbation theory

$$\wp(a_0) = -\frac{\zeta(\omega)}{\omega}, \quad \epsilon = \frac{|p|k}{\wp'(a_0)}, \quad \Gamma = \frac{\wp''(a_0)}{\wp'(a_0)}|\beta|k.$$

From these expressions it can be easily shown that a_0 lines on the line $\mathrm{Im}\, a = \omega'$, on this line $\mathrm{Im}\,\wp'(a) = \mathrm{Im}\,\wp''(a) = 0$ and, therefore, Γ is real. Since the equations are satisfied after the change $a_0 \to -a_0$ the increment changes its sign. $\Gamma > 0$ corresponds to instability of the initial wave.

Evident dependence Γ on k can be found only in the two limiting cases $\omega' \gg \omega$ and $\omega \gg \omega'$. In the intermediate case this dependence can be determined with the help of a computer. The curves on which the increment is defined for $\mathrm{Re}\,\beta = 0$ and $\omega' = 2\omega = 2\pi$ are presented in fig. 1. For small $|\beta|k$ (fig. 1, $|\beta|k < \wp(\omega + \mathrm{i}\omega') - \wp(\mathrm{i}\omega')$) these curves cross the line $\mathrm{Re}\, a = \omega$ at two points symmetric with respect to the line $\mathrm{Im}\, a = \omega'$. When $|\beta|k > \wp(\omega + \mathrm{i}\omega') - \wp(\mathrm{i}\omega')$ (fig. 1c) the curves represent loops. As $|\beta|k$ is growing, loops are narrowing and for $|\beta|k = \wp(\omega) - \wp(\mathrm{i}\omega')$ the loops degenerate into two points $a = \omega$ and $b = \mathrm{i}\omega'$. For this value of parameter $|\beta|k$ the increment Γ turns into zero. When $|\beta|k > \wp(\omega) - \wp(\mathrm{i}\omega')$, the solution of such type is absent, but another stable solutions with $\Gamma^2 < 0$ do exist for which $\mathrm{Re}(a, b) = n\omega'$ (n is

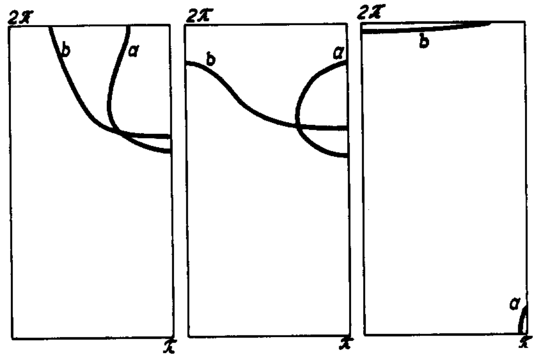

Fig. 1. The curves for parameters a and b, on which the increment is defined, in the fundamental rectangle with the sides $\omega = \pi, \omega' = 2\pi$. Fig. 1a: $|\beta|k = 5 \times 10^{-3} < \wp(\omega + \mathrm{i}\omega') - \wp(\mathrm{i}\omega')$; fig. 1b: $\wp(\omega + \mathrm{i}\omega') - \wp(\mathrm{i}\omega') < |\beta|k = 0{,}01 < \wp(\omega) - \wp(\omega + \mathrm{i}\omega')$; fig. 1c: $\wp(\omega) - \wp(\omega + \mathrm{i}\omega') < |\beta|k = 0.25 < \wp(\omega) - \wp(\mathrm{i}\omega')$.

an integer). Thus, for positive dispersion the wave number k_{cr},

$$|\beta|k_{cr} = \wp(\omega) - \wp(i\omega') , \tag{19}$$

gives the exact boundary of cnoidal wave stability.

In case of $\omega' \gg \omega$ corresponding to small wave amplitude limit the cnoidal wave reduces to the solution of linearized KP equation

$$u \to 4\left(\frac{\pi}{\omega}\right)^2 e^{-\pi(\omega'/\omega)} \cos\frac{\pi}{\omega}\left(x + \frac{\pi^2}{\omega^2}t\right).$$

The small value $b = 4(\pi/\omega)^2 e^{-\pi(\omega'/\omega)}$ plays here the role of the wave amplitude. The perturbations $\delta u(x, y, t)$ within this limit can be represented in the following form (in the system of reference moving with the initial wave):

$$\delta u(x) = \{p\, e^{ipx} + h(p-q)(2 - e^{-iqa} - e^{iqb})\, e^{i(p-q)x} + h(p+q)(2 - e^{iqa} - e^{-iqb})\, e^{i(p+q)x}\}\, e^{iky} ,$$

where $q = \pi/\omega$ is the wave number of the initial wave, $h = e^{-\pi(\omega'/\omega)}$. These perturbations describe the processes of decay and coupling. When the third term in this expression is small and two first ones have the same order of value system (16)–(18) describes the increment of decay instability of the stationary wave (see, e.g. [3]). The relationship (19) in this limit gives the boundary of decay instability.

The limit $\omega \to \infty$ corresponds to transition to soliton solution:

$$u_0 = \frac{2\nu^2}{\mathrm{ch}^2\nu(x - 4\nu^2 t)} ,$$

where $\nu = \pi/2\omega'$. In this case the perturbations can be represented as

$$\delta u(x) \lim_{\omega\to\infty} \frac{d}{dx}[\psi(x)\varphi(x)] = \frac{d}{dx}\left\{e^{-(\varkappa-\eta)x}\left[\frac{\varkappa}{\nu} + \mathrm{th}\,\nu x\right]\left[\frac{\eta}{\nu} - \mathrm{th}\,\nu x\right]\right\} . \tag{20}$$

Here we designate $\kappa = \nu\,\mathrm{cth}\,\nu a$, $\eta = \nu\,\mathrm{cth}\,\nu b$. These eigenfunctions coincide with the solutions of the KP equation linearized on the background of soliton [5]. In this limit conditions (16), (17) are rewritten as

$$\varkappa^2 - \eta^2 = i\beta k , \tag{21}$$

$$\Gamma = -\eta^2 + \varkappa^3 + \nu^2(\eta - \varkappa) . \tag{22}$$

The spectrum definition requires of δu finiteness for $|x| \to \infty$. When $|\varkappa| \neq \nu$ and $|\eta| \neq \nu$ condition (18) corresponding to this requirement has the form

$$\mathrm{Im}\, p = 0, \quad p = i(\varkappa - \eta) .$$

It is not difficult to show that eqs. (21) and (22) independently of the sign of β^2 possess the imaginary value of Γ,

$$\Gamma = ip\left[\frac{3\beta^2 k^2}{p^2} - p^2 - 4\nu^2\right] ,$$

that for $\nu^2 = 0$ transits to the dispersion law for small amplitude wave of KP equation.

For $|\eta| = \nu$ (or $|\varkappa| = \nu$) solution (20) vanishes at infinity when $|\mathrm{Re}\,\varkappa| < \nu$. From (21) for real β corresponding to the negative dispersion we have the following inequality

$$(\mathrm{Re}\,\varkappa)^2 = \nu^2 + (\mathrm{Im}\,\varkappa)^2 > \nu^2 ,$$

which is incompatible with the condition $|\mathrm{Re}\,\varkappa| < \nu$. Thus, for negative dispersion neutral stable solutions takes place only. Note that this conclusion does not agree with the statement of papers [5, 9] about existance of decreasing perturbations for $\beta^2 > 0$ but completely agrees with the result on the neutral stability of periodic wave.

If β is pure imaginary ($i\beta = |\beta|$) the condition of finiteness $|\mathrm{Re}\,\varkappa| < \nu$ does not contradict relationship (21). In this case we obtain the well-known expression for the increment of a soliton [1, 6]:

$$\Gamma = 4|\beta|k(\nu^2 - |\beta|k)^{1/2} ,$$

which is positive for $|\beta|k < v^2$ and that agrees with the general criterion (19). Really $\wp(\omega) - \wp(\mathrm{i}\omega') \to (\pi/2\omega')^2 = v^2$ when $\omega \to \infty$.

Acknowledgement

The authors thank A.I. Chernikh for computation.

References

[1] V.E. Zakharov, S.V. Manakov, S.P. Novikov and L.P. Pitaevski, Theorija Solitonov (Nauka, Moscow, 1980) (in Russian).

[2] V.E. Zakharov and A.B. Shabat, Funk. Analiz. 8 (1974) 33. (in Russian).

[3] G.B. Whitham, Linear and Nonlinear Waves (Wiley-Interscience, New York, London, Sydney, Toronto, 1974).

[4] V.E. Zakharov and E. Schulman, Physica 1D (1980) 192.

[5] V.E. Zakharov, Pis'ma v ZhETF 22 (1975) 7 (in Russian).

[6] E.A. Kuznetsov and A.V. Mikhailov, ZhETF 67 (1974) 1717 (in Russian).

[7] V.E. Zakharov and A.B. Shabat, Funk. Analiz 18 (1979) 13 (in Russian).

[8] V.E. Zakharov, The Inverse Scattering Method, in: Solitons, Topics in Current Physics, v. 17 (Springer, Berlin, Heidelberg, New York, 1980).

[9] B.B. Kadomtsev and V.I. Petviashvili, Soviet Physics Dokl. 15 (1970) 539.

[10] E. Whittaker and G. Watson, A Course of Modern Analyses (Cambridge Univ. Press, Cambridge, 1927).