Wave Turbulence

Kolmogorov Spectra of Turbulence I

Vladimir E. Zakharov,
Victor S. L'vov,
Gregory Falkovich

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Wave Turbulence

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In memory of A. N. Kolmogorov and A. M. Obukhov

Preface

Since the human organism is itself an open system, we are naturally curious about the behavior of other open systems with fluxes of matter, energy or information. Of the possible open systems, it is those endowed with many degrees of freedom and strongly deviating from equilibrium that are most challenging. A simple but very significant example of such a system is given by developed turbulence in a continuous medium, where we can discern astonishing features of universality.

This two-volume monograph deals with the theory of turbulence viewed as a general physical phenomenon. In addition to vortex hydrodynamic turbulence, it considers various cases of wave turbulence in plasmas, magnets, atmosphere, ocean and space. A sound basis for discussion is provided by the concept of cascade turbulence with relay energy transfer over different scales and modes.

We shall show how the initial cascade hypothesis turns into an elegant theory yielding the Kolmogorov spectra of turbulence as exact solutions. We shall describe the further development of the theory discussing stability problems and modes of Kolmogorov spectra formation, as well as their matching with sources and sinks.

This volume is dedicated to developed wave turbulence in different media. It contains a detailed exposition so that the reader can use it as an introductory textbook on wave turbulence theory. Moreover, it also provides an introduction to the general theory of developed turbulence, since wave turbulence at low excitation level is closely related to the Richardson-Kolmogorov-Obukhov cascade picture. In the second volume developed turbulence of incompressible fluids will be described.

This text is based on lecture courses given at the Arizona, Chicago and Novosibirsk Universities, at the Moscow Institute of Space Researches, and the Weizmann Institute of Science in Rehovot.

The book is useful for specialists in hydrodynamics, plasma and solid-state physics, meteorology, and astrophysics. We also hope it will prove instructive for students and young researchers starting their academic careers with studies of the problem of turbulence.

June 1992

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Contents

Main Symbols

$a(\boldsymbol{k},t)$, a_k, $b(\boldsymbol{k},t)$, b_k, $c(\boldsymbol{k},t)$, c_k	wave amplitudes
d	dimension (of the space)
E	energy
$\varepsilon(\boldsymbol{k})$	energy density (in the $\boldsymbol{k}$-space)
$E(k)$	energy density (in the k-space)
g	acceleration of gravity
g_m	magnetic-to-mechanical momentum ratio
$\boldsymbol{H}$, H	magnetic field
$\mathcal{H}$	Hamiltonian
$\hbar$	Planck constant
$I_k\{n(\boldsymbol{k}',t)\}$, $I(\boldsymbol{k})$, I_k	collision integral
$\boldsymbol{k}$	wave vector
k	wave number
m	scaling index of the interaction coefficient
$n(\boldsymbol{k},t)$, n_k	wave density (in the $\boldsymbol{k}$-space)
N	total number of waves
$\boldsymbol{P}$	energy flux (in the $\boldsymbol{k}$-space)
$P(k)$, P_k, P	energy flux (in the k-space)
Π	total wave momentum
$\boldsymbol{R}$	momentum flux (in the $\boldsymbol{k}$-space)
S	entropy
$T(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) = T_{1234}$	coefficient of four-wave interaction
$V(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = V_{123}$	coefficient of three-wave interaction
α	index of wave frequency
$\Gamma(k)$	external increment (or decrement)
$\delta(x)$	(Dirac) delta-function
$\omega(\boldsymbol{k})$, ω_k	wave frequency
ξ	dimensionless variable
$\propto$	proportional
$\approx$	approximately equal
$\simeq$	of the same order

0. Introduction

A central problem of theoretical physics is the construction of a theory of turbulence. This book is the outcome of our continued efforts to solve this problem. The aim of this book is to present several major ideas, which we shall outline in this Introduction.

The basic idea of the book is that turbulence is a general physics problem which requires a comprehensive approach. The problem of turbulence goes far beyond the limits of hydrodynamics and the Navier-Stokes equation. In our view, turbulence is a highly excited state of a system with many degrees of freedom (in most cases a continuous medium) to be described statistically. This excited state is extremely far away from thermodynamic equilibrium and is accompanied by intensive energy dissipation. Such states can be found or created in plasmas, magnets, or nonlinear dielectrics by applying strong electromagnetic fields to them. Quite a few types of turbulence can be observed in hydrodynamics.

Our book is devoted to *developed turbulence*, i.e., to situations with turbulence involving many degrees of freedom. In hydrodynamics this corresponds to extremely large Reynolds numbers. We will not elaborate on the genesis of turbulence and about turbulence described by a low-dimensional attractor. We believe that these topics have already been extensively covered in the literature.

Developed turbulence in general refers to a system where the scales of pumping and effectively damping motions (modes) differ dramatically. The nonlinear interaction allows for an energy redistribution between different modes. The fundamental problem of the theory is therefore to find the stationary spectrum of the turbulence, i.e., the law of energy distribution over the different scales. Since theoretical physics is mainly concerned with universal distributions, the stationary spectrum is sought inbetween the scales of pumping and damping modes (source and sink). In this range (the so-called inertial interval), one can expect the formation of a universal distribution independent of the specific characteristics of the source and sink.

One can figure out two qualitatively different pictures of universal turbulence. The first one, usually associated with the names of Richardson, Kolmogorov and Obukhov, presupposes that the major physical process is a continuing fragmentation that provides a relay energy transfer from the source to the sink. As a result of the large number of fragmentation acts, the distribution "forgets" the details of the energy source. Hence, the turbulence spectrum depends on a single characteristic of pumping: the energy dissipated per unit time and unit volume P. Another picture of universal turbulence, called the structural one, appeared more recently. It is based on the concept of the generation of spatio-temporal structures of universal form (solitons, collapses, etc.)

The main part of this book deals with the first of these two pictures of turbulence. The central idea is the locality of the interaction, as proposed by Kolmogorov in describing turbulence of incompressible fluids: only vortices of spatial extensions of the same order strongly interact with each other. This presumes a constant energy flux P in k-space coinciding with the rate with which the turbulent system dissipates energy. The locality ensures that the stationary spectrum of the energy E_k in the inertial interval may be expressed in terms of the flux P and the current wave vector k only. Dimensional analysis yields

$$E_k = \lambda P^{2/3} k^{-5/3} , \tag{0.1}$$

where λ is a constant. Expression (0.1) is called *the Kolmogorov-Obukhov spectrum* of turbulence of an incompressible fluid [0.1].

Kolmogorov's hypothesis lead to a lot of activities and to an immense scientific literature. Experimenters essentially confirm the validity of (0.1). A rigorous theoretical substantiation of the Kolmogorov spectrum is not yet available. This is mainly due to the absence of a small parameter in the theory of hydrodynamic turbulence. Vortex interaction in incompressible fluids is strong and there exists no small parameter in the hydrodynamic equations. So if one tries to obtain the equations for correlation functions, then some of the mathematical objects in the theory lead not just to asymptotic series, but even to divergent ones. This is reflected for by the fact that it is impossible to perform a consistent linearization of the hydrodynamic equations for an incompressible fluid against a stationary homogeneous background (it yields only the trivial result zero). Except for the free boundary case, there are no homogeneous-background waves in hydrodynamics, whose amplitudes may be taken to be sufficiently small. This is not the case with other media: neither in plasma turbulence and waves on a fluid surface, nor with intensive laser pulses propagating in a nonlinear dielectric, i.e., in all cases in which the system has a consistent linear approximation that describes small-amplitude waves with dispersion. In that case one can consider a situation in which the level of wave excitation is small and effects of the nonlinear interaction are smaller than the linear effects caused by the wave propagation velocity dispersion. Such a turbulence is called a *weak turbulence* and allows a quite efficient theoretical description.

Thus, this book is divided into two volumes. The first volume - which you are reading just now - describes weak wave turbulence; the second is dedicated to *strong turbulence*, essentially to the vortex turbulence of incompressible fluids.

The first volume contains a consistent description of the theory of the developed weak turbulence in different media: fluids, gases, plasmas, and magnets. In this volume we use the terms "wave" and "weak" turbulence as synonyms.

In the theory of weak turbulence, the series that yield the equations for correlation functions contain a small parameter (the nonlinearity level) and are asymptotic, which substantiates the theory sufficiently well. In particular, it may be reliably established that at a low nonlinearity level, turbulence is a set of waves whose phases are close to random. This makes it possible to express higher correlators in terms of lower ones and to ignore higher equations in the chain of equations for correlation functions. As a result, weak turbulence may be described in terms of a closed kinetic equation for the pair correlator, which is the mean square of the wave amplitude. Having established an effective language for describing the phenomenon, Kolmogorov's ideas are very efficiently applied to construct a theory of weak turbulence. For stationary kinetic equations *V.E. Zakharov* found exact power-type solutions identified with the Kolmogorov spectra [0.2]. These solutions correspond to a constant k-space flux of one of the integrals of motion of the system, which often turns out to be the energy. The theory of such spectra including the problems of their stability, formation and matching with sources is quite comprehensive and well advanced. It has not been systematically presented (except for the outline contained in the review by *Zakharov* [0.3]). It has already been applied to the theory of wave turbulence on fluid surfaces, the theory of "optical" turbulence and is expected to find many more physical applications. We believe that acquaintance with this theory is indispensable to everyone who is seriously interested in the theory of turbulence.

In the limit of small nonlinearity, formation of the dynamic structures of the soliton or collapse type is impossible. Therefore universal weak turbulence is most frequently of the Kolmogorov type, and most of its manifestations may be explained in terms of macroscopic characteristics, i.e., fluxes of the integrals of motion. Such an approach has the same relation to the "microscopic" description (in terms of pair correlators) as thermodynamics has to statistical physics. The very possibility of a quasi-thermodynamic description in terms of mean values arises from the fact that the statistics of a weakly turbulent wave field is close to a Gaussian, the major contribution to the average characteristics stemming from the set of most probable events.

The theory of weak turbulence involves a large variety of specific types of turbulence. In order to study this variety from a unified viewpoint, one should adopt a general approach to the description of various nonlinear media. Indeed, according to the aforementioned principle of considering developed turbulence as a universal phenomenon, we ought to invent a universal "language" for its description. Such a language, in our opinion, is the Hamiltonian formalism, which reveals the Hamiltonian structure concealed in the equations of the medium [0.4]. The Hamiltonian theory of equations describing continuous media is an interesting topic of modern mathematical physics.

Chapter 1 of this book can be treated as an elementary introduction to the theory and addresses in particular physicists: our approach being rather pragmatic we intend to show that the dynamic equations for very different media written in normal variables (complex wave amplitudes) acquire a standard form which is quite convenient for application of statistical averaging methods.

Chapter 2 is devoted to the derivation of the averaged kinetic equations. There are several averaging methods. In this volume we shall use only elementary ones based on the hypothesis of phase randomness and on euristically decoupling correlators. [In the second volume we shall describe a more sophisticated diagram technique; we shall pay much attention to substantiating the kinetic equation and shall give two different derivations. We shall treat carefully the question of the range of applicability of the kinetic equation, bearing in mind that it is easy to make an error there. In particular, we shall discuss in detail the problem of acoustic turbulence, i.e., of the statistical description of waves with a linear dispersion law, and clarify the rather sophisticated applicability conditions of the kinetic equation in this case. The derivation of the kinetic equation demonstrates most convincingly the advantages of the Hamiltonian: the kinetic equation has a standard form, the structural functions entering it are simply expressed via the Hamiltonian coefficients.] Classical and quantum kinetic equations are the main mathematical objects of study in the first volume, their general properties being discussed in Chap. 2.

Chapter 3 is the central one of this volume. The exact stationary solutions of the kinetic equations are obtained and shown to be just the Kolmogorov-like spectra referred to in the title of the book. Thus, the Kolmogorov-Obukhov hypothesis is converted into a strict theorem in the theory of weak turbulence. The aforementioned property of interaction locality can be easily verified in that theory for every specific case by calculating a single integral. A general locality criterion (which is the condition for the existence of the Kolmogorov spectrum) is obtained. Solutions of the Kolmogorov type are obtained, not only for scale-invariant isotropic media, but also for anisotropic media and for those close to scale-invariant ones. Boundary conditions for Kolmogorov solutions, i.e., for the matching with sources and sinks are also given.

Chapter 4 deals with the stability problem and the formation of the Kolmogorov spectra. It is interesting that two absolutely different types of instabilities can be dealt with. First, usual instability results in the exponential growth of perturbations; all known spectra are stable with respect to such an instability. Second, there may be a "structural instability" first predicted by *L'vov* and *Falkovich* [0.5]. In the structurally unstable case, a small anisotropy of the pumping caused the stationary spectrum to be substantially anisotropic in the inertial interval. Such an instability can be treated as a manifestation of self-organization in the nonlinear systems. Thus the hypothesis about the local isotropy of the developed-turbulence spectrum (suggested by *Taylor* [0.6] for hydrodynamics) may be incorrect in the case of wave turbulence while the Kolmogorov hypothesis about interaction locality may still be valid. We elaborate in detail on the general stability theory for Kolmogorov spectra of weak turbulence as developed by *Balk and Zakharov* [0.7]. A substantial part of Sect. 4.2 is a translation of the Russian paper [0.7]. We also discuss the different regimes of the nonstationary behavior of wave turbulence systems.

Chapter 5 deals with physical applications of the general theory developed in the preceding chapters. Due to the large variety of such applications it is impossible to discuss every physical system with satisfactory completeness. However, we give answers to the main questions:

– existence of stationary spectra,
– connection between the flux and pump characteristics,
– behavior in the damping region,
– spectrum stability,
– nonstationary regimes.

Throughout this first volume the material is developed in detail since it is addresses students and junior researchers. Some issues which we considered to be rather specialized are printed in small letters. In a first reading these places may be ignored. The first volume is to serve as a simple, yet comprehensive introduction to the general theory of developed turbulence. As far as wave turbulence itself is concerned, we briefly summarize the derivation of the theory in *Chap. 6, the Conclusion*, by giving the recipe of investigation of any new wave system. The reader will see that the recipe is fairly simple. By now the theory has been elaborated to such an extent that answers to most questions may be expressed in terms of the characteristics of a wave system obtained from dimensional analysis or simple asymptotic estimates. Thus, the *Conclusion* contains a methodological guide for the first volume.

Coming to the end of the *Introduction* of this first volume, we briefly expose the prospective contents of the second volume of our monograph.

Volume 2 will be devoted to *strong turbulence* and will be a natural continuation of the first volume, yet containing a new formalism and new ideas. It consistently elaborates on the necessary diagram technique (in doing so, it gives a sufficiently rigorous derivation of the kinetic equation). For hydrodynamics, this technique is developed both in the canonical Clebsch variables and in "natural" variables. The major part of the second volume is devoted to a consistent statistical theory of turbulence of incompressible fluids. This theory proceeds from the Navier-Stokes equation and the diagram approach to perturbations, in which every term of an infinite series is matched with a certain diagram. Thus, two essentially different types of vortex interaction are identified.

The first one is the *sweeping interaction* corresponding to the transfer of a small k-vortex (with dimension $1/k$) as a whole by the spatially homogeneous part of the field of large vortex velocities. This interaction is characterized by the Doppler frequency kv_t, where v_t is the mean-square velocity of turbulent pulsations associated with the spectrum (0.1)

$$v_t^2 \simeq \int_{k_0}^{\infty} E(\boldsymbol{k})\, dk \simeq (PL)^{2/3} \, . \tag{0.2}$$

Here $k_0 = 1/L$ is an "external" boundary of the inertial interval; L, the "energy-containing" or external scale of turbulence coinciding in order of magnitude with the characteristic dimension of the flown-over body. At $k \lesssim k_0$, the spectrum $E(\boldsymbol{k})$ is nonuniversal.

The second type is the *dynamic interaction* of vortices of about the same scale which leads to energy exchange between them and which is responsible for the formation of the turbulence spectrum. This interaction is characterized by the frequency

$$\gamma(k) \simeq P^{1/3} k^{2/3} , \tag{0.3}$$

which we shall call the Kolmogorov frequency. This frequency, as well as the spectrum (0.1), may be determined from dimensional analysis. In the inertial interval the Doppler frequency kv_t is seen to be larger than the Kolmogorov frequency $\gamma(k)$:

$$kv_t \simeq \gamma(k)(kL)^{1/3} .$$

The existence of two types of interaction with different k-dependences implies that the theory of turbulence is not scale-invariant; not even in the inertial interval with $kL \gg 1$. The theory explicitly contains the external scale L. The presence of a large kL-parameter does not simplify the theory, since the weaker dynamic $\gamma(k)$-interaction cannot be discarded. Indeed, it is exactly this interaction that determines the turbulence spectrum. At the same time, the stronger sweeping interaction, which manifests itself as a real physical effect, is of purely kinematic character and bears no relationship to the problem of energy distribution over the scales. This makes the search for scale-invariant energy spectra in the inertial interval rather difficult. Thus, to construct a consistent theory of hydrodynamic turbulence, one has to overcome two major difficulties. The first is associated with the strong interaction and the absence of a small parameter in the theory. The second is due to the existence of two types of interaction and the absence of scale invariance.

To our understanding the second difficulty was overcome by *Belinicher and L'vov* [0.8]. They constructed the statistical theory of developed homogeneous turbulence of incompressible fluids in a coordinate system moving with the velocity of the fluid (in a spatial point r). The transition to this new variable, the so-called quasi-Lagrangian velocity, eliminates any transfer of k-vortices in the region of $1/k$ scale around the reference point r. Elimination of sweeping in a limited region only, turns out to suffice to completely eliminate from the theory its "masking" effect on the dynamic interaction of vortices in the cascade process of energy transfer to small scales. We shall present an analysis of expressions for diagrams of the perturbation theory to an arbitrary order and shall show that the integrals converge both in the infrared and the ultraviolet regions. This proves the Kolmogorov-Obukhov hypothesis about the locality of the dynamic interaction of vortices: the main contribution to the change in the k-vortex energy is made by k_1-vortices of the same scale (k_1 of the order of k) and localized in the $1/k$-region in the vicinity of the given vortex [0.8]. In the limit $kL \to \infty$, one can obtain the scale-invariant solution of the Dyson-Wyld diagram equations which corresponds to the known Richardson-Kolmogorov-Obukhov picture of developed universal turbulence.

As far as the first difficulty is concerned it has not yet been overcome. For this reason, the mathematical objects of the diagram perturbation theory are formal series depending on an external parameter.

The problem of the unambiguous correspondence of the observed physical values to the formal series in the theory of developed hydrodynamic turbulence is, however, open for discussion. Many important problems remain uninvestigated, such as uniqueness and stability of the obtained solution, transition of a nonuniversal solution in the energy-containing interval to the scale-invariant solution in the inertial interval, etc. Thus, there are ample opportunities for further research.

It should be noted, however, that the traditional descriptions of turbulence, including the diagram technique, suggest that the statistics of strong turbulence does not differ much from Gaussian statistics. By now we know that even weak turbulence can contain a non-Gaussian component. Strong turbulence may be essentially non-Gaussian which implies that some rather specific spatio-temporal configurations may contribute unproportionally large to various mean values. This property of turbulence has to do with the known "intermittency" phenomenon. Thus, the energy dissipation density at each moment in time may be distributed in space in a rather inhomogeneous manner, in contrast to the implicit assumption of dissipation homogeneity made in the Richardson-Kolmogorov-Obukhov picture. Numerical and laboratory experiments show that in many physical situations (both in hydrodynamics and in the strong turbulence of plasmas), there are clear-cut short-living zones of dissipation. We associate these zones of higher energy emission with collapses, i.e., the points where the solutions of the original equations describing the medium have singularities.

Let us illustrate this by an example. Quite a universal physical model is the nonlinear Schrödinger equation

$$\frac{\partial \Psi(\boldsymbol{r},t)}{\partial t} + \triangle\Psi + T|\Psi|^2\Psi = 0 , \qquad (0.4)$$

which describes, in particular, the propagation of intensive quasi-monochromatic wave packets in nonlinear dielectrics and the resulting "optical" turbulence. Solutions of (0.4) at $T > 0$ (implying mutual wave attraction) may become singular with time. This corresponds to self-focusing of light in a nonlinear dielectric.

It should be noted that in the case of repulsion ($T < 0$), no singularities can evolve, and the qualitative picture of turbulence must be absolutely different. Meanwhile, in weak turbulence description in the low nonlinearity limit, one can obtain from (0.4) the kinetic equation which contains the quantity $|T|^2$. Thus, in the weak turbulence limit, the sign of the interaction coefficient is insignificant.

The high-frequency part of the spectrum is determined just by the structure of the singularities formed. Therefore the theory of structure turbulence should obviously be built up in an absolutely different way rather than in terms of correlation functions in k-space. In that case, it seems natural to return to treating the different dynamic processes in r-space. From a set of realizations one should choose those making the major contribution to the mean characteristics, and average over this subset of spatio-temporal structures to obtain the wanted statistical characteristics. We shall present a sketch of such a theory using acoustic turbulence as an example. The question which remains open is, how relevant are the collapsing structures to the classical turbulence of incompressible fluids?

1. Equations of Motion and the Hamiltonian Formalism

1.1 The Hamiltonian Formalism for Waves in Continuous Media

Equations describing waves in different media and written in natural variables are diverse. For example, the Bloch equations defining the motion of a magnetic moment are totally different from the Maxwell equations for nonlinear dielectrics. The latter radically differ from the Euler equations for compressible fluids. However all of them as well as many other equations describing nondissipative media, possess an implicit or explicit Hamiltonian structure. This was established empirically and is reflected by the fact that all these models may be derived from initial microscopic Hamiltonian equations of motion.

The Hamiltonian method is applicable to a wide class of weakly dissipative wave systems; it clearly manifests general properties of small-amplitude waves. For example, spin, electromagnetic and sound waves are just waves, i.e., medium oscillations, transferred from one point to another. If we are interested only in small-amplitude wave propagation phenomena, such as diffraction, we do not really need to know what it is that oscillates: magnetic moment, electrical field or density. Their respective dispersion law $\omega(\boldsymbol{k})$ contains all the information about the medium properties that is necessary and sufficient for studying the propagation of noninteracting waves. As we shall see now, the $\omega(\boldsymbol{k})$-function is a coefficient in the term of the Hamiltonian which is quadratic with respect to wave amplitudes, i.e., to complex normal variables. The actual Hamiltonian is a power series in these variables that contains all the information about nonlinear wave interactions. Let us consider the transition to such variables using a simple yet very important example.

1.1.1 The Hamiltonian in Normal Variables

A continuous medium of dimensionality d may be defined in the simplest case by a pair of canonical variables $p(\boldsymbol{r},t)$ and $q(\boldsymbol{r},t)$. The canonical equations of motion are expressed as

$$\frac{\partial q(\boldsymbol{r},t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta p(\boldsymbol{r},t)}\,, \tag{1.1.1}$$

$$\frac{\partial p(\boldsymbol{r},t)}{\partial t} = -\frac{\delta \mathcal{H}}{\delta q(\boldsymbol{r},t)} \,. \tag{1.1.2}$$

The Hamiltonian $\mathcal{H}$ depends on $p(\boldsymbol{r},t)$ and $q(\boldsymbol{r},t)$ as a functional. The symbols $\delta/\delta q$ and $\delta/\delta p$ designate variational derivatives which are extensions of partial derivatives for the continuous case (Sect. A.1). The formal advantage of the Hamiltonian method is that its equations are symmetric in coordinate q and momentum p. To illustrate this advantage, let us first go over to new canonical variables $Q = \lambda q$, $P = p/\lambda$, choosing the dimensional factor λ in such a way that P and Q have the same dimension. Then we introduce complex variables

$$a = (Q + iP)/\sqrt{2} \,, \tag{1.1.3}$$

$$a^* = (Q - iP)/\sqrt{2} \tag{1.1.4}$$

to obtain

$$\sqrt{2}\frac{\partial a}{\partial t} = \frac{\delta \mathcal{H}}{\delta P} - i\frac{\delta \mathcal{H}}{\delta Q}, \qquad \sqrt{2}\frac{\partial a^*}{\partial t} = \frac{\delta \mathcal{H}}{\delta P} + i\frac{\delta \mathcal{H}}{\delta Q} \,. \tag{1.1.5}$$

Substituting for $\mathcal{H}, P$ and Q, we obtain

$$i\frac{\partial a(\boldsymbol{r},t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta a^*(\boldsymbol{r},t)}, \qquad i\frac{\partial a^*(\boldsymbol{r},t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta a(\boldsymbol{r},t)} \,. \tag{1.1.6}$$

The second equation follows from the first by complex conjugation. Hence, we have obtained one complex equation instead of two real ones (1.1.1,2). In quantum mechanics such a substitution of variables corresponds to a transition from the coordinate-momentum representation to a representation using creation and annihilation Bose operators. Their classical analogues are complex canonical variables. Obviously, the canonical variables (1.1.3,4) are by no means the only possible variables to choose. A large choice of transformations from one set of variables to another $a, a^* \to b, b^*$ exists, such that the equations of motion retain their canonical form (1.1.6). For a given set of explicit variables the canonicity condition is expressed through the Poisson brackets of two functions

$$\{f(q), g(q')\} = \int d\boldsymbol{r}'' \left[\frac{\delta f(q)}{\delta a^*(\boldsymbol{r}'')}\frac{\delta g(q')}{\delta a(\boldsymbol{r}'')} - \frac{\delta f(q)}{\delta a(\boldsymbol{r}'')}\frac{\delta g(q')}{\delta a^*(\boldsymbol{r}'')}\right]$$

(see Sect. A.2) and has the simple form

$$\{b(q), b(q')\} = 0, \quad \{b(q), b^*(q')\} = \delta(q - q') \,. \tag{1.1.7}$$

To ensure that this is a one-to-one transformation the index q should cover a "complete set" of values, for example, the space R^d. It should be noted that this wide range of possibilities in selecting canonical variables is an important advantage of the Hamiltonian method. It ensures the choice of the most adequate variables for a specific problem. We shall define the canonical variables $a(p)$ in such a way that they determine wave amplitudes and become zero for vanishing waves.

Let us expand the Hamiltonian $\mathcal{H}$ in a power series of variables $a(\boldsymbol{r})$ and $a^*(\boldsymbol{r})$ assuming these to be small. The zeroth order term is of no interest for us, since it does not occur in the equation of motion. There are no first-order terms as the medium is assumed to be in equilibrium if the amplitudes of the waves were zero, and, consequently, the Hamiltonian to be minimal at $a = a^* = 0$. Thus the $\mathcal{H}$ expansion starts from the second-order terms:

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_{\text{int}} . \tag{1.1.8}$$

The most general form of $\mathcal{H}_2$ is:

$$\begin{aligned}\mathcal{H}_2 = \int d\boldsymbol{r}\,d\boldsymbol{r}'\{&A(\boldsymbol{r},\boldsymbol{r}')a(\boldsymbol{r})a^*(\boldsymbol{r}')\\ &+(1/2)[B^*(\boldsymbol{r},\boldsymbol{r}')a(\boldsymbol{r})a(\boldsymbol{r}')+\text{c.c}]\} .\end{aligned} \tag{1.1.9}$$

Here "c.c." means the complex conjugate of the preceding term.

The $\mathcal{H}_2$ value is real (the Hamiltonian is Hermitian). Therefore

$$A(\boldsymbol{r},\boldsymbol{r}') = A^*(\boldsymbol{r}',\boldsymbol{r}), \quad B(\boldsymbol{r},\boldsymbol{r}') = B(\boldsymbol{r}',\boldsymbol{r}) . \tag{1.1.10}$$

Below we shall consider the medium to be spatially homogeneous. This very important assumption is the basis of all following discussions. Because of space homogeneity, the functions $A(\boldsymbol{r},\boldsymbol{r}')$ and $B(\boldsymbol{r},\boldsymbol{r}')$ depend only on the difference of the arguments $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$. Now

$$A(\boldsymbol{R}) = A^*(-\boldsymbol{R}), \quad B(\boldsymbol{R}) = B(-\boldsymbol{R}) . \tag{1.1.11}$$

The Hamiltonian can be significantly simplified using the Fourier transform:

$$\begin{aligned} a(\boldsymbol{k}) = a_k &= (1/V)\int a(\boldsymbol{r})\exp(-i\boldsymbol{k}\boldsymbol{r})\,d\boldsymbol{r},\\ a(\boldsymbol{r}) &= \sum_{\boldsymbol{k}} a_k \exp(i\boldsymbol{k}\boldsymbol{r}) .\end{aligned} \tag{1.1.12}$$

Here V is the volume of a sample (the propagation medium). We consider the wave vector $\boldsymbol{k}$ to be a discrete variable. This corresponds to imposing periodic space boundary conditions on the wave field $a(\boldsymbol{r}+L) = a(\boldsymbol{r})$, $V = L^d$. If required, one can pass in any conventional manner from the summation over k to the integration

$$(2\pi)^d \sum_{\boldsymbol{k}} = V\int d\boldsymbol{k} . \tag{1.1.13}$$

The Fourier transform (1.1.12) is canonical but not unimodal. This means that the canonical equation of motion (1.1.6) retains its canonical form, but the new Hamiltonian differs from the old one by a factor, which is the inverse sample volume

$$V\mathcal{H}(a_k, a_k^*) = \mathcal{H}\{a(\boldsymbol{r}), a^*(\boldsymbol{r})\} , \tag{1.1.14}$$

$$i\frac{\partial a(\boldsymbol{k},t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta a^*(\boldsymbol{k},t)} \ .$$

It is essential that in the new variables $a(k)$ the quadratic part of the Hamiltonian represents a single integral over $d\boldsymbol{k}$

$$\begin{aligned}\mathcal{H}_2 = \int \{&A(\boldsymbol{k})a(\boldsymbol{k},t)a^*(\boldsymbol{k},t) + \frac{1}{2}[B(\boldsymbol{k})a(\boldsymbol{k},t)a(-\boldsymbol{k},t) \\ &+B^*(\boldsymbol{k})a^*(\boldsymbol{k},t)a^*(-\boldsymbol{k},t)]\}\, d\boldsymbol{k} \ ,\end{aligned} \quad (1.1.15)$$

$$A(\boldsymbol{k}) = \int A(\boldsymbol{R})\exp(i\boldsymbol{k}\boldsymbol{R})\, d\boldsymbol{R} \ , \quad B(\boldsymbol{k}) = \int B(\boldsymbol{R})\exp(i\boldsymbol{k}\boldsymbol{R})\, d\boldsymbol{R} \ .$$

In view of (1.1.10), $A(\boldsymbol{k}) = A^*(\boldsymbol{k})$ is a real function and $B(\boldsymbol{k}) = B(-\boldsymbol{k})$ is an even function. The latter means that we can consider $B(\boldsymbol{k})$ to be real as well. Indeed, if $B(\boldsymbol{k}) = |B(\boldsymbol{k})|\exp[i\psi(\boldsymbol{k})]$, then $\psi(\boldsymbol{k}) = \psi(-\boldsymbol{k})$ and one can dispose of $\psi(\boldsymbol{k})$ by substitution $a(\boldsymbol{k}) \to a(\boldsymbol{k})\exp[-i\psi(\boldsymbol{k})/2]$. Let us pose a question: in which case may the Hamiltonian (1.1.15) be diagonalized with respect to the wave vector using a linear transformation

$$a(\boldsymbol{k},t) = u(\boldsymbol{k})b(\boldsymbol{k},t) + v(\boldsymbol{k})b^*(-\boldsymbol{k},t) \ ; \quad (1.1.16)$$

in other words, is there a way to represent it as

$$\mathcal{H}_2 = \int \omega(\boldsymbol{k})b(\boldsymbol{k},t)b^*(\boldsymbol{k},t)\, d\boldsymbol{k} \ ? \quad (1.1.17)$$

First, let us derive the canonicity conditions for this transformation. On the one hand,

$$\frac{\partial a(\boldsymbol{k},t)}{\partial t} = u(\boldsymbol{k})\frac{\partial b(\boldsymbol{k},t)}{\partial t} + v(\boldsymbol{k})\frac{\partial b^*(-\boldsymbol{k},t)}{\partial t} = i\frac{\delta \mathcal{H}}{\delta a^*(\boldsymbol{k},t)} \ .$$

On the other hand, one should require $\partial b/\partial t$ to equal $i\delta\mathcal{H}/\delta b^*$ and

$$\begin{aligned}\frac{\delta \mathcal{H}}{\delta a^*(\boldsymbol{k})} &= u(\boldsymbol{k})\frac{\delta \mathcal{H}}{\delta b^*(\boldsymbol{k})} - v(\boldsymbol{k})\frac{\delta \mathcal{H}}{\delta b(-\boldsymbol{k})} \\ &= u(\boldsymbol{k})\left[\frac{\delta \mathcal{H}}{\delta a^*(\boldsymbol{k})}u^*(\boldsymbol{k}) + v(-\boldsymbol{k})\frac{\delta \mathcal{H}}{\delta a(-\boldsymbol{k})}\right] \\ &\quad - v(\boldsymbol{k})\left[\frac{\delta \mathcal{H}}{\delta a^*(\boldsymbol{k})}v^*(\boldsymbol{k}) + u(-\boldsymbol{k})\frac{\delta \mathcal{H}}{\delta a(-\boldsymbol{k})}\right] \ .\end{aligned}$$

Thus, the canonicity conditions take the following form

$$|u(\boldsymbol{k})|^2 - |v(\boldsymbol{k})|^2 = 1, \quad u(\boldsymbol{k})v(-\boldsymbol{k}) = u(-\boldsymbol{k})v(\boldsymbol{k}) \ . \quad (1.1.18)$$

The parameter $u(\boldsymbol{k})$ may be chosen to be real without loss in generality, which simply implies a choice of phase for complex variable $b(\boldsymbol{k})$. Since the value of $v(\boldsymbol{k})$ may also be chosen to be real [see (1.1.19)] it is convenient to set

$$u(\boldsymbol{k}) = \cosh[\zeta(\boldsymbol{k})], \quad v(\boldsymbol{k}) = \sinh[\zeta(\boldsymbol{k})].$$

According to (1.1.18), $\zeta(\boldsymbol{k})$ is a real even function. Substituting (1.1.16) into (1.1.15) and comparing to (1.1.17) we obtain after symmetrization with respect to $\boldsymbol{k}$ and $-\boldsymbol{k}$:

$$\omega(\boldsymbol{k}) = A(\boldsymbol{k})\cosh^2[\zeta(\boldsymbol{k})] + A(-\boldsymbol{k})\sinh^2[\zeta(\boldsymbol{k})] + 2B(\boldsymbol{k})\sinh[\zeta(\boldsymbol{k})]\cosh[\zeta(\boldsymbol{k})] \,, \tag{1.1.19a}$$

$$0 = [A(\boldsymbol{k}) + A(-\boldsymbol{k})]\sinh[\zeta(\boldsymbol{k})]\cosh[\zeta(\boldsymbol{k})] + B(\boldsymbol{k})\left[\cosh^2[\zeta(\boldsymbol{k})] + \sinh^2[\zeta(\boldsymbol{k})]\right] . \tag{1.1.19b}$$

Dividing $A(\boldsymbol{k})$ into even and odd parts

$$A(\boldsymbol{k}) = A_1(\boldsymbol{k}) + A_2(\boldsymbol{k}), \qquad A(-\boldsymbol{k}) = A_1(\boldsymbol{k}) - A_2((\boldsymbol{k})$$
$$A_1(-\boldsymbol{k}) = A_1(\boldsymbol{k}), \qquad A_2(-\boldsymbol{k}) = -A(\boldsymbol{k}) \,,$$

and substituting the respective expressions for $B(\boldsymbol{k})$, we obtain

$$\omega(\boldsymbol{k}) = A_2(\boldsymbol{k}) + \frac{A_1(\boldsymbol{k})}{\cosh[2\zeta(\boldsymbol{k})]}$$

for the frequency. Thus, the sign of an even part of $\omega(\boldsymbol{k})$ coincides with that of even part of $A(\boldsymbol{k})$. Expressing $\cosh[2\zeta(\boldsymbol{k})]$ from (1.1.19b) we obtain

$$\omega(\boldsymbol{k}) = A_2(\boldsymbol{k}) + \operatorname{sign} A_1(\boldsymbol{k})\sqrt{A_1^2(\boldsymbol{k}) - B^2(\boldsymbol{k})} \,. \tag{1.1.20}$$

One can see that it is only for real $\omega(\boldsymbol{k})$ possible to find a diagonalizing transformation. In the variables $b(\boldsymbol{k},t)$ the equations of motion become trivial

$$\frac{\partial b(\boldsymbol{k},t)}{\partial t} + i\omega(\boldsymbol{k})b(\boldsymbol{k},t) = 0$$

and have the solution $b(\boldsymbol{k},t) = b(\boldsymbol{k},0)\exp[i\omega(\boldsymbol{k})t]$; thus it is evident that real $\omega(\boldsymbol{k})$ implies the stability of the medium against an exponential growth in the wave amplitudes.

In most physical situations, the Hamiltonian is the wave energy density whose sign coincides, by virtue of (1.1.17), with that of the frequency $\omega(\boldsymbol{k})$.

In general, wave excitation increases the energy of the medium, which implies that the function $\omega(\boldsymbol{k})$ is usually positive. A negative value of $\omega(\boldsymbol{k})$ indicates that the energy of the medium decreases with increasing wave excitation. This is possible in systems that are far from thermodynamic equilibrium, for example, in plasmas containing a flux of particles. In that case a_k describes the negative-energy waves. It should be borne in mind that the Hamiltonian has been formally defined to an accuracy of a sign, since the transformation $\mathcal{H} \leftrightarrow -\mathcal{H}$, $a \leftrightarrow a^*$ is possible. Therefore the negative-energy waves may be considered only in the case when the $\omega(\boldsymbol{k})$ function changes sign in the $\boldsymbol{k}$-space. What is the connection between a change of sign in $\omega(\boldsymbol{k})$ and wave instability? As one can see from

(1.1.20), a change of sign in the even part of $\omega(\boldsymbol{k})$ means that there is a surface on which $A_1(\boldsymbol{k}) = 0$. In the general position on this surface $B(\boldsymbol{k}) \neq 0$. This implies that at least in the vicinity of the zero surface of $A_1(\boldsymbol{k})$ the square of the frequency is negative and the medium is unstable. Thus, if the even part of the $\omega(\boldsymbol{k})$-function is sign-alternating and the $B(\boldsymbol{k})$-function does not identically vanish wherever $A_1(\boldsymbol{k}) = 0$, the k-space contains a field of linear instability. Given this instability, it is impossible to transform the Hamiltonian into the form (1.1.17). In this case, however, the Hamiltonian (1.1.15) reduces to the simple form

$$\mathcal{H}_2 = \int C(\boldsymbol{k})[b(\boldsymbol{k},t)b(-\boldsymbol{k},t) + b^*(\boldsymbol{k},t)b^*(-\boldsymbol{k},t)]\, d\boldsymbol{k} \;,$$

$$C(\boldsymbol{k}) = C(-\boldsymbol{k}) = C^*(\boldsymbol{k}) \;.$$

From the analogy to quantum-mechanics it is evident that such a Hamiltonian describes the creation of a pair of quasi-particles from the vacuum (and the reverse process), thus representing just such an unstable medium. Summarizing, the canonical transformations (1.1.16) allow us to eliminate the term with the least factor (in magnitude) in the Hamiltonian (1.1.15).

Up to now, we have considered the case of a medium containing only one type of waves described by a single dispersion law $\omega(\boldsymbol{k})$. We can examine, without any essential complications, a more general case with a medium having several types of waves. In this case, the medium is described by a set of equations

$$\frac{\partial q_j}{\partial t} = \frac{\delta \mathcal{H}}{\delta p_j}, \quad \frac{\partial p_j}{\partial t} = -\frac{\delta \mathcal{H}}{\delta q_j}, \quad j = 1, \ldots n \;.$$

Going over to complex variables $a_j = (1/\sqrt{2})(q_j + ip_j)$, we have for the quadratic part of the Hamiltonian

$$\begin{aligned}\mathcal{H}_2 = \sum_{i,j}^{n} \int d\boldsymbol{r} \big[& A_{ij}(\boldsymbol{r}-\boldsymbol{r}_1)a_i(\boldsymbol{r},t)a_j^*(\boldsymbol{r}_1,t) \\ & + (1/2)\big(B_{ij}(\boldsymbol{r}-\boldsymbol{r}_1)a_i(\boldsymbol{r},t)a_j(\boldsymbol{r}_1,t) + \text{c.c.}\big)\big] \;.\end{aligned} \tag{1.1.21}$$

Now

$$A_{ij}(\boldsymbol{R}) = A_{ji}^*(-\boldsymbol{R}) \;, \qquad B_{ij}(\boldsymbol{R}) = B_{ji}(-\boldsymbol{R}) \;.$$

Normal variables are introduced by diagonalizing the Hamiltonian (1.1.21), which results in the reduction of the Hamiltonian for a stable medium to the form

$$\mathcal{H} = \sum_j \int \omega_j(\boldsymbol{k}) b_j(\boldsymbol{k},t) b_j^*(\boldsymbol{k},t)\, d\boldsymbol{k} \;. \tag{1.1.22}$$

Diagonalization may be accomplished if all $\omega_j(\boldsymbol{k})$ have the same sign and do not identically coincide.

1.1.2 Interaction Hamiltonian for Weak Nonlinearity

In various problems of nonlinear wave dynamics, the wave amplitude may be defined by a natural dimensionless parameter ξ. For sound waves, this parameter is represented by the ratio of the density variation in the sound wave to the average density of the medium; for fluid surface waves it is the ratio of the wave height to wavelength. For spin waves, ξ is the precession angle of the magnetic moment. For ξ of the order of unity, phenomena specific for each of the above media arise: sound turns into shock waves, fluid surface waves form whitecaps and in ferromagnets an inversion of magnetization occurs, that is, a movable domain wall. Obviously, consideration of all these phenomena from a general viewpoint is not always constructive. If, however, the wave's nonlinearity parameter ξ is small, the characteristic features of the medium are negligible, and the wave dynamics may be described in general terms by expanding the Hamiltonian in terms of canonical variables. Let us now look in greater detail at the expansion we started to analyze in the preceding subsection. Suppose we have only a single wave type in a stable medium. Then the first term of the Hamiltonian expansion has the form (1.1.17), and the corresponding equation of motion may be written as

$$\frac{\partial b(\boldsymbol{k},t)}{\partial t} + i\omega(\boldsymbol{k})b(\boldsymbol{k},t) = 0, \quad b(\boldsymbol{k},t) = b(\boldsymbol{k})\exp[-i\omega(\boldsymbol{k})t] .$$

At this level of sophistication, waves in different media are only distinguished by their dispersion laws $\omega(\boldsymbol{k})$. All information about the wave interaction is contained in the higher coefficients of the expansion of $\mathcal{H}$ in a power series of b:

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_{\text{int}}, \qquad \mathcal{H}_{\text{int}} = \mathcal{H}_3 + \mathcal{H}_4 + \dots . \tag{1.1.23}$$

The physical meaning of $\mathcal{H}_3$ and $\mathcal{H}_4$ is easy to understand by analogy with quantum mechanics. The Hamiltonian $\mathcal{H}_3$ describes three-wave processes:

$$\begin{aligned}\mathcal{H}_3 = &\frac{1}{2}\int(V_q b_1 b_2 b_3 + \text{c.c.})\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \\ &+ \frac{1}{6}\int(U_q b_1^* b_2^* b_3^* + \text{c.c.})\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 .\end{aligned} \tag{1.1.24a}$$

Here and below a shorthand notation is be used: b_1, b_2 are $b(\boldsymbol{k}_1, t), b(\boldsymbol{k}_2, t)$; $q = (\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ and thus $V_q = V_{123} = V(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$. The first term in (1.1.24) defines the decay processes $1 \to 2$ and the reverse confluence processes $2 \to 1$. The second term describes mutual annihilation of three waves or their creation from vacuum.

The Hamiltonian $\mathcal{H}_4$ describes processes involving four waves:

$$\begin{aligned}\mathcal{H}_4 = (1/4)&\int W_p b_1^* b_2^* b_3 b_4 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 \\ &+ \int (G_p b_1 b_2^* b_3^* b_4^* + \text{c.c.})\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 \\ &+ \int (R_p^* b_1 b_2 b_3 b_4 + \text{c.c})\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 + \boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 \,, \\ &\text{with}\quad p = (\boldsymbol{k}_1 \boldsymbol{k}_2 \boldsymbol{k}_3 \boldsymbol{k}_4) \,. \end{aligned} \tag{1.1.24b}$$

The coefficients of the interaction Hamiltonian have the following obvious properties

$$\begin{aligned} V(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) &= V(\boldsymbol{k}_1, \boldsymbol{k}_3, \boldsymbol{k}_2), \\ U(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) &= U(\boldsymbol{k}_1, \boldsymbol{k}_3, \boldsymbol{k}_2) = U(\boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k}_3), \\ G(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) &= G(\boldsymbol{k}_1, \boldsymbol{k}_3, \boldsymbol{k}_2, \boldsymbol{k}_4) = G(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_4, \boldsymbol{k}_3), \\ W(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) &= W(\boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k}_3, \boldsymbol{k}_4) \\ &= W(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_4, \boldsymbol{k}_3) = W^*(\boldsymbol{k}_3, \boldsymbol{k}_4, \boldsymbol{k}_1, \boldsymbol{k}_2), \\ R(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_3) &= R(\boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k}_3, \boldsymbol{k}_4) = R(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_4, \boldsymbol{k}_3) \\ &= R(\boldsymbol{k}_3, \boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k}_4) = R(\boldsymbol{k}_4, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_1) \,. \end{aligned} \tag{1.1.25}$$

The last equation for $W(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)$ follows since the Hamiltonian is real.

But to which order in b, b^* should we expand the Hamiltonian $\mathcal{H}$? The answer turns out to be as general as the question: "$\mathcal{H}_5$ and higher-order terms should normally not be taken into account." This can be explained as follows: Since the expansion uses a small parameter, every subsequent term is smaller than the preceding one, and the dynamics of the wave system is determined by the very first term of $\mathcal{H}_{\text{int}}$, i.e., normally $\mathcal{H}_3$. However, three-wave processes may turn out to be "nonresonant" which means that the spatio-temporal synchronization condition (or, in terms of quasi-particles, the momentum-energy conservation law)

$$\omega(\boldsymbol{k} + \boldsymbol{k}_1) = \omega(\boldsymbol{k}) + \omega(\boldsymbol{k}_1) \tag{1.1.26}$$

cannot be satisfied. Let d be the dimensionality of the medium and $\boldsymbol{k}$ the vector in d-meric space ($d > 1$). Equation (1.1.26) specifies the hypersurface of dimension $2d - 1$ in the $2d$-meric space of vectors $\boldsymbol{k}, \boldsymbol{k}_1$. If this surface does in fact exist [i.e., $\omega(\boldsymbol{k})$ is real], the dispersion law $\omega(\boldsymbol{k})$ is of the decay type. If (1.1.26) has no real solutions, the dispersion law is of the nondecay type.

In isotropic media, $\omega(\boldsymbol{k})$ is a function of k only. Let $\omega(0) = 0$, $\omega' = \partial\omega(\boldsymbol{k})/\partial k > 0$. In this important case a simple criterion for the decay may be formulated: the dispersion law is of the decay type if $\omega'' = \partial^2\omega(\boldsymbol{k})/\partial k^2 > 0$ and of the nondecay type if $\omega'' < 0$. This criterion has a clear geometric meaning. Consider, for example, the case with $d = 2$. The dispersion law $\omega(\boldsymbol{k})$ then specifies the surface of rotation in a three-dimensional space $\omega, \boldsymbol{k}_x, \boldsymbol{k}_y$. In Fig.1.1 this is the surface S for $\omega(\boldsymbol{k})$ and S_1 for $\omega(\boldsymbol{k}_1)$. It is seen that (1.1.26) is satisfied if these surfaces intersect, then all the three points $(\omega, \boldsymbol{k})$, $(\omega_1, \boldsymbol{k}_1)$ and

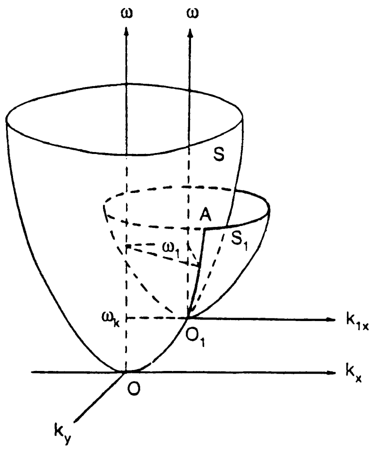

Fig. 1.1. Intersection of the frequency surfaces of the convex function $\omega(k)$

$\big(\omega(\boldsymbol{k}+\boldsymbol{k}_1), \boldsymbol{k}+\boldsymbol{k}_1\big)$ lie on the intersection line O_1A. Surface section requires a convex function, i.e., $\omega'' > 0$. For power functions $\omega(\boldsymbol{k}) \propto k^\alpha$, the dispersion law is of the decay type if $\alpha > 1$. In the limiting case at $\alpha = 1$, decays are only allowed for collinear vectors.

Four-wave processes are always permitted. This is evident from the conservation laws for scattering processes

$$\omega(\boldsymbol{k}) + \omega(\boldsymbol{k}') = \omega(\boldsymbol{k}+\boldsymbol{\kappa}) + \omega(\boldsymbol{k}' - \boldsymbol{\kappa}), \tag{1.1.27}$$

which are allowed at $\kappa \to 0$ for any dispersion laws. Consequently, it is $\mathcal{H}_4$ that will govern the dynamics of wave systems with a nondecay dispersion law. Subsequent expansion terms $\mathcal{H}_5, \ldots$ will define small and, generally, insignificant corrections. However, nature is certainly more interesting than this formal scheme. Some problems may involve an additional small parameter (apart from the nonlinearity level). For example, for spin waves in magnets (Sect.1.4), not only $\mathcal{H}_3$ but also $\mathcal{H}_4$ should be taken into account, even in the decay region of the spectrum. The Hamiltonian $\mathcal{H}_3$ arises only because of the magnetic dipole-dipole interaction and has relativistic smallness as compared with the Hamiltonian $\mathcal{H}_4$ which results from the exchange interaction. The authors are aware of problems where one has to take into consideration the fifth- and even sixth-order terms (for example, while considering an approximate conservation of the wave action integral, see below). These problems are, however, rather specialized and go beyond the frame of this book. Hence, it is usually sufficient to expand the Hamiltonian in the decay region of the spectrum up to third-order, and in the nondecay region up to fourth-order terms.

1.1.3 Dynamic Perturbation Theory. Elimination of Nonresonant Terms

It is intuitively clear that in the case of a nondecay dispersion law, the Hamiltonian $\mathcal{H}_3$ describing three-wave processes may turn out to be irrelevant in some respect. We shall show now that in this case one can go over to new canonical variables c_k, c_k^*, such that $\mathcal{H}_3\{c_k, c_k^*\} = 0$. This is possible because the dynamic system under consideration, the weakly nonlinear wave field, is close to a completely integrable dynamic system (a set of noninteracting oscillators). Traditionally classical perturbation theory is employed to handle systems close to completely integrable ones. In this procedure a canonical transformation is derived by sequentially excluding the nonintegrable terms from the Hamiltonian. It is known that the procedure may encounter the problem of "small resonance denominators"; then the only terms to be excluded from the Hamiltonian are those for which the resonance condition is not satisfied. As shown by *Zakharov* [1.1], we can in this case to some extent apply classical perturbation theory.

Let us demonstrate such a procedure using a simple example. Consider the expansion of the one-wave Hamiltonian:

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 = \omega bb^* + \frac{V}{2}(b^2 b^* + b^{*2} b) + \frac{U}{6}(b^3 + b^{*3})$$
$$+ \frac{W}{4} b^2 b^{*2} + G(b^3 b^* + bb^{*3}) + R(b^4 + b^{*4}).$$

We assume every subsequent term to be smaller than the preceding one, i.e. $\mathcal{H}_2 \gg \mathcal{H}_3 \gg \mathcal{H}_4$. Since one needs to eliminate the $\mathcal{H}_3$ term without changing $\mathcal{H}_2$, the transformation must be close to the unity transformation. Thus it is reasonable to search for the transformation in the form of an expansion, which starts from a linear term:

$$b = c + A_1 c^2 + A_2 cc^* + A_3 c^{*2} + B_1 c^3 + B_2 c^* c^2 + B_3 cc^{*2} + B_4 c^{*3} + \ldots \quad (1.1.28a)$$

Why do we take the following (c^3-order) terms into account? Taking only the linear and quadratic terms in (1.1.28a) into account is indeed sufficient to eliminate the $\mathcal{H}_3$ term. In that case the fourth-order terms govern the nonlinear interaction. Due to the transformation they will acquire additional terms. To derive these new terms c^3-order terms in (1.1.28a) have to be taken into account. Moreover, we will use them to exclude the last two terms describing the $1 \to 3$ and $0 \to 4$ processes in the Hamiltonian (1.1.24b).

Thus we look for seven coefficients: A_1, A_2, A_3, $B_1, \ldots, B_4$. The canonicity condition (1.1.7) is expressed through the Poisson brackets and has the form

$$\{bb^*\} = \frac{\partial b}{\partial c}\frac{\partial b^*}{\partial c^*} - \frac{\partial b^*}{\partial c}\frac{\partial b}{\partial c^*} = 1.$$

Computing the Poisson bracket to an accuracy of c^3-order terms, we obtain three equations

$$A_2 = -2A_1, \quad B_2 = A_3^2 - A_1^2, \quad B_3 + 3B_1 = 2A_2(A_3 - A_1)\ .$$

Substituting (1.1.28a) into the Hamiltonian and demanding that all nonlinear terms (except c^2c^{*2}) vanish, we have four equations

$$2\omega(A_1 + A_2) + V = 0, \quad 6\omega A_3 + U = 0,$$
$$\omega B_4 + \omega A_1 A_3 + \frac{1}{2}VA_3 + \frac{1}{2}UA_1 + R = 0,$$
$$\omega(A_1A_2 + A_2A_3 + B_1 + B_3) + VA_3 + \frac{1}{2}UA_2 + G = 0.$$

From these it is easy to find the transformation coefficients

$$\begin{aligned} A_1 &= \frac{V}{2\omega}, \quad A_2 = -\frac{V}{\omega}, & A_3 &= -\frac{U}{6\omega}, \\ B_1 &= \frac{V^2}{4\omega^2} + \frac{VU}{6\omega^2} + \frac{G}{2\omega}, & B_2 &= \frac{U^2}{36\omega^2} - \frac{V^2}{4\omega^2}, \\ B_3 &= \frac{V^2}{4\omega^2} + \frac{7UV}{12\omega^2} - \frac{3G}{2\omega}, & B_4 &= -\frac{UV}{12\omega^2} - \frac{R}{\omega}. \end{aligned} \tag{1.1.28b}$$

In the new variables the Hamiltonian has the simple form

$$\mathcal{H} = \omega cc^* + \frac{1}{4}Tc^2c^{*2}, \qquad T = W - \frac{3V^2}{\omega} - \frac{U^2}{3\omega}\ .$$

It is easily seen that neglect of the cubic terms in (1.1.28a), would have given wrong values of the additional interaction coefficients supplementing W.

Following the same pattern, let us return to the general case and use a transformation in the form of a power series to eliminate cubic and nonresonant fourth-order terms. In the new variables the Hamiltonian of interaction describes $2 \to 2$ processes (for details see Sect. A.3):

$$\begin{aligned} \mathcal{H}_4 &= \frac{1}{4}\int T_p c_1^* c_2^* c_3 c_4 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4, \\ p &= (\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4), \quad T_p = W_p + T'_p. \end{aligned} \tag{1.1.29a}$$

$$\begin{aligned} T'_p = &-\frac{U_{-1-212}U_{-3-434}}{\omega_3 + \omega_4 + \omega_{3+4}} + \frac{V^*_{1+212}V_{3+434}}{\omega_1 + \omega_2 - \omega_{1+2}} \\ &-\frac{V^*_{131-3}V_{424-2}}{\omega_{4-2} + \omega_2 - \omega_4} - \frac{V^*_{242-4}V_{313-1}}{\omega_{3-1} + \omega_1 - \omega_3} \\ &-\frac{V^*_{232-3}V_{414-1}}{\omega_{4-1} + \omega_1 - \omega_4} - \frac{V^*_{141-4}V_{323-2}}{\omega_{3-2} + \omega_2 - \omega_3}. \end{aligned} \tag{1.1.29b}$$

Here $(j \pm i) = k_j \pm k_i$.

Note that (1.1.29b) is true on the resonant surface

$$\omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) = \omega(\boldsymbol{k}_3) + \omega(\boldsymbol{k}_4), \qquad \boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4$$

only, where the coefficient T'_p has the same properties (1.1.25) as W_p. The necessity of taking cubic terms into account for the transformation to yield the correct value of the four-wave interaction coefficient was first pointed out by *Krasitskii* [1.2].

Let us discuss the singularities of (1.1.29). The denominators become zero on the resonance surfaces of three-wave processes:

$$\boldsymbol{k} + \boldsymbol{k}_1 + \boldsymbol{k}_2 = 0, \qquad \omega_k + \omega_1 + \omega_2 = 0 \tag{1.1.30a}$$

and

$$\boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2, \qquad \omega_k = \omega_1 + \omega_2 \;. \tag{1.1.30b}$$

The conditions (1.1.30a) can only be satisfied if the medium allows for negative-energy waves. Without such waves, denominators of the type (1.1.30a) do not vanish and the corresponding terms in the three-wave Hamiltonian may be eliminated. The condition of nonzero denominators of the type (1.1.30b) coincides with the nondecay condition for the dispersion law $\omega(k)$. In the nondecay case, the cubic terms in the Hamiltonian may thus be completely excluded. The same holds for the terms in the fourth-order Hamiltonian (1.1.24b) differing in their form from (1.1.29). Prohibition of the $1 \to 2$ and $2 \to 1$ processes implies in general that the $1 \to 3$ and $3 \to 1$ processes are not feasible. One can definitely state that the interaction Hamiltonian of type (1.1.29) is a fundamental model for considering nonlinear processes in media that obey a nondecay dispersion law. Additional terms in it may be interpreted as scattering processes that arise in the second order perturbation theory for three-wave processes. In that case, a virtual forced wave appears at an intermediate stage for which the resonance condition is not satisfied. In this interpretation, every term in (1.1.29b) may be juxtaposed with a picture (see Fig. 1.2) to illustrate which particular process is meant.

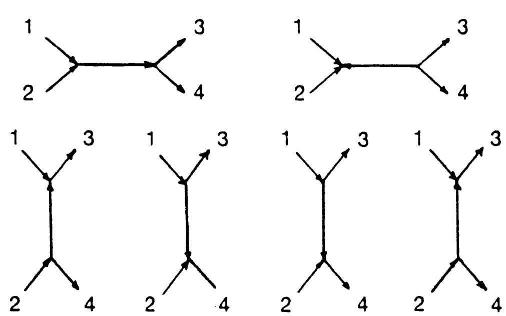

Fig. 1.2. The processes corresponding to different terms in (1.1.29b)

On the resonance manifold (1.1.27) an attempt to exclude the Hamiltonian term (1.1.29) by using a canonical transformation will lead to small denominators describing wave-wave scattering. These processes are allowed irrespective of the type of the dispersion law; hence, it is impossible to exclude this term from the Hamiltonian. The Hamiltonian (1.1.29) may be said to describe wave scattering.

Scattering processes possess an important feature: they do not change the total number of waves. Therefore the equations of motion corresponding to Hamiltonian (1.1.29) preserve one more integral besides energy, namely

$$N = \int c_k^* c_k \, dk ,$$

which gives the total number of quasi-particles. This is known as the wave action integral. A complete system that has to include minor effects associated with higher-order processes generally preserves the N value only approximately.

One can similarly exclude the cubic terms corresponding to nondecay three-wave processes in the Hamiltonian specifying wave interactions of several types.

1.1.4 Dimensional Analysis of the Hamiltonian Coefficients

As stated above, at weak nonlinearity the Hamiltonian of a system of interacting waves has the standard form (1.1.17, 24, 25). Is it possible to evaluate the coefficients ω_k, V_{123}, T_{1234} without considering the specific nature of every particular problem and to understand, e.g., how these values depend on the wave vectors? The answer is positive if the parameters specifying waves of this type do not yield a quantity with the dimension of a length. In this case the problem is said to possess complete self-similarity (first-order self-similarity). The Hamiltonian coefficients are estimated by dimensional considerations.

Let us first obtain the dimensions of the canonical variables b_k and the coefficients of the interaction Hamiltonian V_{123} and T_{1234}. Bearing in mind that $\mathcal{H}$ has the dimension of an energy density and ω_k the one of a frequency, the dimension of b_k is found by (1.1.18).

$$[\mathcal{H}] = \mathrm{g} \cdot \mathrm{cm}^{2-d} \cdot \mathrm{s}^{-2}, \quad [\omega] = \mathrm{s}^{-1}, \quad [b_k] = \mathrm{g}^{1/2} \cdot \mathrm{cm} \cdot \mathrm{s}^{-1/2} . \tag{1.1.31}$$

Here d is the dimensionality of the medium, cm = centimeter, g = gramm, s = second. In view of the fact that $[\omega] = [V_{123} b] = [T_{1234} b^2]$, it is easy to establish

$$[V_{123}] = \mathrm{g}^{-1/2} \cdot \mathrm{cm}^{d/2-1} \cdot \mathrm{s}^{-1/2}, \quad [T_{1234}] = \mathrm{g}^{-1} \cdot \mathrm{cm}^{d-2} . \tag{1.1.32}$$

As to be expected, the dimension of $V b^2$ (here V is the volume of the system!) coincides with the one of Planck's constant $\hbar$. Naturally, our classical approach is true when the quantum-mechanical occupation numbers $N(k) = V b^2/\hbar$ are large compared with unity. On the other hand, wave amplitudes b_k should not be too large for the interaction Hamiltonian $\mathcal{H}_{\mathrm{int}}$ to remain small compared with $\mathcal{H}_2$. This gives an upper estimate for b_k. It may be schematically written as

$$\omega_k \gg V_{kkk} \sum_{k'} b_{k'} . \tag{1.1.33}$$

If we introduce the dimensionless wave amplitude

$$\xi_k = b_k / B_k, \quad B_k = |\omega_k / V_{kkk}|, \tag{1.1.34}$$

the weak nonlinearity condition may be written as

$$\xi_k \ll 1 . \tag{1.1.35}$$

Now we can discuss some particular examples.

Sound in Continuous Media. As parameters the equations of motion for this problem may include only the medium's density ϱ and the elasticity coefficient κ with the respective dimensions $[\varrho] = \mathrm{g}\cdot\mathrm{cm}^{-3}$ and $[\kappa] = \mathrm{g}\cdot\mathrm{cm}^{-1}\cdot\mathrm{s}^{-2}$. These values and the wave vector k combine to yield the dimension of a frequency $[\omega] = \mathrm{s}^{-1} = [\varrho^x\kappa^y k^z] = \mathrm{g}^{x+y}\cdot\mathrm{cm}^{-3x-y+z}\cdot\mathrm{s}^{-2y}$. Equating the exponents at g, cm and s, we have three equations $x+y=0$, $3x+y+z=0$, and $2y=1$. Hence $x=-1/2$, $y=1/2$, $z=1$. Thus the dimensional analysis leads to a linear law for the wave dispersion

$$\omega_k = c_s k, \quad c_s = a(\kappa/\varrho)^{1/2}. \tag{1.1.36}$$

Here c_s is the sound velocity and a a dimensionless parameter of the order of unity. From the parameters of our problem, one can also obtain B_k with the dimension of the canonical variable b_k

$$B_k = (\varrho c_s/k)^{1/2} \tag{1.1.37}$$

and the interaction coefficient

$$V_{123} = \sqrt{\frac{k_1k_2k_3c_s}{\varrho}}\, f\left(\frac{\boldsymbol{k}_1}{k_1}, \frac{\boldsymbol{k}_2}{k_1}, \frac{\boldsymbol{k}_3}{k_1}\right). \tag{1.1.38}$$

Here the dimensionless function f depends on eight dimensionless arguments: the two ratios k_2/k_1 and k_3/k_1, and six angle variables giving the directions of the three vectors. In fact, there are only three angle variables: $\cos\theta_{12}$, $\cos\theta_{13}$, and $\cos\theta_{23}$ [here $\cos\theta_{ij} = (\boldsymbol{k}_i\boldsymbol{k}_j)/k_ik_j$], as our system has no preferred direction.

In the Hamiltonian description, the wave amplitude is proportional to b_k. In the sound wave field, medium density $\varrho(r,t) = \varrho_0 + \varrho_1(r,t)$ and velocity oscillate:

$$\begin{aligned} \varrho_1(r,t) &= \mathrm{Re}[\varrho_k \exp(ikr - i\omega_k t)], \\ v(r,t) &= \mathrm{Re}[v_k \exp(ikr - i\omega_k t)] , \end{aligned} \tag{1.1.39}$$

where ϱ_k, v_k are the respective wave amplitudes in natural variables. In the linear approximation the relationship between natural and normal canonical variables is easily established from dimensional considerations

$$\varrho_k \propto (k\varrho_0/c_s)^{1/2} b_k, \quad v_k \propto (kc_s/\varrho_0)^{1/2} b_k. \tag{1.1.40}$$

The symbol "$\propto$" designates proportionality. In terms of canonical variables the condition of weak nonlinearity is written as

$$\xi_k \simeq \varrho_k/\varrho_0 \simeq v_k/c_s \ll 1. \tag{1.1.41}$$

The symbol "$\simeq$" denotes an estimate to an accuracy of a dimensionless factor of order unity.

Gravitational Waves on a Fluid Surface. These are relatively long waves for which surface tension is insignificant and the force tending to restore the equilibrium state of the surface is the gravitational force. Apart from fluid density ϱ the significant parameters should evidently include the gravitational acceleration g, $[g] = \mathrm{cm} \cdot \mathrm{s}^{-2}$. Following the scheme given in the preceding example and bearing in mind that this is a 2-dimensional problem ($d = 2$), we have:

$$\omega_k = \sqrt{gk}, \quad B_k = \left(\varrho^2 g k^{-5}\right)^{1/4}. \tag{1.1.42}$$

As we see, the dispersion law is of the nondecay type, $\omega_k \propto k^\alpha$, $\alpha = 1/2 < 1$. Therefore the principal interaction is four-wave, with the interaction coefficient

$$T_{k123} = \frac{k^3}{\varrho} f[(k_1/k), (k_2/k), (k_3/k), \cos\theta_{k1}, \cos\theta_{k2}, \cos\theta_{k3}]. \tag{1.1.43}$$

A natural variable describing water waves is $\eta(r)$, the deviation of the fluid surface from the unperturbed state and the dimensionless wave amplitude is $\xi_k = k\eta_k = b_k/B_k$. Whence, we obtain

$$\eta_k = (k/\varrho^2 g)^{1/4} b_k. \tag{1.1.44}$$

Capillary Waves. For sufficiently short waves the restoring force should be entirely determined by surface tension. The significant parameters should in this case instead of g include the surface tension coefficient σ having the dimension of a surface energy $[\sigma] = g \cdot s^{-2}$. Thus,

$$\omega_k = \sqrt{\frac{\sigma k^3}{\varrho}}, \quad B_k = \left(\varrho\sigma/k^3\right)^{1/4}, \quad \eta_k = (\varrho\sigma k)^{-1/4} b_k. \tag{1.1.45}$$

The dispersion law of capillary waves is of the decay type: $\alpha = 3/2 > 1$. Therefore the three-wave interaction remains as the most essential one

$$V_{k12} = \left(\frac{\sigma k^9}{\varrho^3}\right)^{1/4} f(k_1/k, k_2/k, \cos\theta_{k1} \cos\theta_{k2}) . \tag{1.1.46}$$

Comparing the dispersion laws of capillary (1.1.45) and gravitational waves (1.1.42), it is easy to find the boundary value of the wave vector at which these frequencies coincide:

$$k_* = \sqrt{\frac{\varrho g}{\sigma}} . \tag{1.1.47}$$

At $k \ll k_*$, the gravitational energy of a wave is larger than the surface tension energy and the latter may be neglected. Thus long waves on a fluid surface will be gravitational. Accordingly, at $k \gg k_*$ the surface waves will be capillary with the dispersion law (1.1.45). We shall show in Sect. 1.2 below that at arbitrary k's the dispersion law for waves on the surface of a deep fluid is expressed as

$$\omega_k = (gk + \sigma k^3/\varrho)^{1/2}. \tag{1.1.48}$$

Though dimensional estimates usually give answers to the accuracy of a dimensionless factor of the order of unity, the dispersion laws (1.1.42,45) are accurate in the limits of large and small wavelength, respectively.

For water waves at room temperature, $k_* \simeq 4\,\text{cm}$ ($\varrho = 1\text{g/cm}^3$, $\sigma = 70\,\text{g/s}^2$). This corresponds to a wavelength of $\lambda = 2\pi/k_* \simeq 1.6$ cm and frequency $f_* = \omega_*/2\pi \simeq 0.2$ Hz.

Vortex Motions of Incompressible Fluids. From the viewpoint of dimensional analysis, this problem radically differs from the preceding problems in that it has only one significant parameter, the fluid density $[\varrho_0] = \text{g/cm}^3$. Still, this allows to determine the Hamiltonian structure. In particular, since it is impossible to build from ϱ_0 and k a combination with the dimension of a frequency, it should not contain an $\mathcal{H}_2 = \int \omega_k a_k^* a_k \, dk$ term, i.e., $\mathcal{H}_2 = 0$. Using (1.1.31–32), one can easily see that among all factors of the $\mathcal{H}$-expansion into a power series, only the coefficient at a^4 has a dimension containing no time. Therefore, only this factor may be derived from ϱ_0 and k:

$$T_{k123} \simeq \frac{k^2}{\varrho_0}. \tag{1.1.49}$$

It immediately follows that the nonlinearity of incompressible fluid motion is extremely strong, $\xi = \mathcal{H}_4/\mathcal{H}_2 \to \infty$. Another important consequence of dimensional analysis is the nonlinear relationship between fluid velocity v_k and canonical variables a_k. If we formally represent v_k as a power series of a_k:

$$v_k = \sum_{i=1}^{\infty} \phi_i(k) a_k^i \,,$$

the dimension of a single coefficient ϕ_2 will contain no time. From this, $\phi_1 = \phi_3 = \ldots = 0$, $\phi_2 = k/\varrho_0$. Therefore

$$v_k \simeq k a_k^2/\varrho_0 \,. \tag{1.1.50}$$

It is now clear why the Hamiltonian of the problem $\mathcal{H} = (1/2)\int |v(r)|^2 \, dr$ is proportional to the fourth power of the canonical variables a_k, a_k^*.

This section has given the general structure of canonical equations of motion for weakly nonlinear waves. The remaining sections of this chapter deal with various specific systems, the introduction of canonical variables and calculation of Hamiltonian coefficients. Readers who are not interested in the character of waves in different media and the technique for deriving the canonical equations may go over directly to Chap. 2 where the kinetic wave equation is obtained from the dynamic equations given in Sect. 1.1. The paragraphs left out in the first reading, may then be referred to when evaluating the coefficients of the Hamiltonian.

1.2 The Hamiltonian Formalism in Hydrodynamics

The ideal incompressible fluid is the simplest and most important representative of a wide class of dynamic systems of the hydrodynamic type and is thus widely used in physical problems. For zero dissipation all these systems possess an implicit Hamiltonian structure. The description of such structures and the related group-theoretical formulations constitute a formidable mathematical problem extending far beyond the scope of this book (those interested in it are referred to [1.3]). For our purposes it will be sufficient to discuss the introduction of canonical variables only for several cases that are most important for the turbulence theory. Appropriate canonical variables for an imcompressible fluid were first presented by *Clebsh* (see [1.4]) in the last century. Independently *Bateman* [1.5] and later on *Davydov* [1.6] gave the canonical variables for barotropic flows in incompressible fluids with single-valued functions for the pressure. These results will be discussed in Sect. 1.2.1. Further on we shall obtain the Hamiltonians for vortex motion $\mathcal{H}_V$ (Sect. 1.2.2), for small-amplitude (potential) motion of sound $\mathcal{H}_S$ (Sect. 1.2.3) and for sound-vortex interactions $\mathcal{H}_{SV}$(Sect. 1.2.4). Fluid stratification gives rise to new types of motions localized in the regions of maximal inhomogenities. In the extremely nonuniform case of the free surface of a fluid, these are the known surface waves. The canonical variables for them were obtained by *Zakharov* [1.7]. For the general case with arbitrary wavelength and fluid depth the Hamiltonian description of this type of motion will be given in Sects. 1.2.5,6.

1.2.1 Clebsh Variables for Ideal Hydrodynamics

Consider the Euler equations for compressible fluids:

$$\partial\varrho/\partial t + \operatorname{div}\varrho\boldsymbol{v} = 0\,, \tag{1.2.1a}$$

$$\partial\boldsymbol{v}/\partial t + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\nabla p(\varrho)/\varrho\,. \tag{1.2.1b}$$

Here $\boldsymbol{v}(r,t)$ is the Eulerian fluid velocity (in the point $\boldsymbol{r}$ at the moment of time t); $\varrho(\boldsymbol{r},t)$ the density and $p(\boldsymbol{r},t)$ is the pressure which, in the general case, is a function of fluid density and specific entropy s, i.e., $p = p(\varrho, s)$. In ideal fluids where there is neither viscosity nor heat exchange, the entropy per unit volume is carried by the fluid, i.e., it obeys $\partial s/\partial t + (\boldsymbol{v}\nabla)s = 0$. A fluid in which the specific entropy is constant throughout the volume is called *barotropic*. In such a fluid the pressure is a single-valued function of the density $p = p(\varrho)$. In this case, $\nabla p/\varrho$ may be expressed via the gradient of the specific enthalpy of the unit mass $w = E + PV$ and $dw = VdP = dP/\varrho$. Thus, $\nabla p/\varrho = \nabla w$. The enthalpy in turn equals the derivative of the internal energy of the unit volume $\varepsilon(\varrho) = E\varrho$ with respect to the fluid density

$$w = \frac{\delta\varepsilon}{\delta\varrho} . \tag{1.2.1c}$$

Direct differentiation with respect to time shows that (1.2.1) conserves the full energy of the fluid

$$\mathcal{H} = \int [\varrho v^2/2 + \varepsilon(\varrho)]\, d\boldsymbol{r} . \tag{1.2.2}$$

In line with Thomson's theorem, these equations also conserve the velocity circulation around a "fluid" path. This means that there exists a scalar function $\mu(\boldsymbol{r}, t)$ which moves together with the fluid:

$$d\mu/dt = [\partial/\partial t + (\boldsymbol{v}\nabla)]\mu = 0 . \tag{1.2.3}$$

In our search for the canonical variables for the Euler equations (1.2.1), we shall use the Lagrangian approach. For that purpose, we shall consider the known expression for the Lagrangian of a mechanical system (kinetic minus potential energy), generalized for the continuous case and use as external constraints the continuity equation (1.2.1a) and Thomson's theorem (1.2.3):

$$\begin{aligned}\mathcal{L}(t) = \int \Bigg[\varrho\frac{v^2}{2} - \varepsilon(\varrho) + \Phi\left(\frac{\partial\varrho}{\partial t} + \operatorname{div}\varrho\boldsymbol{v}\right) \\ -\lambda\left(\frac{\partial\mu}{\partial t} + (\boldsymbol{v}\nabla)\mu\right)\Bigg] d\boldsymbol{r} .\end{aligned} \tag{1.2.4}$$

Φ and λ are the undetermined Lagrange multipliers. Variation with respect to them leads to (1.2.1a) and (1.2.3). A single integration by parts allows to rewrite the Lagrangian (1.2.4) as

$$\begin{aligned}\mathcal{L} = \int \Big\{ \Phi\frac{\partial\varrho}{\partial t} - \lambda\frac{\partial\mu}{\partial t} + \varrho\frac{v^2}{2} \\ - [\boldsymbol{v}(\varrho\nabla\Phi + \lambda\nabla\mu)] - \varepsilon(\varrho)\Big\}\, d\boldsymbol{r} .\end{aligned} \tag{1.2.5}$$

Consider the action $\mathcal{S} = \int \mathcal{L}\, dt$. Due to its extremality, the condition $\delta\mathcal{S}/\delta\boldsymbol{v} = 0$ should be satisfied, which is equivalent to the condition $\delta\mathcal{L}/\delta\boldsymbol{v} = 0$. From (1.2.5), we have

$$\boldsymbol{v} = \lambda\frac{\nabla\mu}{\varrho} + \nabla\Phi . \tag{1.2.6}$$

As the Lagrangian (1.2.5) and (1.2.6) do not contain time derivatives we can substitute (1.2.6) into (1.2.5) to arrive at the Lagrangian of the Hamiltonian system,

$$\mathcal{L} = \int (\Phi \partial \varrho / \partial t - \lambda \partial \mu / \partial t)\, d\boldsymbol{r} \; - \; \mathcal{H} \; . \tag{1.2.7}$$

$$\partial \varrho / \partial t = \delta \mathcal{H} / \delta \Phi \; , \tag{1.2.8}$$

$$\partial \Phi / \partial t = -\delta \mathcal{H} / \delta \varrho \; , \tag{1.2.9}$$

$$\partial \lambda / \partial t = \delta \mathcal{H} / \delta \mu \; , \tag{1.2.10}$$

$$\partial \mu / \partial t = -\delta \mathcal{H} / \delta \lambda \; , \tag{1.2.11}$$

for which the pairs (ϱ, Φ) and (λ, μ) are pairs of canonically-conjugate variables. Using (1.2.2) and (1.2.6) it is easy to compute the variational derivatives with respect to ϱ and λ,

$$\begin{aligned} \frac{\delta \mathcal{H}}{\delta \varrho} &= \frac{(\nabla \Phi)^2}{2} - \frac{(\lambda \nabla \mu)^2}{2\varrho^2} + \frac{\delta \varepsilon}{\delta \varrho} \\ &= (\boldsymbol{v} \nabla) \Phi - v^2/2 + w \; , \\ \frac{\delta \mathcal{H}}{\delta \lambda} &= \varrho [\boldsymbol{v}, \partial \boldsymbol{v} / \partial \lambda] = (\boldsymbol{v} \nabla) \mu \; , \end{aligned}$$

$[\boldsymbol{a}, \boldsymbol{b}] = \boldsymbol{a} \times \boldsymbol{b}$ denotes the vector product. In calculating the derivatives with respect to Φ and μ, one has to integrate by parts.

$$\delta \mathcal{H} / \delta \Phi = -\mathrm{div}\, \varrho \boldsymbol{v}, \quad \delta \mathcal{H} / \delta \mu = -\mathrm{div}\, \lambda \boldsymbol{v} \; .$$

Thus (1.2.9) and (1.2.10) have the forms

$$\partial \Phi / \partial t + (\boldsymbol{v} \nabla \Phi)/2 - v^2/2 + w = 0 \; ,$$

$$\partial \lambda / \partial t + \mathrm{div}\, \lambda \boldsymbol{v} = 0 \; .$$

As to be expected (1.2.8) and (1.2.11) coincide with the continuity equations (1.2.1a) and (1.2.3), respectively.

Let us consider now to what extent the system of equations (1.2.8–11) is equivalent to the initial hydrodynamic system for the three components of velocity and density. The solvability of (1.2.8–11) should imply that (1.2.1) are satisfied for the velocity given by (1.2.6). This may be verified by direct calculation of the $\partial \boldsymbol{v} / \partial t$ derivative. The question of reverse correspondence reduces to the following one: can we always represent the velocity field $\boldsymbol{v}(\boldsymbol{r}, t)$ in the form of (1.2.6)? To answer this question, we calculate the vorticity. From (1.2.6), we have

$$\mathrm{rot}\, \boldsymbol{v} = [\nabla(\lambda/\varrho), \nabla \mu] = [\nabla \vartheta, \nabla \mu], \quad \vartheta = \lambda / \varrho \; . \tag{1.2.12}$$

Evidently, $\mathrm{div}\, [\nabla \vartheta, \nabla \mu] = 0$. Now we introduce the $q_s = (\boldsymbol{v}\ \mathrm{rot}\, \boldsymbol{v})$ value, the helicity density of the velocity field. From (1.2.6) and (1.2.16), we have

$$q_s = \big(\boldsymbol{v}[\nabla \vartheta, \nabla \mu]\big) = \big(\nabla \Phi [\nabla \vartheta, \nabla \mu]\big) = \mathrm{div}\, \Phi [\nabla \vartheta, \nabla \mu] \; . \tag{1.2.13}$$

The $Q_s = \int q_s \, d\boldsymbol{r}$ value is called the helicity of the velocity field and represents an integral of motion of the hydrodynamic equations. From (1.2.13) it is seen that the possibility of representing the velocity as (1.2.6) means that $Q_s = 0$. As a matter of fact, it is easy to construct examples of the velocity fields for which $Q_s \neq 0$. Let, e.g., $\boldsymbol{v}$ satisfy of the system of equations (α =const)

$$\operatorname{rot} \boldsymbol{v} = \alpha \boldsymbol{v}, \quad \operatorname{div} \boldsymbol{v} = 0 \qquad \text{(Beltramy flow)} .$$

It should be noted that the fields obeying these equations are the stationary solutions of the hydrodynamic equations with the density ϱ =const. It is easily seen that for such flows $Q_s = \alpha \int v^2 \, d\boldsymbol{r} \neq 0$. The condition allowing to represent the velocity field as (1.2.6) may be interpreted geometrically. According to the logic of our construction, $\mu(\boldsymbol{r})$ and $\vartheta(\boldsymbol{r})$ are single-valued functions of the coordinates. It follows from (1.2.12) that the vector $\operatorname{rot} \boldsymbol{v}$ is directed along the intersection line of the level surfaces of these functions. Not every closed line may be represented as an intersection line of level surfaces of single-valued functions. This is, for example, not feasible if the line is knotted, i.e., if it represents the circle image unhomotopic to it. Let such a line be specified by the equation $r = l(y)$ where y is a parameter on the line, and the vorticity field is expressed as (the vortex line)

$$\operatorname{rot} \boldsymbol{v} = \kappa \int \boldsymbol{n}(y) \delta[\boldsymbol{r} - \boldsymbol{l}(y)] dy, \quad |\boldsymbol{n}|^2 = 1 .$$

Topology manuals (see also [1.8]) prove that in this case $Q_s = m\kappa^2$ where m is an integer defining the winding number (knotticity) of the line. A similar formula holds if the line represents a pair of linked circles with $m = 1$.

We shall refer to the variables ϱ, Φ, λ and μ as the Clebsch variables. Evidently, a global definition of the Clebsch variables is not always possible, as it requires zero knotticity of the vortex lines. At least in the vicinity of a regular point of the velocity field, the local introduction of the Clebsch variables is always possible, but attempts to expand the range of validity of their functions may lead to a loss of the single-valuedness of λ and ϑ. Nevertheless, it is possible to introduce several pairs of Clebsch variables (λ_i, μ_i), $i = 1, \ldots N$:

$$\partial \mu_i / \partial t + (\boldsymbol{v} \nabla) \mu_i = 0, \qquad \partial \lambda_i / \partial t + \operatorname{div} (\lambda \boldsymbol{v}) = 0 ,$$

$$\boldsymbol{v} = \left(\frac{1}{\varrho} \sum_{i=1}^{N} \lambda_i \nabla \mu_i + \nabla \Phi \right) .$$

Probably, one can prove that $N = 2$ is sufficient to establish a one-to-one equivalence between the initial hydrodynamic system and a canonical one for arbitrary flows.

1.2.2 Vortex Motion in Incompressible Fluids

In the case of an incompressible fluid with $\partial\varrho/\partial t = 0$, we can set $\varrho = 1$ (to simplify the notation). The velocity may now be written as $\boldsymbol{v} = \lambda\nabla\mu + \nabla\Phi$. The condition $\operatorname{div}\boldsymbol{v} = 0$ yields Φ : $\Phi = -\Delta^{-1}\operatorname{div}\lambda\nabla\mu$. The formula for the velocity may be rewritten as

$$\boldsymbol{v} = -\Delta^{-1}\operatorname{rot}[\nabla\lambda, \nabla\mu] . \tag{1.2.14}$$

Here Δ^{-1} is the inverse operator to the Laplacian. Hence, we have now only one pair of the Clebsch variables λ, μ. Fourier transformation and the transition to complex variables

$$\mu(\boldsymbol{k}) = [a(\boldsymbol{k}) + a^*(-\boldsymbol{k})]/\sqrt{2}, \quad \lambda(\boldsymbol{k}) = [a(\boldsymbol{k}) - a^*(-\boldsymbol{k})]/i\sqrt{2}$$

allow to cast the canonical equations (1.2.10–11) into the standard form (1.1.6):

$$\partial a(\boldsymbol{k}, t)/\partial t = \delta\mathcal{H}_V/\delta a^*(\boldsymbol{k}, t).$$

The Hamiltonian $\mathcal{H}_V$ is obtained by substituting the formula for the velocity into the expression for the kinetic energy (1.2.2)

$$\boldsymbol{v}(\boldsymbol{k}) = \int \varphi_{12} a_1^* a_2 \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2, \quad a_j = a(\boldsymbol{k}_j, t) . \tag{1.2.15a}$$

Here

$$\varphi_{12} = \varphi(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{1}{2\varrho_0(2\pi)^{3/2}}\left[\boldsymbol{k}_1 + \boldsymbol{k}_2 - (\boldsymbol{k}_1 - \boldsymbol{k}_2)\frac{k_1^2 - k_2^2}{|\boldsymbol{k}_1 - \boldsymbol{k}_2|^2}\right] \tag{1.2.15b}$$

which follows from (1.2.14). As a result, we have

$$\mathcal{H}_V = \frac{1}{4}\int T_{12,34} a_1^* a_2^* a_3 a_4 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 , \tag{1.2.16a}$$

where the interaction coefficient is

$$T(\boldsymbol{k}_1\boldsymbol{k}_2, \boldsymbol{k}_3\boldsymbol{k}_4) = \varrho_0[(\varphi_{13}\varphi_{24}) + (\varphi_{14}\varphi_{23})] . \tag{1.2.16b}$$

The expressions (1.2.15, 16) agree with the dimensional estimates (1.1.49–50) obtained in Sect. 1.1.5.

1.2.3 Sound in Continuous Media

As seen from (1.2.6), the case with $\lambda = 0$ or μ =const corresponds to potential fluid motion which is according to (1.2.8–9) defined by a pair of variables (ϱ, Φ). Following the standard scheme presented in Sect. 1.1, we go in the $\boldsymbol{k}$-representation from the real canonical variables $\Phi(\boldsymbol{k})$, $\varrho(\boldsymbol{k})$ over to the complex $b(\boldsymbol{k}), b^*(\boldsymbol{k})$:

$$\Phi(\boldsymbol{k}) = -(i/k)(\omega_k/2\varrho_0)^{1/2}[b(\boldsymbol{k}) - b^*(-\boldsymbol{k})] ,$$

$$\delta\varrho(\boldsymbol{k}) = \boldsymbol{k}(\varrho_0/2\omega_k)^{1/2}[b(\boldsymbol{k}) + b^*(-\boldsymbol{k})] , \tag{1.2.17a}$$

$$\omega(\boldsymbol{k}) = kc_s, \quad c_s^2 = (\partial p/\partial\varrho) . \tag{1.2.17b}$$

Here $\delta\varrho = \varrho - \varrho_0$ is density deviation from the steady state and c_s the sound velocity. The derivative $(\partial p/\partial\varrho)$ is calculated with the entropy s treated as a constant, which corresponds to assuming a barotropic motion of the fluid (without heat exchange). Equations (1.2.17) coincide to an accuracy of a dimensionless multiplier of the order of unity with the dimensional estimates (1.1.36) and (1.1.40). In order to obtain the sound Hamiltonian $\mathcal{H}_S$ one should expand the expression for energy (1.2.2) in terms of $\delta\varrho$ and $\boldsymbol{v} = \nabla\Phi$

$$\mathcal{H} = \mathcal{H}_S = \mathcal{H}_{2S} + \mathcal{H}_{SS} , \tag{1.2.18a}$$

$$\mathcal{H}_{2S} = \frac{1}{2}\int\left[\varrho_0|\nabla\Phi|^2 + c_S^2(\delta\varrho)^2/\varrho_0\right] d\boldsymbol{r} , \tag{1.2.18b}$$

$$\mathcal{H}_{SS} = \frac{1}{2}\int\left[\delta\varrho|\nabla\Phi|^2 + gc_S^2(\delta\varrho)^3\right] d\boldsymbol{r} , \tag{1.2.18c}$$

and substitute Φ and $\delta\varrho$ from (1.2.17) into these expressions. As a result, we see that the quadratic part of the Hamiltonian is diagonal in the variables b_k, b_k^*.

$$\mathcal{H}_{2S} = \int \omega(\boldsymbol{k})b^*(\boldsymbol{k})b(\boldsymbol{k})\, d\boldsymbol{k} . \tag{1.2.19}$$

The Hamiltonian of the sound-sound interaction $\mathcal{H}_{SS}$ has the form (1.1.24a) with the interaction coefficients:

$$\begin{aligned} V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) &= U(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) \\ &= \left(\frac{c_s k k_1 k_2}{4\pi^3 \varrho_0}\right)^{1/2} (3g + \cos\theta_{k1} + \cos\theta_{k2} + \cos\theta_{12}) . \end{aligned} \tag{1.2.20}$$

As expected, this expression is consistent with the result (1.1.38) obtained from a dimensional analysis by specifying the type of angular dependence f.

The supposition that the density of the internal energy $\varepsilon(\boldsymbol{r})$ depends only on $\varrho(\boldsymbol{r})$ is true only in the range of small inhomogeneities. In the general case, the internal energy E_{in} is a density functional which may be represented as a power series in $\nabla\varrho$:

$$E_{\text{in}} = \int\left[\varepsilon(\varrho) + \beta|\nabla\varrho|^2/2 + \ldots\right] d\boldsymbol{r} . \tag{1.2.21}$$

Whence, the expression for the frequency $\omega(k)$ will change from (1.2.17b) to

$$\omega^2(k) = c_s^2k^2 + \beta k^4, \quad \omega(k) = c_s k\left[1 + \frac{\beta k^2}{2c_s^2} + \ldots\right] . \tag{1.2.22}$$

It should be noted, that β may be either positive or negative [see, e.g., (1.2.39, 41) and (1.3.10)]. The expression (1.2.22) is true if the dispersion of sound is small: $\beta k^2 \ll 2c_s^2$. Otherwise, one should take into account the next $\nabla\varrho$-terms.

1.2.4 Interaction of Vortex and Potential Motions in Compressible Fluids

As shown above, in incompressible fluids, the variables λ and μ define vortex motion. A purely potential motion is described by the pair ϱ and Φ. However, in the general case it is wrong to assert that this pair ϱ and Φ describes potential motion and λ and μ vortex motion. Indeed, dividing $\boldsymbol{v}$ into two parts

$$\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2, \quad \operatorname{rot} \boldsymbol{v}_1 = 0, \quad \operatorname{div} \boldsymbol{v}_2 = 0 ,$$

we see that

$$\boldsymbol{v}_1 = \nabla\tilde{\Phi}, \qquad \tilde{\Phi} = \Delta^{-1}\operatorname{div}[(\lambda\nabla\mu)/\varrho] + \Phi . \tag{1.2.23}$$

Therefore the initial *Clebsh* variables are inconvenient for describing the turbulence of compressible fluids: the fields (ϱ, Φ) and (λ, μ) are strongly coupled; even for small fluctuation velocity (with the Mach number M)

$$\langle v^2\rangle / c_s^2 = M^2 \ll 1 . \tag{1.2.24}$$

Formally this manifests itself in the fact that the coefficient of interaction between these fields increases with the sound velocity $\propto \sqrt{c_s}$.

Assuming the Mach number M to be small, *L'vov* and *Mikhailov* obtain a canonical transformation separating potential and vortex motions in the new variables (q,p) and (Q,P) [1.9]. In doing so we shall try to determine the vortex velocity v_2 and the potential velocity v_1 by equations close to (1.2.12) and (1.2.23), respectively. We choose the desired canonical transformation using the generating functional F depending on the new coordinates q, Q and the old momenta Φ, μ (see (A.2.12) in Sect. A.2)

$$\lambda = \delta F/\delta\mu, \quad P = \delta F/\delta Q, \quad \varrho = \delta F/\delta\Phi, \quad p = \delta F/\delta q . \tag{1.2.25a}$$

We write the generating functional as

$$F = F_0 + F_1, \quad F_0 = \int (\tilde{\Phi} q + Q\mu)\, d\boldsymbol{r} \tag{1.2.25b}$$

where F_0 is a functional of the identity transformation chosen in such a way that the pair of canonical variables responsible for potential motion is $q = \varrho$ and $p = \tilde{\Phi}$. The functional F_1 is independent of $\tilde{\Phi}$, bilinear in μ and Q and represents a power series of the variable part of density $\delta\varrho = \varrho(r,t) - \varrho_0$:

$$\varrho_0 F_1 = \int [Q\nabla\mu, \nabla\Delta^{-1}\varrho]\, d\boldsymbol{r} . \tag{1.2.25c}$$

The expansion parameter is

$$\xi = \frac{k_V \delta\varrho}{k_S \varrho_0} \simeq \frac{\lambda_S}{L}\sqrt{\frac{E_S}{\varrho_0 c_s^2}} , \tag{1.2.25d}$$

where $k_S \simeq 1/\lambda_S$ and $k_V = 1/L$ are the characteristic wave vectors of sound waves and vortices, respectively; E_S is the energy of sound motions.

Substituting (1.2.25c) and (1.2.25d) in (1.2.25a) and solving the resulting equations by iterations with regard to the small parameter $\xi \ll 1$, we obtain

$$q = \varrho, \quad p = \tilde{\Phi} = \Phi + \Delta^{-1}\mathrm{div}\,[(\lambda\nabla\mu)/\varrho]\,, \tag{1.2.26a}$$

$$\begin{aligned} \lambda &= Q + (\nabla, \Delta^{-1}\nabla\delta\varrho)Q/\varrho_0 + O(\xi^2)\,, \\ \mu &= P + (\nabla P, \Delta^{-1}\nabla\delta\varrho)/\varrho_0 + O(\xi^2)\,. \end{aligned} \tag{1.2.26b}$$

In the new variables

$$\boldsymbol{v}_1 = \nabla p, \tag{1.2.27a}$$

$$\begin{aligned} \boldsymbol{v}_2 = {}& \Delta^{-1}[\nabla, [\nabla Q, \nabla P]]/\varrho_0 \\ & - [\nabla, [\nabla, [\Delta^{-1}\nabla\varrho, [\nabla P, \nabla Q]]]] + O(\xi^2)\,. \end{aligned} \tag{1.2.27b}$$

Thus we achieved the desired result: the potential motions are defined by the pair (q, p) only; the main contribution to vortex motion is made by the pair Q, P. The last term in (1.2.27b) for v_2 describes the effect of compressibility on the vortex motion.

In the $\boldsymbol{k}$-representation, we go over to the complex variables $b(\boldsymbol{k})$, $b^*(\boldsymbol{k})$, and $a(\boldsymbol{k})$, $a^*(\boldsymbol{k})$ following formulas similar to (1.2.17) and (1.1.3–4). In these variables the hydrodynamic equations have the canonical form

$$i\frac{\partial a(\boldsymbol{k}, t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta a^*(\boldsymbol{k}, t)}\,, \tag{1.2.28a}$$

$$i\frac{\partial b(\boldsymbol{k}, t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta b^*(\boldsymbol{k}, t)}\,, \tag{1.2.28b}$$

with the Hamiltonian

$$\mathcal{H} = \mathcal{H}_S + \mathcal{H}_V + \mathcal{H}_{SV}\,. \tag{1.2.29a}$$

The sound Hamiltonian has the form (1.2.18), the Hamiltonian of vortex motions is specified by (1.2.16), and the Hamiltonian of sound-vortex interaction has two terms:

$$\mathcal{H}_{SV} = \mathcal{H}_{SV1} + \mathcal{H}_{SV2}\,, \tag{1.2.29b}$$

$$\mathcal{H}_{SV1} = \int S_{12,34} a_1^* a_2 b_3^* b_4 \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4\,, \tag{1.2.29c}$$

$$S_{12,34} = \left(k_3 k_4/32\pi^3\right)^{1/2}[(\boldsymbol{n}_3\boldsymbol{\varphi}_{12}) + (\boldsymbol{n}_4\boldsymbol{\varphi}_{12})]\,, \tag{1.2.29d}$$

$$\begin{aligned} \mathcal{H}_{SV2} = {}& \frac{1}{4}\int W_{1234k} a_1^* a_2^* a_3 a_4 \left[b(\boldsymbol{k}) + b^*(-\boldsymbol{k})\right] \\ & \times \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4 - \boldsymbol{k})\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 d\boldsymbol{k}\,, \end{aligned} \tag{1.2.29e}$$

$$W_{1234k} = \sqrt{\frac{\varrho_0 k}{4c_s\pi^3}}\,[(\varphi_{13}\boldsymbol{n}_k)(\varphi_{24}\boldsymbol{n}_k) + (\varphi_{23}\boldsymbol{n}_k + \varphi_{14}\boldsymbol{n}_k)]\ . \tag{1.2.29f}$$

Here $\boldsymbol{n}_j = \boldsymbol{k}_j/k$, and φ_{ij} are determined by (1.2.15b). The term $\mathcal{H}_{SV1}$ describes sound scattering processes and the $\mathcal{H}_{SV2}$-term describes generation and absorption of sound by turbulence. In (1.2.29a) we have not written out the terms $S^{(n)}a_1^*a_2b^{n+2}$, $W^{(n)}a^2a^{*2}b^{n+1}$, $n \geq 1$, which are small in the ξ^n parameter and insignificant for our future considerations.

1.2.5 Waves on Fluid Surfaces

Let us consider potential motion of incompressible fluids with a free surface in a homogeneous gravitational fluid [1.7]. In the quiescent state the fluid surface is a plane $z = 0$ with the bottom at $z = -h$. We describe the surface form by $\eta = \eta(\boldsymbol{r}, t)$ where $\boldsymbol{r} = (x, y)$ is the coordinate in the transverse plane. The full energy of the fluid $\mathcal{H} = T + \Pi$ is a sum of the kinetic energy

$$T = \frac{1}{2}\int d\boldsymbol{r}\int_{-h}^{\eta} v^2\,dz \tag{1.2.30a}$$

and the potential energy

$$\Pi = \frac{1}{2}g\int \eta^2\,d\boldsymbol{r} + \sigma\int\left[\sqrt{1+|\nabla\eta|^2} - 1\right]d\boldsymbol{r}\ . \tag{1.2.30b}$$

Here g is the gravitational acceleration; σ the surface tension coefficient and the free surface element is expressed by $ds = d\boldsymbol{r}\sqrt{1+|\nabla\eta|^2}$.

In constructing the canonical variables we shall proceed from the Hamiltonian (1.2.7) where we set

$$\varrho = \Theta\big(\eta(\boldsymbol{r}) - z\big)\ . \tag{1.2.31}$$

Here $\Theta(\xi) = 1$ at $\xi > 0$, $\Theta(\xi) = 0$ at $\xi < 0$. Further on we shall consider only irrotational fluid flows implying that may use $\lambda = 0$. Substituting (1.2.31) into (1.2.7) and taking advantage of the fact that $\partial\eta(\boldsymbol{r}, z, t)/\partial t = \delta[\eta(\boldsymbol{r}, t) - z]\partial\eta(\boldsymbol{r}, t)/\partial t$, the Lagrangian reads

$$\mathcal{L} = -\int \Psi(\boldsymbol{r}, t)\frac{\partial\eta(\boldsymbol{r}, t)}{\partial t}\,d\boldsymbol{r} - \mathcal{H}\ . \tag{1.2.32}$$

Here $\Psi(\boldsymbol{r}, t) = \Phi(\boldsymbol{r}, z, t)$ at $z = \eta(\boldsymbol{r}, t)$.

The Lagrangian (1.2.32) yields the canonical equations

$$\frac{\partial\eta(\boldsymbol{r}, t)}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\Psi(\boldsymbol{r}, t)}\,, \tag{1.2.33a}$$

$$\frac{\partial\Psi(\boldsymbol{r}, t)}{\partial t} = \frac{\delta\mathcal{H}}{\delta\eta(\boldsymbol{r}, t)}\ . \tag{1.2.33b}$$

Thus the canonical pair of variables is now given by $\eta(\boldsymbol{r},t)$, $\Psi(\boldsymbol{r},t)$. Having specified them, we have now to solve the boundary value problem for the Laplace equation

$$\Delta\Phi(\boldsymbol{r},z,t)=0, \quad \Phi[\boldsymbol{r},\eta(\boldsymbol{r},t),t]=\Psi(\boldsymbol{r},t), \Phi(\boldsymbol{r},-h,t)=0 \tag{1.2.34}$$

in order to determine the fluid's velocity field. Now $\boldsymbol{v}=\nabla\Phi$ and we get

$$T=\frac{1}{2}\int d\boldsymbol{r}\int_{-h}^{\eta} dz|\nabla\Phi|^2$$

for the kinetic energy. The variation $\delta\mathcal{H}/\delta\Psi=\delta T/\delta\Psi$ may be carried out explicitly, but the result is known in advance. To obtain it, we substitute $\varrho=\Theta(\eta-z)$ into the continuity equation (1.2.1) to get

$$\frac{\partial\eta(\boldsymbol{r},t)}{\partial t}+v_{\perp}\nabla\eta=v \tag{1.2.35}$$

as the kinematic condition on the fluid surface which should coincide with (1.2.33a). The physical meaning of this condition is rather simple: the velocity of fluid height-variations should be the same as the velocity of the fluid itself in the given point at the surface.

The explicit solution of the boundary value problem (1.2.34) is not possible but it may be solved in the small nonlinearity limit by expanding the Hamiltonian in a power series with regard to its canonical variables. In coordinate representation, every term in this series is a nonlocal functional of η and Ψ. This is due to the above-mentioned necessity to solve the Laplace equation at every iteration step. Going over to a Fourier representation we obtain

$$\begin{aligned}
\mathcal{H}_2 &= \frac{1}{2}\int\left[(\varrho g+\sigma k^2)|\eta(\boldsymbol{k},t|^2+\varrho k\tanh(kh)|\Psi(\boldsymbol{k},t)|^2\right]d\boldsymbol{k}\ ,\\
\mathcal{H}_3 &= \frac{1}{4\pi}\int L_{123}\Psi(\boldsymbol{k}_1)\Psi(\boldsymbol{k}_2)\eta(\boldsymbol{k}_3)\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\,d\boldsymbol{k}_1d\boldsymbol{k}_2d\boldsymbol{k}_3\ ,\\
L_{123} &= \frac{1}{2}\left(k_1^2+k_2^2-k_3^2\right)-k_1k_2\tanh(k_1h)\tanh(k_2h)\ ,\\
\mathcal{H}_4 &= \frac{1}{(2\pi)^2}\int M_{1234}\Psi(\boldsymbol{k}_1)\Psi(\boldsymbol{k}_2)\eta(\boldsymbol{k}_3)\eta(\boldsymbol{k}_4)\\
&\qquad\times\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4)\,d\boldsymbol{k}_1d\boldsymbol{k}_2d\boldsymbol{k}_3d\boldsymbol{k}_4, \qquad \text{etc}\ .
\end{aligned} \tag{1.2.36}$$

In the case of gravitational waves on deep water the expression for M_{1234} is required, see (1.2.43). In the limit $\sqrt{\varrho g/\sigma}=k_*\gg k\gg h^{-1}$ we obtain

$$\begin{aligned}
M_{1234} &= \frac{1}{4}\sqrt{k_1k_2k_3k_4}[2(k_1+k_2)\\
&\quad -|\boldsymbol{k}_1+\boldsymbol{k}_3|-|\boldsymbol{k}_1+\boldsymbol{k}_4|-|\boldsymbol{k}_2+\boldsymbol{k}_3|-|\boldsymbol{k}_2+\boldsymbol{k}_4|]\ .
\end{aligned}$$

The transition to normal complex variables is given by

$$
\begin{aligned}
\eta(\boldsymbol{k}) &= \sqrt{\frac{\lambda_k}{2}}[a(\boldsymbol{k}) + a^*(-\boldsymbol{k})] \; , \\
\Psi(\boldsymbol{k}) &= -\, i\sqrt{\frac{1}{2\lambda_k}}[a(\boldsymbol{k}) - a^*(-\boldsymbol{k})], \\
\lambda_k &= \frac{\omega(k)}{g + \sigma k^2/\varrho} \; ,
\end{aligned}
\tag{1.2.37}
$$

where $\omega(k)$ is the dispersion law of waves on a fluid surface with depth h:

$$
\omega^2(k) = k[g + \sigma k^2/\varrho] \tanh(kh) \; . \tag{1.2.38}
$$

Written in these variables the canonical equations of motions have the normal form (1.1.14) and the quadratic part of the Hamiltonian is diagonal with respect to $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$. We give the coefficient of the interaction only for the limiting cases in which the problem becomes scale-invariant. As seen from (1.2.38), there are two characteristic scales in k-space: $1/h$ and $k_* = \sqrt{\varrho_0 g/\sigma}$ [see also (1.1.47)]. If the scales strongly differ, the k-space contains the regions of scale-invariant behavior. For example, at $1 \gg k_* h$ we have

1. $k \ll k_*$ are the *shallow-water gravitational-capillary waves*. Their dispersion law is close to that of sound

$$
\omega(k) = \sqrt{gh}\, k[1 + k^2/2k_*^2] \; . \tag{1.2.39}
$$

The velocity of such a wave is determined by gravity only, while the dispersion is determined by surface tension as well. The positive addition to the linear dispersion law makes three-wave processes possible, the interaction coefficients coincide with (1.2.20) where $c_s = \sqrt{gh}$ should be assumed. It should be noted that, despite the absence of complete self-similarity (there is a parameter with the dimension of a length h), the Hamiltonian coefficients are scale invariant. Such cases are usually referred to as incomplete, or second order, self-similarity.

2. $k_* \ll k \ll h^{-1}$ are the *shallow water capillary waves* [1.10]. In this case

$$
\omega(k) = \sqrt{\frac{\sigma h}{\varrho}}\, k^2 \; . \tag{1.2.40a}
$$

The dispersion law is of the decay type. It is sufficient to consider only three-wave processes. Written in normal variables the Hamiltonian $\mathcal{H}_{\text{int}}$ has the standard form (1.1.23) with rather simple interaction coefficients

$$
V_{k12} = U_{k12} = (k^2/8\pi)(\sigma/4\varrho h)^{1/4} \; . \tag{1.2.40b}
$$

3. $h^{-1} \ll k$ are the *deep-water capillary waves* [1.7,11]. In this limit

$$
\omega(k) = \sqrt{\frac{\sigma}{\varrho}}\, k^{3/2} \; , \tag{1.2.41a}
$$

$$V(\boldsymbol{k},\mathbf{12}) = \frac{1}{8\pi}\left(\frac{\sigma}{4\varrho^3}\right)^{1/4}\left[(\boldsymbol{k}_1\boldsymbol{k}_2 + k_1k_2)\left(\frac{k_1k_2}{k}\right)^{1/4} + (\boldsymbol{k}_1\boldsymbol{k} - k_1k)\left(\frac{k_1k}{k_2}\right)^{1/4} + (\boldsymbol{k}\boldsymbol{k}_2 - kk_2)\left(\frac{kk_2}{k_1}\right)^{1/4}\right] . \tag{1.2.41b}$$

The reader should compare these expressions with (1.1.45) and (1.1.46)

However, for $k_*h \simeq 1$ there are only two scale-invariant regions, namely at very short and very long waves, respectively. At $k \to \infty$ we have the case (1.2.41) considered above. At $k \to 0$, we have a dispersion law close to the acoustic one:

$$\omega(k) = \sqrt{ghk}\left(1 + [(1/2k_*) - (h^3/3)]k^2\right) . \tag{1.2.42}$$

In the case of $k_*h > 3/2$, the law (1.2.42) is the nondecay type, and the principal role is played by the four-wave interaction with an interaction coefficient of the form (1.1.30) where $V(k, 12)$ is given by (1.2.20). For $k_*h \gg 1$ the dispersion law (1.2.42) is determined by gravity. These waves are called *shallow water gravitational waves.*

At $k_*h \gg 1$ in the intermediate region $k_* \gg k \gg h^{-1}$ we have *deep-water gravitational waves* with the nondecay dispersion law $\omega(k) = \sqrt{gk}$ (1.1.42) and the interaction coefficient [1.7]

$$U_{k,12} = V_{-k12} = \frac{1}{8\pi}\left(\frac{g}{4\varrho^2}\right)^{1/4}\left[(\boldsymbol{k}_1\boldsymbol{k}_2 + k_1k_2)\left(\frac{k}{k_1k_2}\right)^{1/4} + (\boldsymbol{k}\boldsymbol{k}_1 + kk_1)\left(\frac{k_2}{kk_1}\right)^{1/4} + (\boldsymbol{k}\boldsymbol{k}_2 + kk_2)\left(\frac{k_1}{kk_2}\right)^{1/4}\right] , \tag{1.2.43a}$$

$$W(\boldsymbol{k}\mathbf{1},\mathbf{23}) = \frac{(kk_1k_2k_3)^{1/2}}{64\varrho\pi^2}[R(\boldsymbol{k}\mathbf{123}) + R(\boldsymbol{k}\mathbf{123}) - R(\boldsymbol{k}\mathbf{213}) - R(\boldsymbol{k}\mathbf{312}) - R(\mathbf{12}\boldsymbol{k}\mathbf{3}) - R(\mathbf{13}\boldsymbol{k}\mathbf{2})] , \tag{1.2.43b}$$

$$R(\boldsymbol{k}\mathbf{123}) = \left(\frac{kk_1}{k_2k_3}\right)^{1/4}[2(k + k_1) - |\boldsymbol{k} - \boldsymbol{k}_2| - |\boldsymbol{k} - \boldsymbol{k}_3| - |\boldsymbol{k}_1 - \boldsymbol{k}_2| - |\boldsymbol{k}_1 - \boldsymbol{k}_3|] . \tag{1.2.43c}$$

The resulting coefficient of the four-wave interaction has after elimination of three-wave processes the form (1.1.29b) and possesses the same homogeneity properties as $W(\boldsymbol{k}\mathbf{1},\mathbf{23})$ [see also (1.1.43)].

Thus surface waves obey in different limiting cases either decay or nondecay power dispersion laws [$\omega(k) \propto k^\alpha$, $\alpha = 1/2, 1, 3/2, 2$] and have scale-invariant interaction coefficients.

1.3 Hydrodynamic-Type Systems

1.3.1 Langmuir and Ion-Sound Waves in Plasma

In some situations, a plasma may be regarded as a set of two fluids: electronic and ionic ones, each defined by a system of hydrodynamic equations. If there are no external magnetic fields, this is possible if the wavelengths induced in the plasma are large compared with the Debye length. The simplest model of such a plasma does not take into account the generation of the magnetic field by currents and the electric field in such a plasma is potential.

The equations of motion of the electronic fluid have in this model the form of (1.2.1)

$$\begin{aligned} &\frac{\partial \boldsymbol{v}_e(\boldsymbol{r},t)}{\partial t} + (\boldsymbol{v}_e\nabla)\boldsymbol{v}_e + \nabla\left(\frac{\delta\varepsilon}{\delta\varrho_e}\right) = 0, \\ &\frac{\partial \varrho_e(\boldsymbol{r},t)}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}_e) = 0 \end{aligned} \tag{1.3.1}$$

with the internal energy being the sum of electrostatic and gas kinetic terms

$$\varepsilon = \frac{e^2}{2m^2}\int \frac{\delta\varrho_e(\boldsymbol{r},t)\delta\varrho_e(\boldsymbol{r}',t)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,d\boldsymbol{r}\,d\boldsymbol{r}' + \frac{3T_e}{2m\varrho_0}\int [\delta\varrho_e(\boldsymbol{r},t)]^2\,d\boldsymbol{r}\,. \tag{1.3.2}$$

Here $\delta\varrho_e = \varrho_e - \varrho_0$ is the electronic density variation; e, m and T_e are the charge, mass and temperature of the electrons. In the second term defining the kinetic pressure of the gas the coefficient 3 emphasizes the phenomenological character of the model: it has been chosen to obtain the correct dispersion law of Langmuir waves

$$\omega^2(k) = \omega_p^2(1 + 3k^2 r_D^2) \tag{1.3.3}$$

arising from the precise kinetic description (see e.g. [1.12]). Here ω_p and r_D are, respectively, plasma frequency and Debye length:

$$\omega_p^2 = \frac{4\pi\varrho_0}{m^2}, \quad r_D^2 = \frac{T_e m}{4\pi\varrho_0 e^2}\,. \tag{1.3.4}$$

The Langmuir waves depict a type of plasma motions possessing a potential. In such cases one can introduce in a conventional way the velocity potential $\nabla\Phi = \boldsymbol{v}$. The normal canonical variables are introduced in the same way as for the potential motions of an ordinary fluid, using the formulas (1.2.17a) where $\omega(k)$ should be given by (1.3.3). The coefficients of three-wave interactions are calculated similarly to (1.2.20) and have the form [1.13]

$$\begin{aligned} U_{k12} = V_{k12} = \frac{1}{8\sqrt{2}\pi^3\varrho_0}\Bigg[&\left(\frac{\omega_1\omega_2}{2\omega_k}\right)^{1/2} k\cos\theta_{12} \\ &+ \left(\frac{\omega_k\omega_1}{2\omega_2}\right)^{1/2} k_2\cos\theta_1 + \left(\frac{\omega_k\omega_2}{\omega_1}\right)^{1/2} k_1\cos\theta_2\Bigg]\,. \end{aligned}$$

However, the dispersion law (1.3.3) is valid only in the long wave range $kr_D \ll 1$ and is of the nondecay type. Using the transformation (1.1.28), one can obtain the effective Hamiltonian (1.1.29). For $kr_D \ll 1$ the interaction coefficients U and V become scale-invariant with the scaling index unity and the effective four-wave interaction coefficient (1.1.29b) has the scaling index two since $\omega(k) \approx \omega_p$:

$$\begin{aligned} T_{k123} = \frac{1}{\omega_p}\big[&V_{k+1,k1}V_{2+3,23} - V_{-k-1,k1}V_{-2-3,23} - V_{k,2k-2}V_{3,13-1} \\ &- V_{k,3k-3}V_{2,12-1} - V_{2,k2-k}V_{131-3} - V_{3,k3-k}V_{1,22-1}\big] . \end{aligned} \tag{1.3.5}$$

That expression satisfies symmetry properties and is a homogeneous function of the second degree.

In describing the electronic oscillations we have assumed the ions to be at rest. The slow motion of the ionic fluid will be considered in nonisothermal plasma where the electron temperature is a lot larger than the one of the ion $T_e \gg T_i$. We shall consider the phase velocities of the waves to be much higher than the thermal velocities of the ions but much smaller than the thermal velocities of the electrons. Then, at each moment of time the electrons may be taken to have a Boltzmann distribution $n_e = n_0 \exp(e\phi/T_e)$. The electric field potential $\varphi(\boldsymbol{r},t)$ satisfies the Poisson equation

$$\Delta\varphi = -4\pi e[n - n_0 \exp(e\phi/T_e)] , \tag{1.3.6}$$

where $n(\boldsymbol{r},t)$ is the ion concentration. Neglecting the ion thermal pressure, we obtain a system of ion hydrodynamic equations:

$$\begin{aligned} &\frac{\partial \boldsymbol{v}(\boldsymbol{r},t)}{\partial t} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{e\nabla\varphi}{M} , \\ &\frac{\partial \varrho(\boldsymbol{r},t)}{\partial t} + \operatorname{div}(\varrho\boldsymbol{v}) = 0 . \end{aligned} \tag{1.3.7}$$

Here v and M are ion velocity and mass and $\varrho = Mn$. As (1.2.1) and (1.3.1) the system of equations (1.3.6, 7) may be written in the form of Hamilton equations with the Hamiltonian

$$\begin{aligned} &\mathcal{H} = \frac{1}{2}\int \varrho v^2\, d\boldsymbol{r} + E_{\text{in}} , \\ &E_{\text{in}} = \frac{1}{8\pi}\int |\nabla\varphi|^2\, d\boldsymbol{r} + T_e n_0 \int \left[\left(\frac{e\varphi}{T_e} - 1\right)\exp\left(\frac{e\varphi}{T_e}\right) + 1\right] d\boldsymbol{r} , \end{aligned} \tag{1.3.8}$$

i.e., to a sum over the ion kinetic, electrostatic field and thermal energies of the electron gas. The canonically conjugate variables are ϱ and velocity potential Φ: $\boldsymbol{v} = \nabla\Phi$; thus $\partial\varrho/\partial t = -\delta\mathcal{H}/\delta\Phi$ yields the first of the equations (1.3.7). To obtain the right-hand side of the first equation (1.3.7) one should calculate the internal-energy variational derivative which, by virtue of the Poisson equation (1.3.6), equals

$$\frac{\delta E_{\text{in}}}{\delta\varrho(\boldsymbol{r})} = \int \varphi(\boldsymbol{r}') \left[-\frac{1}{4\pi}\Delta[\delta\varphi(\boldsymbol{r}')/\delta\varrho(\boldsymbol{r})] + \frac{\delta\varphi(\boldsymbol{r}')}{\delta\varrho(\boldsymbol{r})} e^2 n_0 \exp\left(\frac{e\varphi}{T_e}\right)\right] d\boldsymbol{r}' = \frac{e}{M}\delta(\boldsymbol{r}-\boldsymbol{r}') \, .$$

Thus the second Hamilton equation $\partial\Phi/\partial t = \delta\mathcal{H}/\delta\varrho$ coincides with the second of relations (1.3.7). Assuming the unperturbed plasma to be quasi-neutral, $n(r,t) = n_0$ holds for small perturbations $n, \boldsymbol{v} \propto \exp[i(\boldsymbol{kr} - i\Omega(k)t]$ and the dispersion law reads

$$\Omega^2(k) = \frac{k^2 T_e}{M(1+k^2 r^2)} \, . \tag{1.3.9}$$

In the long-wave range $kr_D \ll 1$, the dispersion law (1.3.9) is almost linear

$$\Omega(k) \approx kc_s\left(1 - k^2 r_D^2/2\right) \, . \tag{1.3.10a}$$

Such oscillations are called *ion sound*, they are only at $T_e \gg T_i$ well defined (i.e., they are weakly damped). The sound velocity $c_s^2 = T_e/M$ is determined by electron temperature and ion mass (inertia). A correction to the linear term in $\Omega(k)$ is negative, therefore the dispersion law (1.3.10) [as in (1.3.3)] is of the nondecay type, the resonance interaction of ion-sound waves with each other is specified by the four-wave Hamiltonian (1.1.29) where U and V are computed in a similar way as for ordinary sound and are given by (1.2.20) with $g = -1/3$:

$$T_{1234} = \frac{V_{1+212}V_{3+434}}{\omega_1 + \omega_2 - \omega_{1+2}} - \frac{2V_{131-3}V_{424-2}}{\omega_{4-2} + \omega_2 - \omega_4} - \frac{2V_{232-3}V_{414-1}}{\omega_{4-1} + \omega_1 - \omega_4} \, . \tag{1.3.10b}$$

Here we neglected the small terms which do not contain small denomitators $\propto r_D^2$. Thus three-wave processes are forbidden for systems containing either Langmuir or ion-sound waves. However, there should be an interaction between Langmuir and ion sound waves. The physical reason for it is in the joint action of two mechanisms: the slow ion-sound density variations alter the plasmon frequency, and the high-frequency field creates a mean ponderomotive force (proportional to the gradient of the square of the field) which affects ions. Such phenomena can be described in the framework of the so called Zakharov equations

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - c_s^2\Delta\right)\delta n &= \frac{1}{16\pi M}\Delta|E|^2 \\ \Delta\left(2i\frac{\partial\varphi}{\partial t} + 3r_D^2\Delta\varphi\omega_p\right) &= \frac{\omega_p}{n_0}\operatorname{div}(\delta n\nabla\varphi) \, . \end{aligned} \tag{1.3.11}$$

Here

$$E = \nabla\varphi, \quad c_s = \sqrt{(T_e + 5T_i/2)/M}\ .$$

These equations correspond to the two fluid plasma model [1.13]. The canonical variables are introduced similarly to the above cases. So we obtain plasmons with the dispersion law (1.3.3), ion sound with that of (1.3.10a) and the interaction Hamiltonian

$$\mathcal{H}_{\text{int}} = \int \left(V_{123} b_1 a_2 a_3^* + \text{c.c.}\right) \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3)\ , \tag{1.3.12}$$

where b and a are amplitudes of ion-sound and Langmuir waves, respectively, and the interaction coeffecient is equal to [1.13]

$$V_{k12} = \frac{\omega_p \sqrt{k}}{4\pi\sqrt{4\pi M n_0 c_s}} \frac{(\boldsymbol{k}_1 \boldsymbol{k}_2)}{k_1 k_2}\ . \tag{1.3.13}$$

This Hamiltonian describes plasmon decay with sound wave emission

$$\omega(\boldsymbol{k}) = \omega(\boldsymbol{k} - \boldsymbol{k}_1) + \Omega(\boldsymbol{k}_1)\ .$$

a process which is sometimes called Cherenkov emission, as it is analogous to wave emission by a particle moving in a medium with a velocity exceeding the phase velocity of waves. Similarly, the process (1.3.12) is allowed if the group velocity of the plasmons is larger than the sound velocity.

The ion interaction also contributes to the four-plasmon interaction. For example, in an isothermal plasma, the interaction coefficient of Langmuir waves with virtual ion-sound waves is [1.14]

$$T_{k123} = \frac{\omega_p(\cos\theta_{k2}\cos\theta_{13} + \cos\theta_{k3}\cos\theta_{12})}{8\pi^3 n_0 T}\ . \tag{1.3.14}$$

Its scaling index is zero. Curiously enough T_{k123} vanishes for one-dimensional motion if $\boldsymbol{k} + \boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3$.

In a constant external magnetic field H, the Hamiltonian coefficients depend on the angles in $\boldsymbol{k}$-space. In particular, the dispersion laws of both Langmuir and ion-sound waves in strong enough fields are of the decay type. Scale-invariance of $\omega(\boldsymbol{k})$ and $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$ is observed separately for the two components of the wave vector: one parallel to the field k_z and the other perpendicular $k_\perp$.

Let us consider ion sound in a magnetized plasma [1.15]. The presence of magnetic field will give rise to a Lorentz force in the Euler equation, and instead of (1.3.7), we have

$$\begin{aligned} &\frac{\partial \boldsymbol{v}(\boldsymbol{r},t)}{\partial t} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{e\nabla\varphi}{M} + \frac{e[\boldsymbol{v}, \boldsymbol{H}]}{Mc}\ , \\ &\frac{\partial \varrho(\boldsymbol{r},t)}{\partial t} + \operatorname{div}(\varrho\boldsymbol{v}) = 0\ . \end{aligned} \tag{1.3.15}$$

Assuming the motion to be quasi-neutral (the criterion for this will be obtained below), we shall consider the ions, like the electrons, to obey the Boltzmann distribution law

$$\varrho = \varrho_0 \exp\left(\frac{e\varphi}{T}\right) . \tag{1.3.16}$$

Equations (1.3.15, 16) form a closed system. Having obtained φ from (1.3.16), we can rewrite the term $e\nabla\varphi/M$ in (1.3.15) in the standard form $e\nabla\varphi/M = \nabla w$, where w is the enthalpy

$$w = (T/M)\ln(\varrho/\varrho_0) = c_s^2 \ln(\varrho/\varrho_0) .$$

The Hamiltonian has the form known from hydrodynamic-type systems (1.2.2) where the internal energy $\varepsilon(\varrho)$ is related to the enthalpy: $w = \delta\varepsilon/\delta\varrho$. As usual, the canonical variables (λ, μ) and (ϱ, Φ) are introduced and similarly as in (1.2.5a) and (1.2.23) the velocity $\boldsymbol{v}$ reads then

$$\boldsymbol{v} = (\lambda\nabla\mu - \mu\nabla\lambda)/2\varrho + \nabla\Phi - e\boldsymbol{A}/Mc$$

(considering that in the magnitic field the generalized momentum is renormalized to vector potential $\boldsymbol{p} \to \boldsymbol{p} - e\boldsymbol{A}/c$).

The vector potential of the constant field is $2\boldsymbol{A} = [\boldsymbol{H}\boldsymbol{r}]$, $\boldsymbol{r} = (x, y)$. In the new variables the equations have the form of the Hamilton equations (1.2.6, 7) containing the coordinate r in an explicit form. In order to eliminate $\boldsymbol{r}$ from $\boldsymbol{v}$, we perform the canonical transformation

$$\lambda \to \lambda + \sqrt{\varrho\Omega_H}\, x, \quad \mu \to \mu - \sqrt{\varrho\Omega_H}\, y ,$$

$$\Phi \to \Phi - \sqrt{\varrho\Omega_H}(\lambda y + \mu x) .$$

Here $\Omega_H = eH/Mc$ is the Larmor frequency of ion rotation in the magnetic field H. Going then over to normal variables and expanding the Hamiltonian, we obtain $\omega(\boldsymbol{k})$ and $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$.

In this case it will be more convenient to derive first a truncated equation describing the waves in the range involved and to go then over to normal variables. Supposing the magnetic energy to be much greater than the thermal one ($8\pi nT \ll H^2$) and considering low-frequency waves $\big(\omega(\boldsymbol{k}) \ll \Omega(\boldsymbol{k})\big)$ with weak dispersion ($k_\perp c_s \ll \Omega_H$), we get by virtue of (1.3.15, 16) the dispersion law for magnetized ion sound

$$\omega(k) = c_s k_z (1 - k_\perp^2 c_s^2 / 2\Omega_H^2) . \tag{1.3.17}$$

Since the deviations from quasi-neutrality lead, as seen from (1.3.10), to the correction $k^2 r_D^2/2$, (1.3.7) holds for not too small $k_\perp$'s, when $k_\perp/k > r_D^2 \Omega_H / c_s$. The dispersion law (1.3.17) is of the decay type. It should be noted that the group velocity of such waves is directed along the magnetic field. If we restrict our consideration to unidirectional waves and the quadratic nonlinearity, then for the value $u = \partial\Phi/\partial z$ from (1.3.15, 16), we obtain

$$\frac{\partial u(\boldsymbol{r},t)}{\partial t} - c_s \frac{\partial}{\partial z}\left[u(\boldsymbol{r},t) + \frac{c_s^2}{2\Omega_H^2}\Delta_\perp u(\boldsymbol{r},t) - \frac{u^2(\boldsymbol{r},t)}{2c_s}\right] = 0 , \tag{1.3.18}$$

whose linear part corresponds to (1.3.17). Going over to a reference system moving along the z-axis with speed c_s yields

$$\frac{\partial u(\boldsymbol{r},t)}{\partial t} = c_s \frac{\partial}{\partial z}\left[\frac{c_s^2}{2\omega_H^2}\Delta_\perp u(\boldsymbol{r},t) - \frac{u^2(\boldsymbol{r},t)}{2c_s^2}\right] . \tag{1.3.19}$$

After transition to normal variables according to

$$u_k = c_s\sqrt{k_z/2}[a(k_z,k_\perp) + a^*(-k_z,-k_\perp)] ,$$

(1.3.19) corresponds to the standard Hamiltonian $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3$ [see (1.1.17, 23)] with the coefficients

$$\begin{aligned} \omega(k_z,k_\perp) &= \frac{c_s^3 k_z k_\perp^2}{2\Omega_H^2}, \\ V(k,k_1,k_2) &= \frac{1}{4\pi}\sqrt{\frac{c_s k_z k_{1z} k_{2z}}{2\varrho_0}} . \end{aligned} \tag{1.3.20}$$

In conclusion, we give, without calculations the formulas defining the dispersion law and the matrix elements of the decay interaction for Langmuir waves in magnetized plasma [1.13]. The system of equations differs from (1.3.1) in the Lorentz force substituted for the gas kinetic term on the right-hand-side of the Euler equation. The resulting dispersion law has two branches

$$\omega_\pm^2(\boldsymbol{k}) = \frac{1}{2}\left(\omega_p^2 + \omega_H^2\right) \pm \frac{1}{2}\sqrt{\omega_p^4 + \omega_H^4 - 2\omega_p^2\omega_H^2\cos 2\theta_k} , \tag{1.3.21}$$

$\omega_H = eH/mc$ is the electronic Larmor frequency and θ_k the angle between the wave vector $\boldsymbol{k}$ and the constant magnetic field. We shall be concerned with the lower branch in the two limiting cases when the problem becomes scale-invariant:

1. $\omega_H \gg \omega_p$ is *the strong field case*. Considering waves propagating almost perpendicular to the field, one can get

$$\omega_k = \omega_p|k_z/k_\perp| . \tag{1.3.22a}$$

For the angular range $\cos\theta_k \ll \omega_p/\omega_H$ we obtain

$$\begin{aligned} V_{k12} = {} & i\frac{\omega_p}{\omega_H}\frac{(\boldsymbol{k}_{1\perp}[\boldsymbol{h},\boldsymbol{k}_{2\perp}])}{\sqrt{32\varrho_0}}\sqrt{\frac{\omega_1\omega_2}{\omega_k}}\operatorname{sign} k_z \\ & \times\left[\frac{\operatorname{sign} k_z}{k_\perp}\left(\frac{k_{1\perp}}{k_{2\perp}} - \frac{k_{2\perp}}{k_{1\perp}}\right) + \frac{\operatorname{sign} k_{1z}}{k_{2\perp}} + \frac{\operatorname{sign} k_{2z}}{k_{1\perp}}\right] . \end{aligned} \tag{1.3.22b}$$

Here $\boldsymbol{h} = \boldsymbol{H}/H$.

At larger angles $\omega_p/\omega_H \ll \cos\theta_k \ll 1$ we have

$$\begin{aligned} V_{k12} = {} & \operatorname{sign}(k_z k_{1z} k_{2z})\sqrt{\frac{\omega_k\omega_1\omega_2}{32\varrho_0\omega_p^2}} \\ & \times (k_\perp \operatorname{sign} k_z + k_{1\perp}\operatorname{sign} k_{1z} k_{2\perp}\operatorname{sign} k_{2z}) . \end{aligned} \tag{1.3.22c}$$

2. $\omega_H \ll \omega_p$ is *the weak field case* ($\cos\theta_k \ll 1$):

$$
\begin{aligned}
&\omega_k = \omega_H \frac{k_z}{k_\perp}, \\
&V_{k12} = \sqrt{\frac{\omega_1\omega_2}{32\varrho_0\omega_k}} \frac{(\boldsymbol{k}_{1\perp}[\boldsymbol{h}, \boldsymbol{k}_{2\perp}])}{k_{1\perp}k_{2\perp}} \\
&\qquad \times \left(k_\perp + k_{2\perp}\mathrm{sign}\,(k_z k_{2z}) + k_{1\perp}\mathrm{sign}\,(k_z k_{1z})\right) .
\end{aligned} \tag{1.3.23}
$$

The scaling indices of $\omega(\boldsymbol{k})$ and $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$ are the same as for (1.3.22a,b).

1.3.2 Atmospheric Rossby Waves and Drift Waves in Inhomogeneous Magnetized Plasmas

Drift Rossby waves propagate in atmospheres of planets and in oceans. Their frequencies are small as compared to the frequency of global rotation of planets Ω_0, and the lengths are rather large compared to the extensions of the medium L (depth of the ocean or height/thickness of the atmosphere). At large amplitudes these waves become planetary vortices. The largest among them is the Big Red Spot of Jupiter. The planetary waves (vortices) are named after the Swedish geophysicist Rossby who revealed, in the 1930–40, their important role in the processes of global circulation of the atmosphere [1.16], although theoretically they had been known since the end of the last century [1.17]. Those waves are successfully simulated in laboratories [1.18–19], observed in the atmosphere of the Earth and in oceans [1.20]. Rossby waves are analogues to the drift waves in inhomogeneous magnetized plasmas [1.21–22]. They may have some relation to the generation of magnetic fields in nature [1.23–24].

We distinguish barotropic and baroclinic Rossby waves. The former allow to treat phenomena observed in nature (atmosphere or ocean) as occuring in a quasi-two-dimensional medium where the Rossby waves have a wave length λ which is much larger than the vertical extension L. In describing the baroclinic waves, one should take into account the vertical inhomogeneity of the density of an ocean (which is due to the vertical temperature profile and salt concentration) or atmosphere. This inhomogeneity gives rise to vertical oscillations of the fluid which is stable against convection. In barotropic Rossby waves there are no oscillations and the medium may be regarded as two-dimensional. We shall deal with this simple case in more detail.

We suppose the planet to rotate with angular velocity Ω_0. Proceeding from the Euler equation complemented by the Coriolis force one can obtain the equation which describes the atmospheric Rossby waves:

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\Delta\psi - k_0^2\psi) + \beta\frac{\partial\psi}{\partial x} = \frac{\partial\Omega}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\Omega}{\partial y}\frac{\partial\psi}{\partial x} \\
&\qquad\qquad \Omega = \Delta\psi - k_0^2\psi .
\end{aligned} \tag{1.3.24}
$$

This equation and its derivation are described in details in various monographs (see, e.g.[1.25–28]). So we shall not dwell upon that matter. We shall only explain the notation used and give the applicability criterion for (1.3.24). The stream function $\psi(x, y, t)$ is via

$$v_y = -\frac{\partial \psi}{\partial y}, \quad v_x = \frac{\partial \psi}{\partial x}.$$

related to the velocity $v = (v_x, v_y)$. The parameter β describes the dependence of the Coriolis force $f = 2\Omega_0 \cos\alpha$ on the latitude

$$\beta = \frac{\partial f}{\partial y} = -R^{-1}\frac{\partial f}{\partial \alpha} . \tag{1.3.25}$$

R is the radius of the planet's curvature, $\alpha = \pi/2 - \phi$ where ϕ is the geographical latitude, $k_0 = 1/r_R$ and the Rossby-Obukhov radius equals

$$r_R = \frac{\sqrt{gL}}{f} . \tag{1.3.26}$$

In the literature two names are assigned to (1.3.24): in hydrodynamics it is called the Charney-Obukhov equation [1.26] and in the plasma physics the Hasegawa-Mima equation [1.27]. It describes not only Rossby waves but also some other phenomena.

Some of them are:

1. Drift waves in inhomogeneous magnetized plasmas [1.21, 27, 28]. In this case $\psi = e\phi/\Omega_H M$ holds where e is the electron charge; ϕ is the potential of the electric field. The Rossby-Obukhov radius is $r_R = \sqrt{T_e/M}/\Omega_H$, the Larmor radius for ions calculated by the electron temperature. As elsewhere in this section, M, Ω_H are the mass and the Larmor frequency of ion rotation, respectively. The parameter β is the plasma inhomogeneity $\beta = \Omega_H \partial[\ln(n_0/\Omega_H)]/\partial y$ and is calculated at $y = 0$. Here n_0 is the equilibrium concentration of the plasma.
2. Low-hybrid drift waves in plasmas of compact toruses, pinches with reverse field and ionospheric F-stratum [1.29]. For these waves, $\psi = e\phi/\omega_H m$, $r_R = T_i/(m\omega_H^2)$, $\beta = \omega_H \partial[\ln(n_0/\omega_H)]/\partial y$. Here m and ω_H are mass and Larmor frequency of electron rotation, respectively.
3. Electromagnetic oscillations of the electronic component in inhomogeneous magnetized plasmas occur in z-pinches and other pulsed high-current discharges [1.30]. Here $\psi = \phi cm/H_0 m$, $r_R = c/\omega_p$, $\beta = \omega_H \partial[\ln(n_0/\omega_H)]/\partial y$, where c is the velocity of light; H_0 is the equilibrium magnetic field and ω_p the plasma frequency.
4. Density waves in rotating gas disks of galaxies [1.31]. In this case, ψ is the gravitational potential, $\beta = 2\Omega\partial[\ln(\varrho/\Omega)]/\partial r$, where Ω is the frequency of gas disk rotation; ϱ is the unperturbed density of galactic gas as a function of the radial coordinate r.

Equation (1.3.24) is applicable for intermediate wavelengths λ larger than the depth L (shallow water approximation) but less than R so we may regard the medium under consideration to be plane (not spheric). To provide a feel for the orders of magnitude of the characteristic lengths Table 1.1 gives the values of R, r_R and the mean height of atmosphere L for the Earth, Jupiter and Saturn.

Table 1.1. Characteristic lengths determined the Rossby waves: planet's radius R, Rossby-Obukhov radius $r_R = \sqrt{gL}/f$, and mean height of atmosphere L

Planet	R in km	r_R in km	L in km
Earth	6400	3000	8
Jupiter	71000	6000	25
Saturn	≃70000	6000	80

We see that there is a large interval of wave lengths λ where, on the one hand, $\lambda \gg L$ and the motion may be considered to be two-dimensional, and on the other hand, $R \gg \lambda$, so that one can neglect the planet's curvature. Thus we arrive at the "β-plane approximation" used, in effect, in deriving (1.3.24). This approximation considers waves to be on a "β-plane" tangent to the planet's surface rather than on its spheric surface. The dependence of the coefficient β (1.3.25) on y (or α) is not taken into account.

From (1.1.24) follows the dispersion law of the Rossby waves:

$$\omega(k) = -\frac{\beta k_x}{k^2 + k_0^2}, \quad k^2 = k_x^2 + k_y^2 \,. \tag{1.3.27}$$

The phase velocity of Rossby waves is directed westwards, against the global rotation of the planet. The phase velocity decreases with increasing k, its maximal value

$$v_R = \beta k_0^{-2} = \beta r_R^2 \tag{1.3.28}$$

is called the Rossby velocity. The wave frequency (1.3.27) $\omega(k) \to 0$ at $k \to 0, \infty$. It reaches its maximal value

$$\omega_R = \frac{\beta}{2k_0} \tag{1.3.29}$$

at $\boldsymbol{k} \parallel k_x$, $k = k_0$. It is interesting to compare the Rossby velocity v_R with the linear velocity of the planet's surface motion $\Omega_0 R$, and the maximal frequency of Rossby waves ω_R with the planet's rotation frequency Ω_0. From (1.3.26–29), we get the estimates

$$\frac{v_R}{\Omega_0 R} \simeq \left(\frac{r_R}{R}\right)^2, \qquad \frac{\omega_R}{\Omega_0} \simeq \frac{r_R}{R},$$

whence follows

$$\frac{r_R}{R} \simeq \frac{\sqrt{gL}}{\Omega_0 R}.$$

For the atmosphere, $\sqrt{gL} \simeq c_s$ is the velocity of sound near the planet's surface.

Baroclinic Rossby Waves. As mentioned earlier, in view of the vertical inhomogeneity of the medium one should not only consider horizontal wave motion (as in the case of barotropic waves), but also vertical wave motion. This complicates the equation of motion which will be given here without derivation (see, e.g., [1.26]). The function ψ now depends also on the vertical coordinate z: $\psi(x, y, z)$, and the desired equation is obtained by substituting into (1.3.24) an expression for the vortex density Ω different from the previous one

$$\Omega = \Delta\psi + \frac{\partial}{\partial z}\frac{f^2}{N^2}\frac{\partial\psi}{\partial z} . \tag{1.3.30a}$$

The equation for $\Omega(x, y, z, t)$ can thus be written

$$\frac{\partial\Omega}{\partial t} + \beta\frac{\partial\psi}{\partial x} = \frac{\partial(\Omega, \psi)}{\partial(x, y)} = \frac{\partial\Omega}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\Omega}{\partial y}\frac{\partial\psi}{\partial x} . \tag{1.3.30b}$$

Here N is the frequency of vertical oscillations of a convection-stable inhomogeneous fluid. The fluid density is assumed to decrease in the vertical direction. In an incompressible medium

$$N = -\frac{g}{\varrho}\frac{\partial\varrho}{\partial z} , \tag{1.3.31}$$

whereas for a compressible fluid, the term g^2/c_s^2 should be added to the right-hand-side of (1.3.31), where c_s is the sound velocity in the medium.

It is seen from (1.3.30) that the dispersion relation for the baroclinic waves has the same form as for the barotropic ones (1.3.27). But instead of the Rossby-Obukhov radius (1.3.26) (it would be natural to call it barotropic) as a characteristic dispersion scale, it contains the so-called baroclinic Rossby radius r_l:

$$k_0 \to k_l = r_l^{-1} = \frac{\pi l f}{NL} . \tag{1.3.32}$$

Here l is the number of the vertical mode: $\psi \propto \exp(i\pi l z/L)$. In the Earth' ocean at $l = 1$ $r_1 \simeq 50\,\text{km}$, which is much less then the barotropic radius $r_R \simeq 3000\,\text{km}$.

Hamiltonian Description of Barotropic Rossby Waves. It has been shown independently by *Weinstein* [1.32] and *Zakharov* and *Kuznetsov* [1.3] that (1.3.24) is a Hamiltonian system and may be represented as:

$$\frac{\partial\Omega}{\partial t} = \{\Omega, \mathcal{H}\}. \tag{1.3.33a}$$

Here $\mathcal{H}$ is the energy of the system

$$\mathcal{H} = \frac{1}{2}\int\left[(\nabla\psi)^2 + k_0^2\psi^2\right] dx dy . \tag{1.3.33b}$$

which is the Hamiltonian, and the symbol $\{F, G\}$ denotes the Poisson bracket determined on functionals of $\Omega(x, y)$ by

$$\{F, G\}_\Omega = \int(\Omega + \beta y)\frac{\partial(\delta F/\delta\Omega, \delta G/\delta\Omega)}{\partial(x, y)} dx dy . \tag{1.3.33c}$$

The equivalence of (1.3.24) and (1.3.33) may be checked by the direct calculation.

To introduce the canonical variables for the system (1.3.33) means to diagonalize the Poisson bracket, i.e., to represent it in a form involving constant coefficients. This problem has been solved by *Zakharov and Piterbarg* [1.33]. They introduce a function $\xi(x,y)$ related to $\Omega(x,y)$ by two equivalent equations

$$\Omega(x,y) = \xi\big(x, y + \beta^{-1}\Omega(x,y)\big), \tag{1.3.34a}$$

$$\xi(x,y) = \Omega\big(x, y - \beta^{-1}\xi(x,y)\big) . \tag{1.3.34b}$$

Then they prove that

$$\{F,G\}_\Omega = \beta \int \frac{\delta F}{\delta \xi}\frac{\partial}{\partial x}\frac{\delta G}{\delta \xi}\, dx dy, \tag{1.3.35}$$

i.e., in the variables $\xi(x,y)$ the bracket (1.3.33c) becomes a bracket with constant coefficients. Here we repeat the proof:
Equation (1.3.34a) is represented as

$$\Omega(x,w) = \int \xi(x,z)\delta\big(z - w - \beta^{-1}\Omega(x,w)\big)\, dz ,$$

hence

$$\frac{\delta\Omega(x,w)}{\delta\xi(x,y)} = \frac{\beta}{\beta - \xi_y}\delta\big(y - w - \beta^{-1}\Omega(x,w)\big) = \delta\big(w - y + \beta^{-1}\xi(x,y)\big)$$

and, consequently,

$$\frac{\delta F}{\delta\xi(x,y)} = \int \frac{\delta}{\delta\Omega(x,w)}\frac{\delta\Omega(x,w)}{\delta\xi(x,y)}\, dw = \frac{\delta F}{\delta\Omega(x,w)} .$$

Here

$$w = y - \beta^{-1}\xi(x,y), \qquad dy = \frac{\beta\, dw}{\beta - \xi_y} . \tag{1.3.36}$$

Now we calculate

$$\begin{aligned}\{F,G\}_\xi &= \beta \int \frac{\delta F}{\delta\xi(x,y)}\frac{\partial}{\partial x}\frac{\delta G}{\delta\xi(x,y)}\, dx\, dy \\ &= \beta \int \frac{1}{1 - \beta^{-1}\xi_y}\frac{\delta G}{\delta\Omega(x,w)} \\ &\quad \times \left[\frac{\partial}{\partial x}\frac{\delta F}{\delta\Omega(x,w)} - \frac{\xi_x}{\beta}\frac{\partial}{\partial w}\frac{\delta F}{\delta\Omega(x,w)}\right] dx\, dw .\end{aligned}$$

Differentiating this equation with respect to x and y, we find [the w and y points are related by (1.3.36)]

$$\frac{\beta\xi_y}{\beta - \xi_y} = \Omega_w, \quad \frac{\beta\xi_x}{\beta - \xi_y} = \Omega_x . \tag{1.3.37}$$

Substituting (1.3.37) into the previous equation, we get

$$\begin{aligned}\{F,G\}_\xi &= \beta\int\frac{\delta G}{\delta\Omega}\frac{\partial}{\partial x}\frac{\delta F}{\delta\Omega}\,dx\\ &+\int\left(\Omega_x\frac{\partial}{\partial w}\frac{\delta F}{\delta\Omega}-\Omega_w\frac{\partial}{\partial x}\frac{\delta F}{\delta\Omega}\right)\frac{\delta G}{\delta\Omega}\,dx\,dw\\ &=\{F,G\}_\Omega .\end{aligned}$$

We used in the proof the local reversibility of (1.3.36) and the inverse equation

$$y = w + \beta^{-1}\Omega(x,w)\ . \tag{1.3.38}$$

Reversibility of (1.3.38) has a clear topological meaning, indicating the in-closed character of isocurlus lines of a full planetary vortex $\omega+\beta y$. It is known that this condition precludes the existence of solitons or localized vortices in the solution of (1.3.24) and is satisfied for the waves of sufficiently small amplitude. At small ξ,Ω equations (1.3.34) may be expanded in a series

$$\Omega(x,y)=\sum_{n=0}^{\infty}\frac{(-1)^n\beta^n}{(n+1)!}\frac{\partial^n}{\partial y^n}\xi^{n+1}\ , \tag{1.3.39a}$$

$$\xi(x,y)=\sum_{n=0}^{\infty}\frac{\beta^n}{(n+1)!}\frac{\partial^n}{\partial y^n}\Omega^{n+1}\ . \tag{1.3.39b}$$

The statement we have proved shows that after the transition to the ξ-variable, (1.3.33a) reduces to

$$\frac{\partial\xi}{\partial t}=\beta\frac{\partial}{\partial x}\frac{\delta\mathcal{H}}{\delta\xi}\ . \tag{1.3.40}$$

It is now easy to see that use of the representation

$$\xi(x,y)=\frac{\sqrt{\beta/2}}{\pi}\int_{p>0}\sqrt{p}\left[a(p,q)e^{i(px+qy)}+a^*(p,q)e^{-i(px+qy)}\right]dpdq\ , \tag{1.3.41}$$

converts (1.3.40) into the Hamiltonian form

$$\frac{\partial a}{\partial t}=-i\frac{\delta\mathcal{H}}{\delta a^*}\ . \tag{1.3.42}$$

The Hamiltonian $\mathcal{H}$ is an infinite power series of the variables $a(\boldsymbol{k})$, $a^*(\boldsymbol{k})$. The first terms of this series have the form

$$\begin{aligned}\mathcal{H}=2\int\omega(k)|a(\boldsymbol{k})|^2\,d\boldsymbol{k}+\int V_{k12}\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\\ \times\left[a^*(\boldsymbol{k})a(\boldsymbol{k}_1)a(\boldsymbol{k}_2)-a(\boldsymbol{k})a^*(\boldsymbol{k}_1)a^*(\boldsymbol{k}_2)\right]d\boldsymbol{k}d\boldsymbol{k}_1d\boldsymbol{k}_2\ .\end{aligned} \tag{1.3.43}$$

Here $\boldsymbol{k}_i = (p_i, q_i)$, the integration domain is $p_i > 0$ and $\omega(\boldsymbol{k}) = \beta p(k^2 + k_0^2)^{-1}$ is the dispersion law of Rossby waves. The interaction coefficient has the simple form

$$V_{k12} = \frac{i}{\pi}\sqrt{\frac{\beta p p_1 p_2}{2}}\left(\frac{q}{k^2 + k_0^2} - \frac{q_1}{k_1^2 + k_0^2} - \frac{q_2}{k_2^2 + k_0^2}\right) . \tag{1.3.44}$$

These expressions may be considered from two different viewpoints. On the one hand, if (1.3.24) is a postulate, then the corresponding canonical equation is (1.3.42) with the Hamiltonian (1.3.43–44). But in deriving (1.3.24) we can consider the limit $k > k_0$. Strictly speaking, it is only in this region that we can handle the three-wave matrix element for Rossby waves

$$V_{k12} \to \frac{i}{\pi}\sqrt{\frac{\beta p p_1 p_2}{2}}\left(\frac{q}{k^2} - \frac{q_1}{k_1^2} - \frac{q_2}{k_2^2}\right) . \tag{1.3.45}$$

Hamiltonian Description of Baroclinic Rossby Waves. As shown by *Zakharov et al.* [1.34], the canonical variables for baroclinic Rossby waves are introduced similarly to the barotropic case. At first one can carry out a direct calculation to see that the equations of motion (1.3.30) reduce to a Hamiltonian form similar to (1.3.33a):

$$\frac{\partial \Omega}{\partial t} = \{\Omega, \mathcal{H}\}_\Omega , \tag{1.3.46a}$$

where Ω is given by (1.3.30a). The Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int\left[(\nabla\psi)^2 + \frac{f^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2\right] d\boldsymbol{r}\,dz , \tag{1.3.46b}$$

differs from (1.3.33b) and the Poisson bracket between two functionals F and G,

$$\{F, G\}_\Omega = \int(\Omega + \beta y)\frac{\partial(\delta F/\delta\Omega, \delta G/\delta\Omega)}{\partial(x, y)}\,dx\,dy\,dz . \tag{1.3.46c}$$

has a form which deviates from (1.15c) by an integration over z. Similarly to the barotropic case, one can prove [1.34] that in the ξ-variables determined like (1.3.39),

$$\begin{aligned} \Omega(x, y, z) &= \xi\left(x, y, +\beta^{-1}\Omega(x, y, z), z\right) , \\ \xi(x, y, z) &= \Omega\left(x, y - \beta^{-1}\xi(x, y, z), z\right) , \end{aligned} \tag{1.3.47}$$

(1.3.46) reduces to (1.3.40). For functions of x, y, z we shall use representations of the form

$$\xi(x, y, z) = \frac{1}{2\pi}\sum_m \varphi_m(z)\int\limits_{k_x>0}\left[\xi_m(k)e^{i(\boldsymbol{k}\boldsymbol{x})} + \xi_m^*(k)e^{-i(\boldsymbol{k}\boldsymbol{x})}\right] dk . \tag{1.3.48}$$

where $\boldsymbol{x} = (x, y)$, $\{\varphi_m(z)\}$ is an orthonormal system of eigenfunctions of the operator $\hat{L}$:

$$\hat{L} = \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\right)\frac{\partial}{\partial z}\,,$$
$$\hat{L}\varphi_m = -k_m^2\varphi_m\,, \quad \varphi_m(0) = \varphi_m(L) = 0\,, \tag{1.3.49}$$
$$\frac{1}{L}\int_0^L \varphi_n\varphi_m\left(\frac{f_0^2}{N^2}\right)dz = \delta_{nm}\,.$$

In such terms the Hamiltonian (1.3.46) reduces to a form

$$\mathcal{H} = \frac{1}{2}\sum_m (k_m^2 + k^2)|\varphi_m(k)|^2\,dk = \frac{1}{2}\sum_m \int \frac{|\Omega_m(k)|^2\,dk}{k_m^2 + k^2}\,. \tag{1.3.50}$$

Now we go over to normal canonical variables,

$$a_m(k) = (-2\beta k_x)^{1/2}\xi_m(k)\,,$$

in which the Hamilton equation (1.3.40) takes the canonical form (1.3.42), and all that is left to do is to express the Hamiltonian (1.3.50) in terms of the normal variables. In view of the fact that for the Rossby waves, three-wave resonance interactions are possible, in calculating $\mathcal{H}$ we shall restrict ourselves to those terms that are quadratic and cubic in $a_m(k)$. This allows us to use, instead of (1.3.47), an approximate relationship between Ω and ξ which holds to an accuracy of the order of β^{-2}:

$$\Omega(x,y,z) = \xi(x,y,z) + \frac{1}{2\beta}\frac{\partial}{\partial y}\xi^2(x,y,z)\,, \tag{1.3.51a}$$

$$\Omega_m(k) = \xi_m(k) + \frac{1}{4\pi\beta}\int \exp[-i(kx)]\varphi_m(z)\left(\frac{f_0^2}{N^2}\right)\frac{\partial\xi^2}{\partial y}\,dxdydz$$
$$= \xi_m(k) + \frac{ik_y}{4\pi\beta}\sum_{m_1,m_2} B(m,m_1,m_2)\xi_{m_1}(k_1)\xi_{m_2}(k_2) \tag{1.3.51b}$$
$$\times\,\delta(k-k_1-k_2)\,dk_1dk_2\,,$$
$$B(m,m_1,m_2) = L^{-1}\int_0^L \varphi_m(z)\varphi_{m_1}(z)\varphi_{m_2}(z)\left(f_0^2/N^2\right)dz\,. \tag{1.3.51c}$$

Substitution of (1.3.51b) into (1.3.50) gives

$$\mathcal{H} = 2\sum_m\int \sigma_m(k)|a_m(k)|^2\,dk$$
$$+\ 2\sum_{mm_1m_2}\int [V_{k12}\left(a^*a_1a_2 - \text{c.c.}\right)\delta(k-k_1-k_2)$$
$$+\,U_{k12}\left(a^*a_1^*a_2^* - \text{c.c}\right)\delta(k+k_1+k_2)]\,dkdk_1dk_2\,,$$
$$V_{k12} = \frac{i}{4\pi}\sqrt{-2\beta k_xk_{1x}k_{2x}}[S(k,k_1,k_2)$$
$$-\ S(k_2,-k_1,k) - S(k_1,k,-k_2)]\,,$$
$$U_{k12} = -\frac{i}{4\pi}\sqrt{-2\beta k_xk_{1x}k_{2x}}\,S(k,k_1,k_2)\,,$$
$$S(k,k_1,k_2) = S(m,m_1,m_2,k,k_1,k_2) = \frac{B(m,m_1,m_2)k_y}{k_m^2+k^2}\,,$$
$$\sigma_m(k) = -\frac{\beta k_x}{k_m^2+k^2}\,.$$

If the vertical inhomogeneity is linear, then according to (1.3.31) N =const holds and

$$k_m = \frac{\pi m f}{NL}, \qquad \varphi_m(z) = \sin(\pi m z/L) .$$

and for the coefficients (1.3.51c) we obtain

$$B(m, m_1, m_2) = \frac{f^2}{N^2} \times \frac{\left[(-1)^{m+m_1+m_2} - 1\right] m m_1 m_2}{(m+m_1+m_2)(m-m_1+m_2)(-m-m_1+m_2)(-m+m_1+m_2)} .$$

It should be noted that in the case of the barotropic waves, this expression is replaced by unity and k_m by the constant k_0.

1.4 Spin Waves

1.4.1 Magnetic Order, Energy and Equations of Motion

To date, a large variety of magnetically-ordered substances are known: dielectrics, semiconductors and metals, both crystalline and amorphous [1.35]. Their structure includes paramagnetic atoms (ions) with uncompensated spin S, thus having magnetic moment μS (μ is the Bohr magneton here). Such atoms give rise to the exchange interaction which is of electrostatic nature and is to be associated with the Pauli principle prohibiting more than one electron to be in a given quantum-mechanical state [1.36]. At low temperatures, this interaction leads to magnetic ordering orienting the magnetic moments of the atoms in a definite manner. The simplest type of magnetic ordering is the ferromagnetic state in which the magnetic moments of all atoms are parallel. This results in a macroscopic magnetic moment with density $\boldsymbol{M}$. In contrast to ferromagnets, the total magnetic moment of antiferromagnets is zero. In the simplest case, an elementary cell of crystalline antiferromagnet has two magnetic atoms whose moments are antiparallel and equal in magnitude. In describing antiferromagnets one uses the notion of a magnetic sublattice, which contains the translational-invariant magnetic atoms, i.e., the positions of them differ by an integer number of elementary translations of the crystalline lattice. In the simplest antiferromagnet, there are two magnetic sublattices with the moments $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, with $\boldsymbol{M} = \boldsymbol{M}_1 + \boldsymbol{M}_2 = 0$.

At low temperatures the long-wave magnetic excitations may be described classically, using the functions $\boldsymbol{M}_j(\boldsymbol{r}, t)$. These excitations are spin waves or precession waves of the magnetic moment. The equation of motion for $\boldsymbol{M}(\boldsymbol{r}, t)$ (the Bloch equation) describes the precession of a vector with a fixed length $|\boldsymbol{M}(\boldsymbol{r}, t)|^2 = M^2(T)$ =const in an effective magnetic field $\boldsymbol{H}_{\text{eff}}(\boldsymbol{r}, t)$ (see, e.g., [1.37]:

$$\partial \boldsymbol{M}(\boldsymbol{r}, t)/\partial t = g_m[\boldsymbol{H}_{\text{eff}}(\boldsymbol{r}, t), \boldsymbol{M}(\boldsymbol{r}, t)] , \qquad (1.4.1a)$$

$$\boldsymbol{H}_{\text{eff}}(\boldsymbol{r}, t) = -\delta W/\delta \boldsymbol{M}(\boldsymbol{r}, t) . \qquad (1.4.1b)$$

Here $g_m = \mu/\hbar$ is the ratio of magnetic to mechanical moment of the electrons, and W, the energy of the system. The g_m value is approximately equal to $2\pi 2.8$ MHz/Ö. The energy W is a functional of $\boldsymbol{M}(\boldsymbol{r},t)$. In ferromagnets it includes W_0, the interaction energy of a spin subsystem with an external field $\boldsymbol{H}_0$, the exchange energy W_{ex} and a number of the terms of relativistic origin. The main contributions stem from the energy of the magnetic dipole-dipole interaction W_{dd} and the energy of the crystalline anisotropy W_a:

$$W = W_0 + W_{ex} + W_{dd} + W_a \ , \tag{1.4.2a}$$

$$W_0 = -g_m \int (\boldsymbol{H}\boldsymbol{M})\, d\boldsymbol{r} \ , \tag{1.4.2b}$$

$$W_{ex} = \frac{1}{2}\kappa_{ik} \int \frac{\partial M_j}{\partial x_i}\frac{\partial M_j}{\partial x_k}\, d\boldsymbol{r} \ , \tag{1.4.2c}$$

$$W_a = K \int M_z^2\, d\boldsymbol{r} \ , \tag{1.4.2d}$$

$$W_{dd} = -\frac{1}{2} \int (\boldsymbol{H}_m \boldsymbol{M})\, d\boldsymbol{r} \ . \tag{1.4.2e}$$

Here κ_{ik} and K are material constants; $\boldsymbol{H}_m$ is the static magnetic field created by the magnetic moment distribution. The phenomenological expression (1.4.2) for the energy of a ferromagnet is discussed in detail in [1.37–38]. Here we shall only comment on it in brief. The physical meaning of W_0 is obvious, it is the magnetic dipole energy in the external field $\boldsymbol{H}$; the integral in the expression for W_{dd} defines the dipole energy in the self-magnetic field $\boldsymbol{H}_m$, with the interaction energy of each pair taken into account twice. The factor $\frac{1}{2}$ in (1.4.2e) compensates for this double counting. The expression (1.4.2c) for W_{ex} is general enough. Indeed, if we suggest that (i) W_{ex} is independent of the magnetization relative to the crystal axes; (ii) the crystal has an inverted symmetry element; (iii) the energy dependence on the magnetization is quadratic. For the derivation of (1.4.2c) see, e.g., [1.37]. Item (i) follows from the nature of the exchange interaction which is invariant relative to the total rotation of all spins. (ii) is satisfied since most magnets are just of this kind. (iii) holds strictly speaking only for magnetic atoms with the spin $S = 1/2$. At $S > 1/2$ this suggestion is invalid, but experiment shows that corrections to (1.4.2c) are small and this has been theoretically substantiated [1.39]. The term (1.4.2d) for the crystalline anisotropy energy appears in second order perturbation theory with regard to the spin-orbit interaction as a weak one. The constant K is nonzero for uniaxial crystals. At $K < 0$, the energetically favorable orientation of magnetization is that along the crystal's symmetry axis (z axis). At $K > 0$ the orientation in the plane perpendicular to the z axis is favored. In the former case, the anisotropy is referred to as being of the *easy axis* type, and in the latter case, it is said to be of the *easy plane* type. Finally, in crystals with a cubic symmetry, the expression for κ_{ik} is simplified compared with that for the isotropic continuous medium $\kappa_{ik} = \kappa\delta_{ik}$.

1.4.2 Canonical Variables

In (1.4.1) we go over to circular variables

$$M_\pm = M_x + iM_y, \quad \frac{\partial M_\pm}{\partial t} = 2ig_m M_z, \quad M_z^2 = M_0^2 - M_+ M_- \,. \tag{1.4.3}$$

We choose the z axis along the equilibrium direction of magnetization. Then at small oscillation amplitudes of the magnetic moment the $M_\pm$ values will be small, and M_z will be close to the length of $\boldsymbol{M}$, i.e., M_0. Comparing (1.4.3) and (1.1.6), we see that these equations have in the $M_\pm$-linear approximation the form of Hamilton equations if we take as the canonical variables

$$a(\boldsymbol{r},t) = \frac{M_+}{\sqrt{2g_m M_0}}, \quad a^*(\boldsymbol{r},t) = \frac{M_-}{\sqrt{2g_m M_0}} \,.$$

Therefore it is reasonable to write these canonical variables as

$$\begin{aligned} M_+ &= af(a^*a)\sqrt{2g_m M_0}\,, \\ M_- &= a^* f(a^*a)\sqrt{2g_m M_0}, \quad f(0) = 1 \,. \end{aligned} \tag{1.4.4}$$

Substituting (1.4.4) into (1.4.3), we obtain an equation for $\partial a(\boldsymbol{r},t)/\partial t$:

$$\frac{\partial a(\boldsymbol{r},t)}{\partial t} = -\frac{iM_z(a^*a)}{(f^2 + 2aa^* ff')M_0}\frac{\delta W}{\delta a^*}, \tag{1.4.5a}$$

Here

$$M_z(a^*a) = M_0\sqrt{1 - 2f^2 g_m a^*a/M_0}. \tag{1.4.5b}$$

Demanding these equations to coincide with the canonical equations (1.1.6), we obtain the differential equation for the function $f(x)$

$$f^2 + 2xff' = 2g\sqrt{M_0^2 - xf^2}, \tag{1.4.6a}$$

of which the only solution satisfying the condition $f(0) = 1$ is

$$f(x) = \sqrt{1 - g_m a^*a/2M_0} \tag{1.4.6b}$$

Thus we have expressed the natural variables of the ferromagnet's spin subsystem $M_z, M_\pm$ through the canonical ones:

$$\begin{aligned} M_+ &= a\sqrt{2g_m M_0[1 - g_m a^*a/2M_0]}, \quad M_- = M_+^*, \\ M_z &= M_0 - g_m a^*a \,. \end{aligned} \tag{1.4.7}$$

This equation is nonlinear and valid if $g_m a^*a < 2M_0$. The ferromagnet's energy W expressed via the canonical variables becomes the Hamiltonian $\mathcal{H}(a^*, a)$. In quantum mechanics, the Holstein-Primakoff representation has long been known [1.37–38]. It gives the spin operators in terms of Bose operators. The formulas (1.4.7) are the classical analogue of this representation. They were first used by

Schlömann for the analysis of nonlinear processes in a spin wave system [1.40]. The choice of canonical variables is certainly not unique. For (1.4.1) one can introduce other canonical variables to express the vector M as follows

$$M_z + iM_x = M_0\sqrt{1 + \frac{g_m \mid b^* - b \mid}{2M_0}} \exp\left[i(b^* + b)\sqrt{g_m/2M_0}\right] , \qquad (1.4.8)$$
$$M_y = i\sqrt{g_m M_0/2}(b^* - b) .$$

These formulas are the classical analogue of the spin operator representation via Bose operators as suggested by *Baryakhtar* and *Yablonsky* [1.41].

Comparing (1.4.7) and (1.4.8), one can see that the Holstein-Primakoff a^*, a and Baryakhtar-Yablonsky b^*, b variables coincide in the linear approximation. The specific character of the problem under consideration determines the type of representation to be preferred.

1.4.3 The Hamiltonian of a Heisenberg Ferromagnet

The procedure for calculating the Hamiltonian coefficients is standard. The magnetization (1.4.8) is expanded into a series of canonical variables, the result is substituted into (1.4.2) for the energy to go then over to the k-representation. Neglecting the relativistic interaction W_{dd} and W_a, one thus finds that the quadratic part of the Hamiltonian is diagonal in $a^*(\boldsymbol{k}, t)$, $a(\boldsymbol{k}, t)$, $\mathcal{H}_3 = 0$, and that out of the fourth-order terms, only those terms in (1.1.24) that are proportional to W and describe $2 \rightarrow 2$ scattering are nonzero. In the isotropic case,

$$\omega(k) = \omega_0 + \beta k^2, \quad \omega_0 = gH, \quad \beta = 2\kappa g_m M_0 , \qquad (1.4.9a)$$

$$W_{12,34} \equiv W_{ex}(\boldsymbol{k}_1, \boldsymbol{k}_2; \boldsymbol{k}_3, \boldsymbol{k}_4) = -\kappa g_m[(\boldsymbol{k}_1\boldsymbol{k}_2) + (\boldsymbol{k}_3\boldsymbol{k}_4)]. \qquad (1.4.9b)$$

As mentioned in Sect. 1.1, the scattering processes do not change the total number of waves, therefore the equations of motion corresponding to such a Hamiltonian preserve an additional integral of motion

$$N = \int a^*(\boldsymbol{k}, t)a(\boldsymbol{k}, t)\, d\boldsymbol{k}. \qquad (1.4.10)$$

As seen from (1.4.7), conservation of N means the constancy of the z-projection of the magnetization $M_z = M_0 - gN$. This is a consequence of the fact that the operator $\hat{M}_z$ commutes with the Heisenberg Hamiltonian of the ferromagnet [1.37]. Inclusion of relativistic interactions violates this relation, leading to the terms in $\mathcal{H}_{\text{int}}$ describing processes that do not conserve the number of waves: those of type $1 \rightarrow 2$, $2 \rightarrow 1$, $1 \rightarrow 3$, etc. In cubic ferromagnets, which were the object of extensive experimental research on nonlinear spin wave dynamics [1.38], the dominant relativistic interaction is of the magnetic dipole-dipole type (1.4.2c). The consequence of such interaction is the fact that the "circular" (circularly polarized) canonical variables $a^*(\boldsymbol{k}, t)$, $a(\boldsymbol{k}, t)$ are no longer normal for the quadratic Hamiltonian $\mathcal{H}_2$. Diagonalization of $\mathcal{H}_2$ with the help of the

(u,v)-transformation (1.1.16) leads to the "elliptical" (elliptically polarized) variables $b^*(\boldsymbol{k},t)$, $b(\boldsymbol{k},t)$ and the expression for the frequency (1.4.9a) is replaced by

$$\omega^2(k) = \left[\omega_0 - \omega_M N_z + \beta k^2 + \frac{1}{2}\omega \sin^2\theta\right]^2 - \frac{1}{4}\omega_M^2 \sin^4\theta \ . \qquad (1.4.11a)$$

Here $\omega_M = 4\pi g_m M_0$, θ is the angle between vectors $\boldsymbol{M}$ and $\boldsymbol{k}$; N_z is the demagnetization factor (equal to $1/3$ for a spherical sample, to 0 for a longitudinally magnetized cylinder and 1 for a tangentially magnetized disk). Computing the contribution of the dipole-dipole interaction to the three- and four-wave Hamiltonian does not present any special difficulties, but the computation procedure and the result are rather cumbersome. Monograph [1.38] gives them in full. We restrict our considerations to the case of a relatively small dipole-dipole interaction where one can do without the (u,v)-transformation in the expression for $\mathcal{H}_3$ and $\mathcal{H}_4$:

$$V_{1,23} = \frac{1}{2}(V_2 + V_3), \quad V_k = -\omega_M\sqrt{\frac{g}{2M_0}}\sin\theta\cos\theta\exp(i\varphi_k) \ . \qquad (1.4.12)$$

$$\begin{aligned} W_{12,34} &= W_{ex}(12,34) + \frac{1}{4}(C_{14} + C_{13} + C_{23} + C_{24}) \\ &\quad - \frac{1}{4}(D_1 + D_2 + D_3 + D_4) \ , \\ C_{ij} &\equiv C(\boldsymbol{k}_i,\boldsymbol{k}_j) = C(\boldsymbol{k}_i - \boldsymbol{k}_j) \ , \\ C(\boldsymbol{k}) &= 4\pi g_m^2\cos^2\theta, \quad C(0) = 4\pi g_m^2 N_z \ , \\ D_i &= D(\boldsymbol{k}_i) = 4\pi g_m^2 \sin^2\theta\exp(2i\varphi_k) \ . \end{aligned} \qquad (1.4.13)$$

We see that the problem of spin waves in ferromagnets has no complete self-similarity. This is due to the presence of two interactions, the exchange and dipole-dipole interactions characterized by frequencies βk^2 and ω_M with different dependences on the wave vector. Nevertheless, one can single out regions in k-space where the Hamiltonian coefficients are scale-invariant. At $\beta k^2 \gg \omega_M$, the dispersion law is quadratic with a gap (1.4.9a), and the coefficient of the four-wave interaction (1.4.9b) has the homogeneity degree two. At $\omega_0 - \omega_M N_z + \beta k^2 \ll \omega_M \sin^2\theta \ll \omega_M$ for the waves propagating at small angles with the z axis, like in the case of magnetized plasma, a separate self-similarity in $k_z, k_\perp$ results:

$$\omega(\boldsymbol{k}) = \sqrt{\omega_M(\omega_0 - \omega_M N_z)}\ \mid k_z/k_\perp \mid \ . \qquad (1.4.14a)$$

It is of the decay type, like (1.3.22) so that the interaction coefficient (1.4.12) becomes

$$V_{1,2,3} = \frac{1}{2}\omega_M\sqrt{g_m/2M_0}[\exp(i\varphi_1)k_{1\perp}/k_{1z} + \exp(i\varphi_2)k_{2\perp}/k_{2z}]. \qquad (1.4.14b)$$

Other cases of self-similarity are given in [1.42].

1.4.4 The Hamiltonian of Antiferromagnets

The simplest antiferromagnets have two magnetic sublattices and, accordingly, two spin wave branches. The quadratic part of the Hamiltonian has the standard form (1.1.17)

$$\mathcal{H}_2 = \int \left[\omega(\boldsymbol{k})a_k a_k^* + \Omega(\boldsymbol{k})b_k b_k^*\right] d\boldsymbol{k}$$

We give this expression here to introduce the notations for the spin wave frequencies in the two branches $\omega(\boldsymbol{k})$, $\Omega(\boldsymbol{k})$ and the normal canonical variables $a_k = a(\boldsymbol{k}, t)$, $a_k^* = a^*(\boldsymbol{k}, t)$. In the uniaxial ferromagnets with an "easy axis"-type anisotropy, the (crystalline) anisotropy field $\boldsymbol{H}_a$ tends to keep the magnetization parallel to that axis (usually called the z axis).

By analogy with ferromagnets, spin wave frequencies with $k \to 0$ would be expected to correspond to sublattice magnetization precession in the field $\boldsymbol{H}_a$, i.e.

$$\omega_0 = \Omega_0 = \omega_a,$$

where

$$\omega_a = gH_a \ . \tag{1.4.15}$$

However this is not so. In fact, the magnetization of the sublattice $\boldsymbol{M}_1$ oriented upwards is affected by the anisotropy field $\boldsymbol{H}_{a1}$ which is also oriented upwards: $\boldsymbol{H}_{a1} = \boldsymbol{H}_a$. The second sublattice $\boldsymbol{M}_2 = -\boldsymbol{M}_1$ is affected by another field $\boldsymbol{H}_{a2} = -\boldsymbol{H}_{a1}$. As a result, the sublattices tend to precess in opposite directions. In this case, the antiparallel arrangement of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ will inevitably be broken up, which is prevented by the strong exchange interaction between sublattices. As a result we have [1.37]

$$\omega_0^2 = 2\omega_{ex}\omega_a - \omega_H^2, \quad \Omega_0^2 = 2\omega_{ex}\omega_a + \omega_H^2, \quad \omega_H = gH \ . \tag{1.4.16}$$

Here $\omega_{ex} = g_m BM$ characterizes the antiferromagnetic exchange between sublattices. The order of magnitude of the dimensionless exchange constant is $B \simeq 10^3$.

In uniaxial antiferromagnets with "easy plane"-type anisotropy, the anisotropic field rotates the sublattice moments into the plane perpendicular to the z axis. The possibility of almost free oscillations of moments in this plane leads to the fact that one of the spin wave frequencies at $k = 0$ turns out to be small, i.e., $\omega_0 \simeq \omega_H$. The upper branch gap lies much higher:

$$\Omega_0^2 = 2\omega_{ex}\omega_a + \omega_0^2 \ . \tag{1.4.17}$$

This situation is of special interest for experiments on the nonlinear properties of spin waves: the investigated frequency range below 50 GHz includes the lower branch ω_k. Thus for "easy-plane" antiferromagnets we write down the results for the Hamiltonian coefficients in question [1.38]:

$$\omega_k^2 = \omega_0^2 + (vk)^2, \quad \Omega_k^2 = 2\omega_{ex}\omega_a + \omega_k^2 \,. \tag{1.4.18}$$

$$\begin{aligned}
\mathcal{H}_3 &= \frac{1}{2}\int \Big[\big(V_1 b_1 a_2^* a_3^* + 2V_2 a_1 b_2^* a_3^* + \text{c.c}\big)\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) \\
&\quad + \big(U b_1^* a_2^* a_3^* + \text{c.c}\big)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\Big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \,, \\
\mathcal{H}_4 &= \frac{1}{4}\int W a_1^* a_2^* a_3 a_4 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 \,, \\
V_1 &= -\frac{1}{4}\sqrt{2g_m\omega_{ex}/M\omega_2\omega_3\Omega_1}(\Omega_1 + \omega_2 + \omega_3)\omega_H \,, \\
V_2 &= -\frac{1}{4}\sqrt{2g_m\omega_{ex}/M\omega_1\omega_3\Omega_2}(\omega_1 - \Omega_2 + \omega_3)\omega_H \,, \\
U &= -\frac{1}{4}\sqrt{2g_m\omega_{ex}/M\omega_2\omega_3\Omega_1}(\Omega_1 - \omega_2 - \omega_3)\omega_H \,, \\
W &= 9\omega_{ex}\frac{\left[\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 - \omega_{(1-3)}^2 - \omega_{(2-3)}^2 - \omega_{(1+2)}^2\right]}{4M\sqrt{\omega_1\omega_2\omega_3\omega_4}} \,.
\end{aligned} \tag{1.4.19}$$

We used the shorthand notation

$$\Omega_i = \Omega(k_i),\ \omega_i = \omega(k_i),\ \omega_{i-j} = \omega(k_i - k_j), \quad \text{etc} \,.$$

It should be borne in mind that a scattering process of type $2 \to 2$ does not just require to take into account the Hamiltonian $\mathcal{H}_4$ but also $\mathcal{H}_3$ in second order perturbation theory. It may be shown that the dominant contribution to the ω_k/ω_{ex} parameter will be due to processes involving the upper-branch virtual wave, as explicitly given in (1.4.19). Using (1.1.29), we can derive an expression for the effective coefficient of the lower branch four-wave processes T [1.38]. The general expression for T is rather cumbersome. For $k_1 = k_3,\ k_2 = k_4,\ \omega_1 = \omega_2 = \omega_3 = \omega_4$ it reads

$$T = -\frac{g^2\omega_{ex}}{8\omega_k^2}\left[\omega_0^2 + \omega_H^2(3\Omega_0^2 - 4\omega_k^2)/(\omega_0^2 - 4\omega_k^2)\right] . \tag{1.4.20}$$

It is evident from these formulas that the problem of spin waves in antiferromagnets does not possess complete self-similarity, like in ferromagnets. However, a second-order self-similarity does exist. Thus in the dispersion law (1.4.18), three regions may be singled out. At $k \to 0$, the ω_k and Ω_k branches have a gap with a quadratic addition. In the region of large k, the functions ω_k and Ω_k follow a linear law. Since usually $\omega_a\omega_{ex} \gg \omega_H^2$, there is an intermediate range of the k values where the frequency ω_k has already become asymptotically linear and $\Omega_k = \Omega_0 + \beta k^2$. We shall not go through the simple but cumbersome analysis of the asymptotics of the interaction coefficient (1.4.16, 17). We shall only note that at $k \to 0$, the coefficient of the four-wave interaction becomes constant. In this case the Hamiltonian of the problem for low-frequency waves has an especially simple form:

$$\mathcal{H} = \int \omega(k) b^*(\boldsymbol{k}) b(\boldsymbol{k})\, d\boldsymbol{k} + \frac{T_0}{4} \int b^*(\boldsymbol{k}_1) b^*(\boldsymbol{k}_2) b(\boldsymbol{k}_3) b(\boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 \,, \quad \omega_k = \omega_0 + \beta k^2 \,. \tag{1.4.21}$$

Why have we taken into account the term $\propto k^2$ in the expression for ω_k but neglected it in the expression for $T = T_0 + O(k^2)$ although it has in fact the same relative order of magnitude? The answer is: the constant ω_0 is not involved in the problem and is eliminated by the time-dependent canonical transformation

$$c(\boldsymbol{k}, t) = b(\boldsymbol{k}, t) \exp(i\omega_0 t/2) \,. \tag{1.4.22}$$

In the variables $c(\boldsymbol{k}, t)$, the Hamiltonian (1.4.21) has the form:

$$\mathcal{H} = \beta \int k^2 c^*(\boldsymbol{k}) c(\boldsymbol{k}) d\boldsymbol{k} + \frac{T_0}{4} \int c^*(\boldsymbol{k}_1) c^*(\boldsymbol{k}_2) c(\boldsymbol{k}_3) c(\boldsymbol{k}_4) d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 \,. \tag{1.4.23}$$

The respective dynamic equation $i\partial c(\boldsymbol{k}, t)/\partial t = \delta\mathcal{H}/\delta c(\boldsymbol{k}, t)$ has after the inverse Fourier transform $c(\boldsymbol{k}, t) \to \psi(\boldsymbol{r}, t)$ the form of the nonlinear Schrödinger equation

$$i\partial\psi/\partial t + \beta \triangle \psi + \frac{T_0}{2} \mid \psi \mid^2 \psi = 0 \,. \tag{1.4.24}$$

Indeed, that is the quantum mechanical Schrödinger equation with $|\psi|^2$ as a potential.

1.5 Universal Models

The nonlinear Schrödinger equation which we encountered at the end of the previous section is a member of a small family of famous universal equations that arise in the nonlinear wave theory. Every such equation describes a large variety of physical systems belonging to different topics of physics. As we shall show just now, the nonlinear Schrödinger equation describes the behavior of a narrow wave packet envelope for most nonlinear wave systems, see Sect. 1.5.1. If we consider another physical situation, namely long-wave perturbations of the acoustic type (which can exist in most media), then we obtain the well-known Korteweg - de Vries (KdV) equation in the one-dimensinal case and the rather famous Kadomtsev-Petviashvili equation for weakly two-dimensional distributions, see Sect. 1.5.2. In Sect. 1.5.3 we shall consider a third rather universal physical situation: the interaction of three-wave packets in media with a decay dispersion law. All three of these models have extremely wide applications, ranging from solid state physics to hydrodynamics, plasma physics, etc.

1.5.1 Nonlinear Schrödinger Equation for Envelopes

This equation, like the Hamiltonian (1.4.23), has a wide range of applications. Basically, these applications are associated with the fact that (1.4.24) describes the behavior of an envelope of quasi-monochromatic waves in isotropic nonlinear media. Equation (1.4.24) should be regarded as written in a reference system moving with the group velocity of a wave packet.

Now let us show how the dynamical equation (1.4.24) can be obtained for a narrow wave packet. If the carrier wave vector is denoted by $\boldsymbol{k}_0$ and $|\boldsymbol{k} - \boldsymbol{k}_0| = |\boldsymbol{\kappa}| \ll k_0$, we assume

$$\omega(\boldsymbol{k}) \approx \omega(\boldsymbol{k}_0) + (\boldsymbol{\kappa v}) + \frac{1}{2}\kappa_i\kappa_j\left(\frac{\partial^2\omega}{\partial k_i \partial k_j}\right)_{k=k_0} \tag{1.5.1}$$

to hold. Here $\boldsymbol{v} = (\partial\omega/\partial\boldsymbol{k})_{k=k_0}$ is the group velocity. In isotropic media, with the frequency depending only on the modulus of the wave vector, the third term in (1.5.1) can be simplified

$$\kappa_i\kappa_j\left(\frac{\partial^2\omega}{\partial k_i \partial k_j}\right)_{k=k_0} = \frac{v}{k_0}\kappa_\perp^2 + \omega''\kappa_\parallel^2, \quad \kappa_\parallel = (\boldsymbol{\kappa k}_0)/k_0 \ .$$

In the propagation of a single narrow envelope the three-wave interaction is not important (but it must be taken into account in higher-order perturbation theory). Waves with approximately equal wave vectors are involved in the four-wave interaction. The dynamic equation has the form

$$\begin{aligned}\frac{\partial c(\boldsymbol{k},t)}{\partial t} + i\omega(k)c(\boldsymbol{k},t) = -i\int T(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\\ \times c^*(\boldsymbol{k}_1,t)c(\boldsymbol{k}_2,t)c(\boldsymbol{k}_3,t)\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \ .\end{aligned} \tag{1.5.2}$$

For a narrow packet it can be assumed that

$$T(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) \approx T(\boldsymbol{k}_0,\boldsymbol{k}_0,\boldsymbol{k}_0,\boldsymbol{k}_0) \equiv T(2\pi)^{-3}.$$

Introducing the envelope of the quasi-monochromatic wave

$$\psi(\boldsymbol{r},t) = (2\pi)^{-3/2}\exp[i(\boldsymbol{k}_0\boldsymbol{r}) - i\omega(k_0)t]\int c(\boldsymbol{k}_0+\boldsymbol{\kappa},t)\exp[i(\boldsymbol{\kappa r})]d\boldsymbol{\kappa}$$

and making the inverse Fourier transform in (1.5.2), we arrive at

$$i\frac{\partial\psi}{\partial t} + iv\frac{\partial\psi}{\partial z} + \frac{v}{2k_0}\Delta_\perp\psi + \frac{\omega''}{2}\frac{\partial^2\psi}{\partial z^2} - T|\psi|^2\psi = 0 \ . \tag{1.5.3}$$

The z-axis is chosen in the direction of the wave propagation. The second term in (1.5.3) can be eliminated by the transition to a reference system moving with the group velocity: $z \to z - vt$. Expanding the z-coordinate according to $z \to z(k_0\omega''/v)^{1/2}$ and using $\Delta = \Delta_\perp + \partial^2/\partial z^2$, one can reduce (1.5.3) to

(1.4.24). It should be noted that (1.5.3) can be considered both for three- and two-dimensional media. The latter applies mainly to water wave envelopes.

In deriving (1.5.3) we assume that the coefficient $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ is a continuous function when all the arguments tend to $\boldsymbol{k}_0$. This is not always true; in some cases, this limit depends on the direction of the vector $\boldsymbol{k}$ with respect to the direction of $\boldsymbol{k}_0$. In such cases (1.5.3) should be replaced by a more complicated equation.

However in most of the isotropic media the behavior of envelopes is governed by (1.5.3). In particular, this equation describes self-focussing of light in nonlinear dielectrics [1.43] and the quasi-classical limit of a weakly nonhomogeneous Bose gas [1.44–45]. It should also be noted that the coefficient T_0 may be either positive (corresponding to the effective attraction of quasi-particles) or negative (corresponding to their repulsion). Scale transformations of the coordinates and the field ψ may be used to obtain unity for the values of the parameters β and T_0. The turbulence of envelopes is sometimes called *optical turbulence* because nonlinear optics is a field with a multitude of applications for nonlinear wave theory.

1.5.2 Kadomtsev-Petviashvili Equation for Weakly Dispersive Waves

Let us consider a hydrodynamic type system with the usual Hamiltonian

$$\mathcal{H} = \int \frac{\varrho v^2}{2}\, d\boldsymbol{r} + E_{\text{in}} ,$$

where the internal energy E_{in} is connected with the density variation $\varrho = \varrho_0 + \delta\varrho$ and has the form (1.2.18,21)

$$E_{\text{in}} = \frac{1}{2} \int \left[c_s^2 (\delta\varrho)^2/\varrho_0 + g c_s^2 (\delta\varrho)^3 + \beta |\nabla\varrho|^2 \right] d\boldsymbol{r} .$$

Such a Hamiltonian expansion using the smallness of nonlinearity and dispersion is quite universal [for example, we obtain the equations of the ion sound in plasmas (1.3.10) with such an approximation]. Respective dynamical equations have the standard form (1.2.8–9):

$$\frac{\partial \delta\varrho}{\partial t} + \varrho_0 \operatorname{div} \boldsymbol{v} = -\operatorname{div}(\boldsymbol{v}\,\delta\varrho) ,$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{c_s^2}{\varrho_0} \nabla\delta\varrho = -(\boldsymbol{v}\nabla)\boldsymbol{v} - \beta\nabla\Delta\delta\varrho .$$

The right-hand-sides of these equations contain the small effects of nonlinearity and dispersion while the left-hand-side describes the main phenomenon: the motion of a perturbation with the sound velocity ($\boldsymbol{v}_{tt} - c_s^2 \Delta \boldsymbol{v} = 0$). We can eliminate the latter by going over to a moving reference frame. Let the x-axis coincide with the direction of motion of the sound velocity. We suppose

$v \approx v_x = v$, $\partial v/\partial x \gg \partial v/\partial y$, $\partial v/\partial z$ and the same inequality for $\delta\varrho$. Substituting into the terms on the right the zero-order relations $\delta\varrho_t \approx -\varrho_0$ div $v \approx \varrho_0 v_x$, $v_t \approx -c_s^2 \nabla\delta\varrho/\varrho_0$, we obtain

$$v_{tt} - c_s^2 v_{xx} = c_s^2 \Delta_\perp v + c_s \frac{\partial}{\partial x}(v v_x) + \beta\varrho_0 \Delta^2 v$$

Here the terms on the left-hand-side are much greater than those on the right-hand-side. We introduce instead of x, t the slow variables $\xi = x - vt$ and slow time τ, so we can substitute $v_{tt} - v_{xx} = (\frac{\partial}{\partial x} - c_s\frac{\partial}{\partial t})(\frac{\partial}{\partial x} + c_s\frac{\partial}{\partial t})v = 2\frac{\partial}{\partial\xi}\frac{\partial}{\partial\tau}v$ to obtain finally the Kadomtsev-Petviashvili equation [1.46]

$$\frac{\partial}{\partial\xi}\left(v_\tau + v v_\xi + \frac{\beta\varrho_0}{2c_s} v_{\xi\xi\xi}\right) = \frac{c_s}{2}\Delta_\perp v \ . \tag{1.5.4}$$

As it is clear from its derivation, this equation is valid in the case of weak nonlinearity ($v \ll c_s$), weak dispersion ($\beta v_{xx} \ll v$) and small deviations from one-dimensionality ($v_x \gg \nabla_\perp v$). In the truncated equation all terms are generally of the same order, since linear effects are excluded. Following (1.2.17), one can introduce for (1.5.4) normal canonical variables with $v = \nabla\Phi$. So we obtain

$$\omega(k_x, k_\perp) = \frac{\beta\varrho_0}{2c_s}k_x^3 + \frac{c_s}{2k_x}k_\perp^2 \ , \tag{1.5.5a}$$

$$V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) = 3\sqrt{\frac{c_s k_x k_{1x} k_{2x}}{4\pi^3 \varrho_0}}\,\Theta(k_x)\Theta(k_{1x})\Theta(k_{2x}) \ . \tag{1.5.5b}$$

For purely one-dimensional motions the Kadomtsev-Petviashvili equation returns to the famous KdV equation:

$$v_\tau + v v_\xi + v_{\xi\xi\xi} = 0 \ .$$

1.5.3 Interaction of Three Wave Packets

Let us discuss the interaction of three narrow wave packets with characteristic wave vectors $\boldsymbol{k}_1, \boldsymbol{k}_2$ and $\boldsymbol{k}_3$. For this interaction to be essential, it is necessary that these vectors should lie in the vicinity of the resonant surface $\omega(\boldsymbol{k}_1 + \boldsymbol{k}_2) = \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2)$. Let us represent the wave amplitude $c(\boldsymbol{k}, t)$ as

$$c(\boldsymbol{k}, t) = c_1(\boldsymbol{k}_1 + \boldsymbol{\kappa}, t) + c_2(\boldsymbol{k}_2 + \boldsymbol{\kappa}, t) + c_3(\boldsymbol{k}_3 + \boldsymbol{\kappa}, t) \tag{1.5.6}$$

and, making use of the narrowness of the packets ($\kappa \ll k_j$), expand the function $\omega(\boldsymbol{k})$:

$$\omega(\boldsymbol{k}_j + \boldsymbol{\kappa}) = \omega_j + \boldsymbol{\kappa}\boldsymbol{v}_j, \quad \omega_j = \omega(\boldsymbol{k}_j), \quad \boldsymbol{v}_j = \frac{\partial\omega(\boldsymbol{k}_j)}{\partial\boldsymbol{k}_j}, \quad j = 1,2,3 \ . \tag{1.5.7}$$

The value $c(\boldsymbol{k}, t)$ obeys the usual dynamic equation for the three-wave case

$$i\frac{\partial c(\boldsymbol{k},t)}{\partial t} - \omega_k c(\boldsymbol{k},t) = \int \left[\frac{1}{2}V_{k12}c_1c_2\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2) + V^*_{1k2}c_1c^*_2\delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2)\right] d\boldsymbol{k}_1 d\boldsymbol{k}_2\ . \quad (1.5.8)$$

Neglecting its dependence on κ we shall regard the interaction coefficient to be a constant. Then we shall use

$$(2\pi)^{3/2}c_j(\boldsymbol{r},t) = \exp(-i\omega_j t)\int c_j(\boldsymbol{k}_j+\boldsymbol{\kappa})\exp[-i(\boldsymbol{\kappa r})]\,d\boldsymbol{\kappa}\ ,$$

which also includes a procedure for eliminating the fast time dependence to go over to the r-representation. As a result, we obtain the known equation for the three-wave resonant interaction

$$\begin{aligned}\left[\frac{\partial}{\partial t} + (\boldsymbol{v}_1\nabla)\right]c_1(\boldsymbol{r},t) &= -(2\pi)^{3/2}iV_{123}c_2c_3.\\ \left[\frac{\partial}{\partial t} + (\boldsymbol{v}_2\nabla)\right]c_2(\boldsymbol{r},t) &= (2\pi)^{3/2}iV^*_{123}c_2c^*_3.\\ \left[\frac{\partial}{\partial t} + (\boldsymbol{v}_3\nabla)\right]c_3(\boldsymbol{r},t) &= (2\pi)^{3/2}iV^*_{123}c_1c^*_2\ .\end{aligned} \quad (1.5.9)$$

These relations show that the wave packets move in the r-space with group velocities v_j, the characteristic time of their amplitude and slow-phase variation being

$$t_{\text{int}} \simeq \frac{1}{2\pi}\left|V_{123}c_{\max}\right|\ . \quad (1.5.10)$$

2. Statistical Description of Weak Wave Turbulence

> It is a thing which you can easily explain twice before anybody knows what you are talking about
>
> *A. Milne "The House at Pooh Corner"*

In this chapter we shall go over from the dynamic description of wave systems to the statistical one. This will be done in terms of pair correlators of a wave field. They represent the occupation numbers (density) of waves in $\boldsymbol{k}$-space. In Sect. 2.1 we shall obtain the kinetic equation for the occupation numbers of waves as the main mathematical objects dealt with in the first volume of this book. In Sect. 2.2 we shall study the general properties of these equations and derive the stationary equilibrium solutions. Section 2.2.3 deals with the general necessary conditions for the existence of Kolmogorov spectra.

2.1 Kinetic Wave Equation

2.1.1 Equations of Motion

In the preceding chapter we have applied the Hamiltonian method to the motion in continuous media like fluids, plasmas, solids and the planetary atmosphere, have considered the structure of the Hamiltonian for weak nonlinearities and given its coefficients for some physical examples. The canonical equation of motion in terms of complex wave amplitudes $c(\boldsymbol{k}, t)$, i.e.,

$$i\frac{\partial c(\boldsymbol{k},t)}{\partial t} - \omega_k c(\boldsymbol{k},t) = \frac{\delta H_{\text{int}}}{\delta c^*(\boldsymbol{k},t)} \tag{2.1.1}$$

is the main result. After application of the procedure outlined in Sect. 1.1.3, the interaction Hamiltonian $\mathcal{H}_{\text{int}}$ includes only the terms describing real processes for which the energy and momentum conservation laws are valid. If a medium supports propagating waves for which three-wave decay processes are allowed, we have:

$$\mathcal{H}_{\text{int}} = \mathcal{H}_3 = \frac{1}{2}\int \left[V_{123} c_1^* c_2 c_3 + \text{c.c.}\right] \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \ . \tag{2.1.2}$$

In this case

$$i\frac{\partial c(\boldsymbol{k},t)}{\partial t} - \omega_k c(\boldsymbol{k},t) = \int \Big[\frac{1}{2} V_{k12} c_1 c_2 \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2) + V^*_{1k2} c_1 c^*_2 \delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2)\Big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 \,. \tag{2.1.3}$$

The first term of the right-hand-side of this equation describes the $k \to 1{+}2$ decay processes with the conservation laws

$$\omega_k = \omega_1 + \omega_2 \quad \text{and} \quad \boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2 \tag{2.1.4a}$$

and the reverse $1+2 \to k$ processes. The second term in (2.1.3) describes the confluence $k+2 \to 1$ processes with

$$\omega_k + \omega_2 = \omega_1 \quad \text{and} \quad \boldsymbol{k} + \boldsymbol{k}_2 = \boldsymbol{k}_1 . \tag{2.1.4b}$$

If the three-wave processes (2.1.4) are forbidden, the Hamiltonian (2.1.2) vanishes under the nonlinear canonical transformation (1.1.28). In that case, $\mathcal{H}_{\text{int}}$ includes only four-wave scattering processes of the type $2 \to 2$:

$$\mathcal{H}_{\text{int}} = \mathcal{H}_4 = \frac{1}{4}\int T_{12,34} c^*_1 c^*_2 c_3 c_4 \delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 \,. \tag{2.1.5}$$

and the canonical equations of motion have the form:

$$i\frac{\partial c(\boldsymbol{k},t)}{\partial t} - \omega_k c(\boldsymbol{k},t) = \frac{1}{2}\int T_{k123} c^*_1 c_2 c_3 \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3))\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \,. \tag{2.1.6}$$

The right-hand-side of this equation describes the dynamics of scattering processes with

$$\omega_k + \omega_1 = \omega_2 + \omega_3 \quad \text{and} \quad \boldsymbol{k} + \boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3 \,.$$

2.1.2 Transition to the Statistical Description

The dynamic equations (2.1.3, 6) describe the time evolution of the wave amplitudes $|c(\boldsymbol{k},t)|$ and of their phases $\varphi(\boldsymbol{k},t)$, i.e., $c(\boldsymbol{k},t) = |c(\boldsymbol{k},t)|\exp[i\varphi(\boldsymbol{k},t)]$. For weak nonlinearity and a large number of excited waves, such a description is in general highly redundant: it includes the slow evolution of amplitudes (constant in the linear approximation) and the fast but uninteresting phase dynamics $\varphi(\boldsymbol{k},t) \approx \omega(\boldsymbol{k})t$ which leave the amplitude evolution virtually unaffected. This redundancy is eliminated by the transition from the dynamic description of a wave system in terms of $|c(\boldsymbol{k},t)|$ and $\varphi(\boldsymbol{k},t)$ to the statistical one in terms of the correlation functions of the field $c(\boldsymbol{k},t)$. Consistent statistical averaging consists of two steps, the construction of an ensemble of solutions of the dynamic equation and specification of the rule for using them to compute averages. Then one should consider only those averaging conditions (i.e., correlation properties of the field) which are compatible with the dynamic equations and independent of time. In other words, it is necessary to introduce an invariant measure on the manifold of wave field configurations. Using the smallness of the nonlinearity we can go over to a statistical description in terms of perturbation theory in $\mathcal{H}_{\rm int}$.

In the zeroth approximation in $\mathcal{H}_{\rm int}$ we have a free wave field with the trivial evolution that the amplitudes $|c(\boldsymbol{k},t)| = c(\boldsymbol{k})$ are constant and the phases are given by $\varphi(\boldsymbol{k},t) = \omega(\boldsymbol{k})t$. For the "slow" wave phase $\psi(\boldsymbol{k},t) = \varphi(\boldsymbol{k},t) - \omega(\boldsymbol{k})t$ by virtue of (2.1.1) we get $\partial\psi(\boldsymbol{k},t)/\partial t = 0$ at $\mathcal{H}_{\rm int} = 0$, i.e., the $\psi(\boldsymbol{k})$ phases are in a state of indifferent equilibrium. Consequently, any small random perturbation like medium ingomogeneities and the interaction with a thermostat drives them into a chaotic regime. Apart from external reasons there is also an "internal" reason for phase chaotization: the dispersion (i.e., the $\boldsymbol{k}$-dependence) of wave frequencies. Because of dispersion even initially correlated harmonics with different $\boldsymbol{k}$'s undergo phase randomization as time progresses. Thus, in describing a free wave field it would be natural to average over the ensemble of chaotic (random) phases, i.e., to use the random phase approximation. After such an averaging, only correlators that are independent of the wave phase will be nonzero. For example,

$$\begin{aligned}
&\langle c_k\rangle = \langle |c_k| exp(i\varphi_k)\rangle = 0\\
&\langle c_k c_{k'}\rangle = \langle |c_k||c_{k'}| exp(i\varphi_k + i\varphi_{k'})\rangle = 0\\
&\langle c_k c^*_{k'}\rangle = \langle |c_k||c_{k'}| exp(i\varphi_k - i\varphi_{k'})\rangle = n(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{k}')\ .
\end{aligned} \tag{2.1.7a}$$

Evidently, all odd-order correlators vanish. They will only be nonzero if the number of $c(k,t)$'s coincides with the number of $c^*(k,t)$'s and if the wave vector at one of the $c(k,t)$ coincides with that for (any) other $c^*(k,t)$. Only then will the averaged expression being be phase-independent. Later on we shall need the fourth- and sixth-order correlators

$$\langle c_1^* c_2^* c_3 c_4\rangle = n(\boldsymbol{k}_1)n(\boldsymbol{k}_2)[\delta(\boldsymbol{k}_1-\boldsymbol{k}_3)\delta(\boldsymbol{k}_2-\boldsymbol{k}_4)+\delta(\boldsymbol{k}_1-\boldsymbol{k}_4)\delta(\boldsymbol{k}_2-\boldsymbol{k}_3)]\ , \tag{2.1.7b}$$

$$\begin{aligned}
\langle c_1^* c_2^* c_3^* c_4 c_5 c_6\rangle = n(\boldsymbol{k}_1)n(\boldsymbol{k}_2)n(\boldsymbol{k}_3)\big[&\delta(\boldsymbol{k}_1-\boldsymbol{k}_4)[\delta(\boldsymbol{k}_2-\boldsymbol{k}_5)\delta(\boldsymbol{k}_3-\boldsymbol{k}_6)\\
&+\delta(\boldsymbol{k}_2-\boldsymbol{k}_6)\delta(\boldsymbol{k}_3-\boldsymbol{k}_5)] + \delta(\boldsymbol{k}_1-\boldsymbol{k}_5)\\
&\times[\delta(\boldsymbol{k}_2-\boldsymbol{k}_4)\delta(\boldsymbol{k}_3-\boldsymbol{k}_6)+\delta(\boldsymbol{k}_2-\boldsymbol{k}_6)\delta(\boldsymbol{k}_3-\boldsymbol{k}_4)]\\
&+\delta(\boldsymbol{k}_2-\boldsymbol{k}_4)[\delta(\boldsymbol{k}_3-\boldsymbol{k}_5)\delta(\boldsymbol{k}_2-\boldsymbol{k}_4)\\
&+\delta(\boldsymbol{k}_2-\boldsymbol{k}_5)\delta(\boldsymbol{k}_3-\boldsymbol{k}_4)]\big]\ .
\end{aligned} \tag{2.1.7c}$$

It is seen from these examples that the random fourth-order correlator decomposes into various products of pair correlators. This implies Gaussian statistics for the free wave field. Of course, the above considerations are not a real proof of this fact. With more rigor the statistical properties of free and weakly interacting fields will be discussed in the second volume of this book.

We consider only ergodic systems for which $\langle\,\rangle$ corresponds equally well to averaging over the ensemble and over times that are much longer than the time of fast phase evolution and much smaller than the time of slow amplitude evolution.

In (2.1.7), the wave field was taken to be statistically homogeneous in space, therefore the correlator $\langle c(\boldsymbol{k},t)c^*(\boldsymbol{k}',t)\rangle$ contains $\delta(\boldsymbol{k}-\boldsymbol{k}')$. The classical pair correlator $n(\boldsymbol{k},t)$ is proportional to the quantum-mechanical occupation numbes $N(\boldsymbol{k},t)$:

$$n(\boldsymbol{k},t) = \hbar N(\boldsymbol{k},t), \tag{2.1.8}$$

i.e., to the "number of the quanta" of the respective Bose field. Therefore we shall call $n(\boldsymbol{k},t)$ the number of waves, in spite of the fact that $n(\boldsymbol{k},t)$ is a dimensional quantity. For weak nonlinearity we may use perturbation theory in $\mathcal{H}_{\rm int}$ to obtain an equation closed with respect to $n(\boldsymbol{k},t)$.

2.1.3 The Three-Wave Kinetic Equation

To calculate $\partial n(\boldsymbol{k},t)/\partial t$, we shall multiply (2.1.3) by $c^*(\boldsymbol{k},t)$, the complex conjugate equation by $c(\boldsymbol{k}',t)$, subtract the latter from the former and average. Setting $\boldsymbol{k}=\boldsymbol{k}'$, we obtain:

$$\begin{aligned}\frac{\partial n(\boldsymbol{k},t)}{\partial t} = \mathrm{Im}\int\big[&V_{k12}J_{k12}\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\\ &-2V_{1k2}J_{1k2}\delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2)\big]\,d\boldsymbol{k}_1 d\boldsymbol{k}_2\ .\end{aligned} \tag{2.1.9a}$$

Here

$$J_{123}\delta(\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3) = \langle c_1^* c_2 c_3\rangle \tag{2.1.9b}$$

is the triple correlation function. For a free field, $J_{123}=J_{123}^{(0)}=0$. This implies that in first order perturbation theory in $\mathcal{H}_{\rm int}$ we have $\partial n(\boldsymbol{k},t)/\partial t=0$. In order to calculate $\partial n(\boldsymbol{k},t)/\partial t$ in second order perturbation theory in $\mathcal{H}_{\rm int}$, one should know J_{123} in the first order. Using definition (2.1.9b) and the equations of motion (2.1.3), we calculate $\partial J/\partial t$:

$$\begin{aligned}&\left[i\frac{\partial}{\partial t}+(\omega_1-\omega_2-\omega_3)\right]J(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3;t)\\ &\quad=\int\Big[-\frac{1}{2}V_{145}^*J_{4523}\delta(\boldsymbol{k}_1-\boldsymbol{k}_4-\boldsymbol{k}_5)\\ &\qquad+V_{425}^*J_{1534}\delta(\boldsymbol{k}_4-\boldsymbol{k}_2-\boldsymbol{k}_5)\\ &\qquad+V_{435}J_{1524}\delta(\boldsymbol{k}_4-\boldsymbol{k}_3-\boldsymbol{k}_5)\Big]\,d\boldsymbol{k}_4 d\boldsymbol{k}_5\ .\end{aligned} \tag{2.1.10a}$$

Here

$$J_{1234}\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4) = \langle c_1^* c_2 c_3 c_4\rangle \tag{2.1.10b}$$

is a quadruple correlation function. As we are interested in $J_{123}^{(1)}$ (in the first order), we should substitute into the right-hand-side of (2.1.10a) J_{1234} in the zeroth order in $\mathcal{H}_{\rm int}$, i.e., $J_{1234}^{(0)}$ for the free field. In accordance with (2.1.7), this correlator is expressed via the pair correlators $n(\boldsymbol{k},t)$. As a result we have

$$\left[i\frac{\partial}{\partial t} + (\omega_1 - \omega_2 - \omega_3)\right] J_1(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3; t) \\ = V^*_{1,23}[n_1 n_3 + n_1 n_2 - n_2 n_3] = A_{123} \ . \tag{2.1.10c}$$

Remaining consistantly within second order perturbation theory, one should neglect the time dependence of $n(\boldsymbol{k}, t)$ on the right-hand-side of (2.1.10c), i.e., set A =const. Then this equation may be solved

$$J(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_1; t) = B \exp(-i\Delta\omega t) + A_{123}/\Delta\omega, \\ \Delta\omega = \omega_1 - \omega_2 - \omega_3 \ . \tag{2.1.10d}$$

Substituting the first term into (2.1.9a), we get at $t \neq 0$ an integral of a fast oscillating function. Its contribution decreases with increasing t and becomes nonessential at times larger than $1/\omega(\boldsymbol{k})$. The second term of (2.1.10d) gives for J_1 an expression depending via $n(\boldsymbol{k}_j, t)$ parametrically on the slow time:

$$J_1(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3; t) = \frac{V^*_{123}(n_1 n_2 + n_1 n_3 - n_2 n_3)}{\omega_1 - \omega_2 - \omega_3 + i\delta} \ . \tag{2.1.11}$$

To the denominator we have added the term $i\delta$ to circumvent the pole. It may be obtained in a consistent procedure by considering the free wave field at $t \to -\infty$ and adiabatically slowly including the interaction [for $t \gg 1/\omega(\boldsymbol{k})$]. The sign of δ can also be determined by accounting for the presence of small damping. Substituting (2.1.11) into (2.1.9a) and using $\mathrm{Im}\,\{\omega + i\delta\} = -\pi\delta(\omega)$, we obtain the three-wave kinetic equation

$$\frac{\partial n(\boldsymbol{k}, t)}{\partial t} = \pi \int \Big[|V_{k12}|^2 f_{k12} \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\delta(\omega_k - \omega_1 - \omega_2) \\ + 2|V_{1k2}|^2 f_{1k2} \delta(\boldsymbol{k}_1 - \boldsymbol{k} - \boldsymbol{k}_2)\delta(\omega_1 - \omega_k - \omega_2)\Big] \, d\boldsymbol{k}_1 d\boldsymbol{k}_2, \tag{2.1.12a}$$

$$f_{k12} = n_1 n_2 - n_k(n_1 + n_2), \quad n_j = n(\boldsymbol{k}_j, t). \tag{2.1.12b}$$

Most of the first volume of this book is devoted to studying the solutions of this equation. Now we shall discuss at a qualitative level the conditions for applications of it.

2.1.4 Applicability Criterion of the Three-Wave Kinetic Equation (KE)

Let us consider a distribution $n(\boldsymbol{k}, t)$ in the form of a broad packet with $\Delta k \simeq k$. The characteristic time $1/\gamma(\boldsymbol{k})$ for variations of the occupation numbers $n(\boldsymbol{k}, t)$ may be estimated from the kinetic equation (2.1.12)

$$\gamma(\boldsymbol{k}) \simeq \left|V_{k,kk}\right|^2 \frac{n_k k^d}{\omega_k} \simeq \left|V_{kkk}\right|^2 \frac{N}{\omega_k} \ , \\ N = \int n(\boldsymbol{k})\, d\boldsymbol{k} \ . \tag{2.1.13}$$

Perturbation theory and random phase approximation are applicable if this characteristic time is significantly larger than the time required for phase chaotization estimated to equal roughly $1/\Delta\omega_k \simeq 1/\omega_k$. Here $\Delta\omega_k$ is the difference between the frequencies of the wave packet. Comparison of these times yields the following applicability criterion for the kinetic equation

$$\xi_1(\boldsymbol{k}) = \left|V_{kkk}\right|^2 N\omega_k \ll 1 . \tag{2.1.14}$$

Let us derive the same criterion by calculating directly the ratio

$$\xi_2(\boldsymbol{k}) = \frac{\langle\mathcal{H}_{\text{int}}\rangle}{\langle\mathcal{H}_2\rangle} = \frac{\operatorname{Re}\int V_{123} J_{123}\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3}{\int \omega(\boldsymbol{k}) n(\boldsymbol{k})\, d\boldsymbol{k}} . \tag{2.1.15}$$

Substituting here (2.1.11) for J_1, we get

$$\xi_2(\boldsymbol{k}) = \frac{\int \left|V_{123}\right|^2 f_{123}(\omega_1 - \omega_2 - \omega_3)^{-1}\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3}{\int \omega(\boldsymbol{k}) n(\boldsymbol{k})\, d\boldsymbol{k}} . \tag{2.1.16}$$

Rough estimates of this equation for a wave packet with $\Delta k \simeq k$ yield $\xi_1 \simeq \xi_2$. Thus the criterion (2.1.14) ensures the smallness of the interaction Hamiltonian $\langle\mathcal{H}_{\text{int}}\rangle$ as compared to the Hamiltonian of the free wave field $\langle\mathcal{H}_2\rangle$. This is exactly the same as the applicability condition of perturbation theory. For narrow wave distributions the applicability of the kinetic equation should be treated more accurately. For example, the interaction of three narrow packets should obey (1.5.9), so the nonlinear interaction time is by virtue of (1.5.10)

$$t_{\text{int}} \simeq \frac{1}{2\pi}\left|V_{123} c_{\max}\right| . \tag{2.1.17}$$

Let all three packets in the initial state have approximately equal amplitudes $c_1 \simeq c_2 \simeq c_3 \simeq c_{\max}$ and widths $l_1 \simeq l_2 \simeq l_3 \simeq l$ and let them evolve in such a way that they collide at a certain moment, i.e., that they are at that moment in the same spatial region. All three packets overlap each other during the time of collision

$$t_{\text{col}} \simeq \frac{1}{v_{123}}, \quad v_{123} = \max\left[\left|\boldsymbol{v}_1 - \boldsymbol{v}_2\right|, \left|\boldsymbol{v}_1 - \boldsymbol{v}_3\right|, \left|\boldsymbol{v}_2 - \boldsymbol{v}_3\right|\right] . \tag{2.1.18}$$

If $t_{\text{int}} \gg t_{\text{col}}$, the amplitude and phase of each packet will change only slightly during one collision. It is clear that the interaction of an ensemble of such packets may be described statistically assuming their phases to be almost random, i.e., using the kinetic equation. The applicability criterion for the kinetic equation thus derived has a form

$$\xi_{123} = \frac{t_{\text{col}}}{t_{\text{int}}} \simeq (2\pi)^{3/2}\left|v_{123} c_{\max}\right| \frac{l}{V_{1,23}} \ll 1 . \tag{2.1.19}$$

Using (1.5.8), we calculate

$$\langle |c_j(\boldsymbol{r},t)|^2 \rangle = \frac{1}{(2\pi)^3} \int n_j(\boldsymbol{\kappa}_j + \boldsymbol{k}_j)\, d\boldsymbol{\kappa}_j = \frac{N_j}{(2\pi)^3} \,. \tag{2.1.20}$$

Taking into account that the width of the packet in the coordinate representation l is related to Δk through $l \simeq 1/\Delta k$, we obtain

$$\xi_{123} \simeq |V_{123}| \frac{\sqrt{N}}{\Delta k v_{123}} \ll 1 \,. \tag{2.1.21}$$

If we set here $\Delta k \simeq k$, $v_{123} \simeq v_j \simeq \partial\omega(k_j)/\partial k_j$, $kv_j \simeq \omega_j$, the expression for ξ_{123} will coincide with (2.1.14) and (2.1.16). This is, of course, not accidental, but a consequence of the fact that all three inferences are equivalent. Our reasoning in the coordinate representation was the most detailed one, therefore the applicability criterion (2.1.21) allows us to clarify more subtle things. In particular, it yields that for waves with the linear dispersion law $\omega(k) = ck$, the classical kinetic equation is inapplicable at any excitation level of the system. Indeed, at $\omega(k) \propto k$, it follows from the conservation laws (2.1.4) that $\boldsymbol{k}_1 \parallel \boldsymbol{k}_2 \parallel \boldsymbol{k}_3$. Then $\boldsymbol{v}_1 = \boldsymbol{v}_2 = \boldsymbol{v}_3 = c\boldsymbol{k}_j/k_j$ and, by (2.1.18), $v_{123} = 0$. In the coordinate language, the inapplicability of the kinetic equation to this case may be easily understood: all interacting packets travel into the same direction with identical velocities and therefore they cannot diverge. As a result an arbitrary weak interaction will lead to substantial correlations of the phases of the wave packet.

As to be demonstrated later, there exists a difference between two- and three-dimensional cases in the applicability of the kinetic equation to acoustic turbulence (see Sect. 3.2.1). In the three-dimensional case the kinetic equation formally undergoes a transition to the nondispersive limit and it may be shown that at low excitation level the kinetic equation is applicable [see also remark after (2.1.25) below].

Now let us consider a wave system with the power dispersion law $\omega(k) = ak^\alpha$, $\alpha > 1$, and let $\boldsymbol{k}_1 = 2\boldsymbol{k}$, $\boldsymbol{k}_2 = \boldsymbol{k} + \boldsymbol{\kappa}$, $\boldsymbol{k}_3 = \boldsymbol{k} - \boldsymbol{\kappa}$. It follows from the conservation law (2.1.4) that

$$\begin{aligned} &\omega(2k) = 2\omega(k) + \hat{L}\kappa^2, \\ &\hat{L}\kappa^2 = \sum_{i,j} \kappa_i \kappa_j \frac{\partial^2 \omega}{\partial k_i \partial k_j} = \frac{v\kappa^2}{k} + \frac{\omega''(\kappa k)^2}{k^3}, \\ &v = \frac{\partial \omega(k)}{\partial k}, \quad \omega'' = \frac{\partial^2 \omega(k)}{\partial k^2} \end{aligned} \tag{2.1.22}$$

and whence

$$(\kappa/k)^2 \simeq \alpha - 1 \,. \tag{2.1.23a}$$

It is easy to see that all the differences in the group velocities in (2.1.20b) are of the same order and that

$$v_{123} \simeq v(\kappa/k). \tag{2.1.23b}$$

Thus criterion for the applicability of the kinetic equation has the form

$$\xi_k \simeq \frac{|V_{123}|\sqrt{N}k}{v\kappa\Delta k} \ll 1 \,, \tag{2.1.24a}$$

or, after substitution of (2.1.23a) for κ,

$$\xi_k \simeq \frac{|V_{123}|}{v\Delta k}\sqrt{\frac{N}{\alpha-1}} \ll 1 \,. \tag{2.1.24b}$$

But for the dispersion law $\omega(k) = c_s k(1+\mu k^2)$, criterion (2.1.24a) remains valid; but for κ we have, instead of (2.1.23a),

$$(\kappa/k)^2 \simeq \mu k^2 \,.$$

The applicability criterion for the kinetic equation is in this case

$$\xi_k \simeq \frac{|V_{123}|\sqrt{N}}{c_s k\sqrt{\mu}\Delta k} \ll 1 \,. \tag{2.1.25}$$

As we shall show in the second volume of this book, the strict criterion for weakly dispersive waves depends on the dimensionality d of the medium (see also Sect. 3.2 bellow). The criterion (2.1.25) holds only for $d = 2$. For the case $d = 3$, (2.1.25) should be replaced by the criterion $|V_{123}|\sqrt{N}/c_s k \ll ln^{-1}(\mu k^2)^{-1}$

In addition to the motion of packets limiting the time of their interaction, one should also take into account the dispersion diffusion of packets. The time of such a diffusion process is

$$t_{\rm dif} \simeq 1/\hat{L}(\Delta k)^2 \,, \tag{2.1.26}$$

where the L operator is given by (2.1.22). When the diffusion is dominant (at $t_{\rm dif} < t_{\rm col}$) we obtain instead of criterion (2.1.21) [and (2.1.24-25) following from it] the inequality

$$\xi_k \simeq t_{\rm int}/t_{\rm dif} \simeq \frac{|V_{1,23}|\sqrt{N}}{\hat{L}(\Delta k)^2} \ll 1 \,. \tag{2.1.27}$$

It should be borne in mind that all our deductions about the applicability criterion of the kinetic equation are of qualitative character. More rigorous this question will be discussed in terms of the Wyld's diagram technique in the second volume of this book.

2.1.5 The Four-Wave Kinetic Equation

In the nondecay case the resonant conditions (2.1.4) cannot be satisfied so that the right-hand-side of the kinetic equation (2.1.12) vanishes identically. Consequently, four-wave processes should be taken into account. The respective kinetic equation is derived in the same way as the one for three-wave processes. Namely, using (2.1.6), we calculate $\partial n(\boldsymbol{k},t)/\partial t$:

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = \mathrm{Im} \int T_{k123} J_{k123} \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \ , \qquad (2.1.28a)$$

where J_{k123} is the fourth-order correlation function (2.1.10b). In the zeroth order in the interaction it is determined by (2.1.7b). This gives

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = n(\boldsymbol{k},t)\mathrm{Im} \int T_{kk'kk'} n(\boldsymbol{k}')\, d\boldsymbol{k}' \ . \qquad (2.1.28b)$$

As $T_{1234} = T^*_{3412}$, the right-hand-side becomes zero and $\partial n/\partial t = 0$ to first order in the interaction. Using the scheme employed in Sect. 2.1.3 for the derivation of $J(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$, we obtain a first-order addition to $J(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)$:

$$J_1(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4; t) = T^*_{1234} f_{k123} \left(\omega_1 + \omega_2 - \omega_3 - \omega_4 + i\delta\right)^{-1}, \qquad (2.1.28c)$$

$$f_{k123} = n_2 n_3 (n_1 + n_k) - n_1 n_k (n_2 + n_3) \ . \qquad (2.1.29a)$$

In deriving that expression, we have represented the sixth-order correlator for the noninteracting field in terms of the product of pair correlators by (2.1.7c) and neglected the time derivative $\partial J_1(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4; t)/\partial t$ like in the three-wave case. Note that we have neglected the nonlinear renormalization of the frequency $\mathrm{Re} \int T_{kk'kk'} n_{k'} d\boldsymbol{k}'$ since its contribution to $\partial n(\boldsymbol{k},t)/\partial t$ is only of the next order. Substituting (2.1.28c) into (2.1.28a), we get the four-wave kinetic equation

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = \frac{\pi}{2} \int \left|T_{k123}\right|^2 f_{k123} \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3) \times \delta(\omega_k + \omega_1 - \omega_2 - \omega_3)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \ . \qquad (2.1.29b)$$

Its applicability criterion may also be obtained in terms of interacting packets. In contrast to (2.1.20a), t_{int} for the four-wave processes has the form

$$\frac{1}{t_{\mathrm{int}}} = (2\pi)^3 |Tc_{\mathrm{max}}|^2 \simeq TN \ . \qquad (2.1.30a)$$

The most rigorous applicability criterion of the four-wave kinetic equation will be obtained in the case of the self-interaction within a single wave packet which is narrow in $\boldsymbol{k}$-space. We denote its center by k_0 and its width by Δk. In the $\boldsymbol{r}$-representation this is a smooth packet with an envelope of a width $l \simeq 1/\Delta k$. The interaction process must be restricted by the diffusion of the packet during the time t_{dif} (2.1.26). Thus, a necessary condition for the applicability of the kinetic equation (2.1.29) is

$$\xi_k \simeq t_{\mathrm{int}}/t_{\mathrm{dif}} \simeq TN/L(\Delta k)^2 \ll 1 \ , \quad L \simeq \omega'' \ . \qquad (2.1.30b)$$

2.1.6 The Quantum Kinetic Equation

The above derivation of the kinetic equation is valid in the classical limit, when $n(\boldsymbol{k}) \gg \hbar$, i.e., for quantum-mechanical occupation numbers (2.1.8) $N(k) \gg 1$. In deriving the quantum kinetic equation, it is convenient to start from the Hamiltonian operator for the wave system in the second quantization representation. One can go over from the Hamilton function $\mathcal{H}$ expressed in terms of complex canonical variables to the operator $\hat{\mathcal{H}}$ by substituting for c_k, c_k^* the Bose creation and annihilation operators $\hat{c}$ and $\hat{c}^+$ of wave field quanta, respectively (see, e.g., [2.2]):

$$c_k^* \to \sqrt{n}\hat{c}_k^+, \quad c_k \to \sqrt{n}\hat{c}_k. \tag{2.1.31}$$

The anticommutator of $\hat{c}, \hat{c}^+$ is known to be nonzero:

$$\left\{\hat{c}(\boldsymbol{k}), \hat{c}^+(\boldsymbol{k}')\right\} = \hat{c}(\boldsymbol{k})\hat{c}^+(\boldsymbol{k}') - \hat{c}^+(\boldsymbol{k})\hat{c}(\boldsymbol{k}') = \delta(\boldsymbol{k} - \boldsymbol{k}') \, , \tag{2.1.32}$$

therefore we encounter in $\mathcal{H}_2$ a problem related to the order of the operators which amounts to choosing a reference point for the energy. If we assume

$$\mathcal{H}_2 = \frac{1}{2} \int \hbar\omega(\boldsymbol{k}) \left[\hat{c}^+(\boldsymbol{k})\hat{c}(\boldsymbol{k}) + \hat{c}(\boldsymbol{k})\hat{c}^+(\boldsymbol{k})\right] d\boldsymbol{k} \, , \tag{2.1.33}$$

then the $\hat{\mathcal{H}}$-eigenvalues will coincide with the energy of a system of noninteracting oscillators

$$E = \int \hbar\omega_k (N_k + 1/2) \, d\boldsymbol{k} \, . \tag{2.1.34}$$

The presence of $\hbar\omega_k/2$ terms in this integral indicates that even in the ground state (vacuum), when all the occupation numbers N_k are zero, there is a quantum uncertainty in the coordinates and velocities of the atoms comprising our "continuous" medium. This quantum-mechanical effect is called zero oscillations, and the terms $\hbar\omega_k/2$, accordingly, the energy of zero oscillations. It should be noted that the integral in (2.1.34) defining the full energy of zero oscillations of a continuous medium diverges in the region of large k's. But as any medium (a solid, fluid, gas or plasma) consists of separate particles, the range of integration in (2.1.34) is cut off at some $k_{\max} \simeq 1/l$, where l is an average distance between the particles. It is customary to write all annihilation operators on the right-hand-side of the interaction operator $\mathcal{H}_{\rm int}$, then its average value in the vacuum state will be zero. Thus, we obtain from (2.1.2, 5, 31)

$$\mathcal{H}_3 = \frac{\hbar^{3/2}}{2} \int \left[V_{123}\hat{c}_1^+\hat{c}_2\hat{c}_3 + \text{H.c.}\right] \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) \, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3. \tag{2.1.35}$$

$$\mathcal{H}_4 = \frac{\hbar^2}{2} \int T_{1234}\hat{c}_1^+\hat{c}_2^+\hat{c}_3\hat{c}_4 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4) \, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4. \tag{2.1.36}$$

The quantum-mechanical states of a medium may be classified according to the states of a noninteracting wave field whose Hamiltonian $\mathcal{H}_2$ (2.1.33) is the sum of the Hamiltonians of noninteracting harmonic oscillators. The energy spectrum of an oscillator with frequency ω is known to be $\hbar\omega(N+1/2)$, where $N = 0, 1, 2, \dots$ with all states N being nondegenerate. Therefore for an oscillator there exists a complete set of functions $\psi_0, \psi_1, \dots, \psi_N, \dots$. They may be denoted by $\psi_N = |N\rangle$. Such a choice of functions is called the second quantization representation. The matrix elements of the operators $\hat{c}$ and $\hat{c}^+$ are [2.2]

$$\langle N-1|\hat{c}|N\rangle = \langle N|\hat{c}^+|N-1\rangle = \sqrt{N}, \tag{2.1.37}$$

being zero for matrix elements between other states. It is seen that the creation operator $\hat{c}$ increases the number of quanta N by unity while $\hat{c}^+$ reduces it by unity. One can introduce an operator describing the number of quanta $\hat{N} = \hat{c}^+\hat{c}$, which is diagonal in the $|N\rangle$ representation:

$$\langle N|\hat{N}|N\rangle = N. \tag{2.1.38}$$

The question is whether the oscillator description using the average value of the occupation number N is complete or not. The answer may be found using an analogy with classical mechanics where an oscillator is characterized by the energy $\propto N$ and the phase ϕ. In quantum mechanics, the energy of an ocsillator and its phase may not be measured simultaneously. In a state of a definite energy the phase is absolutely random. Thus the description of a quantum system using the distribution function N_k is completely equivalent to the random phase approximation (2.1.7) used by us in the classical approach. A more complete description of a quantum system allowing for its phase correlations is possible in terms of a density matrix (where the numbers N_k are its diagonal elements). Thus, in the random phase approximation we obtain an equation describing the variation rate of the number of quanta of a field N_k with a given wave vector $\boldsymbol{k}$. The value of N_k decreases in the decay processes

$$|N_k, N_1, N_2\rangle \longrightarrow |N_k - 1, N_1 + 1, N_2 + 1\rangle, \quad \boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2 \tag{2.1.39a}$$

and the confluence processes

$$|N_k, N_1, N_2\rangle \longrightarrow |N_k - 1, N_1 - 1, N_2 + 1\rangle, \quad \boldsymbol{k} + \boldsymbol{k}_1 = \boldsymbol{k}_2, \tag{2.1.39b}$$

and increases in the reverse processes. According to quantum-mechanical perturbation theory, the probability of these processes equals the product of $2\pi/\hbar$ with the squared modulus of a matrix element of the interaction Hamiltonian and with the δ-function of the energy difference between initial and final states. For example, for the sum of the decay processes (2.1.39a) with different $\boldsymbol{k}_1$ and with $\boldsymbol{k}_2 = \boldsymbol{k} - \boldsymbol{k}_1$, the probability is

$$\begin{aligned} W_{\mathrm{a}} = 2\pi \int &\left|\langle N_k, N_1, N_2|\hat{c}_k\hat{c}_1^+\hat{c}_2^+|N_k - 1, N_1 + 1, N_2 + 1\rangle_{\mathrm{a}}\right|^2 \\ &\times \left|V_{k12}\right|^2 \times \hbar^2\delta(\omega_k - \omega_1 - \omega_2)\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2\ . \end{aligned} \tag{2.1.40}$$

Here the matrix element of the operator $\hat{c}_k\hat{c}_1^+\hat{c}_2^+$ should be calculated for the free field. It decomposes into the product of matrix elements from each degree of freedom of the field

$$\langle\,\rangle_a = \langle N_k|\hat{c}_k|N_k - 1\rangle\,\langle N_1|\hat{c}_1^+|N_1 + 1\rangle\,\langle N_2|\hat{c}_2^+|N_2 + 1\rangle\ . \qquad (2.1.41a)$$

This procedure is, in effect, equivalent to the uncoupling of the classical correlators (2.1.7) used above in deducing the expression (2.1.11) for the classical triple correlator. Using (2.1.37), we obtain from (2.1.41a)

$$|\langle\,\rangle_a|^2 = N_k(N_1 + 1)(N_2 + 1)\ , \qquad (2.1.41b)$$

which, together with (2.1.40), gives the final expression for the probability W_a. Clearly,

$$\frac{\partial N(\boldsymbol{k},t)}{\partial t} = -(W_a + W_b) + (\tilde{W}_a + \tilde{W}_b)\ .$$

W_a and W_b are the probabilities of the processes (2.1.39a, b), which reduce N_k by unity. $\tilde{W}_a$ and $\tilde{W}_b$ are the probabilities of the reverse processes increasing N_k by a unity. Computing in a similar way W_b, $\tilde{W}_a$ and $\tilde{W}_b$, we get finally

$$\frac{\partial N(\boldsymbol{k},t)}{\partial t} = \pi\hbar\int\Big[|V_{k12}|^2F_{k12}\delta(\omega_k - \omega_1 - \omega_2)\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)$$
$$- 2|V_{1k2}|^2F_{1k2}\delta(\omega_1 - \omega_k - \omega_2)\delta(\boldsymbol{k}_1 - \boldsymbol{k} - \boldsymbol{k}_2)\Big]\,d\boldsymbol{k}_1 d\boldsymbol{k}_2 \qquad (2.1.42a)$$

with

$$F_{k12} = (N_1 + 1)N_2N_3 - N_1(N_2 + 1)(N_3 + 1)\ . \qquad (2.1.42b)$$

The factor 1/2 before V_{k12} eliminates double counting of individual processes. The expression (2.1.42b) may be rewritten in the form

$$F_{123} = N_2N_3 - N_1(N_2 + N_3 + 1)\ . \qquad (2.1.42c)$$

Neglecting unity in this expression and replacing $N(k_j)$ by $\hbar n(k_j)$, the quantum kinetic equation (2.1.42) goes over into the classical one (2.1.12). In a similar way we obtain the quantum kinetic equation for scattering processes:

$$\frac{\partial N(\boldsymbol{k},t)}{\partial t} = \frac{\pi}{2}\int|T_{k123}|^2F_{k123}\delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)$$
$$\times\ \delta(\omega_k + \omega_1 - \omega_2 - \omega_3)\,d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3\ , \qquad (2.1.43)$$
$$F_{k123} = (N_k + 1)(N_1 + 1)N_2N_3 - N_kN_1(N_2 + 1)(N_3 + 1)$$
$$= N_2N_3(N_1 + N_k + 1) - N_1N_k(N_2 + N_3 + 1)\ .$$

For $N_k \gg 1$ it goes over to the classical kinetic equation (2.1.29).

For those familiar with solid state theory, it is customary to use the kinetic equation to describe weak nonequilibrium kinetics (see, for example, *Ziman* [2.3]). Here we shall use the kinetic equation to describe turbulence, i.e., states which are rather far from equilibrium. As we have seen, this use of the kinetic equation is possible since its applicability is ensured by the weakness of the nonlinearity (rather than the proximity to equilibrium); averaging is possible due to the statistics of the free wave field (but not the thermodynamic equilibrium).

In the opposite (quantum) limit $N_k \ll 1$, waves behave like particles and (2.1.43) goes over into the well-known kinetic Boltzmann equation for classical particles with energy $\varepsilon = \hbar\omega_k$ and momentum $\boldsymbol{p} = \hbar\boldsymbol{k}$

$$\frac{\partial N_p}{\partial t} = \frac{\pi}{2}\hbar^{1-2d}\int |T_{p123}|^2 \left(N_2N_3 - N_1N_p\right)\delta(\boldsymbol{p}+\boldsymbol{p}_1-\boldsymbol{p}_2-\boldsymbol{p}_3) \times \delta(\varepsilon_p+\varepsilon_1-\varepsilon_2-\varepsilon_3)\,d\boldsymbol{p}_1 d\boldsymbol{p}_2 d\boldsymbol{p}_3 \tag{2.1.44}$$

2.2 General Properties of Kinetic Wave Equations

2.2.1 Conservation Laws

The existence of conservation laws for the kinetic equations is an important problem which turns out to be more involved than expected. Let us first attempt to obtain the energy integral. As seen above, the initial dynamic equations for the wave amplitudes conserve the full Hamiltonian $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_{\text{int}}$. The kinetic equation is valid in the limit $\mathcal{H}_{\text{int}} \ll \mathcal{H}_2$; the interaction is in this case described by the collision integral whose form is entirely determined by the resonance processes. In this case, every elementary interaction preserves the sum over the wave frequencies. Thus, in the weakly nonlinear limit the role of the energy should be played by the quadratic term of the Hamiltonian $\mathcal{H}_2$. The kinetic equation should retain the averaged value $\langle\mathcal{H}_2\rangle$. However, for a statistically homogeneous field we have $\langle c(\boldsymbol{k})c^*(\boldsymbol{k}_1)\rangle = n_k\delta(\boldsymbol{k}-\boldsymbol{k}_1)$ and the Hamiltonian $\mathcal{H}_2$ is infinite. The finite value is its density per unit volume. Considering a wave system in a box with sides of length L then we have for $\delta(\boldsymbol{k})$ at $k = 0$ $\quad(2\pi/L)^d$ and see that the value

$$E = \int \omega(k)n(\boldsymbol{k},t)\,d\boldsymbol{k} \tag{2.2.1}$$

is the spatial density of the average wave energy. Indeed, it is easy to see that (2.2.1) may be an integral of motion in (2.1.12, 29) if some additional conditions are satisfied. The formal reason for the vanishing of dE/dt is the presence of the frequency δ-functions ensuring energy conservation in every elementary act of wave interaction. For example, in (2.1.12):

$$\frac{dE}{dt} = \int (\omega_k - \omega_1 - \omega_2)\left|V_{k12}\right|^2\delta(\omega_k - \omega_1 - \omega_2)\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \times(n_1n_2 - n_kn_1 - n_kn_2)\,d\boldsymbol{k}d\boldsymbol{k}_1d\boldsymbol{k}_2 = 0\ . \tag{2.2.2}$$

This equation is valid if its integrals converge. Assuming that there are no singularities of $n(\boldsymbol{k},t)$ at $k=0$ and a power law of the functions $\omega(k)$ and V_{k12} at $k \to \infty$, we see that the integrals in (2.2.2) converge if in the region of large k the occupation number $n(\boldsymbol{k},t)$ decreases more rapidly than k^{-m-d}. Here m is the homogeneity index of V_{k12} at $k \to \infty$ and d the dimension of the $\boldsymbol{k}$-space. In order to explain the physical meaning of this condition, we should introduce the idea of energy flux in the $\boldsymbol{k}$-space. Equation (2.2.2) allows us to write the kinetic equation (2.1.12) in the form of a differential equation, i.e., as a continuity equation for energy density $\varepsilon(\boldsymbol{k},t) = \omega(k)n(\boldsymbol{k},t)$

$$\frac{\partial \varepsilon(\boldsymbol{k},t)}{\partial t} + \operatorname{div} \boldsymbol{p}(\boldsymbol{k},t) = 0 \,, \qquad (2.2.3)$$

where $\boldsymbol{p}(\boldsymbol{k},t)$ is the energy flux in $\boldsymbol{k}$-space given by

$$\operatorname{div} \boldsymbol{p}(\boldsymbol{k},t) = -\omega(k) I(\boldsymbol{k},t) \,. \qquad (2.2.4)$$

The symbols $I(\boldsymbol{k},t) = I_k\{n(\boldsymbol{k},t)\} = I_k$ denote here and below the *collision integral*, i.e., the right-hand-side of the kinetic equation (2.1.12) or (2.1.29). The variation rate of the full energy of the wave system dE/dt, according to (2.2.3) must be equal to the integral over a spheric surface of infinite radius from the normal component of the vector $\boldsymbol{p}(\boldsymbol{k},t)$. The condition of a faster fall-off of $n(\boldsymbol{k},t)$ as compared to k^{-m-d} is necessary to ensure that this integral goes to zero at $k \to \infty$, see (2.2.4). It may well be that this condition is not satisfied. Moreover, even if the initial distribution is rather localized, the power distributions corresponding to a constant flux are formed in a natural way in the evolution process at $k \to 0$ or $k \to \infty$. Such asymptotics may set in at an infinite interval during the finite period of time (as it will be described in Chap. 4). Due to this, the problems related to the existence of integrals of motion of the kinetic eqution are not quite trivial. Thus, some value may be conserved only for a limited period of time (see Sect. 4.3 below). One of the main goals of this book is to demonstrate that given the infinite k-space the "naively" determined integrals of motion may prove to be fictitious and are not really conserved.

Violation of the conservation law of the kinetic equation surely implies also its violation for the initial dynamic equation which is, however, more difficult to show.

The four-wave kinetic equation (2.1.29) may also be represented in the form of a continuity equation like (2.2.3). In this case, full energy of the wave system is conserved if at $k \to \infty$, the value $n(\boldsymbol{k},t)$ decreases with k more rapidly than $k^{-d-2m/3}$. As mentioned in the previous chapter, four-wave scattering processes do not change the number of waves. Therefore, the kinetic equation (2.1.29) as well as the dynamic one (2.1.5b), have an additional integral of motion

$$N = \int n(\boldsymbol{k},t)\, d\boldsymbol{k} \,. \qquad (2.2.5)$$

Integrating (2.1.29) over $d\boldsymbol{k}$ this is readily seen to hold. The integrand changes sign after the interchange $k \leftrightarrow k_1$ or $k_1 \leftrightarrow k_2$ of dummy integration variables.

Owing to the conservation of N, the four-wave kinetic equation may be written in the divergence form

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} + \operatorname{div} \boldsymbol{q}(\boldsymbol{k},t) = 0\,, \quad \operatorname{div} \boldsymbol{q}(\boldsymbol{k},t) = -I(\boldsymbol{k},t)\ . \tag{2.2.6}$$

The quantity $\boldsymbol{q}(\boldsymbol{k},t)$ is called the flux of the wavenumber or the wave action flux. The latter name is motivated by the fact that $n(\boldsymbol{k},t) = \varepsilon(\boldsymbol{k},t)/\omega(k)$ has the meaning of the density of an adiabatic wave invariant in phase space.

Besides the frequency δ-functions, the kinetic equations also contain the δ-functions of wave vectors leading to the conservation of the full momentum of the wave system

$$\begin{aligned} \boldsymbol{\Pi} &= \int \boldsymbol{k} n(\boldsymbol{k},t)\, d\boldsymbol{k} = \int \boldsymbol{\pi}(\boldsymbol{k},t)\, d\boldsymbol{k}\ , \\ &\frac{\partial \boldsymbol{\pi}(\boldsymbol{k},t)}{\partial t} + \operatorname{div} \boldsymbol{R}(\boldsymbol{k},t) = 0\ . \end{aligned} \tag{2.2.7}$$

Here the momentum flux $\boldsymbol{R}$ is a second order tensor.

It should be pointed out that, contrary to E, the quantities N and $\boldsymbol{\Pi}$ are exact integrals of motion, i.e., they are preserved within the framework of the starting dynamic equations (for distributions that go fast to zero).

In the general (nondegenerate) case the quantities E, N and $\boldsymbol{\Pi}$ form a complete set of integrals of motion for the kinetic equations. Additional integrals may only appear in degenerate cases. Let us consider, for example, the three-wave kinetic equation (2.1.12). The resonance surface for interaction in the space of vectors $\boldsymbol{k}_1, \boldsymbol{k}_2$ is given by the condition

$$\omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) = \omega(\boldsymbol{k}_1 + \boldsymbol{k}_2)\ . \tag{2.2.8a}$$

If the function $\omega(k)$ is not uniquely retrieved from the form of this surface, i.e., if there is another function $f(\boldsymbol{k})$ $[f(\boldsymbol{k}) \not\equiv A\omega(k) + (\boldsymbol{B}\boldsymbol{k})]$ satisfying on the surface (2.2.8a) the same condition

$$f(\boldsymbol{k}_1) + f(\boldsymbol{k}_2) = f(\boldsymbol{k}_1 + \boldsymbol{k}_2)\ , \tag{2.2.8b}$$

then the dispersion law $\omega(\boldsymbol{k})$ is called degenerate. Here the quantity

$$F = \int f(\boldsymbol{k}) n(\boldsymbol{k},t)\, d\boldsymbol{k}$$

will be the integral of motion of the kinetic equation (2.1.12). The nondegeneracy criterion for $\omega(\boldsymbol{k})$ and examples of the degenerate dispersion laws are given in [2.4].

For example, let us consider shallow-water gravitational-capillary waves with the dispersion law (1.2.39)

$$\omega(k) = ck(1 + k^2/2k_*^2)\ .$$

If the motion is almost one-dimensional with $k_x \gg k_y$, the dispersion law has the form

$$\omega(k_x, k_y) \approx ck_x(1 + k_x^2/2k_*^2 + k_y^2/2k_x^2) = Ap + p^3/16 + 3q^2/p \ .$$

This dispersion law corresponds to the Kadomtsev-Petviashvili equation (1.5.4) and coincides with (1.5.5a) with $\beta > 0$ and $p = 2k_x(c/k_*^2)^{1/3}$ and $q = k_y(c^2/k_*)^{1/3}/\sqrt{3}$. In the four-dimensional space of vectors k_1, k_2 the resonance condition (2.2.8a) defines a three-dimensional surface which can be parametrized by [2.4]

$$p_1 = 2(\xi_1 - \xi_2), \quad q_1 = \xi_1^2 - \xi_2^2, \quad p_2 = 2(\xi_2 - \xi_3), \quad q_2 = \xi_2^2 - \xi_3^2 \ ,$$

$$\omega(p_1, q_1) = \frac{A}{2}(\xi_1 - \xi_2) + 2(\xi_1^3 - \xi_2^3) \ ,$$

$$\omega(p_2, q_2) = \frac{A}{2}(\xi_2 - \xi_3) + 2(\xi_2^3 - \xi_3^3) \ .$$

Thus, if we introduce

$$f(p, q) = \varphi(\xi_1) - \varphi(\xi_2) = \varphi\left(\frac{q}{p} + \frac{p}{4}\right) - \varphi\left(\frac{q}{p} - \frac{p}{4}\right)$$

with the arbitrary even function φ, then f satisfies condition (2.2.8b). Therefore we have an infinite set of integrals of motion for this case which corresponds to the (well-established) integrability of the Kadomtsev-Petviashvili equation.

2.2.2 Boltzmann's H-Theorem and Thermodynamic Equilibrium

The dynamic Hamilton equations are invariant with respect to time reversal, i.e., to the transformations $t \to -t$, $\mathcal{H} \to -\mathcal{H}$. The kinetic equations obtained after the averaging procedure, however, describe an irreversible evolution towards thermodynamic equilibrium. The mathematical statement of irreversibility is the theorem of entropy growth which is similar to the Boltzmann's H-theorem for gas kinetics [2.5]. Indeed, let us consider, for example, the time dependence of the entropy

$$S(t) = \int \ln[n(\boldsymbol{k}, t)]\, d\boldsymbol{k} \ , \tag{2.2.9}$$

of a wave system that obeys the three-wave kinetic equation (2.1.12). We obtain

$$\begin{aligned} \frac{dS}{dt} &= \int \frac{\partial n(\boldsymbol{k},t)}{\partial t} \frac{d\boldsymbol{k}}{n(\boldsymbol{k},t)} = \int \frac{I(\boldsymbol{k},t)}{n(\boldsymbol{k},t)}\, d\boldsymbol{k} \\ &= \int \left|V_{k12}\right|^2 \delta(\omega_k - \omega_1 - \omega_2)\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \\ &\times \frac{\left(n_1 n_2 - n_k n_1 - n_k n_2\right)^2}{n_k n_2 n_3}\, d\boldsymbol{k} d\boldsymbol{k}_1 d\boldsymbol{k}_2 > 0 \ . \end{aligned} \tag{2.2.10}$$

Similar inequalities may be derived for the four-wave and quantum kinetic equations. In the latter case, one should proceed from the exact expression for the entropy of a Bose gas

$$S_B(t) = \int \left\{[1 + N(\boldsymbol{k}, t)] \ln[1 + N(\boldsymbol{k}, t)] - N(\boldsymbol{k}, t) \ln N(\boldsymbol{k}, t)\right\} d\boldsymbol{k} \ , \tag{2.2.11}$$

whose classical limit [at $N(\boldsymbol{k}) = n(\boldsymbol{k})/\hbar \gg 1$] is (2.2.9). Differentiating (2.2.11) and substituting $N(\boldsymbol{k})$ from (2.1.42) we get

$$\begin{aligned}\frac{dS_B}{dt} = \int & \left|V_{k12}\right|^2 \delta(\omega_k - \omega_1 - \omega_2)\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \\ & \times \left\{N(\boldsymbol{k},t)[N(\boldsymbol{k}_1,t)+1][N(\boldsymbol{k}_2,t)+1]\right. \\ & \left. - N(\boldsymbol{k}_1,t)N(\boldsymbol{k}_2,t)[N(\boldsymbol{k},t)+1]\right\} \\ & \times \ln \frac{[N(\boldsymbol{k}_1,t)+1][N(\boldsymbol{k}_2,t)+1]N(\boldsymbol{k},t)}{N(\boldsymbol{k}_1,t)N(\boldsymbol{k}_2,t)[N(\boldsymbol{k},t)+1]} \, d\boldsymbol{k} d\boldsymbol{k}_1 d\boldsymbol{k}_2 > 0 \; .\end{aligned}$$

The inequality follows since $(x - y)\ln(x/y) > 0$ holds at any x, y.

Thus, the entropy of a closed wave system can only increase. The thermodynamic equilibrium corresponds to the maximum of entropy given a constant total energy. Making use of the method of Lagrange multipliers, we obtain from (2.2.11)

$$\frac{\delta}{\delta N(k)}(S - \lambda E) = \ln \frac{N(k)+1}{N(k)} = 0 \; .$$

The parameter λ is given by the temperature of the system $\lambda = T/\hbar$, hence we have the Planck distribution

$$N(k) = \{\exp[\hbar\omega(k)/T] - 1\}^{-1} \; . \tag{2.2.12a}$$

In the quantum limit [when $\hbar\omega = \varepsilon \gg T$ and $N(k) \ll 1$] (2.2.12a) gives the *Maxwell distribution* $N(\varepsilon) = \exp(-\varepsilon/T)$, which is the stationary solution of the Boltzmann kinetic equation (2.1.44). In the classical limit $\hbar\omega(k) \ll T$ and (2.2.12a) goes over to the *Rayleigh-Jeans distribution*

$$n(k) = \hbar N(k) = \frac{T}{\omega(k)} \, , \tag{2.2.12b}$$

which may be obtained directly by varying (2.2.9). The equilibrium solution (2.2.12b) corresponds to the equipartition of energy over the degrees of freedom $\varepsilon(k) = T$.

If we demand that, apart from energy, the total momentum of the system be also nonzero and constant, we obtain the *drift equilibrium distribution*

$$n(k) = \frac{T}{\omega(k) - (\boldsymbol{k}\boldsymbol{u})} \; . \tag{2.2.13}$$

Here $\boldsymbol{u}$ is the drift velocity of the wave system relative to the chosen reference system.

Upon direct substitution it is seen that (2.2.12a) is a stationary solution of (2.1.42, 43), and (2.2.12b, 13) are the stationary solutions of (2.1.12,29). In this case every term in the collision integral vanishes separately. The energy flux of equilibrium distributions is identically zero $\boldsymbol{p}(\boldsymbol{k}) = 0$.

The four-wave kinetic equation has three integrals of motion, therefore the general equilibrium distribution depends on three constants:

$$n(k) = \frac{T}{\omega(k) - (\boldsymbol{k}\boldsymbol{u}) - \mu} \ . \tag{2.2.14}$$

Here μ is the chemical potential. As (2.1.29) is invariant relative to the substitution $\omega(k) \to \omega(k) - \mu$, one can set $\mu = 0$, which corresponds to the definite choice of the zero energy.

2.2.3 Stationary Nonequilibrium Distributions

Under the influence of external effects the wave system deviates from thermodynamic equilibrium. Under sufficiently powerful effects all wave excitation mechanisms in different media lead to instability and exponential growth of wave occupation numbers at the linear stage. Damping of waves is possible due to interactions with inhomogeneities of the medium, quasi-particles of other types etc. We shall describe all the effects of wave generation and relaxation by the same function $\Gamma(\boldsymbol{k})$:

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = I(\boldsymbol{k},t) + \Gamma(\boldsymbol{k})n(\boldsymbol{k},t) \ .$$

In the regions of the $\boldsymbol{k}$-space where $\Gamma(\boldsymbol{k})$ is positive it defines the growth-rate of the wave instability and for $\Gamma(k) < 0$, the damping decrement.

We shall call turbulence a highly nonequilibrium state of the wave system when the occupation numbers deviate from equilibrium in some regions of the $\boldsymbol{k}$-space. The kinetic equation, which is valid at weak nonlinearity, describes the so-called weak turbulence.

Let us discuss the necessary requirements to be satisfied by $\Gamma(\boldsymbol{k})$ to ensure the existence of the stationary distribution $n(\boldsymbol{k})$

$$\Gamma(\boldsymbol{k})n(\boldsymbol{k}) + I_k\{n(\boldsymbol{k}')\} = 0 \tag{2.2.15}$$

[2.6]. These requirements are based on the general properties of the collision integral $I(\boldsymbol{k})$ which is a functional of $n(\boldsymbol{k}_1)$. First, it follows from the H-theorem that $\int I(\boldsymbol{k})n^{-1}(\boldsymbol{k})\,d\boldsymbol{k} > 0$ [see (2.2.10)]. Thus, it is necessary that

$$\int \Gamma(\boldsymbol{k})\,d\boldsymbol{k} < 0 \tag{2.2.16}$$

to satisfy (2.2.15). The physical meaning of this condition is obvious: for the existence of a nonequilibrium steady state the environment should provide a constant output of entropy from the system.

Since energy and momentum are conserved as waves interact with each other, in the stationary distribution we have

$$\int \Gamma(\boldsymbol{k})\omega(\boldsymbol{k})n(\boldsymbol{k})\,d\boldsymbol{k} = 0 \ , \tag{2.2.17}$$

$$\int \Gamma(\boldsymbol{k})\boldsymbol{k}n(\boldsymbol{k})\,d\boldsymbol{k} = 0 \ . \tag{2.2.18}$$

These conditions should be satisfied under convergence of their integrals, i.e., with the fluxes becoming zero at $k \to \infty$.

It is seen from (2.2.16) that the function $\Gamma(\boldsymbol{k})$ ensuring a steady state should be sign-alternating, i.e., describe both sources and sinks of wave energy. The mutual disposition of sources and sinks in k-space should not at all be arbitrary. Let us consider, for example, an isotropic arrangement. Let us introduce the energy density in the space of wave numbers

$$E(k) = (2k)^{d-1}\pi\varepsilon(k) = (2k)^{d-1}\pi\omega(k)n(k)$$

and the respective flux $P(k)$ (also in spherical normalization):

$$\frac{dP(k)}{dk} = -(2k)^{d-1}\pi\omega(k)n(k) \ .$$

Then the stationary kinetic equation (2.2.15) may be written as

$$\frac{dP(k)}{dk} = \Gamma(k)I(k) \ . \tag{2.2.19}$$

Let us integrate it from some $k_{\rm m}$ to infinity:

$$P(\infty) - P(k_{\rm m}) = \int_{k_{\rm m}}^{\infty} \Gamma(k)E(k)\,dk \ . \tag{2.2.20}$$

Assuming the occupation numbers to decrease rather rapidly [faster than k^{-m-d} for three-waves KE (2.1.12) and $k^{-d-2m/3}$ for four-waves KE (2.1.29)], we have $P(\infty) = 0$. Nearly in all cases of wave turbulence (see Chap. 1) we have $m+d > \alpha$ and $2m/3+d > \alpha$ where m, α are the indices of interaction coefficient and dispersion law at $k \to \infty$. These inequalities imply that with the growth of k in the region of large k the nonequilibrium stationary distributions should decrease more rapidly than the equilibrium ones. This means that at sufficiently large k the energy flux is positive $P(k) > 0$. Indeed, for the equilibrium solution we have $\varepsilon(k)$ =const and $P(k) = 0$ while more localized distributions have a flux directed towards the regions with smaller energy density $\varepsilon(k)$, i.e., towards large k [positiveness of the flux may also be shown strictly, see below Sect. 3.1 (3.1.13)]. Returning to (2.2.20), we see that for the stationary distribution $\exists k_{\rm m}\, \forall k > k_{\rm m}$:

$$\int_k^{\infty} \Gamma(k')E(k')\,dk' = \int_k^{\infty} (2k')^{d-1}\pi\Gamma(k')\omega(k')n(k')\,dk' < 0 \ . \tag{2.2.21}$$

Thus, a necessary condition for the existence of a nonequilibrium steady state is energy damping in the region of large k. For the four-wave kinetic equation the condition

$$\int \Gamma(\boldsymbol{k})n(\boldsymbol{k})\,d\boldsymbol{k} = 0 \tag{2.2.22}$$

following from the conservation of the total number of waves should be satisfied in addition to (2.2.17–18). From this one can easily obtain (in isotropic situation too) the necessity for the existence of at least one more sink, i.e., a region with negative $\Gamma(k)$. Indeed, if the function Γ changes its sign only once, it is easy to prove that it is impossible to satisfy (2.2.16) and (2.2.22) simultaneously, since $\omega(k)$ is assumed to be a monotonic function.

A very common situation is that the regions with pumping and damping are in k-space well separated (by a large region with $\Gamma(\boldsymbol{k}) \approx 0$, called the inertial interval). In this region the stationary turbulent distribution should satisfy

$$I_k\{n(\boldsymbol{k}_1)\} = 0 \ . \tag{2.2.23}$$

At the ends of the inertia interval the solution $n(\boldsymbol{k})$ should match source and sink. Clearly, the equilibrium distributions (2.2.12–14), though reducing the collision integral $I(\boldsymbol{k})$ to zero, cannot match external sources and sinks.

Determination of nonequilibrium stationary distributions is based on the universality hypothesis, according to which the form of the solution in the inertial interval should be independent of the structure of $\Gamma(\boldsymbol{k})$ in the pumping and damping regions. The expression for $n(\boldsymbol{k})$ may contain only the integral characteristics of the source that determine the fluxes of the integrals of motion. Such a universality implies a step-by-step (cascade) transfer of energy (or another integral) over the different scales from the source to the sink. The necessary condition is locality of the interaction in k-space; motions with strongly differing scales should produce a weak effect on each other.

Thus, the program for determining stationary weak turbulent distributions is:

1) find the universal solutions of (2.2.23) transmitting the fluxes of conserved quantities;
2) verify the locality of the resulting distributions, i.e., the convergence of the collision integrals;
3) match the universal solutions with the source and the sink.

In order to make sure that the obtained solutions are physically realistic, one should check whether they are stable.

The first three items are covered in Chap. 3, the third in Sect. 3.4 and the stability problem is solved in Chap. 4.

3. Stationary Spectra of Weak Wave Turbulence

This is the key chapter of the first volume. Here we deal with stationary distributions of weak turbulence. In Sects. 3.1–4 we describe the universal Kolmogorov-like spectra in the inertial interval. We obtain the Kolmogorov spectra as exact solutions of kinetic equations for scale-invariant media, both isotropic and nonisotropic, and for nearly scale-invariant ones. Section 3.1.5 deals with the structure of a stationary spectrum in the pumping and damping regions.

3.1 Kolmogorov Spectra of Weak Turbulence in Scale-Invariant Isotropic Media

What I tell you three times is true.

L. Caroll "The Hunting of the Shark"

This part of the book is the highlight of the first volume. We shall transform the qualitative arguments about turbulence spectra into exact formulas and the locality hypothesis into a strict theorem. This will be done in three steps. First, we shall start from dimensional analysis and, using a specific form of the kinetic equation, find the form of the stationary turbulence spectrum in the case of complete self-similarity. If this is not the case despite the scale invariance of Hamiltonian coefficients, the form of the spectrum can still be obtained quite easily. For that purpose it is sufficient to demand the flux to be a constant. That means that the flux expressed in terms of the collision integral must be proportional to the zeroth power of the wave vector. Such procedures cannot always be relied upon or taken for granted; so they will not guarantee the existence of the stationary spectrum which can only be ensured by proving locality of the interaction. Thus, as a second step, we shall discuss the structure of the asymptotic form of the kinetic equation and work out the locality criterion. Finally we shall acquaint the reader with a little miracle of the theory of wave turbulence theory, the so-called Zakharov transformations. They factorize the collision integral. As a result one can (i) prove directly that Kolmogorov spectra reduce the collision integral to zero and (ii) find that the Rayleigh-Jeans and Kolmogorov distributions are the only universal stationary power solutions of the kinetic equation.

3.1.1 Dimensional Estimations and Self-Similarity Analysis

This section deals with universal flux distributions corresponding to constant fluxes of the integrals of motion in k-space. In this subsection we shall show that for scale-invariant media, these solutions may be obtained from dimensional analysis (see also [3.1,2]).

For complete self-similarity we shall first discuss the possible form of universal flux distributions $n(k)$ and the corresponding energy spectra $E(k) = (2k)^{d-1}\omega(k)n(k)$. We shall recall how to find the form of the spectrum $E(k)$ for the turbulence of an incompressible fluid: in this case there is only one relevant parameter, the density ϱ; and $E(k)$ may be expressed via ϱ, k and the energy flux P. Comparing the dimensions of these quantities, we obtain

$$E(k) \simeq P^{2/3}k^{-5/3}\varrho^{1/3} \tag{3.1.1}$$

which is the famous Kolmogorov-Obukhov "5/3 law" [3.3,4].

As seen in Sect. 1.1, in the case of wave turbulence there are always two relevant parameters. We can choose the medium density as the first one. In contrast to eddies, waves have frequencies which may be chosen as the second parameter. The frequency enables us to arrange these quantities to give the dimensionless parameter

$$\xi = \frac{Pk^{5-d}}{\varrho\omega^3(k)} ,$$

so $E(k)$ may be determined from dimensional analysis up to an unknown dimensionless function $f(\xi)$:

$$E(k) = \varrho\omega_k^2 k^{d-6} f\left(Pk^{5-d}/\varrho\omega_k^3\right) . \tag{3.1.2}$$

In particular, if we demand that $\omega(k)$ be eliminated from (3.1.2), we obtain $f(\xi) \propto \xi^{2/3}$, and (3.1.2) coincides with (3.1.1). In the case of weak wave turbulence the connection between $P(k)$ and $n(k)$ follows from the stationary kinetic equation

$$dP(k)/dk = -(2k)^{d-1}\pi\omega(k)I(k) \tag{3.1.3}$$

which holds in the limit $\xi \ll 1$.

For the three-wave kinetic equation (2.1.12) $I(k) \propto n^2(k)$ and $n(k) \propto P^{1/2}$, and for the four-wave one, $n(k) \propto P^{1/3}$. These expressions may be unified to

$$n(k) \propto P^{1/(j-1)} ,$$

here j is the number of waves participating in an elementary interaction act ($j = 3$ for decay case or $j = 4$ for nondecay one). From this one can easily find the form of the function of the dimensionless argument at $\xi \to 0$

$$f(\xi) \propto \xi^{1/(j-1)}$$

and the Kolmogorov spectrum for weak turbulence

$$\begin{aligned} E(k) &\simeq \varrho^{(j-2)/(j-1)} \omega_k^{(2j-5)(j-1)} P^{1/(j-1)} k^{(dj-2d-6j+1)/(j-1)} \\ n(k) &\simeq \varrho^{(j-2)/(j-1)} \omega_k^{(j-4)/(j-1)} P^{1/(j-1)} k^{(10-d-5j)/(j-1)} \end{aligned} \quad . \tag{3.1.4}$$

Separately for the three-wave processes the nonequilibrium distribution $n(k)$ reads

$$n(k) \simeq (P\varrho)^{1/2} \omega_k^{-1/2} k^{-(5+d)/2} \ . \tag{3.1.5}$$

In the case of a four-wave interaction, apart from the distribution with energy flux derived from (3.1.4),

$$n(k) \simeq (P\varrho^2)^{1/3} k^{-(10+d)/3} \ , \tag{3.1.6a}$$

a solution describing the constant flux of wave action Q may exist [in spheric normalization: $Q(k) = (2k)^{d-1}\pi q(k)$]. It is evident from dimensional analysis that such a distribution may be obtained by substituting in (3.1.6a) for P the quantities $\omega(k)Q$

$$n(k) \simeq (Q\varrho^2)^{1/3} \omega_k^{1/3} k^{-(10+d)/3} \ . \tag{3.1.6b}$$

Let us discuss the somewhat more general situation when the dispersion law and the interaction coefficients are scale-invariant:

$$\begin{aligned} &\omega(\lambda k) = \lambda^\alpha \omega(k), \quad V(\lambda \boldsymbol{k}, \lambda \boldsymbol{k}_1, \lambda \boldsymbol{k}_2) = \lambda^m V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) \ , \\ &T(\lambda \boldsymbol{k}, \lambda \boldsymbol{k}_1, \lambda \boldsymbol{k}_2, \lambda \boldsymbol{k}_3) = \lambda^m T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) \ . \end{aligned} \tag{3.1.7a}$$

As shown in the previous chapter, this is possible not only in the case of complete self-similarity (without a length parameter), but also when a parameter with the dimension of a length is present, i.e., in the case of second-order self-similarity.

It follows from (3.1.7) that $\omega(k)$ is a power function

$$\omega(k) = \beta k^\alpha \tag{3.1.7b}$$

and the interaction coefficients may be written as

$$\begin{aligned} |V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)|^2 &= V_0^2 k^{2m} f_1(\boldsymbol{k}_1/k, \boldsymbol{k}_2/k) \ , \\ |T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)|^2 &= T_0^2 k^{2m} f_2(\boldsymbol{k}_1/k, \boldsymbol{k}_2/k, \boldsymbol{k}_3/k) \ . \end{aligned} \tag{3.1.7c}$$

Here V_0, T_0 are the dimensional constants and f_1, f_2 are dimensionless functions.

The exponent s_0 of the Kolmogorov stationary solution

$$n(k) = Ak^{-s_0}$$

with a constant energy flux is found directly from (3.1.3). For example, for the three-wave case

$$P(k) = \pi \int_0^k \omega(k) I(k) (2k)^{d-1}\, dk$$
$$= k^{2(m+d-s_0)} V_0^2 A^2 a(m,d,\alpha,s_0) \ . \qquad (3.1.8a)$$

Here $a(m,d,\alpha,s_0)$ is the dimensionless integral defined below, see (3.1.13). Demanding that the flux be constant $P(k) = P$, we obtain the stationary distribution

$$n(k) = (P/aV_0^2)^{1/2} k^{-s_0}, \quad s_0 = m + d \ . \qquad (3.1.9)$$

Likewise we get for the four-wave kinetic equation

$$n(k) = (P/aT_0^2)^{1/3} k^{-s_0}, \quad s_0 = 2m/3 + d \ . \qquad (3.1.10a)$$

It should be pointed out that the indices of the Kolmogorov distribution that transports the energy flux are independent of the index of the dispersion law. On the contrary, the equilibrium distributions, as shown in Sect. 2.2, depend neither on the interaction coefficients nor on the space dimension and are entirely determined by the wave dispersion law.

We have already encountered the $(m + d)$ and $(2m/3 + d)$ power indices in the previous section when we discussed the conditions for energy conservation of wave systems. As we have seen, if $n(k) \to k^{-s}$ at $k \to \infty$ and $s > m + d$ (or $2m/3 + d$ in the four-wave case), then the flux $P \to 0$ at $k \to \infty$. The distributions (3.1.9–10a) with $s = s_0$ correspond to a constant energy flux in k-space. The direction of the flux is determined by the sign of the integral $a(m,d,\alpha)$ — see below (3.1.13).

Similarly to (3.1.10a), from the stationary kinetic equation $\partial Q(k)/\partial k = (2k)^{d-1}\pi I(k)$ one can obtain the stationary distribution with the flux of wave action:

$$n(k) = (\beta Q/aT_0^2)^{1/3} k^{-x_0}, \quad x_0 = \frac{2m}{3} + d - \frac{\alpha}{3} \ . \qquad (3.1.10b)$$

In the case of complete self-similarity, the interaction coefficients may be expressed in terms of $\varrho, \omega(k), k$:

$$V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) = \varrho^{-1/2} \omega_k^{1/2} k^{(5-d)/2} f_1(\boldsymbol{k}_1/k, \boldsymbol{k}_2/k),$$

$$T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \varrho^{-1} k^{5-d} f_2(\boldsymbol{k}_1/k, \boldsymbol{k}_2/k, \boldsymbol{k}_3/k) \ .$$

The distributions (3.1.9, 10) will go over to (3.1.5, 6), respectively.

3.1.2 Exact Stationary Solutions of the Three-Wave Kinetic Equation

Let us verify directly that (3.1.9, 10) are the solutions of the stationary kinetic equation (2.2.23) and determine the constant $a(m,d,\alpha)$. Integrating (2.2.23) from zero to k and taking into account the boundary condition (the presence of the source $\Gamma(k)$ at $k_0 \ll k$), we obtain (3.1.8a) which we here represent as:

$$\pi \int_0^k (2k)^{d-1} \omega(k) \Gamma(k) n(k)\, dk = P = -\pi \int_0^k \omega(k) I(k) (2k)^{d-1}\, dk\,. \qquad (3.1.8b)$$

In this section we shall consider the case of the three-wave collision integral $I(k)$ (2.1.12). If a medium is isotropic, the interaction coefficient V_{k12} is invariant with regard to rotations in $\boldsymbol{k}$-space and, consequently, like the frequency $\omega(k)$, may be considered to be a function of only the absolute values of wave vectors $\boldsymbol{k}, \boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Since the Kolmogorov solution (3.1.9) is also isotropic, only the δ-function of wave vectors is to be integrated over the angles in the $\boldsymbol{k}_1, \boldsymbol{k}_2$ spaces. This may be conveniently done by a direct angular integration. For $d = 2$

$$\int \delta(k - k_1 \cos\theta_1 - k_2 \cos\theta_2)\delta(k_1 \sin\theta_1 + k_2 \sin\theta_2)\, d\theta_1 d\theta_2$$
$$= (k k_1 \sin\theta_1^0)^{-1} = \Delta_2^{-1}\,,$$

where Δ_2 is the area of the triangle formed by the vectors k, k_1, k_2:

$$\Delta_2 = (1/2)\left[2\left(k^2 k_1^2 + k^2 k_2^2 + k_1^2 k_2^2\right) - k^4 - k_1^4 - k_2^4\right]^{1/2}.$$

In the same way, for $d = 3$ introducing the spherical coordinates (k, θ, φ), we obtain

$$\int \delta(k - k_1 \cos\theta_1 - k_2 \cos\theta_2)\delta(k_1 \sin\theta_1 \sin\varphi_1 + k_2 \sin\theta_2 \varphi_2)$$
$$\times\, \delta(k_1 \sin\theta_1 \cos\varphi_1 + k_2 \sin\theta_2 \cos\varphi_2)\, d\cos\theta_1 d\varphi_1 d\cos\theta_2 d\varphi_2$$
$$= \int \delta(k - k_1 \cos\theta_1 - k_2 \cos\theta_2)\delta(k_1 \sin\theta_1 + k_2 \sin\theta_2)\frac{\sin\theta_1\, d\theta_1 d\theta_2}{2k_2}$$
$$= \frac{1}{2k k_1 k_2} = \frac{\Delta_3^{-1}}{2}\,.$$

The functions $\Delta_d(k, k_1, k_2)$ are independent of the signs of the terms in the arguments of the δ-function, are invariant relative to rearrangements of its arguments and nonzero if the vectors with lengths k, k_1, k_2 can be used to form a triangle. Let us now go over from the integration over the absolute values of k_1, k_2 to the integration over the frequencies $\omega(k_1) = \omega_1$, $\omega(k_2) = \omega_2$

$$\pi (2k)^{d-1} \frac{I(k)}{v(k)} = I(\omega) = \int_0^\infty \int_0^\infty d\omega_1 d\omega_2 \Big[(R(\omega, \omega_1, \omega_2) \qquad (3.1.11a)$$
$$- R(\omega_1, \omega, \omega_2) - R(\omega_2, \omega, \omega_1)\Big]\,.$$

Here

$$
\begin{aligned}
R(\omega,\omega_1,\omega_2) = \beta^{-2}|V(k,1,2)|^2 2^{d-1}\Delta_d^{-1}(\omega\omega_1\omega_2)^{-1+d/\alpha} \\
\times\, \delta(\omega-\omega_1-\omega_2)\Theta(\omega-\omega_2)\Theta(\omega-\omega_1) \\
\times\, [n_1 n_2 - n(\omega)(n_1+n_2)] \\
v(k) = d\omega(k)/dk, \quad n_j = n(\omega_j), \quad \omega_j = \omega(k_j)\ .
\end{aligned}
\qquad (3.1.11b)
$$

The expression for the flux (3.1.8b) may also be rewritten as a frequency integral

$$
\begin{aligned}
P(\omega) &= -\int_0^{\omega} \omega' I(\omega')\, d\omega' \\
&= \int_0^{\omega} \omega'\, d\omega' \int_0^{\infty}\int_0^{\infty} d\omega_1 d\omega_2 [R(\omega',\omega_1,\omega_2) - R(\omega_1,\omega',\omega_2) - R(\omega_2,\omega',\omega_1)]\ .
\end{aligned}
$$

After rearrangements in the second and third terms $\omega_1 \leftrightarrow \omega'$ and $\omega_2 \leftrightarrow \omega'$, we obtain

$$
\begin{aligned}
P(\omega) = \int_0^{\omega} \omega'\, d\omega' \int_0^{\infty} W(\omega',\omega_1)\big[n(\omega')n(\omega_1) \\
- n(\omega')n(\omega'-\omega_1) - n(\omega')n(\omega'+\omega_1)\big]\, d\omega_1\ .
\end{aligned}
\qquad (3.1.11c)
$$

The form of the homogeneous function W is specified by comparison with (3.1.11a)

$$
W(\omega_1,\omega_2) = \omega_2^{\gamma} f(\omega_1/\omega_2)\,, \quad \gamma = \frac{2m+2d}{\alpha} - 3\ .
$$

Indeed, regarding f_1 (3.1.7c) as a function of ω_1/ω we obtain, e.g., for $d = 3$

$$
f(x) = [x(1-x)]^{(d-1-\alpha)/\alpha} f_1(x, 1-x)\ .
$$

So all the information about the interaction is in fact contained in one number m and in one dimensionless function $f(x)$ of a single variable, $f(x)$ is a structural function expressed via f_1. To avoid possible misunderstandings we should point out that $n(\omega)$ in (3.1.11) and below denotes the wave density in the space of wave numbers (but not frequencies). It is taken to be a function of ω, namely $n(\omega) = n[\omega(k)] = n(k)$.

Thus, we have to solve the nonlinear integral equation

$$
P(\omega) = \text{const} = P\ .
$$

At $P = 0$ the equation has an equilibrium solution $n(\omega) = T/\omega$. The general stationary solution may depend both on P and T (see Sect. 4.1.2 below). Let us derive a turbulent solution for $P \neq 0$ and $T = 0$. Indeed, the existence of a damping region is necessary for the turbulence to be steady. According to the Kolmogorov concept, the expression for the spectrum does not depend on the details of damping, while in the presence of damping it is necessary to set $T = 0$, since no thermal reservoir can exist in this case. Of course, the temperature of the turbulent fluid is not zero, otherwise the fluid would be frozen. One should neglect T in some frequency range (where the turbulent distribution exists). Indeed, according to the very definition of turbulence, the level of excitation is supposed to be much higher than the equilibrium level so that the latter is negligible. In Sect. 3.1.2 we shall elaborate on the relation between P and T in turbulence.

Locality of the Interaction. Let us first discuss the convergence of the collision integral (3.1.11a) for power distributions $n(\omega_k) = k^{-s} = \omega^{-s/\alpha}$. We have a sum of integrals over power functions, obviously each of them diverges either at $\omega_1 \to 0$ or at $\omega_1 \to \infty$. However, the structure of the kinetic equation ensures the reduction of these divergences by a power of ω_1 at $\omega_1 \to \infty$

$$n(\omega_1 - \omega) - n(\omega_1) \approx \omega \frac{\partial n(\omega_1)}{\partial \omega_1} \propto n(\omega_1) \frac{\omega}{\omega_1}$$

and by two powers (i.e., by ω_1^2) at $\omega_1 \to 0$.

Indeed, let at $x \to 0$, the structural function (3.1.7c) be $f_1(x, 1-x) \equiv f_1(x) \to x^{m_1}$. This implies that at $k_1 \ll k$ the interaction coefficient behaves like

$$|V(k, k_1, k_2)|^2 \propto k_1^{m_1} k^{2m-m_1} .$$

Then the second and third terms in (3.1.11a) converge at $\omega_1 \to \infty$ if the condition

$$s > s_2 = 2m - m_1 + d + 1 - 2\alpha \qquad (3.1.12a)$$

is satisfied. Such a reduction takes place at $\omega_1 \to 0$

$$n(\omega - \omega_1) - n(\omega) \approx \omega_1 \frac{\partial n(\omega)}{\partial \omega} \propto n(\omega) \frac{\omega_1}{\omega} .$$

The divergences at $\omega_1 \to 0$ are also mutually cancelled with divergences at $\omega_1 \to \omega$ [here one should take into account that, owing to the symmetry of the matrix elements $V_{k12} = V_{k21}$, the structural function $f_1(k_1/k, k_2/k) = f_1(x, y)$ satisfies the condition $f_1(x^\alpha, 1 - x^\alpha) = f_1(1 - x^\alpha, x^\alpha)$ on the resonance surface $\omega(k_1) + \omega(k_2) = \omega(k)$ (that is, $x^\alpha + y^\alpha = 1$)]. As a result, an additional factor ω_1/ω arises and the convergence criterion of the collision integral at small frequencies is

$$s < s_1 = m_1 + d - 1 + 2\alpha . \qquad (3.1.12b)$$

Thus, if $s_1 > s_2$, i.e.,

$$2m_1 > 2m + 2 - 4\alpha , \qquad (3.1.12c)$$

then there exists an interval of s exponents ensuring locality of the interaction. It is important to note that the Kolmogorov exponent $s_0 = m + d$ always lies exactly in the middle of the "locality interval": $s_0 = (s_1 + s_2)/2$ [3.5]. This is due to the fact that for the Kolmogorov distribution, the contributions to interactions of all scales, from small to large ones, level out implying the "balance of interactions". The main contribution to the collision integral $I(\omega)$ comes from the frequency range $\omega_1 \simeq \omega$, indicating locality of the interaction. Thus, the collision integral over the Kolmogorov spectrum either converges at $\omega_1 \to 0$ and $\omega_1 \to \infty$ or diverges at both limits.

It should be pointed out that we demanded convergence of the collision integral only for isotropic distributions, in particular, on the Kolmogorov solution itself. That property will be referred to as stationary locality. For stationary locality the kinetic equation has a stationary Kolmogorov solution. One can, however, raise the question about arbitrary (especially anisotropic) perturbations of the Kolmogorov spectrum. In Sects. 4.1, 2 we shall see that there are cases when the collision integral describing the behavior of weak anisotropic perturbations against the background of the stationary solution has no locality interval at all. It is also possible that the evolution of perturbations of Kolmogorov distributions is determined by interaction with the ends of the inertial interval (for details see Sect. 4.2). It would be natural to call that property "evolution nonlocality". Unless stated otherwise, interaction locality will be used below in the sense of stationary locality. It should be noted, that realization of equilibrium spectra does not presuppose interaction locality.

Signs of Fluxes. Let us now substitute into (3.1.11) $n(\omega) = A\omega^{-s/\alpha}$. Under the locality condition the collision integral over the power distribution is also a power function

$$I(\omega) = \omega^{\sigma-2}\left(V_0 A\right)^2 I(s), \quad \sigma = \frac{2(m+d-s)}{\alpha},$$

where $I(s)$ is a dimensionless integral. Hence, the expression for the flux is

$$P = \omega^{\sigma}\left(V_0 A\right)^2 \frac{I(s)}{\sigma}. \tag{3.1.13a}$$

As we see, at $s = s_0 = m + d$ the expression (3.1.13a) contains an indeterminacy of the form 0/0, as the collision integral over the Kolmogorov solution should vanish, $I(m+d) = 0$, see below (3.1.14c). Evaluating the indeterminacy using L'Hospital's rule, we obtain an expression where the energy flux is proportional to the derivative of the collision integral with respect to the index of the solution [3.6, 7]. From (3.1.11a, b) we get

$$\begin{aligned} P &= \left(V_0 A\right)^2 a(m,d,\alpha) = \left(V_0 A\right)^2 \left(\frac{dI(s)}{ds}\right)_{s_0} \\ &= \left(V_0 A\right)^2 \int_0^\infty \ln(1-x)[x(1+x)]^{-s_0/\alpha} f_1(x) \\ &\qquad \times \left[(1+x)^{s_0/\alpha} - x^{s_0/\alpha} - 1\right] dx \, . \end{aligned} \tag{3.1.13b}$$

Such an expression can be obtained from (3.1.11c) by changing the order of integration. The sign of P is determined by the last square bracket which is positive if $s_0/\alpha > 1$; as mentioned earlier, the energy flux is positive, i.e., directed towards large k if the Kolmogorov distribution decays for growing k more rapidly than the equilibrium one. In the other case with $s_0/\alpha < 1$, the Kolmogorov solution does not exist since it would correspond to imaginary value of the constant A.

The interaction locality ensures the convergence of the integral in (3.1.13b), i.e., the finiteness of the flux. Thus, the power solution (3.1.9) corresponds to a constant energy flux and owing to (2.3.1) should be a solution of the kinetic equation in the inertial interval.

Zakharov Transformations. Using the conformal transformations suggested by *Zakharov* [3.8] can be directly verified that the collision integral becomes zero on the Kolmogorov distribution. The method of conformal transformations allows to obtain all power solutions of the stationary kinetic equation. Let us substitute into (3.1.11a) the distribution $n(\omega) = \omega^{-\nu}$. Then we rearrange the second term

$$\omega_1 = \omega\frac{\omega}{\omega_1'}, \qquad \omega_2 = \omega_2'\frac{\omega}{\omega_1'} \tag{3.1.14a}$$

using the factor ω_1'/ω and perform a similar manipulation with the third term (with substitution $\omega_1 \leftrightarrow \omega_2$):

$$\omega_1 = \omega_1'\frac{\omega}{\omega_2'}, \qquad \omega_2 = \omega\frac{\omega}{\omega_2'} . \tag{3.1.14b}$$

In (3.1.14) the integration limits 0 and ∞ change places, therefore these relations are only valid for converging integrals, i.e., in the case of locality. After those rearrangements the collision integral is factorized and acquires the simple form

$$\begin{aligned} I(\omega) &= \int\int_0^\infty d\omega_1 d\omega_2 \left[1 - (\omega/\omega_1)^x - (\omega/\omega_2)^x\right] R(\omega,\omega_1,\omega_2) \\ &= \int_0^\infty \left[1 - \left(\frac{\omega}{\omega_1}\right)^x - \left(\frac{\omega}{\omega-\omega_1}\right)^x\right] \\ &\quad \times \left[1 - \left(\frac{\omega}{\omega_1}\right)^\nu - \left(\frac{\omega}{\omega-\omega_1}\right)^\nu\right] \\ &\quad \times \omega_1^\nu(\omega-\omega_1)^\nu W(\omega_1,\omega-\omega_1)\,d\omega_1, \quad x = 2\frac{m+d}{\alpha} - 2\nu - 1 , \end{aligned} \tag{3.1.14c}$$

where W is a positive function. As we see, the integrals vanish at $\nu = 1$ and $\nu = (m+d)/\alpha$, for all other ν they retain their signs. Thus the Rayleigh-Jeans and Kolmogorov distributions are the only universal stationary power solutions of the kinetic equation (2.1.12). Each of them is a one-parameter solution. The question with regard to general solutions of the stationary kinetic equation which depend on several parameters is still open. Partly, this problem will be discussed in Sect. 4.1 and for the particular case of acoustic turbulence in Sect. 5.1.

The expression (3.1.13b) for the flux can be obtained as a derivative of the collision integral (3.1.14c).

Examples. Let us now consider particular examples of wave systems in isotropic media with decay (power) dispersion laws. There are two such examples among the wave systems mentioned in Chap. 1: capillary waves on shallow (1.1.39) and on deep (1.1.40) water. In both examples we have $d = 2$.

Let us start with the shallow-water case. The dispersion law (1.1.39a) is quadratic: $\alpha = 2$. The interaction coefficient has an extremely simple form (1.1.39b), $m = 2$, $f_1(x) = 1$, $m_1 = 0$. The Kolmogorov index is $s_0 = m + d = 4$, $n(k) \propto k^{-4}$ and the locality conditions (3.1.12) are satisfied. The dimensionless constant a equals the integral from (3.1.13b) which here has the form

$$a = \int_0^\infty x^{-3/2} \ln(1+x)\, dx = 2\pi \ .$$

Thus, the Kolmogorov solution (3.1.9) for shallow-water waves is:

$$n(k) = P^{1/2} 8\sqrt{\pi} \left(\frac{\varrho h}{\sigma}\right)^{1/4} k^{-4} \ . \qquad (3.1.15a)$$

It was first obtained by *Kats* and *Kontorovich* [3.9].

For waves on the surface of deep water $\alpha = 3/2, m = 9/4$. Taking into account

$$\lim_{k_1 \to 0} |V(k, k_1, k_2)|^2 \propto k_1^{m_1} k^{2m - m_1} \ ,$$

(3.1.13b) yields $m_1 = 7/2$ and the locality conditions (3.1.12) are also satisfied. The index of this solution is $s_0 = 17/4$. In this case one should insert the function

$$\begin{aligned} f_1(x) = {} & [x(1-x)]^{2/3} \{(1 - x^{2/3})^2 (1-x)^{-1/3} \\ & + [1 - (1-x)^{2/3}]^2 x^{-1/3} - [x^{2/3} - (1-x)^{2/3}]^2\}^2 \\ & \times \{4x^{4/3}(1-x)^{4/3} - [1 - x^{4/3} - (1-x)^{4/3}]^2\}^{-1/2} \end{aligned}$$

into the integral (3.1.13b) specifying a. The respective Kolmogorov solution was obtained by *Zakharov* and *Filonenko* [3.10]:

$$n(k) = \left(\frac{P}{a}\right)^{1/2} 8\pi \left(\frac{4\varrho^3}{\sigma}\right)^{1/4} k^{-17/4} \ . \qquad (3.1.15b)$$

From dimensional analysis it follows that the Kolmogorov distribution for capillary waves on the surface of an arbitrary-depth fluid may be written as

$$n(k) = P^{1/2} \left[\frac{k}{\omega(k)}\right]^{1/2} k^{-4} \phi(kh) \qquad (3.1.15c)$$

where $\phi(x)$ is a dimensionless function. At $x \to 0$, $\phi \to x^{1/2}$ and (3.1.15c) goes over to (3.1.15a); at $x \to \infty$, $\phi \to$const and (3.1.15c) goes over to (3.1.15b).

3.1.3 Exact Stationary Solutions for the Four-Wave Kinetic Equations

Let us now consider the kinetic equation (2.1.29) and assuming the distribution to be isotropic $n(\boldsymbol{k}) = n[\omega(k)] = n(\omega)$ average in it over the angles to get

$$\begin{aligned}\frac{\partial N(\omega,t)}{\partial t} = \int\int_0^\infty\int & U(\omega,\omega_1,\omega_2,\omega_3)[n(\omega_1)n(\omega_2)n(\omega_3) \\ & + n(\omega)n(\omega_2)n(\omega_3) - n(\omega)n(\omega_1)n(\omega_2) \\ & - n(\omega)n(\omega_1)n(\omega_3)]\delta(\omega+\omega_1-\omega_2-\omega_3) \\ & \times d\omega d\omega_1 d\omega_2 d\omega_3 \\ = & \; I(\omega)\ .\end{aligned} \tag{3.1.16}$$

Here $N(\omega)$ is the wave density in the frequency space

$$N(\omega) = (2k)^{d-1}\pi\left(\frac{dk}{d\omega}\right)n(\omega)\ ,$$

and $U(\omega,\omega_1,\omega_2,\omega_3)$ is the result of angle averaging of the function

$$2^d\pi^2|T_{k123}|^2\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)(kk_1k_2k_3)^{d-1}v_kv_1v_2v_3,\quad v_i = \frac{d\omega(k_i)}{dk_i}\ .$$

In spite of the isotropy of the medium, the coefficient of the four-wave interaction $T_{k123} = T(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$ cannot be reduced to a function of the wave numbers only (as in the three-wave case). This is due to the fact that the triangle is uniquely specified by the lengths of its sides, whereas the quadrangle is not.

As the constant in the dispersion law may be eliminated by a simple rearrangement (1.4.22) we shall use below $\omega(0) = 0$.

Exact Solutions. Let us find the stationary power-solution $n(\omega)$ of the kinetic equation (3.1.16). Integration over $d\omega_1$ gives

$$\begin{aligned}I(\omega) = \int_\Omega & d\omega_2 d\omega_3 U(\omega,\omega_2+\omega_3-\omega,\omega_2,\omega_3) \\ & \times n(\omega_2)n(\omega_2+\omega_3-\omega)n(\omega)n(\omega_3) \\ & \times\left[n^{-1}(\omega)+n^{-1}(\omega_2+\omega_2-\omega)-n^{-1}(\omega_2)-n^{-1}(\omega_3)\right]\ .\end{aligned} \tag{3.1.17}$$

In (3.1.17) we integrate over the (shaded) region Ω in Fig. 3.1. It has a form of a square $\omega_2 > 0$, $\omega_3 > 0$ without its lower left-side angle. This region is divided into four subregions: in the one designated by 2 in Fig. 3.1 ($\omega_2, \omega_3 > \omega$) we substitute the variables as follows

$$\omega_2 = \frac{\omega\omega_2'}{\omega_2'+\omega_3'-\omega},\quad \omega_3 = \frac{\omega\omega_3'}{\omega_2'+\omega_3'-\omega}\ , \tag{3.1.18a}$$

and in subregions 3, 4

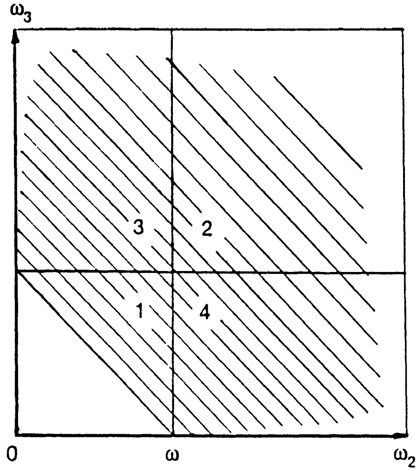

Fig. 3.1. The integration region for the integral given by (3.1.17)

$$\omega_2 = \frac{\omega^2}{\omega_2'}, \quad \omega_3 = \frac{\omega\omega_3'}{\omega_2'}, \qquad \omega_3 = \frac{\omega^2}{\omega_3'}, \quad \omega_2 = \frac{\omega\omega_2'}{\omega_3'}. \tag{3.1.18b}$$

In these transformations the subregions 2, 3, 4 will go over to subregion 1.

The function $U(\omega, \omega_1, \omega_2, \omega_3)$ obeys the same symmetry conditions as T_{k123}

$$U(\omega, \omega_1, \omega_2, \omega_3) = U(\omega_1, \omega, \omega_2, \omega_3) = U(\omega, \omega_1, \omega_3, \omega_2) = U(\omega_2, \omega_3, \omega, \omega_1).$$

Besides, U is a homogeneous function, i.e.,

$$U(\lambda\omega, \lambda\omega_1, \lambda\omega_2, \lambda\omega_3) = \lambda^{\gamma} U(\omega, \omega_1, \omega_2, \omega_3), \quad \gamma = \frac{2m + 3d}{\alpha} - 4$$

that we can write in the form

$$U(\omega, \omega_1, \omega_2, \omega_3) = \omega^{\gamma} f_3(\omega_1/\omega, \omega_2/\omega, \omega_3/\omega)$$

where f_3 is a dimensionless function of three variables. Therefore the Zakharov transformation (3.1.18) will convert (3.1.17) into

$$\begin{aligned} I(\omega) = \int_1 = \int_0^{\omega} d\omega_3 \int_{\omega-\omega_3}^{\omega} & d\omega_2 U(\omega, \omega_2 + \omega_3 - \omega, \omega_2, \omega_3) \\ & \times \left[\omega^x + (\omega_2 + \omega_3 - \omega)^x - \omega_2^x - \omega_3^x)\right] \\ & \times [\omega(\omega_2 + \omega_3 - \omega)\omega_2\omega_3]^{-x} \\ & \times \left[1 + \left(\frac{\omega_2 + \omega_3 - \omega}{\omega}\right)^y - \left(\frac{\omega_2}{\omega}\right)^y - \left(\frac{\omega_3}{\omega}\right)^y\right] = 0 \end{aligned} \tag{3.1.19}$$

with $y = 3x - 3 - \gamma$. The integrand in (3.1.19) becomes zero at the four points

$$x = 0, \quad x = 1, \quad 3x - 3 - \gamma = 0, \quad 3x - 3 - \gamma = 1$$

and due to positiveness of the function U it retains its sign at all other x.. Thus (3.1.17) has four universal power-solutions

$$n(\omega) = C_1/\omega, \qquad n(\omega) = C_2\,; \tag{3.1.20a, b}$$

$$n(\omega) = C_3\omega^{-(3+\gamma)/3}, \qquad n(\omega) = C_4\omega^{-(4+\gamma)/3}\,. \tag{3.1.21a, b}$$

The solutions (3.1.20) correspond to thermodynamical equilibrium. They are the limiting cases of solutions (2.2.14) with $u = 0$ at $\mu \to 0$ and $\mu \to \infty$. The solutions (3.1.21) are the Kolmogorov solutions, rewriting the exponents, they coincide with (3.1.10): function (3.1.21a)

$$n(\omega) \propto \omega^{-x_Q}, \qquad x_Q = \frac{2m + 3d}{3\alpha} - \frac{1}{3}$$

corresponds to the flux of the action spectrum and (3.1.21b)

$$n(\omega) \propto \omega^{-x_P}, \qquad x_P = x_Q + \frac{1}{3} = \frac{2m + 3d}{3\alpha}$$

to the flux of the energy spectrum.

The Kolmogorov solution holds for local turbulence, i.e., the integrals in (3.1.19) should converge. This is easy to check directly in every particular case.

Signs of the Fluxes. Let us now discuss the directions the transmitted fluxes have in the ω-space. It is clear that [similarly to the three-wave case (3.1.8, 13)] the fluxes will be proportional to the derivatives of the collision integral with respect to the index of solutions x taken at the respective $x = x_Q$ or $x = x_P$

$$\begin{aligned} Q = \int_0^1 dy_2 \int_{1-y_2}^1 & dy_3 f_3(y_2, y_3)(y_2 + y_3 - 1)^{-x_Q}(y_2 y_3)^{-x_Q} \\ & \times \ln\left(\frac{y_2 + y_3 - 1}{y_2 y_3}\right)\left[1 + (y_2 + y_3 - 1)^{x_Q} - u_2^{x_Q} - y_3^{x_Q}\right], \end{aligned} \tag{3.1.22a}$$

$$\begin{aligned} P = \int_0^1 dy_2 \int_{1-y_2}^1 & dy_3 f_3(y_2, y_3)(y_2 + y_3 - 1)^{-x_P}(y_2 y_3)^{-x_P} \\ & \times [(y_2 + y_3 - 1)\ln(y_2 + y_3 - 1) - y_2 \ln y_2 - y_3 \ln y_3] \\ & \times \left[1 + (y_2 + y_3 - 1)^{x_Q} - u_2^{x_Q} - y_3^{x_Q}\right], \end{aligned} \tag{3.1.22b}$$

Thus, the index m and the dimensionless function $f(y_2, y_3)$ defined on the triangle $y_2, y_3 < 1$, $y_2 + y_3 > 1$ exhaust in this case all the information about the interaction. Here $y_2 = \omega_2/\omega$ and $y_3 = \omega_3/\omega$. Since U and f_3 do not change sign, it is sufficient to analyze the behavior of the last square bracket in that expression because

$$\ln\left(\frac{y_2+y_3-1}{y_2y_3}\right)<0 \qquad \text{and}$$
$$[(y_2+y_3-1)\ln(y_2+y_3-1)-y_2\ln y_2-y_3\ln y_3]>0$$

at $y_2, y_3 < 1$, $y_2+y_3 > 1$. Therefore, it is necessary to consider

$$\phi(x,y_2,y_3)=1+(y_2+y_3-1)^x-y_2^x-y_3^x \ .$$

It is easy to establish sign ϕ = sign $x(x-1)$. Thus (see also [3.7]),

$$\operatorname{sign} Q=\operatorname{sign}\left(\frac{\partial\phi}{\partial x}\right)_{x=x_Q}=-\operatorname{sign}(x_Q-1)x_Q \ , \tag{3.1.22c}$$

$$\operatorname{sign} P=\operatorname{sign}\left(\frac{\partial\phi}{\partial x}\right)_{x=x_P}=\operatorname{sign}(x_P-1)x_P \ . \tag{3.1.22d}$$

Most of the physical examples known to us, including those given in Chap. 1, correspond to Kolmogorov spectra decaying with the growth of ω more rapidly than the equilibrium spectrum: $x_Q > 1$. In terms of the initial indices the latter condition is written as $2m+3d > 4\alpha$. The only exception is given by the two-dimensional Schrödinger equation which corresponds to $m = 0$, $d = 2$, $\alpha = 2$ (see Sect. 5.3.2 below).

Equation (3.1.16) may be cast into the divergence form of a continuity equation for $n(\omega)$:

$$\frac{\partial N(\omega,t)}{\partial t}=\frac{\partial^2 K(\omega,t)}{\partial\omega^2} \ , \tag{3.1.23}$$

where

$$\begin{aligned}K(\omega,t)=\int&(\omega+\omega_1-\omega_2-\omega_3)U(\omega_2+\omega_3-\omega_1,\omega_1,\omega_2,\omega_3)\\&\times\left\{\left[n(\omega_2+\omega_3-\omega_1)+n(\omega_1)\right]n(\omega_3)n(\omega_2)-n(\omega_1)\right.\\&\times\left.n(\omega_2+\omega_3-\omega_1)\left[n(\omega_2)+n(\omega_3)\right]\right\}d\omega_1d\omega_2d\omega_3 \ .\end{aligned} \tag{3.1.24}$$

The integration in (3.1.24) is performed over the range limited by the inequalities

$$\omega_1>0, \quad \omega_2>0, \quad \omega_3>0, \qquad \omega_1<\omega_2+\omega_3<\omega+\omega_1 \ .$$

It follows from (3.1.23) that the wave action flux is $Q(\omega,t) = -\partial K(\omega,t)/\partial\omega$. Because of energy conservation, (3.1.16) may also be written in the form

$$\frac{\partial E(\omega,t)}{\partial t}=\omega\frac{\partial N(\omega.t)}{\partial t}=-\frac{\partial}{\partial\omega}\left[K(\omega,t)-\omega\frac{\partial K(\omega,t)}{\partial\omega}\right]$$

[differing from (3.1.23)] which is a continuity equation for $E(\omega)$. For the energy flux $P(\omega,t)$ we have then

$$P(\omega,t)=K(\omega,t)-\omega\frac{\partial K(\omega,t)}{\partial\omega}=K(\omega,t)+\omega Q(\omega,t) \ .$$

A general isotropic stationary solution of (2.3.23) may depend on the four parameters T, μ, P, Q (see Sect. 4.1.2 below). For an equilibrium system one should set $P = Q = 0$ which corresponds to $K \equiv 0$. The general turbulent solution of (3.1.23) corresponds to $T = \mu = 0$ and to the constant flux

$$K(\omega) = P - \omega Q \ . \tag{3.1.25}$$

The physical meaning of this solution is that at $\omega = 0$ there is an energy source of intensity P and at $\omega = \infty$ a wave action source of intensity $-Q$. The fluxes of these quantities flowing into opposite directions: $P > 0$, $Q < 0$. If the intensity of one of the sources is zero, then the solution (3.1.25) is of the power type in the whole space and coincides with one of the solutions of (3.1.21).

In a real situation, the existence of a stationary distribution requires the presence of damping regions both at large and small ω (see Sect. 2.2), even if there is only one source. Let us consider, for example, the situation schematically indicated in Fig. 3.2: at $\omega = \omega_2$ there is a wave source generating N_2 waves per unit time and at $\omega = \omega_1$ and $\omega = \omega_3$ there are two sinks absorbing N_1 and N_3 waves.

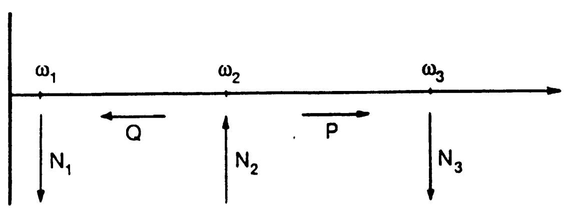

Fig. 3.2. A system with one source and two sinks; the directions of the fluxes are indicated by the arrows

By virtue of conservation of energy and of the total number of waves, the equations

$$N_2 = N_1 + N_3, \qquad \omega_2 N_2 = \omega_1 N_1 + \omega_3 N_3 \tag{3.1.26}$$

should be satisfied in the steady state. From this we easily obtain

$$N_1 = N_2 \frac{\omega_3 - \omega_2}{\omega_3 - \omega_1}, \qquad N_3 = N_2 \frac{\omega_2 - \omega_1}{\omega_3 - \omega_1} \ .$$

It is seen that at a sufficiently large left inertial interval (i.e., at $\omega_1 \ll \omega_2 < \omega_3$), the whole energy is absorbed by the right sink: $\omega_2 N_2 \approx \omega_3 N_3$. Similarly, at $\omega_3 \gg \omega_2 > \omega_1$, we have $N_1 \approx N_2$, i.e., the wave action is absorbed at small ω. At $\omega_1 \ll \omega_2 \ll \omega_3$, the solution (3.1.10a) should serve as an intermediate asymptotic at $\omega_2 \ll \omega \ll \omega_3$, and the solution (3.1.10b), at $\omega_1 \ll \omega \ll \omega_2$. This simple deduction shows that energy and action fluxes move into opposite directions: $P > 0$ and $Q < 0$. The distributions with $Q > 0$ ($x_Q < 1$) or with $P < 0$ ($x_P < 1$), which are the formal solutions of (2.1.29) seem to exclude matching with boundary conditions (i.e., sources and sinks). This may be proved with the help of the following simple argument suggested by *Fournier* and *Frish* [3.11]. Let us consider, for example, formation of the distribution on the right-hand-side of the source located at $k = k_0$ (see Fig. 3.3 adopted from [3.12]). We assume that in an interval from the source to some k_1 a stationary spectrum has been established. Then, the following condition has to be satisfied to allow the stationary spectrum $n_k = Ak^{-s_0}$ to extend still further into the region of large k:

If the distribution decays for growing k at $k > k_1$ faster than the stationary distribution then the occupation numbers $n(k,t)$ should increase with time. Conversely, for the less steep distributions the derivative of the occupation numbers with respect to time should be negative. Thus we come to the condition that $\forall k_2\ (k_2 > k_1)$

$$\operatorname{sign}\left[\frac{\partial n(k_2,t)}{\partial t}\right] = \operatorname{sign}(s - s_0)\ .$$

Since k_2 is located in the inertial interval, the evolution of the occupation numbers is determined only by the interaction of waves with each other, i.e., by the collision integral

$$\frac{\partial n(k_2,t)}{\partial t} = I(k_2,t)\ .$$

It is readily seen from the last two equations that formation of the stationary spectrum $n_k = Ak^{-s_0}$ in the region of large k requires for power distributions a positive derivative of the collision integral with respect to index of the solution

$$\left[\frac{\partial I(s)}{\partial s}\right]_{s_0} > 0\ .$$

But according to (3.1.13), the sign of this derivative determines the sign of the flux of the corresponding integral of motion (in this case, of energy).

It is proved in a similar way that on the left-hand-side of the source, there can only be a spectrum with a negative flux of the integral of motion.

Thus, only those Kolmogorov solutions may be realized which transfer the flux of the integral of motion from the source to the sink but not into the opposite direction. This is obvious. We note that the above reasoning is true irrespective of the type of collision integral.

Examples. Out of all examples of wave turbulence with the nondecay dispersion law, gravitational waves on the surface of a deep fluid (1.1.42), (2.2.42) are the most important physical case. This is the kind of waves that may be excited by the wind on the surface of seas and oceans (see, e.g., [3.13] and references). For these waves we have $d = 2$ and $\alpha = 1/2$ [see (1.1.43) and (1.2.42)]. There are two Kolmogorov solutions. The first one obtained by *Zakharov* and *Filonenko* [3.14] transmits the flux of the wave action to a low-frequency region

$$n(k) = g^{1/6}\varrho^{2/3}(Q/a_1)^{1/3}k^{-23/6} \qquad (3.1.27)$$

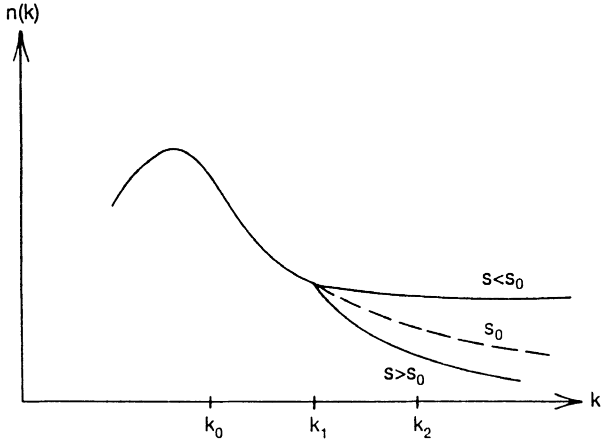

Fig. 3.3. The function $n(k)$ is plotted for different parameters S

and another one, the energy flux to high frequencies

$$n(k) = \varrho^{2/3}(P/a_2)^{1/3}k^{-4} \ . \tag{3.1.28}$$

Here g is the acceleration due to gravity and ϱ the fluid density. It is seen from the analysis of the (rather complicated) integrals for the dimensionless constants a_1 and a_2 that both distributions are local.

Other examples of Kolmogorov solutions (3.1.10) are Langmuir and spin waves (see Sects. 1.3.4). The dispersion laws (1.3.3) and (1.4.9a,21) have the same form, quadratic with a gap, so that $\alpha = 2$ holds. Let us set $d = 3$. With regard to the interaction coefficients there are two possibilities:
i) $m = 2$, which corresponds to the direct interaction of spin waves (1.4.9b);
ii) $m = 0$, which corresponds to the interaction of plasmons via virtual ion-sound waves (1.3.14) and the interaction of magnons in antiferromagnets (1.4.20). Case ii) also corresponds to the nonlinear Schrödinger equation (1.4.24, 1.5.3) which defines the "turbulence of envelopes". In particular, the equation describes the "turbulence of light" in nonlinear dielectrics.

In case i), a solution with energy flux is

$$n(k) = Ak^{-13/3} \ , \tag{3.1.29a}$$

where we have for the Langmuir waves [see (1.3.5)]

$$A = (P/a_1)^{1/3}n_0^{2/3} \tag{3.1.29b}$$

and for the spin waves [see (1.4.9b)]

$$A = (P/a_2)^{1/3}(\beta g_m)^{-2/3} \ . \tag{3.1.29c}$$

The Kolmogorov solution possessing an action flux is

$$n(k) = Bk^{-11/3} \ , \tag{3.1.30a}$$

where B reads for plasmons

$$B = (Q/b_1)^{1/3}\omega_p^{1/3}(r_D n_0)^{2/3} \tag{3.1.30b}$$

and for magnons

$$B = (Q/b_2)^{1/3}(M/\beta g_m)^{1/3} . \tag{3.1.30c}$$

The analysis shows that the dimensionless constants a_1, a_2, b_1, b_2 are given by convergent integrals, i.e., the solutions (3.1.29–30) are local.

In case ii), only distributions (3.1.10b) with wave action flux are local [3.8]:

$$n(k) = Ck^{-7/3} . \tag{3.1.31a}$$

The quantity C reads for Langmuir waves [see (1.3.14)]

$$C = (Q/c_1)^{1/3}(r_D n_0 T)^{2/3}\omega_p^{-1/3} , \tag{3.1.31b}$$

and for spin waves

$$C = (Q/c_2)^{1/3} g_m^{-4/3}\omega_{ex}^{-1/2}\omega_a^{1/6} . \tag{3.1.31c}$$

As far as distribution (3.1.10a) is concerned, it is in this case nonlocal. The solution carrying a constant energy flux has the form

$$n(k) = DP^{1/3}k^{-3}\ln^{-2/3}[k/k_0]$$

with k_0 being the pumping frequency [3.15].

Coming to the end, let us discuss the applicability range of the weak-turbulence approximation for the Kolmogorov solutions (3.1.9–10). For power distributions $n(k) \propto k^{-s}$, the characteristic nonlinear interaction time depends on k in the following way

$$t_{NL} \propto k^z, \qquad z = \alpha + (s_0 - d)(j - 2) - 2m . \tag{3.1.32}$$

This formula is valid for both (2.1.12) and (2.1.29), j is the number of participants in the elementary interaction act. The applicability parameter of the weak-turbulence approximation (2.1.14) is:

$$\xi^{-1}(k) = \omega(k)t_{NL} \propto k^{z+\alpha} . \tag{3.1.33a}$$

For a distribution possessing an energy flux we have $s_0 = d + 2m/(j-1)$ and

$$\xi^{-1}(k) \propto k^y, \qquad y = 2\alpha - 2m/(j-1) . \tag{3.1.33b}$$

At $\alpha(j-1) > m$, the applicability criterion of the theory of weak turbulence is violated at small k and in the case of $\alpha(j-1) < m$, at large k. For distributions possessing a wave action flux we have $j = 4$, $s_0 = d + 2m/3 - \alpha/3$, and the ξ-parameter behaves as follows

$$\xi^{-1}(k) \propto k^x, \qquad x = 2(2\alpha - m)/3 . \tag{3.1.33c}$$

At $2\alpha > m$, the turbulence becomes strong for small k and at $2\alpha < m$ for large k.

3.1.4 Exact Power Solutions of the Boltzmann Equation

The Maxwell equilibrium distribution $N(\varepsilon) = \exp(-\varepsilon/T)$ decreases exponentially with energy. It is interesting that nonequilibrium power distributions can be obtained as exact stationary solutions of the kinetic Boltzmann equation [3.16]. Moreover, the power-law spectra of particles have been observed in various experiments: for cosmic rays [3.17], for emission curent from metal irradiated by strong laser impulse [3.18], etc.

Let us consider the kinetic Boltzmann equation (2.1.44)

$$\begin{aligned}\frac{\partial N(\boldsymbol{p},t)}{\partial t} = I(\boldsymbol{p}) = \frac{\pi}{2}\hbar^{1-2d} & \\ \times \int \left|T(\boldsymbol{p},\boldsymbol{p}_1;\boldsymbol{p}_2,\boldsymbol{p}_3)\right|^2 & \left[N(\boldsymbol{p},t)N(\boldsymbol{p}_1,t) - N(\boldsymbol{p}_2,t)N(\boldsymbol{p}_3,t)\right] \\ \times\, \delta(\boldsymbol{p}+\boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{p}_3) & \delta(\varepsilon_p + \varepsilon_1 - \varepsilon_2 - \varepsilon_3)\, d\boldsymbol{p}_1 d\boldsymbol{p}_2 d\boldsymbol{p}_3 \ . \end{aligned} \tag{3.1.34}$$

We also suppose the scattering probability $\left|T(\boldsymbol{p},\boldsymbol{p}_1;\boldsymbol{p}_2,\boldsymbol{p}_3)\right|^2$ and energy $\varepsilon(p)$ to be scale-invariant functions of momenta with $2m$ and α indices, respectively. For isotropic distributions we can integrate over the angles. Going over from $\boldsymbol{p}$ to ε-variables, we obtain

$$\begin{aligned}\frac{\partial N(\varepsilon,t)}{\partial t} &= I(\varepsilon) \\ &= \int U(\varepsilon,\varepsilon_1;\varepsilon_2,\varepsilon_3)\left[N(\varepsilon,t)N(\varepsilon_1,t) - N(\varepsilon_2,t)N(\varepsilon_3,t)\right] \\ &\quad \times \delta(\varepsilon+\varepsilon_1-\varepsilon_2-\varepsilon_3)\, d\varepsilon_1 d\varepsilon_2 d\varepsilon_3 \ . \end{aligned} \tag{3.1.35}$$

Here the particle density in ε-space $N(\varepsilon) = N(\boldsymbol{p})\pi(2p)^{d-1}(dp/d\varepsilon)$ and U is a homogeneous function with the same scaling index

$$\gamma = \frac{2m+3d}{\alpha} - 4$$

as for the four-wave kinetic equation (3.1.16). Following (3.1.16–19) we integrate over $d\omega_1$ and perform the Zakharov transformations. Thus we find for the power distribution $N(\varepsilon) = A\varepsilon^{-x}$

$$\begin{aligned} I(\varepsilon) = A^2 \int_0^{\varepsilon} d\varepsilon_3 \int_{\varepsilon-\varepsilon_3}^{\varepsilon} & d\varepsilon_2 U(\varepsilon,\varepsilon_2+\varepsilon_3-\varepsilon;\varepsilon_2,\varepsilon_3) \\ \times & \left[\varepsilon^{-x}(\varepsilon_2+\varepsilon_3-\varepsilon)^{-x} - \varepsilon_2^{-x}\varepsilon_3^{-x}\right] \\ \times & \left[1 + \left(\frac{\varepsilon_2+\varepsilon_3-\varepsilon}{\varepsilon}\right)^y - \left(\frac{\varepsilon_2}{\varepsilon}\right)^y - \left(\frac{\varepsilon_3}{\varepsilon}\right)^y\right] \end{aligned} \tag{3.1.36}$$

with $y = 2x - 3 - \gamma$.

Considering the last square bracket in (3.1.36) we see that both $y = 0$ and $y = 1$ imply that the collision integral $I(\varepsilon)$ vanishes. So the stationary power solutions have the form [compare with (3.1.10, 21)]

$$N_1(\varepsilon) = A_1\varepsilon^{-x_1}, \quad x_1 = \frac{2m + 3d - \alpha}{2\alpha}, \tag{3.1.37a}$$

$$N_2(\varepsilon) = A_2\varepsilon^{-x_2}, \quad x_2 = \frac{2m + 3d}{2\alpha}. \tag{3.1.37b}$$

The first solution evidently corresponds to the constant flux of particles Q and the second to a constant energy flux P. Following the same procedure, the fluxes can be expressed in terms of $\partial I/\partial x$ so that we have $A_1 \propto \sqrt{Q}$, $A_2 \propto \sqrt{P}$. For an analysis of the locality of the different cases see [3.19].

Thus, the quantum kinetic equation has power nonequilibrium solutions in both limits: $N_k \gg 1$ and $N_k \ll 1$. The respective solutions have different indices. This is naturally connected with the absence of scaling invariance in the quantum kinetic equation which contains terms of different order in N_k.

3.2 Kolmogorov Spectra of Weak Turbulence in Nearly Scale-Invariant Media

As mentioned above, the behavior of weakly nonlinear waves is completely specified by two functions, the dispersion law $\omega(k)$ and the interaction coefficients $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$ or $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ characterizing the given medium. Under the condition of scale invariance of these functions and in the absence of preferred directions, the stationary flux solutions of the kinetic equations are power functions of the modulus of the wave vector (as shown in the preceding section). Clearly, in the case of arbitrary functions $\omega(\boldsymbol{k})$, $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$, $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ it is impossible to find the form of the flux spectra. In this section we shall describe the Kolmogorov solutions which may be obtained for the nearly scale-invariant situations.

3.2.1 Weak Acoustic Turbulence

Separate consideration is necessary for the case of weakly dispersive waves whose dispersion law

$$\omega(k) = ck + \Omega(k) \tag{3.2.1}$$

deviates only a little bit from the linear one: $\Omega(k) \ll ck$. As a rule, $\Omega(k)$ corresponds to the next term of the expansion of $\omega(k)$ in powers of a small parameter. If the dispersion law is close to a linear one in the short-wave region, then the small parameter is $(ak)^{-1}$, where a is a characteristic scale. The expansion contains usually even powers and $\Omega(k) \propto ck(ak)^{-2}$. Thus, for example, the dispersion law of spin waves in antiferromagnets (1.4.18) and of ultrarelativistic particles, $\omega^2(k) = \omega_0^2 + (vk)^2$, reduces at $k \gg \omega/v$ to the form $\omega(k) \approx vk + \omega_0^2/(2vk)$. But if the long-wave region is considered (acoustic type systems), the small parameter is (ak) and $\Omega(k) \propto ca^2k^3$ [see, e.g., (1.2.22), (1.2.39), (1.3.10)].

Let us consider wave turbulence with a nearly acoustic decay dispersion law

$$\omega(k) = ck(1 + a^2k^2)$$

which, in the region of small k [the region where (3.2.1) is valid], is close to a linear one, i.e., to a power law with the exponent $\alpha = 1$. At first sight it seems as if one should expect in this region a flux spectrum close to the power Kolmogorov spectrum (3.1.9): $n(k) \propto k^{-m-d}$ where the index of interaction coefficient for sound is $m = 3/2$ [see (1.1.38, 1.2.20)]. However, we have to bear in mind that the simple criteria for weak turbulence obtained in Sect. 2.1.4 are not applicable to nondispersive sound waves. The kinetic equation [whose solution is (3.1.9)] is not suitable for a description of that case. Thus, one can expect the realization of a weakly turbulent Kolmogorov spectrum in an intermediate scale region where the dispersion term ca^2k^3 is much smaller than the frequency $\omega(k)$ but much greater than the characteristic reverse interaction time, see (2.1.25).

Under these conditions, weak acoustic turbulence is described by the kinetic equation (2.1.12) into which one should insert the $\omega(k)$ and $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$ given by (1.2.22) and (1.2.20), respectively. It is readily seen that in the small dispersion region, $ak \ll 1$, the space-time synchronization condition $\omega(\boldsymbol{k}) = \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k} - \boldsymbol{k}_1)$ allows interaction only between waves propagating at close angles:

$$\theta(k, k_1) \approx \sqrt{6}|k - k_1|a \ . \tag{3.2.2}$$

Thus, to first order in ak, ak_1 [in which the dispersion parameter should be retained only in the argument of the δ-function] one can set $n(|\boldsymbol{k} - \boldsymbol{k}_1|) = n(k - k_1)$ and exchange the scalar products in interaction coefficient against modulus products. Assuming an isotropic distribution $n(\boldsymbol{k}) = n(k)$, we can perform all integrations except one to obtain

$$\begin{aligned}\frac{\partial n(k,t)}{\partial t} = 2^{d-1}b\pi(\sqrt{6}a)^{d-3}\Bigg\{ &\int_0^k dk_1 \left[k_1(k-k_1)\right]^{d-1} \\ &\times \left[n(k_1)n(k-k_1) - n(k)n(k_1) - n(k)n(k-k_1)\right] \\ &- 2\int_k^\infty dk_1 \left[k_1(k_1-k)\right]^{d-1}\left[n(k)n(k_1-k) - n(k_1)n(k)\right. \\ &\left.- n(k_1)n(k_1-k)\right]\Bigg\} \\ = I_d(k) \ .\end{aligned} \tag{3.2.3}$$

Here b is the dimensional constant in the interaction coefficient

$$|V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)|^2 = bkk_1k_2$$

and $d = 2, 3$, the dimension of the $\boldsymbol{k}$-space. In addition to the three-dimensional case, we have here also the two-dimensional one corresponding to gravitational-capillary waves on the surface of a shallow fluid, see (1.3.19).

The presence of the dispersion parameter a with a length dimension [see (3.2.3)] changes structure and index of the angle-averaged interaction coefficient as compared to the scale-invariant expression (3.1.11a). That is why the Kolmogorov solution differs in general from (3.1.9). Indeed, the index of the flux spectrum s_0 is easy to find by computing the energy flux index and equating it to zero:

$$P = P(k) = -2^{d-1}\pi \int_0^k \omega(k) k^{d-1} I_d(k)\, dk \propto k^{3d-2s_0} , \qquad (3.2.4a)$$

$$n(k) = \lambda P^{1/2} k^{-s_0}, \qquad s_0 = 3d/2 . \qquad (3.2.4b)$$

We see that the resulting expression $s_0 = d + d/2$ coincides with (3.1.9) $s_0 = d + m = d + 3/2$ only in the three-dimensional case for which the dispersion length a disappears from the kinetic equation (3.2.3).

Thus, the kinetic equation (3.2.3) formally experiences the transition to the nondispersive limit $a \to 0$ for $d = 3$. The three-dimensional case is intermediate. For $d > 3$, the mean nonlinear interaction time increases when $a \to 0$, so turbulence is effectively weak even in that limit. On the other hand for $d < 3$, the interaction time decreases and turbulence becomes strong when $a \to 0$. As we shall show in the second volume of this book, the case with $d = 3$ corresponds to the first one (for $d > 3$), since the interaction time decreases for $a \to 0$ slower then that of dispersion.

The fact that the solution (3.2.5) yields a vanishing collision integral (3.2.3) may be verified by the aid of the Zakharov transformations (3.1.14). In this case the second term in (3.2.3) should be split up into two identical terms: in the first one we make the substitution $k_1 \to k^2/k_1$, in the second one $k_1 \to kk_1/(k - k_1)$, and at $n_k \propto k^{-3d/2}$ we get $I_d(k) \equiv 0$. Substituting (3.2.5) into $P_k = P$ (3.2.4), one can find the λ-constant. By changing the order of integration as done for (3.1.13b), we obtain

$$\lambda^{-2} = b 2^{2d-2} \pi^2 (\sqrt{6}a)^{d-3} I_d,$$

$$I_d = \int_0^\infty \ln(1+x)[x(1+x)]^{-1-d/2} \left[(1+x)^{3d/2} - x^{3d/2} - 1\right] dx .$$

The dimensionless integral I_d for $d = 3$ was calculated approximately in [3.20] to yield $I_3 \approx 0.2$ and for $d = 2$ it was evaluated exactly in [3.5] as $I_2 = 2$. In the two-dimensional case, $\lambda \propto \sqrt{a}$, i.e., the amplitude of the stationary flux solution vanishes as the dispersion parameter goes to zero.

Let us consider now the case with a nondecay dispersion law. If the dispersion law of waves is nearly acoustic

$$\omega(k) = ck - \Omega(k), \quad 0 < \Omega(k) \ll ck, \quad \frac{\partial^2 \Omega}{dk^2} > 0$$

with the dispersion addition being a negative and convex function, then the three-wave processes are prohibited. The dominant role in the interaction is played by the four-wave processes with the interaction coefficient (1.1.29) arising in the second order perturbation theory at the expense of triple interactions via a virtual but almost real (due to smallness of Ω) intermediate wave.

$$\begin{aligned} T_p = &- \frac{U_{-1-212}U_{-3-434}}{\omega_3 + \omega_4 + \omega_{3+4}} + \frac{V^*_{1+212}V_{3+434}}{\omega_1 + \omega_2 - \omega_{1+2}} \\ &- \frac{V^*_{131-3}V_{424-2}}{\omega_{4-2} + \omega_2 - \omega_4} - \frac{V^*_{242-4}V_{313-1}}{\omega_{3-1} + \omega_1 - \omega_3} \\ &- \frac{V^*_{232-3}V_{414-1}}{\omega_{4-1} + \omega_1 - \omega_4} - \frac{V^*_{141-4}V_{323-2}}{\omega_{3-2} + \omega_2 - \omega_3} . \end{aligned}$$

Here $(j \pm i) = k_j \pm k_i$.

However such a coefficient $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ is not scale-invariant because of the presence of two terms with different k-dependences in the functions $\omega(k_j)$ in the denominator. How can we find the Kolmogorov solution in this case? An answer to this question was given in [3.21] where it was shown that in this case the four-wave kinetic equation also has isotropic Kolmogorov solutions $n(k) \propto k^{-s}$ in the weak dispersion region. Really, at $\Omega(k) \ll ck$ the main contribution to the interaction should stem from small-angle scattering when all four vectors $\boldsymbol{k}$, $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$ are almost parallel and the denominators in $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ are small. In this case one can set, for example:

$$\omega(k) \pm \omega(k_j) - \omega(k \pm k_j) \approx ck \frac{k_j \theta_j^2}{2|k \pm k_j|} + \Omega(k) \pm \Omega(k_j) - \Omega(k \pm k_j) ,$$

where θ_j is the angle between the wave vectors $\boldsymbol{k}_j$ and $\boldsymbol{k}$. If $\Omega(k)$ is scale-invariant, i.e., $\Omega(\lambda k) = \lambda^\beta \Omega(k)$, one can perform transformations that leave the collision integral

$$\begin{aligned} I(\boldsymbol{k}) = \int &\left|T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\right|^2 \delta(\omega_k + \omega_1 - \omega_2 - \omega_3)\delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) \\ &\times n_k n_1 n_2 n_3 \left(n_k^{-1} + n_1^{-1} - n_2^{-1} - n_3^{-1}\right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \end{aligned}$$

unchanged. In addition to a dilatation of the wave vector by a factor of $\lambda_j = k/k_j$ [similar to (3.1.18)] these transformations also involve multiplications of the angle by $\lambda_j^{(\beta-1)/2}$, i.e., $\theta'_j = \lambda_j^{(\beta-1)/2}\theta_j$, necessary to ensure the invariance of the denominators in T and of the arguments of the frequency δ-function. In addition, the transformations should contain a rotation which matches the vector $\boldsymbol{k}_j$ with vector $\boldsymbol{k}$. (Such transformations consisting of a dilatation and rotations in the $\boldsymbol{k}$-space were suggested in 1971 by *Katz* and *Kontorovich* [3.22]. We shall consider them in detail below, in Sect. 4.1, where we will use them to obtain the stationary anisotropic additions to Kolmogorov distributions.) For example, in the two-dimensional case the transformation converting $\boldsymbol{k}_2$ to $\boldsymbol{k}$ ($j = 2$) but leaving $I(k)$ invariant [except for an arbitrary factor] has the form

$$k_2' = \lambda_2 k = \lambda_2^2 k_2, \quad k_1' = \lambda_2 k_3, \quad k_3' = \lambda_2 k_1 ,$$

$$\theta_2' = -\lambda_2^{(\beta-1)/2}\theta_2, \quad \theta_1' = \lambda_2^{(\beta-1)/2}(\theta_3 - \theta_2), \quad \theta_3' = \lambda_2^{(\beta-1)/2}(\theta_1 - \theta_2) .$$

Splitting the collision integral up into four identical terms and subjecting three of them to transformations with $j = 1, 2, 3$, we obtain in the weak-dispersion region

$$I(\boldsymbol{k}) = \frac{k^\nu}{2} \int |T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)|^2 \delta\Big[\Omega(k) + \Omega(k_1) - \Omega(k_2) - \Omega(k_3)$$
$$+ \frac{c}{2}\left(k_1\theta_1^2 - k_2\theta_2^2 - k_3\theta_3^2\right)\Big] \delta(k + k_1 - k_2 - k_3)$$
$$\times \delta(k_1\theta_1\boldsymbol{\kappa}_1 - k_2\theta_2\boldsymbol{\kappa}_2 - k_3\theta_3\boldsymbol{\kappa}_3) n_k n_1 n_2 n_3 \left(n_k^{-1} + n_1^{-1}\right.$$
$$\left. - n_2^{-1} - n_3^{-1}\right)\left(k^{-\nu} + k_1^{-\nu} - k_2^{-\nu} - k_3^{-\nu}\right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 .$$

Here $\boldsymbol{\kappa}_j = \boldsymbol{k}_{j\perp}/k_{j\perp}$, where $\boldsymbol{k}_{j\perp}$ is the component of the $\boldsymbol{k}_j$ vector orthogonal to the $\boldsymbol{k}$ vector (in the two-dimensional case $\kappa_j = 1$). The index ν equals

$$\nu = -3s + 4m - 3\beta - 1 - \frac{(\beta+1)(d-1)}{2} + 3\frac{(\beta-1)(d-1)}{2} + 4d .$$

In this expression m is the scaling index of the three-wave interaction coefficient V_{k12} and according to (1.2.20) we have $m = 3/2$. The stationary Kolmogorov solution with constant energy flux will be obtained for $\nu = -1$, the distribution index will then be equal to:

$$s_P = (8 + 2d + \beta d - 4\beta)/3 . \tag{3.2.5a}$$

The choice of $\nu = 0$ reduces the collision integral to zero and yields the index

$$s_Q = (7 + 2d + \beta d - 4\beta)/3 \tag{3.2.5b}$$

of the Kolmogorov solution with constant wave energy flux.

Further consideration requires concretization of the form of $\Omega(k)$, in particular, specification of the value of β. We shall restrict ourselves to systems of the acoustic type for which we have

$$\Omega(k) = ca^2k^3, \quad \beta = 3 .$$

Below we shall be concerned with this case (see Sects. 3.4, 4.3 and 5.1). Thus, we see that for a three-dimensional medium and in the nondecay case, the Kolmogorov distributions of weak acoustic turbulence are equal to

$$n_k \propto k^{-s_Q} = k^{-10/3} , \tag{3.2.6a}$$

$$n_k \propto k^{-s_P} = k^{-11/3} . \tag{3.2.6b}$$

Those solutions can be obtained using the general formulas (3.1.10) $s_P = d + 2m/3$, $s_Q = s_P - \alpha/3$ with $\alpha = 1$, $m = 1$. Thus the index of the effective interaction coefficient is unity.

In agreement with the criterion (3.1.22), the solution (3.2.6a) transfers the wave action flux to the long-wave region and the solution (3.2.6b) transfers the energy flux to the short-wave region. It is interesting to compare (3.2.6b) with the distribution (3.2.5) transmitting the energy flux in the decay case: at $d = 3$ (3.2.5) gives $n_k \propto k^{-9/2}$. If we write the almost acoustic dispersion law in the form of (1.2.22), $\omega^2(k) = c^2k^2 + vk^4$, then, according to (3.2.5,6b), the s index of the stationary turbulent distribution depends in the short-wave region on v, as shown in Fig. 3.4.

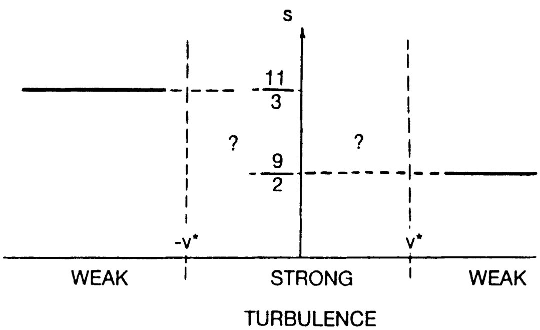

Fig. 3.4. The index of the Kolmogorov spectrum for acoustic turbulence

The structure of the stationary spectrum at $|v| \lesssim |v^*|$ (in particular, in the nondispersion case with $v = 0$), when the weak turbulence criterion (2.1.25) is violated, is currently unknown. We shall discuss the available theoretical approaches to the problem of acoustic turbulence in the second volume of this book.

The two-dimensional case (e.g., the turbulence of shallow-water gravitational waves) requires special consideration. This is due to the fact that in this region and for waves propagating in a narrow angular cone, the dynamic equations (1.3.11–33) may naturally be reduced to the known Kadomtsev-Petviashvili (KP) equation (1.5.4) (see also [3.23]):

$$\frac{\partial}{\partial x}\left[\frac{\partial \eta}{\partial t} + \sqrt{gh_0}\left(\frac{\partial \eta}{\partial t} + \frac{h_0^2}{6}\frac{\partial^3 \eta}{\partial x^3} + \eta\frac{\partial \eta}{\partial x}\right)\right] = -\frac{1}{2}\sqrt{gh_0}\frac{\partial^2 \eta}{\partial y^2}$$

Here h_0 is the depth, the dynamic variable $\eta(x, y, t)$ describes the deviation of fluid surface from the unperturbed state (see Sect. 1.2.5); the motions are assumed to be weakly non-one-dimensional: $|\partial\eta/\partial x| \gg |\partial\eta/\partial y|$. The Kadomtsev-Petviashvili equation possesses a number of remarkable properties owing to its integrability, in particular, an infinite set of conservation laws (for details see [3.24]). The latter generate a series of integrals of motion of the corresponding kinetic equation. However, as shown by *Zakharov* and *Shulman* [3.25], the dispersion law of the weakly non-one-dimensional sound with negative dispersion (corresponding to the KP equation)

$$\omega(k_x, k_y) = ck_x\left(1 - \frac{k^2 h_0^2}{6}\right) + \frac{ck_y^2}{2k_x}$$

is nondegenerate (see Sect. 2.2.1). This implies that it is impossible to find a function $f(k)$ that is on the resonance surface given by

$$\omega(\boldsymbol{k}) + \omega(\boldsymbol{k}_1) - \omega(\boldsymbol{k}_2) - \omega(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2) = 0\ ;$$

the same equation should be obeyed by $f(k)$

$$f(\boldsymbol{k}) + f(\boldsymbol{k}_1) - f(\boldsymbol{k}_2) - f(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2) = 0\ .$$

For a nondegenerate dispersion law (in contrast to the positive dispersion case), the existence of a sequence of integrals of motions of the kinetic equations leads to a degenerate interaction coefficient, viz., to its vanishing on the resonance surface, see [3.25]. Via direct (and rather cumbersome) computation it may be verified that the interaction coefficient (1.1.29b) with $V(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \theta(k_{1x})\theta(k_{2x})\theta(k_{3x})\sqrt{k_{1x}k_{2x}k_{3x}}$, vanishes on the resonance surface. Thus, in the two-dimensional case, one should take into account the following terms of the expansion of the interaction Hamiltonian expansion in small parameter kh_0. As a result, the coefficient of the four-wave interaction will gain the factor h_0 increasing its scaling index by unity. In view of this the Kolmogorov indices are $s_P = 10/3$ and $s_Q = 3$ [3.26].

3.2.2 Media with Two Types of Interacting Waves

This is a rather widespread case. It is observed in the interaction of electromagnetic and Langmuir waves in plasmas, in forced Mandelstam-Brillouin scattering on interacting of Langmuir and ion-sound waves [3.27–29], on interacting spin and sound waves in magnets [3.30] and in some other physical systems.

Kinetic Equations and Their Solutions. Following *Zakharov* and *Kuznetsov* [3.28], we shall consider the interaction of high-frequency (HF) waves with the dispersion law $\omega(k)$ and low frequency (LF) waves with the dispersion law $\Omega(k)$ in the three-wave decay processes described by (1.3.11–12). The Hamiltonian of such a system

$$\mathcal{H} = \int \omega_k a_k a_k^* d\boldsymbol{k} + \int \Omega_k b_k b_k^* d\boldsymbol{k} + \int \left(V_{k12} b_k a_1 a_2^* + \text{c.c.}\right)\delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2)\, d\boldsymbol{k} d\boldsymbol{k}_1 d\boldsymbol{k}_2$$

leads to equations of motion linear in the amplitudes of the low frequency waves b_k. Therefore, it is for the applicability of the weak-turbulence approximation (2.1.14c) sufficient to require the smallness of the HF wave amplitudes. Going over to the statistical description and introducing $\langle a_k a_k^{*\prime}\rangle = N_k\delta(\boldsymbol{k} - \boldsymbol{k}')$, $\langle b_k b_k^{*\prime}\rangle = n_k\delta(\boldsymbol{k} - \boldsymbol{k}')$, one can obtain the kinetic equations

$$\frac{\partial N(\boldsymbol{k},t)}{\partial t} = \int \left[T(\boldsymbol{k}_2;\boldsymbol{k},\boldsymbol{k}_1) - T(\boldsymbol{k}_2;\boldsymbol{k}_1,\boldsymbol{k})\right] d\boldsymbol{k}_1 d\boldsymbol{k}_2 \,, \tag{3.2.7a}$$

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = -\int T(\boldsymbol{k};\boldsymbol{k}_1,\boldsymbol{k}_2)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 \,, \tag{3.2.7b}$$

where

$$T(\boldsymbol{k}_2;\boldsymbol{k},\boldsymbol{k}_1) = 2\pi\left|V(k_2,k,k_1)\right|^2\left[N(k_1)n(k_2) - N(k)n(k_1) - N(k)n(k_2)\right]$$
$$\times\, \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\delta(\omega_k - \omega_1 - \Omega_2) \,.$$

Let us assume the interaction coefficient to be scale-invariant

$$V(\lambda k, \lambda k_1, \lambda k_2) = \lambda^m V(k, k_1, k_2) \,,$$

and the dispersion laws to be of the power-type

$$\omega(k) = \omega_0 + c_1 k^\alpha, \qquad \Omega(k) = c_2 k^\beta \,. \tag{3.2.8}$$

Obviously, with such a choice of $\omega(k)$ the frequency ω_0 does not appear in (3.2.7) so that we may without loss of generality use $\omega_0 = 0$.

Equations (3.2.7) conserve the complete number of HF waves and the total wave energy $\varepsilon_k = \omega_k N_k + \Omega_k n_k$ and, consequently, support the thermodynamically equilibrium stationary solutions $N_k = T/(\omega_k + \mu)$, $n_k = T/\Omega_k$. We shall now deal with the nonequilibrium (flux) solutions of these equations.

Let us first examine the case of complete scale invariance when $\alpha = \beta$. Such a coincidence occurs, for example, upon the interaction of electromagnetic and sound waves. We shall search for stationary solutions of (3.2.7) in the power form

$$N_k = Ak^x, \qquad n_k = Bk^x \,. \tag{3.2.9}$$

In (3.2.7a), we shall map the integration domain of the second integral as determined by the decay conditions into the integration domain of the first integral. For that purpose it is convenient to introduce the coordinates k_x, k_y and the complex variable $w = k_x + ik_y$. The mapping is [3.28]

$$w = w'\frac{w}{w'}, \quad w_1 = w\frac{w}{w'}, \quad w_2 = w''\frac{w}{w'} \,. \tag{3.2.10}$$

As a result, the integrand is factorized and in the stationary case we have

$$\int T(\boldsymbol{k}_2;\boldsymbol{k},\boldsymbol{k}_1)\left[1-(k/k_1)^\gamma\right] d\boldsymbol{k}_1 d\boldsymbol{k}_2 = 0 \,,$$

where $\gamma = 2m + 2d - \alpha - 2x$.

Equation (3.2.7b) is transformed in a similar way. Multiplication by Ω_k and addition of (3.2.7a) multiplied by ω_k yields a second equation for the stationary case

$$\int T(\boldsymbol{k}_2; \boldsymbol{k}, \boldsymbol{k}_1)\left[\omega_k - \omega_1(k/k_1)^{\gamma+\alpha} - \Omega_2(k/k_2)^{\gamma+\alpha}\right] d\boldsymbol{k}_1 d\boldsymbol{k}_2 = 0 \ .$$

From these equations, two nonequilibrium solutions may be obtained with $\gamma = 0$ and $\gamma = -\alpha$ or $x_1 = -m - d + \alpha/2$ and $x_2 = -m - d$. One of the equations coincides with (3.2.7b) in the stationary case

$$\int T(\boldsymbol{k}; \boldsymbol{k}_1, \boldsymbol{k}_2)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 = 0 \ . \tag{3.2.11}$$

That equation defines the relationship between the constants A and B appearing in the distributions (3.2.9).

Analysis of Solutions of the Kinetic Equations. Let us demonstrate that the first solution ($\gamma = 0$) corresponds to the constant flux of the number of HF waves and the second one ($\gamma = -\alpha$) to the constant energy flux.

By integrating (3.2.7a) we find the wave flux

$$P_N = -2^{d-1}\frac{\pi k^{\gamma+d}}{\gamma}\int T(\boldsymbol{k}_2; \boldsymbol{k}, \boldsymbol{k}_1)\left(k^{-\gamma} - k_1^{-\gamma}\right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 \tag{3.2.12}$$

in the case of power distributions (3.2.9).

For the energy $\varepsilon(k,t)$ one can obtain from (3.2.7)

$$\begin{aligned}\frac{\partial\varepsilon(k,t)}{\partial t} = \int \big[\omega_k T(\boldsymbol{k}_2; \boldsymbol{k}, \boldsymbol{k}_1) &- \omega_k T(\boldsymbol{k}_2; \boldsymbol{k}_1, \boldsymbol{k}) \\ &- \Omega_k T(\boldsymbol{k}; \boldsymbol{k}_1, \boldsymbol{k}_2)\big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 \ .\end{aligned} \tag{3.2.13}$$

By integrating (3.2.13) we find the energy flux

$$\begin{aligned}P_\varepsilon = &-2^{d-1}\frac{\pi k^{\gamma+d+\alpha}}{\gamma+\alpha}\int T(\boldsymbol{k}_2; \boldsymbol{k}, \boldsymbol{k}_1) \\ &\times\left(\omega_k k^{-\gamma-\alpha} - \omega_1 k_1^{-\gamma-\alpha} - \omega_2 k_2^{-\gamma-\alpha}\right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 \ .\end{aligned} \tag{3.2.14}$$

Going over to the limits ($\gamma \to 0$ and $\gamma + \alpha \to 0$) in (3.2.12, 14) and taking into account the conditions (3.2.11) we see that the first solution corresponds to a Kolmogorov spectrum with a constant flux of the number of HF waves with $P_\varepsilon = 0$. The second solution corresponds to another one with the energy flux $P_\varepsilon \neq 0$ and $P_N = 0$. Hence it appears that $A, B \propto P^{1/2}$, in full agreement with dimensional analysis.

Finally, let us determine the direction of the fluxes. For this purpose, it should be remarked that one can explicitly find n_k from (3.2.11). Substitution of this distribution into (3.2.12, 14) and symmetrization of the integrand give rise to

$$P_N = \int R_{k123} N_k N_1 N_2 N_3 \left(N_k^{-1} + N_1^{-1} - N_2^{-1} - N_3^{-1}\right) \ln\frac{k k_1}{k_2 k_3} d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3,$$

$$\begin{aligned}P_\varepsilon = \int & R_{k123} N_k N_1 N_2 N_3 \left(N_k^{-1} + N_1^{-1} - N_2^{-1} - N_3^{-1}\right) \\ &\times \left(\omega_k \ln k + \omega_1 \ln k_1 - \omega_2 \ln k_2 - \omega_3 \ln k_3\right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \ ,\end{aligned}$$

where

$$R_{1234} = T_{1324} + T_{3142} + T_{1423} + T_{4132} ,$$

$$T_{1234} = T(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) = \int d\boldsymbol{k} \frac{U(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) U(\boldsymbol{k}, \boldsymbol{k}_3, \boldsymbol{k}_4)}{\int U(\boldsymbol{k}, \boldsymbol{k}_5, \boldsymbol{k}_6)[N(k_6) - N(k_5)]\, d\boldsymbol{k}_5 d\boldsymbol{k}_6} ,$$

$$U_{123} = (2k)^d \frac{\pi^2}{8} \left| V_{123} \right|^2 \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 + \boldsymbol{k}_3) \delta(\Omega_1 - \omega_2 + \omega_3) ,$$

$$\omega_i = \omega(k_i), \qquad \Omega_j = \Omega(k_j), \qquad N_i = N(k_i) .$$

The resulting expressions formally coincide with the fluxes in the four-wave interaction; the sign of the interaction coefficient R_{1234} is opposite to sign $(x+\alpha)$. Therefore, using a similar procedure as in Sect. 3.1, we obtain

$$\operatorname{sign} P_N = \operatorname{sign}(x_1 + \alpha) . \tag{3.2.15}$$

For the interaction of electromagnetic and sound waves this procedure leads to ($\alpha = 1$, $m = -1/2$, $d = 3$) [3.27, 28]

$$x_1 = -2, \quad P_N < 0, \qquad x_2 = -5/2, \quad P_\varepsilon > 0 .$$

Thus, we have in this case one Kolmogorov solution (with the constant flux of the number of HF waves) in the long-wave region where P_N is directed inwards and another one in the short-wave region where the energy flows outwards.

Diffusion Approximation. The homogeneity of the kernels of the kinetic equations (3.2.8) and, as a consequence, the existence of exact solutions in the form of Kolmogorov spectra are first of all due to the coincidence of the indices α and β. This property is not observed in most examples important for applications. Among these applications is first of all the interaction of HF Langmuir waves with long frequency ion-sound waves. A typical situation for such turbulence corresponds to the frequency $\omega_k \gg \Omega_k$, i.e., the characteristic scale of the variation of the frequency of the HF wave is small. In this scale the integrands in (3.2.7) are nearly scale-invariant. In such a situation (the so-called diffusion approximation) the equations may be simplified.

Expanding in (3.2.7) the frequency δ-function in a series of Ω and assuming isotropy of the wave distributions N_k and n_k, we obtain

$$\frac{\partial n_k}{\partial t} = \int \tilde{T}(\boldsymbol{k}; \boldsymbol{k}_1, \boldsymbol{k}_2)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 , \tag{3.2.16}$$

$$\frac{\partial N_k}{\partial t} + \operatorname{div} \boldsymbol{p}_N = 0 , \tag{3.2.17}$$

$$\frac{\partial \varepsilon_k}{\partial t} = -\operatorname{div}(\omega_k \boldsymbol{p}_N) + \int \left(\Omega_k \tilde{T}_{k12} - \Omega_1 \tilde{T}_{1k2} \right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 , \tag{3.2.18}$$

where

$$\tilde{T}_{k12} = 2\pi |V_{k12}|^2 \left(N_1^2 + n_1 \Omega_k \frac{\partial N_1}{\partial \omega_1} \right) \delta(\boldsymbol{k} - \boldsymbol{k}_1 + \boldsymbol{k}_2) \delta(\omega_1 - \omega_2) ,$$

$$\boldsymbol{p}_N = \frac{\boldsymbol{k}}{k} \int \frac{\Omega_2}{\omega_k} \tilde{T}_{2k1} \, d\boldsymbol{k}_1 d\boldsymbol{k}_2$$

is the density of the flux of HF quasi-particles. These equations represent two independent systems (3.2.16, 17) and (3.2.17, 18) whose identity is readily verified by direct calculations. Like the starting system (3.2.7, 13), equations (3.2.16–18) have solutions in the form of the Rayleigh-Jeans distribution, with vanishing fluxes P_N and P_ε and $\tilde{T}(\boldsymbol{k}; \boldsymbol{k}_1, \boldsymbol{k}_2) = 0$.

Let us consider the solutions with constant fluxes P_N and P_ε. In the case of the power solutions

$$N_k = Ak^x, \qquad n_k = Bk^x \omega_k / \Omega_k \tag{3.2.19}$$

these quantities are constant. For the spectra corresponding to P_N = const, the x-index is directly determined by calculating the powers in (3.2.17)

$$x_1 = -m - d + \alpha - \beta/2 \, . \tag{3.2.20}$$

Here $A, B \propto P^{1/2}$.

To find a second solution corresponding to P_ε =const, we shall subject (3.2.18) to a transformation similar to (3.2.10) which gives rise to

$$x_2 = -m - d - (\beta - \alpha)/2, \quad A, B \propto P^{1/2} \, . \tag{3.2.21}$$

For both solutions the relationship between constants A and B is determined from the stationary equation (3.2.17). It yields a vanishing flux $P_\varepsilon = 0$ at P_N =const and vice versa. The expressions for the signs which do not include the index β remain the same. This is due to the fact that from (3.2.17, 18) the $\Omega_k n_k$ value is eliminated instead of n_k at $\alpha = \beta$.

Turbulence Locality. Our solutions of both diffusion and exact equations are valid, (i) only in the inertial interval, which is an intermediate range of k-space where there is neither damping nor pumping and (ii) when the wave interaction in this region is local. Therefore, the local Kolmogorov spectra are independent of the detailed structures of instability growth-rate and damping decrement, but are entirely determined by the flux magnitudes.

For a spectrum to be local, it is sufficient to prove the convergence of the integrals in the diffusion equations (3.2.16–18). This in turn requires the necessity to know the asymptotics of the interaction coefficients $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$. We shall assume

$$V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) \to Dk^{m-m_1} k_1^{m_1} \quad \text{at} \quad k_1 \gg k$$

to hold. Then the upper convergence (at $k \to \infty$) of the integrals is achieved for

$$x + 2m_1 + d - 2\alpha < -1 \, ,$$

and the lower convergence for

$$x + \alpha + \beta + d + 2(m - m_1) > 1 \ .$$

It should be noted that the index (3.2.21) is situated exactly in the middle of the locality strip.

Some Particular Examples. We shall consider the interaction of HF Langmuir and spin waves with the sound waves:

$$\omega_k = \omega_0 + \mu k^2, \quad \Omega_q = cq \ . \tag{3.2.22}$$

In the diffusion approximation, $\mu k \gg c$ and in each elementary act $\omega(\boldsymbol{k}) = \omega(\boldsymbol{k} - \boldsymbol{q}) + \Omega(\boldsymbol{q})$ the HF wave with the wave vector $\boldsymbol{k}$ interacts with sound with the wave vector $q \approx 2k$, directed backwards.

For spin waves in ferromagnets, the interaction coefficient with sound is equal to [3.30]

$$|V_{2k1}|^2 = \frac{V_1^2}{k} \left\{ \lambda_1 \left[(\boldsymbol{p}\boldsymbol{k}_1)(\boldsymbol{k}\boldsymbol{k}_2) + (\boldsymbol{p}\boldsymbol{k}_2)(\boldsymbol{k}\boldsymbol{k}_1) \right] + \lambda_2 (\boldsymbol{k}_1 \boldsymbol{k}_2)(\boldsymbol{k}\boldsymbol{p}) \right\}^2 \ .$$

Here V_1 is a dimensional constant; $\boldsymbol{p}$ is the vector of sound polarization; λ_1, λ_2 are the dimensionless phenomenological constants of the magnetoelastic interaction, they are of the order of unity. In this case we have $\alpha = 2$, $\beta = 1$, $m = 5$, $m_1 = 1$. The distributions with a constant spin wave flux into the region of large scales are

$$N_k \simeq P_N^{1/2} V_1^{-1} k^{-13/2}, \quad n_k \simeq \frac{\mu}{c} P_N^{1/2} V_1^{-1} k^{-11/2} \ ,$$

the dispersive part of the energy flux $\varepsilon_k = \mu k^2 + ck$:

$$N_k \simeq P_\varepsilon^{1/2} V_1^{-1} k^{-15/2}, \quad n_k \simeq \frac{\mu}{c} P_\varepsilon^{1/2} V_1^{-1} k^{-13/2} \ .$$

All these distributions are local.

In antiferromagnets, the coefficient of the magnon-phonon interaction is

$$|V_{2k1}|^2 = V_2^2 k \varphi(\boldsymbol{p}, \boldsymbol{k}/k) \ , \tag{3.2.23}$$

where φ is a dimensionless function depending on the mutual orientation of external magnetic field, wave vector of the sound and its polarization [3.30]; $m = 1/2$ and $m_1 = 0$. The dispersion law for spin waves (1.4.18)

$$\omega_k = \sqrt{\omega_0^2 + v^2 k^2}$$

is only in the regions $k \to 0$ and $k \to \infty$ reduced to the cases considered in this subsection. At $k \gg \omega_0/v$ we have the scale-invariant case $\alpha = \beta$ with the Kolmogorov solutions having the form (3.2.9), where $x - 7/2$ holds for the distribution transferring the "energy" flux to the short-wave region. Another solution (with magnon flux) corresponds to $x = -3$. Both distributions are local. At $\omega_0 \gg kv \gg c$, we have the case (3.2.2). The distributions with magnon flux to the long-wave region have the form (3.2.19, 20)

$$N_k \simeq P_N^{1/2} V_2^{-1} k^{-2}, \quad n_k \simeq \frac{v^2}{\omega_0 c} P_N^{1/2} V_2^{-1} k^{-1} , \tag{3.2.24}$$

and those possessing an "energy" flux

$$N_k \simeq P_\epsilon^{1/2} V_2^{-1} k^{-3}, \quad n_k \simeq \frac{v^2}{\omega_0 c} P_\epsilon^{1/2} V_2^{-1} k^{-2} . \tag{3.2.25}$$

The power distributions for the interacting Langmuir and ion-sound waves have exactly the same form. Really, the interaction coefficient (1.3.13) has the same form as (3.2.23) and the dispersion law coincides with (3.2.22). It is easy to see that the solutions (3.2.24, 25) are local [3.28].

The Kolmogorov Spectrum Close to Thermodynamical Equilibrium. One interesting fact should be noted: the solutions (3.2.24) $N_k \propto k^{-2}$ and $n_k \propto k^{-1}$ coincide with the ones for thermodynamic equilibrium and can consequently transmit only zero fluxes. What is the structure of the flux spectra in this case? An answer to this question will be given following *Kanashov* and *Rubenchik* [3.31]. Let us substitute into (3.2.16, 17) the interaction coefficient (1.3.13), introduce a new variable $\omega = k^2$ and use $z(\omega) = n(2k)2kc/\mu$. Then renormalization of time for the dimensional constant in the interaction coefficient gives:

$$\frac{\partial N}{\partial t} = \frac{2}{\sqrt{\omega}} \frac{d}{d\omega} \left(2\frac{\partial N}{\partial \omega} \int_0^\omega z(\omega')\omega' \, d\omega' + \omega^2 N^2 \right) , \tag{3.2.26a}$$

$$\frac{\partial z}{\partial t} = V = \frac{\sqrt{\omega}}{2} \left(-zN + \int_\omega^\infty N^2 \, d\omega' \right) . \tag{3.2.26b}$$

To obtain (3.2.26), we have used unity for the angular dependence of the interaction coefficient $\cos^2\theta_{12}$. However, since we shall deal only with the behavior of the distributions in the region $\omega \to 0$, the resulting asymptotics will also satisfy the exact equations (3.2.16, 17). Let us consider the stationary solutions of (3.2.26); due to conservation of the number of HF waves, (3.2.26a) is integrated once to yield

$$2\frac{\partial N}{\partial \omega} \int_0^\omega z(\omega')\omega' \, d\omega' + \omega^2 N^2 = P_N , \tag{3.2.27a}$$

$$zN = \int_\omega^\infty N^2 \, d\omega' . \tag{3.2.27b}$$

Here the constant P_N is the flux of the number of HF waves. It is seen that at $P_N = 0$, the system (3.2.27) has the simple solution $z = 1$, $N = 1/\omega$, corresponding to (3.2.24). Equations (3.2.27) may be reduced to a system of ordinary differential equations. Let us differentiate (3.2.27b) with regard to ω, then substitute the N^2 thus found into (3.2.27a) and integrate this equation from 0 to ω:

$$\begin{aligned} \omega P_N &= 2N \int_0^\omega z(\omega')\omega' \, d\omega' - N\omega^2 z \\ &\quad - N\left(2\int_0^\omega z(\omega')\omega' \, d\omega' - \omega^2 z \right)_{\omega=0} . \end{aligned} \tag{3.2.28}$$

Expecting that the resulting spectra will not grow too quickly at $\omega \to 0$ [see below (3.2.31)], we shall assume the last term to become zero. Then, introducing the dimensionless variables $z \to z\sqrt{P_N}$, $N = y\sqrt{P_N}/\omega$, we obtain

$$\frac{d}{d\omega}\frac{\omega^2}{y} = -\omega^2\frac{dz}{d\omega}\,, \quad \frac{d}{d\omega}\frac{zy}{\omega} = -\frac{y^2}{\omega^2}\,. \tag{3.2.29}$$

The system (3.2.29) is homogeneous with regard to the variable ω and, therefore, introducing the variable $t = \ln\omega$, we shall reduce it to the form

$$\frac{dz}{dy} = \frac{z+y}{y(y^2 - zy - 2)}\,, \tag{3.2.30a}$$

$$\frac{dz}{dy} = -\frac{z+y}{zy+1} \tag{3.2.30b}$$

admitting a simple investigation. If the wave vector goes to zero then $t \to -\infty$. It is seen from (3.2.30b) that in this case z increases monotonically. The behavior of the solutions of (3.2.30a) is shown in Fig. 3.5

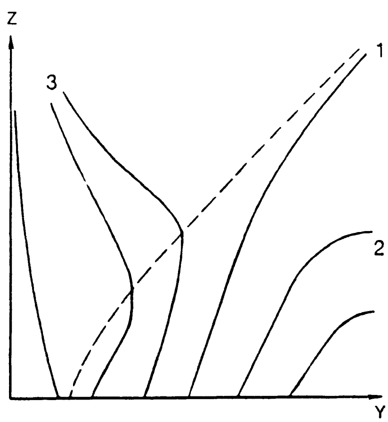

Fig. 3.5. For different integration constants z is plotted as a function of y

The dotted line corresponds to $y^2 - zy - 2 = 0$. Above it the slopes of the curves are negative and below they are positive. It is easy to see that the curves 2 having the asymptotics $y \gg z$, describe the spectra of Langmuir oscillations with singularities at finite k. On the curves of type 3 the inequality $y \ll z$ is satisfied. Their spectrum of Langmuir oscillations tends to zero, $N \propto \omega$ at $\omega \to 0$. Thus the considered flux solution corresponds to a separatrix (curve 1). Asymptotically, $y \simeq 2\sqrt{\ln k^2}$, $z \simeq y + 4/y$, $y \gg 1$. This asymptotic solution also satisfies the initial equations (3.2.16, 17) with exact interaction coefficients this may be verified by direct substitution.

We have shown that there are stationary nonpower solutions (3.2.26) corresponding to the spectra of high-frequency and sound waves growing while k decreases. In terms of the initial variables we have

$$N_k \propto \frac{y}{\omega}\,, \qquad n_k \propto \frac{z}{k}\,. \tag{3.2.31}$$

Since z and y vary logarithmically, the forms of the spectra differ only slightly from (3.2.24), though, they provide in contrast to them an energy flux to small k.

Let us now discuss how the solutions (3.2.31) and (3.2.20) may be matched. The latter should represent asymptotics at $\omega \to \infty$

$$N \to 2A\omega^{-3/2} , \qquad z \to A\omega^{-1/2} .$$

We shall consider the excitation of Langmuir oscillations by a small increment $\gamma_k = \gamma\delta(k - k_0)$. The solutions obtained by us contain four arbitrary constants: P_N, A and the coordinates of the solution on the integral curves at $k = k_0$. Since the spectra are described by the second-order differential equation (3.2.26a), n_k and N_k should be continuous at $k = k_0$. Integrating the kinetic equation for the Langmuir oscillations, we obtain the condition $P_N = \gamma k_0^2 N(k_0)$. The solutions (3.2.25) correspond to a vanishing flux of the wave number P_N. Similarly to (3.2.28), one obtains for them

$$N\left(-2\int_0^\omega z(\omega')\omega'\,d\omega' + \omega^2 z\right) + \frac{2}{3}A^2 = 0 .$$

Comparing this equation with (3.2.28), we find:

$$A = \sqrt{3P_N\omega(k_0)/2} .$$

The four conditions obtained allow us to match the solutions. Of course, since the integral curves may be found only numerically, it is impossible to give explicit formulas to match the solutions, whereas the last formula provides an unambiguous relationship between the asymptotic solutions in the regions with $k \gg k_0$ and $k \ll k_0$.

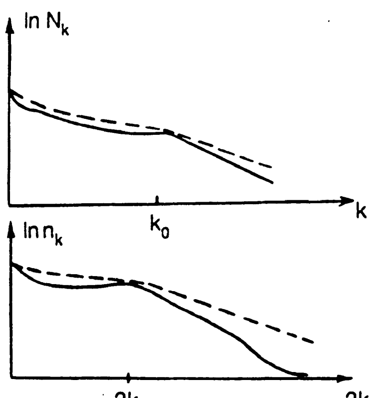

Fig. 3.6. The steady state distributions of Langmuir waves N_k and of ion sound n_k as obtained by numeric simulation [3.29]

Fig. 3.6 shows the numerical solution of (3.2.7) with the interaction coefficients (3.1.13), (3.2.23) for isotropic excitation of HF waves with $k = k_0$ and dissipation at $k > k_0$ (simulating the Landau damping of waves on particles) [3.29]. Despite the narrowness of the source, the distributions turn out to be rather smooth and correspond qualitatively to the solutions (3.2.25) and (3.2.31) depicted by the dotted lines in Fig. 3.6. The substantial deviation from the Kolmogorov solutions at $k > k_0$ is due to the presence of dissipation.

3.3 Kolmogorov Spectra of Weak Turbulence in Anisotropic Media

3.3.1 Stationary Power Solutions

In this section we shall discuss situations where the Kolmogorov solutions in anisotropic media may be calculated explicitly. In the absence of isotropy, the dispersion law and the interaction coefficient depend differently on the components of the wave vector. The Kolmogorov solutions can be found if in each component of k these dependences are separately scale-invariant. Such solutions were first obtained by *Kuznetsov* [3.32]. Let us consider, for example, an axially symmetric medium, like a plasma in a magnetic field. We shall direct the z-axis of the chosen cylindric coordinate system along the specified direction in $\boldsymbol{k}$-space. Let p denote the longitudinal component of the wave vector k_z and $\boldsymbol{q}$ the transverse component $\boldsymbol{k}_\perp$. We assume the dispersion law and the interaction coefficient of the three-wave interaction to be *bihomogeneous functions* of p and q (see Sects. 1.3,4):

$$\omega(p,q) = p^a q^b \tag{3.3.1a}$$

$$V(\lambda p, \lambda p_1, \lambda p_2, \mu \boldsymbol{q}, \mu \boldsymbol{q}_1, \mu \boldsymbol{q}_2) = \lambda^u \mu^v V(p, p_1, p_2, \boldsymbol{q}, \boldsymbol{q}_1, \boldsymbol{q}_2) . \tag{3.3.1b}$$

This is just the case when the stationary Kolmogorov distributions are power functions of the components of the wave vector and may be found as exact solutions of the kinetic equation.

Let us averaging over the angles in our cylindric coordinate system. The kinetic equation acquires then the form

$$\begin{aligned} 2\pi q \frac{\partial n(p,q,t)}{\partial t} &= \int [\mathcal{R}(p,p_1,p_2,q,q_1,q_2) - \mathcal{R}(p_1,p,p_2,q_1,q,q_2) \\ &\quad - \mathcal{R}(p_2,p_1,p,q_2,q_1,q)]\, dp_1 dp_2\, dq_1 dq_2 \\ &\equiv 2\pi q I(p,q) , \end{aligned} \tag{3.3.2a}$$

where

$$\begin{aligned} \mathcal{R}(p,p_1,p_2,q,q_1,q_2) = U(p,p_1,p_2,q,q_1,q_2)[&n(p_1,q_1)n(p_2,q_2) \\ &- n(p,q)n(p_1,q_1) - n(p,q)n(p_2,q_2)] \\ &\times \delta(p-p_1-p_2)\delta(p^a q^b - p_1^a q_1^b - p_2^a q_2^b) . \end{aligned} \tag{3.3.2b}$$

Without loss of generality one can consider U to depend only on p_i and the modulus of $\boldsymbol{q}_i$. Then

$$U = (2\pi)^3 |V|^2 \frac{q q_1 q_2}{\Delta_2} ,$$

where the quantity $2\pi/\Delta_2$ is the result of averaging over the angles of the two-dimensional δ-function of wave vectors which we encountered already in Sect. 3.1.2. In the given case we have

$$2\Delta_2 = \sqrt{2(q_1^2 q_2^2 + q^2 q_1^2 + q^2 q_2^2) - q^4 - q_1^4 - q_2^4} \ . \tag{3.3.3}$$

Thus, U is a function homogeneous in p_i of $2u$ power and in q_i of $2v$ power. Besides, $U = 0$ if segments of lengths q, q_1 and q_2 cannot be used to construct a triangle with sides of these lengths. Due to this, integration in the plane q_1, q_2 is performed over the shaded region shown in Fig. 3.7.

Due to the presence of two δ-functions (3.3.2) has two integrals of motion: the energy $\int p^a q^b n(p,q)\, dpdq$ and the z-component of momentum $\int pn(p,q)\, dpdq$). Consequently, in addition to the Kolmogorov solutions with a constant energy flux

$$P \propto |p|^{2+2u} q^{4+2v} n^2(p,q), \quad n(p,q) \propto P^{1/2} |p|^{-1-u} q^{-2-v} \tag{3.3.4}$$

one can expect here stationary distributions carrying the constant momentum flux

$$\begin{aligned} R &\propto |p|^{3+2u-a} q^{4+2v-b} n^2(p,q), \\ n(p,q) &\propto R^{1/2} |p|^{(a-2u-3)/2} q^{(b-2v-4)/2} \ . \end{aligned} \tag{3.3.5}$$

We shall seek the stationary solutions of (3.3.2) in the power form:

$$n(p,q) \propto |p|^{-x} q^{-y} \ . \tag{3.3.6}$$

Throughout this section, we shall take the z-components of the wave vectors to be positive ($p, p_1, p_2 > 0$), i.e., we consider only waves traveling in one direction along the specified axis. In this case, the distributions of the form (3.3.6) have a nonzero momentum

$$\int dq \int_0^\infty pn(p,q)\, dp \neq 0 \ ,$$

and both stationary solutions (3.3.4–5) must exist.

Indeed, the bihomogeneity of the collision integral (3.3.2) in p and q, like in the isotropic case, allows the transformation of the second and third terms to the first one (to the accuracy of a a factor). These transformations found by *Kuznetsov* [3.32] represent an analogue of separate Zakharov transformations for the longitudinal component of the wave vector and the modulus of the transverse components [cf. (3.1.14)]:

$$p_1 = p^2/p_1', \quad p_2 = (p/p_1')p_2' \ , \tag{3.3.7a}$$

$$q_1 = q^2/q_1', \quad q_2 = (q/q_1')q_2' \ . \tag{3.3.7b}$$

Besides, it is clear that

$$p = (p/p_1')p_1', \quad q = (q/q_1')q_1' \ .$$

One can verify that (3.3.7b) transforms the integration domain over q_1, q_2 into itself. In this case the second term in (3.3.2) is transformed to the first one with

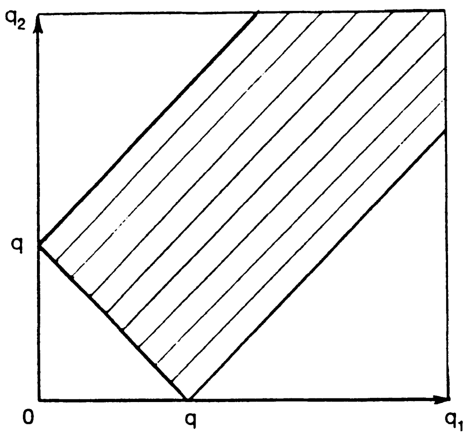

Fig. 3.7. The integration region for the collision integral (3.3.2) is indicated

the accuracy of a factor. A similar transformation with the third term gives, in the stationary case, the equation

$$0 = \int U(p, p_1, p_2, q, q_1, q_2)\delta(p - p_1 - p_2)\delta(p^a q^b - p_1^a q_1^b - p_2^a q_2^b)$$
$$\times \left[(p_1 p_2)^{-x}(q_1 q_2)^{-y} - (pp_1)^{-x}(qq_1)^{-y} - (pp_2)^{-x}(qq_2)^{-y}\right] \tag{3.3.8}$$
$$\times \left[1 - \left(\frac{p}{p_1}\right)^r \left(\frac{q}{q_1}\right)^s - \left(\frac{p}{p_2}\right)^r \left(\frac{q}{q_2}\right)^s\right] dp_1 dp_2 dq_1 dq_2$$

where $r = 2(1 + u - x) - a$, $s = 2(2 + v - y) - b$.

Equation (3.3.8) has four solutions. Indeed, the first square bracket vanishes either due to the frequency δ-function by choosing

$$x_1 = a, \quad y_1 = b, \qquad n(p,q) \propto \omega^{-1}(p,q)\,, \tag{3.3.9a}$$

or due to the δ-function in the z-components of the wave vector

$$x_2 = 1, \quad y_2 = 0, \qquad n(p,q) \propto p^{-1}\,. \tag{3.3.9b}$$

The solutions (3.3.9) are the limiting cases of the general thermodynamically equilibrium solution $n(p, q, T) = T/[\omega(p, q) + cp]$, see (2.2.13).

Using the frequency δ-function to evaluate the second square bracket to zero, we set $r = -a$, $s = -b$, which gives

$$x_3 = 1 + u, \quad y_3 = 2 + v. \tag{3.3.10a}$$

In fact we have obtained (3.1.9) for the relationship between the solution indices and the interaction coefficients separately for different wave vector components. Indeed, for the longitudinal component p the dimension of the k-space is equal to a unity, and for q it is equal to two.

Finally, using the δ-function in p, we obtain a fourth solution corresponding to $r=-1$, $s=0$:

$$x_4=(3+2u-a)/2, \quad y_4=(4+2v-b)/2\,. \tag{3.3.10b}$$

The solutions corresponding to (3.3.9) are the Kolmogorov solutions. Comparing them with (3.3.4–5), we see that (3.3.10a) corresponds to the constant energy flux

$$n(p,q)=\lambda_1 P^{1/2}|p|^{-1-u}q^{-2-v} \tag{3.3.10c}$$

and the solution (3.3.10b), to the constant flux of the z-component of momentum

$$n(p,q)=\lambda_2 R^{1/2}p^{(a-2u-3)/2}q^{(2v+4-b)/2}\,. \tag{3.3.10d}$$

It should be noted that the solution of (3.3.10c) also exists in the case in which p varies from $-\infty$ to ∞.

The distributions (3.3.10) are only for local systems solutions of (3.3.2), i.e., when the integral converges for all singularities of the subintegrand. This has ro be verified directly for every particular case.

3.3.2 Fluxes of Integrals of Motion and Families of Anisotropic Power Solutions

Let us now discuss the directions and values of the transferred fluxes. Let us start with the distributions transferring an energy flux [3.33]. Symmetery considerations show the energy flux vector to have the two components P_z and $P_\perp$ ($P_\phi=0$). The stationary kinetic equation for power solutions (3.3.6) may be written in the form

$$\begin{aligned}\operatorname{div}\boldsymbol{P}_k &= \frac{\partial P_z}{\partial p}+\frac{1}{q}\frac{\partial}{\partial q}(qP_\perp)=-\omega(p,q)I(p,q)\\ &= -\lim_{(x,y)\to(x_3,y_3)}\left[q^{2(y_3-y)-2}|p|^{2(x_3-x)-1}I(x,y)\right],\end{aligned} \tag{3.3.11a}$$

where $I(x,y)$ is a p- and q-independent integral derived from $I(p,q)$ (3.3.2,8) similarly to $I(s)$ in (3.1.13):

$$I(x,y)=\omega(p,q)I(p,q)|p|^{2(x-x_3)+1}q^{2(y-y_3)+2}\,.$$

Using the δ-function representation mentioned in Sect. 3.1 $\lim_{\epsilon\to0}|x|^{\epsilon-1}=2\delta(x)$, we shall find the explicit form of the singularities in (3.3.11a);

$$\frac{\partial P_z}{\partial p}+\frac{1}{q}\frac{\partial}{\partial q}(qP_\perp)=\frac{2}{q^2}\frac{\partial I}{\partial x}\delta(p)+\frac{2\pi}{p}\frac{\partial I}{\partial y}\delta(q)\,. \tag{3.3.11b}$$

The derivatives $\partial I/\partial x$ and $\partial I/\partial y$ in (3.3.11b) are calculated at $x=x_3=1+u$, $y=y_3=2+v$.

We see that the external sources must be nonzero in the $p=0$-plane and on the line with $q=0$ where the distribution (3.3.10a, c) has singularities. In the remaining $\boldsymbol{k}$-space we have $\operatorname{div}\boldsymbol{P}=0$.

Due to the homogeneous dependence of all quantities on p and q, the components of the energy flux should also be power functions of p and q. Equation (3.3.11b) has a single solution of such a form

$$P_\perp = \frac{A}{q|p|}, \quad P_z = \frac{Bp}{q^2|p|}, \quad A = \left(\frac{\partial I}{\partial y}\right)_{x_3,y_3}, \quad B = \left(\frac{\partial I}{\partial x}\right)_{x_3,y_3} . \tag{3.3.12}$$

Thus, the dependence of the components of the energy flux vector on the ones of the wave vector proved to be universal, in every particular case only the A and B constants are different. The complete energy flux vector in $\boldsymbol{k}$-space is directed at some angle to the wave vector

$$\frac{P_z}{P_\perp} = \frac{p}{q}\frac{A}{B} = \frac{k_z}{k_\perp}\frac{\partial I/\partial y}{\partial I/\partial x} . \tag{3.3.13}$$

This is similar to the case of the momentum flux components. Indeed, in this second-rank tensor, only two components are nonzero: R_{zz} and $R_{z\perp}$, the fluxes of the z-component of momentum, respectively, along and across the z-axis. Taking the z-component of the equation for momentum conservation $(\operatorname{div}\boldsymbol{R})_z = -pI_k$ one can construct an equation for them which coincides with (3.3.11)

$$\begin{aligned}\frac{\partial R_{zz}}{\partial p} + \frac{1}{q}\frac{\partial}{\partial q}\left(qR_{z\perp}\right) &= -\lim_{(x,y)\to(x_4,y_4)}\left[q^{2(y_4-y)-2}p^{2(x_4-x)-1}I(x,y)\right]\\ &= \frac{2}{q^2}\frac{\partial I}{\partial x}\delta(p) + \frac{2\pi}{p}\frac{\partial I}{\partial y}\delta(q) .\end{aligned}$$

Thus, the components of the momentum flux tensor also show a universal dependence (3.3.12) on the components of the wave vector

$$R_{z\perp} = \frac{A}{q|p|}, \quad R_{zz} = \frac{Bp}{q^2|p|}, \quad A = \left(\frac{\partial I}{\partial y}\right)_{x_4,y_4}, \quad B = \left(\frac{\partial I}{\partial x}\right)_{x_4,y_4} . \tag{3.3.14}$$

As we see, in the anisotropic case the fluxes of conserving quantities are also specified by the derivatives of the collision integral with regard to the solution index.

The presence of two indices (x and y) in the power distribution (3.3.6) implies that the family of stationary solutions of (3.3.8) is richer than the set (3.3.9–10), as pointed out by *Kanashov* [3.33]. Let us consider the plot of the function $I = I(x,y)$ representing a surface in the (I,x,y) space. The points in which the surface touches or intersects the plane $I = 0$ will be the power indices of stationary solutions of the kinetic equation. It is easily and directly verified that in the points (x_i, y_i) with $i = 1,\dots,4$ (see 3.3.9–10), the derivatives $\partial I/\partial x$ and $\partial I/\partial y$ in general are nonzero. This means that in these points the surface $I = I(x,y)$ does not touch the plane $I = 0$ but intersects it at certain lines. In a small neighborhood of the point (x_i, y_i), the intersection curve may be assumed to be a straight line given by

$$(x - x_i)\left(\frac{\partial I}{\partial x}\right)_{x_i,y_i} + (y - y_i)\left(\frac{\partial I}{\partial y}\right)_{x_i,y_i} = 0 \; . \tag{3.3.15}$$

The set of indices of stationary solutions situated on the intersection curve is really limited by the locality condition specifying a certain region in the plane (x, y). However, if a solution with indices (x_i, y_i) is local, the solutions with indices close to it will also be local.

These families of power solutions $n \propto |p|^{-x} q^{-y}$ also correspond to the family of solutions of the equation div $\boldsymbol{P} = 0$ at $p \neq 0$, $q \neq 0$ generalizing (3.3.12, 14):

$$\begin{aligned} P_\perp &= A|p|^{2(x_3-x)-1} q^{2(y_3-y)-1} \; , \\ P_z &= Bp|p|^{2(x_3-x)-1} q^{2(y_3-y)-2} \; . \end{aligned} \tag{3.3.16}$$

It is easy to see that (3.3.16) satisfies div $\boldsymbol{P} = 0$ at $p \neq 0$, $q \neq 0$ if

$$A(y - y_3) + B(x - x_3) = 0 \; .$$

This condition coincides with (3.3.15).

It should be pointed out that the locality region in the (x, y)-plane may be either two- or one-dimensional. In the latter case (as for Rossby waves – see below Sect. 5.3.5), it may be impossible to define a stationary solution with indices (x, y) different from (x_i, y_i). The definition of fluxes may be impossible as well because the derivatives $\partial I/\partial x$ or $\partial I/\partial y$ of the collision integral may not exist. Nevertheless, the collision integral converges on the Kolmogorov solution so that we have stationary locality.

To conclude this subsection we list some naturally arising questions, the answers to which are currently not available. The set of stationary indices (x, y) has been computed only in a particular case. In the work by *Balk* and *Nazarenko* [3.34], the set of stationary indices was found numerically for the drift-type waves described in Sect. 1.3.2. Two curves were obtained on the (x, y) plane. One of the curves passes through the two points corresponding to the equilibrium solutions (3.3.9) and another one through the points corresponding to the Kolmogorov solutions (3.3.10). (The authors of [3.34] also obtained curves with a stationary index outside the locality region. In doing so they exchanged the integral against a sum of integrals which they used in computer calculations.)

Which of the stationary solutions is realized in the presence of a particular source? As seen from (3.3.12–14), everything depends on the relation between powers of the sources located at $q = 0$ and at $p = 0$. For the ratio A/B of these powers the solution (3.3.10c) is realized; for C/D it is given by (3.3.10d). When the power ratio is equal to c out of the interval $(A/B, C/D)$ then one of the solutions of the type (3.3.16) with the indices x and y located on the curve passing through (x_3, y_3) and (x_4, y_4) should be realized. The relation $(\partial I/\partial x)/(\partial I/\partial y) = c$ must be fulfilled. But if the power ratio can not be represented in the form of a ratio of derivatives in some point (x, y) on the stationary curve in the locality region, the stationary solution must be, generally speaking, of the non-power type. We note that physical meaning and occurrence of the $n \propto p^{-x} q^{-y}$-type distribution are not quite clear, since a source capable of exciting waves simultaneously on a plane and a line in the $\boldsymbol{k}$-space [see (3.3.11)] is still considered to be rather exotic. However, it is possible that on wave excitation by quite an arbitrary source localized in the region $p \simeq p_0$, $q \simeq q_0$, the asymptotics of a stationary distribution at $p \gg p_0$, $q \gg q_0$ will be described by some of the solutions (3.3.10). For a source generating the fluxes of both conserved quantities, i.e., energy and momentum, it follows from a dimensional analysis that the two-flux solution may be written in the form

$$n(p,q) = P^{1/2}|p|^{-1-u}q^{-2-v}f(\xi) \tag{3.3.17}$$

where $\xi = R|p|^{a-1}q^b/P$ is a dimensionless parameter specifying the relation between the fluxes. At $\xi \to 0$, (3.3.17) should coincide with (3.3.10c), therefore $f(\xi) \to \text{const}$. In the opposite limit, $n(p,q)$ depends only on the momentum flux $f(\xi) \to \sqrt{\xi}$ at $\xi \to \infty$. Thus, the two-flux solution has power asymptotics only in the regions of the $\boldsymbol{k}$-space that are sufficiently remote both from the source and the surface given by the equality $\xi = 1$. The function $f(\xi)$ has not yet been calculated for any particular case either.

In Sects. 1.3,4 we gave examples of physical systems in which the dispersion law and the interaction coefficient are bihomogeneous functions of the wave vector components, i.e., they satisfy (3.3.1). Examples of Kolmogorov spectra in anisotropic media will be described in Chap. 5.

The attentive reader has probably noted that in all the examples listed in Sects. 1.3, 4 the bihomogeneity (3.3.1) was observed only for the waves whose wave vectors are located within a narrow angular range in $\boldsymbol{k}$-space. For the Kolmogorov solutions (3.3.10) to be valid, one should prove in every case that the waves inside this angular range interact with each other by far stronger than with waves from the remaining part of the $\boldsymbol{k}$-space.

3.4 Matching Kolmogorov Distributions with Pumping and Damping Regions

In the previous sections (3.1–3), we have obtained the universal nonequilibrium distributions (3.1.5–6), (3.3.3–6, 9, 19–21), (3.3.10) which reduce the corresponding collision integrals to zero. For each of these solutions we have found a relationship between the amplitude of the distribution and the amount of the flux it transmits [see (3.1.13, 22), (3.2.12, 14), and (3.3.13)]. In this section we shall go further and discuss, first of all, the question of how to find the flux carried away by the Kolmogorov solution for given amplitude and spectral characteristics of the wave source. The role of the source will be played, as mentioned in Sect. 2.2.3, by the increment of some instability, i.e., the positive part of the function $\Gamma(k)$ [see below (3.4.1)]. To answer this question, it is necessary to find the form of the distribution n_k satisfying the stationary kinetic equation (2.2.15)

$$\Gamma_k n_k + I_k\{n_{k'}\} = 0 \tag{3.4.1}$$

in the region of k-space where $\Gamma_k > 0$. In other words, one should match the Kolmogorov solution (realized far from the regions where $\Gamma_k \neq 0$) with the one for the pumping region. As we shall see in this section, spectrally narrow pumping gives rise to a very interesting intermediate solution, the *pre-Kolmogorov asymptotics*, in a wide range of scales up to those strongly differing from the pumping region scale. Secondly, we shall discuss the effect of wave damping on the structure of the stationary distribution, i.e., we shall take into consideration that, starting from some k_d, the function $\Gamma_k < 0$ and the occupation numbers n_k should fall off faster than the Kolmogorov law.

Consideration of at least one of the finite scales, the pumping region k_0 or the damping region k_d, is physically absolutely indispensable. This follows from the fact that the Kolmogorov solutions obtained in Sections 3.1–3 in an infinite interval $k \in (0, \infty)$ are the power solutions $n_k \propto k^{-s_0}$ with the wave energy density per unit volume of a medium in such distributions being infinite. Indeed, the integral

$$E = \int \omega_k n_k \, dk \propto \int_0^{\infty} k^{\alpha+d-s_0-1} \, dk \tag{3.4.2}$$

diverges either at the upper or lower limit, depending on the sign of the quantity

$$h = \alpha + d - s_0 \ . \tag{3.4.3}$$

As seen, at $h < 0$, the energy-containing region is the pumping region $k \simeq k_0$, while it is for $h > 0$ the damping region $k \lesssim k_d$. The h-index determines, in particular, the type of the dependence of energy flux on the characteristic scale of the pumping region (Sect. 3.4.1) and the type of nonstationary behavior of weakly turbulent distributions (Chapt. 4).

Thus, in this section we shall study the stationary kinetic equation (3.4.1). It is a nonlinear integral equation and it is only in degenerate cases that its complete analytical solution may be obtained. Therefore in this section, along with analytical methods, we shall use order-of-magnitude estimates and numeric simulations.

3.4.1 Matching with the Wave Source

Let us consider the isotropic turbulence in the decay case in which the only Kolmogorov solution has the form

$$n_k^0 = \lambda P^{1/2} k^{-m-d} \tag{3.4.4}$$

and the h-index is equal to

$$h = \alpha + d - s_0 = \alpha - m \ . \tag{3.4.5}$$

The power absorbed by a wave system (that is the flux transmitted) P is easily estimated for sources with a broad spectrum as depicted in Fig. 3.8. This is accomplished as follows: the energy flux directed into a medium by pumping

$$P = \int \Gamma(k)\omega(k)n(k)\,dk \simeq \Gamma_0\omega(k_0)n(k_0)k_0^d \tag{3.4.6}$$

is expressed via the values of the occupation numbers in the pumping region $n(k_0)$. Assuming the distribution to be moderately distorted by a smooth source (not more than by a magnitude of the same order), one can estimate $n(k_0)$, extending the Kolmogorov solution (3.4.4) to small k's

$$n(k_0) \simeq \lambda P^{1/2} k_0^{-m-d} \ . \tag{3.4.7}$$

Substituting (3.4.7) into (3.4.6), we find the flux

$$P \simeq \Gamma_0^2\omega^2(k_0)k_0^{-2m}\lambda^2 \propto \gamma_0^2 k_0^{2h} \ . \tag{3.4.8}$$

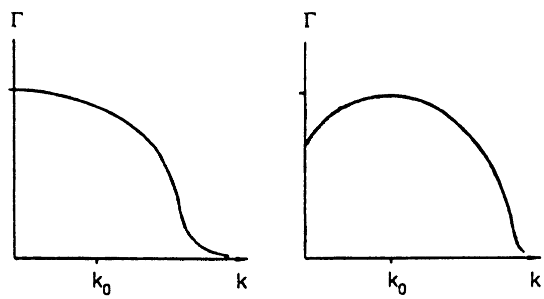

Fig. 3.8. Typical behavior of Γ_k for sources with a broad spectrum around the single characteristic wave number k_0

The assumption of small distortions of the Kolmogorov distribution by a broad source is supported by Fig. 3.9 taken from the work by *Zakharov* and *Musher* [3.35] where the Kolmogorov spectrum of wave turbulence has first been observed in a numeric simulation. Equation (3.2.3) for three-dimensional acoustic turbulence was modelled, and the source was chosen to be $\Gamma_k = \exp[-(k-10)^2/4]$. The resulting distribution shown in Fig. 3.9 is close to the Kolmogorov solution $n_k^0 \propto k^{-9/2}$ in the $5 < k < 50$ range (Fig. 3.9 represents the $k^2 n_k$ value diminishing as $k^{-5/2}$).

Expression (3.4.8) has a clear physical meaning: the energy flux increases when the source is shifted into the energy-containing region. Indeed, at $h > 0$, P is an increasing function of k_0, at $h < 0$, a decreasing one. This formula was obtained by *Falkovich* and *Shafarenko* [3.36]. In the same way we obtain for the four-wave case

$$P \propto \Gamma_0^{3/2} k_0^{3h_1/2} \quad \text{and} \quad Q \propto \Gamma_0^{3/2} k_0^{3h_2/2} \ .$$

The quantity $h_2 = d - s_0$ indicates the positions at which the action integral $N = \int n_k\,dk$ diverges; the value h_1 corresponds to (3.4.3) with $s_0 = 2m/3 + d$.

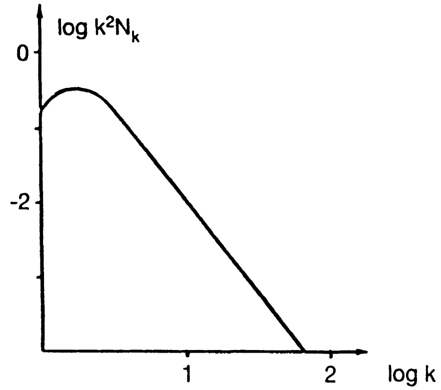

Fig. 3.9. Steady state distribution of acoustic waves excited by broad sources

Let us now consider the excitation of turbulence by a source which is nonzero in the small neighborhood Δk in the vicinity of $k = k_0$, i.e., is narrow in the modulus of k ($\Delta k \ll k_0$) and, accordingly, in frequencies: $\Delta\omega_k \ll \omega(k_0) \equiv \omega_0$.

The question with regard to the structure of a stationary distribution excited by a spectrally narrow source is not quite trivial. The stationary kinetic equation (3.4.1) may be written in the form

$$\Gamma_k n_k + I_k\{n_{k'}\} = \Gamma_k n_k - \gamma_k\{n_{k'}\}n_k + S_k\{n_{k'}\} = 0 .$$

Here the collision integral is split up into two parts. The first one, $\gamma_k n_k$, contains n_k outside the integral. The quantity $\gamma_k\{n_{k'}\}$ has the meaning of a nonlinear damping decrement of a wave with wave number k due to the presence of the distribution $n_{k'}$. This quantity is a functional of $n_{k'}$ and is in the three-wave case given by the single integral over $d\boldsymbol{k}$

$$\begin{aligned}\gamma_k = {} & 2\int |V(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}-\boldsymbol{k}')|^2\delta\big(\omega(k)-\omega(k')-\omega(|\boldsymbol{k}-\boldsymbol{k}'|)\big)n(k')\,d\boldsymbol{k}' \\ & +2\int |V(\boldsymbol{k}',\boldsymbol{k},\boldsymbol{k}'-\boldsymbol{k})|^2\delta\big(\omega(k')-\omega(k)-\omega(|\boldsymbol{k}'-\boldsymbol{k}|)\big) \\ & \qquad \times [n(\boldsymbol{k}'-\boldsymbol{k})-n(k')]\,d\boldsymbol{k}'\end{aligned}$$

and in the four-wave case by the double integral

$$\begin{aligned}\gamma_k = \int & |T(\boldsymbol{k},\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{k},\boldsymbol{k}_2,\boldsymbol{k}_3)|^2\delta\big(\omega(k)+\omega(|\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{k}|)-\omega_2-\omega_3\big) \\ & \times [(n_2+n_3)n(\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{k})-n_2n_3]\,d\boldsymbol{k}_2 d\boldsymbol{k}_3 \ .\end{aligned}$$

As usual, $\omega_i = \omega(k_i)$, $n_i = n(k_i)$.

In the three-wave case, the remaining part of the collision integral equals

$$S_k = \int |V(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}-\boldsymbol{k}_1)|^2 \delta\big(\omega(k)-\omega(k_1)-\omega(\boldsymbol{k}-\boldsymbol{k}_1)\big) n_1 n(\boldsymbol{k}-\boldsymbol{k}_1)\, d\boldsymbol{k}_1$$
$$+2\int |V(\boldsymbol{k}_1,\boldsymbol{k},\boldsymbol{k}_1-\boldsymbol{k})|^2 \delta\big(\omega(\boldsymbol{k}_1)-\omega(k)-\omega(\boldsymbol{k}_1-\boldsymbol{k})\big)$$
$$\times\, n_1 n(\boldsymbol{k}_1-\boldsymbol{k})\, d\boldsymbol{k}_1$$

and in the four-wave case

$$S_k = \int |T(\boldsymbol{k},\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{k},\boldsymbol{k}_2,\boldsymbol{k}_3)|^2 \delta\big(\omega(k)+\omega(\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{k})$$
$$-\,\omega_2-\omega_3\big) n(\boldsymbol{k}_2+\boldsymbol{k}_2-\boldsymbol{k}) n_2 n_3\, d\boldsymbol{k}_2 d\boldsymbol{k}_3 \ .$$

It is important to emphasize that, being distribution integrals, the quantities γ_k and S_k should be smooth functions of the $\boldsymbol{k}$-vector for sufficiently arbitrary $n_{k'}$. Let us write the stationary kinetic equation in the form

$$n_k = \frac{S_k\{n_{k'}\}}{\gamma_k\{n_{k'}\}-\Gamma_k} \equiv \frac{S_k}{\gamma_k-\Gamma_k} \ .$$

As a consequence of the smoothness of γ_k, S_k it follows clearly that if in any region of k-space the external pumping Γ_k has a sharp anomaly, the occupation numbers n_k should also have an anomaly in this region. In other words, since $\partial\gamma_k/\partial k$ and $\partial S_k/\partial k$ are small, the derivative $\partial n_k/\partial k$ is large in regions where $\partial\Gamma_k/\partial k$ is large. This conclusion is correct both for the decay and nondecay cases. But there arises the question whether the anomaly is essential, i.e., if there is a sharp peak of occupation numbers in the pumping region. The answer may vary with types of interaction (three- or four-wave ones).

Let the source excite waves in a narrow interval $\Delta\omega$ around ω_0. In the case of three-wave interactions, by $2 \to 1$ confluence processes these waves will generate a peak of the distribution of occupation numbers at $\omega = 2\omega_0$. By the induced $1 \to 2$ decay processes waves from a second peak can generate waves in the first (pumping) interval rather than outside of it. Indeed, in the second integral of the kinetic equation (2.1.12) for $\partial n(\omega,t)/\partial t$, the terms differing from zero will be those containing $n(2\omega_0)n(2\omega_0-\omega)$. Besides, waves from the first peak will merge with waves from the second peak to generate a third peak, etc. However, due to the locality of the interaction, the influence of remote peaks on the first one (and the resulting peak broadening) should be small. Thus, in the three-wave case it would be natural to expect a sharp peak of occupation numbers in the pumping region and a chain of peaks (at multiple frequencies) generated by it. This subsection mainly deals with the properties of such solutions.

In the nondecay case, wave excitations by a narrow source should occur quite differently. Indeed, now every $2 \to 2$ interaction involves four waves. If two out of four interacting waves have frequencies close to the pumping frequency, the frequencies of the other two waves can vary over a wide range. For example, scattering of two waves with frequency ω_0 can give waves with frequencies from zero to $2\omega_0$. As a result, the energy of the source might be distributed

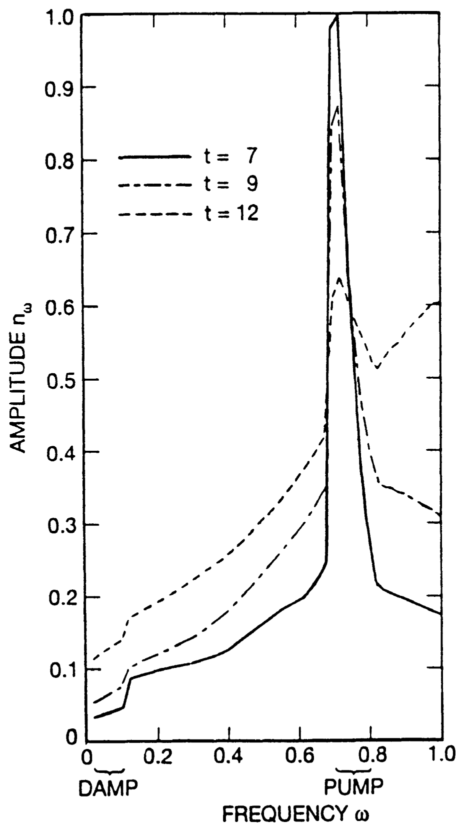

Fig. 3.10. Numerical simulations [3.12] of the distribution of Langmuir turbulence for different intermediate times

over a wide region of k-space instead of being concentrated in narrow intervals. As indicated by computer simulations of weak Langmuir turbulence, spectrally narrow pumping does not generate a sharp distribution. Figures 3.10 and 3.11 obtained by *Hansen* and *Nicholson* [3.12] shows the formation of a Langmuir turbulence spectrum by pumping (at $0.7 < \omega < 0.8$). At first pumping excites a sharply peaked distribution which is then smeared out (Fig. 3.10); in the quasi-steady state (Fig. 3.11 corresponds to $t = 100$) it is no longer pronounced.

A distribution of the same kind had been obtained by *Falkovich* and *Ryzhenkova* [3.15] for optical turbulence. Figures 3.12a, b obtained by numeric simulation of the kinetic equation corresponding to nonlinear Schrödinger equation (see Sect. 1.4.3) show the steady behavior of the distribution and its index, respectively. A weak anomaly of the distribution and a strong one of the derivative $s(\omega) = d \ln n(\omega)/d\omega$ were observed.

The question with regard to presence or absence of a sharp peak generated by pumping in a stationary distribution is also related to whether the energy flux depends on the integral power of the source $\Gamma = \int \Gamma_k \, dk$ or only on its maximal value $\Gamma(k_0) = \Gamma_0$. It should be noted that the absence of a sharp peak in the narrow pumping region could also be connected with a large value of γ_k (formally speaking, with the divergency of the integral for γ_k).

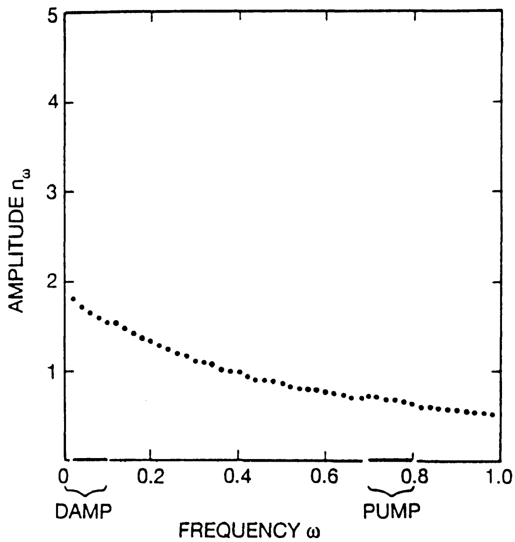

Fig. 3.11. Steady distribution of Langmuir turbulence excited by spectrally narrow pumping

Let us now return to the decay case. If one assumes also for narrow pumping a smooth distribution then instead of (3.4.6), one obtains

$$P = \int \Gamma_k \omega_k n_k \, dk \simeq k_0^{d-1} \omega(k_0) n(k_0) \int \Gamma_k \, dk \equiv \Gamma \omega_0 n(k_0) k_0^{d-1} \; .$$

Using in here the $n(k_0)$ from (3.4.7) yields then

$$P \simeq \Gamma^2 \omega_0^2 k_0^{-2m-2} \lambda^2 \propto \Gamma^2 k_0^{2(h-1)} \tag{3.4.9}$$

which differs from (3.4.8) by the factor $(\Delta k / k_0)^2$ and contains Γ instead of Γ_0. This estimate (3.4.9) which has also been given in the literature [3.2, 8] is incorrect.

Following [3.36], we shall now show the dependence $P \propto k_0^{2h}$ to hold also for spectrally narrow pumping. An error made in deriving (3.4.9) is the unjustified assumption that the occupation numbers at $k \approx k_0$ exhibit a smooth behavior. As a matter of fact, the sharp function Γ_k may generate a sharp peak n_k at $k \approx k_0$, $\omega_k \approx \omega_0$. As described above, the properties of the three-wave interaction ensure that the peak will give rise to a chain of peaks at multiple frequencies $\omega_j = j\omega_0$ where j is a natural number. To illustrate this, we shall give an example where the limiting case of such a solution may be found analytically. Consider (3.2.3) for two-dimensional acoustic turbulence. Instead of $\Gamma_k n_k$ we shall take an external force F_k independent of n_k

$$\begin{aligned} I(k) = &\int_0^k k_1(k-k_1)(n_1 n_2 - n_k n_1 - n_k n_2) \, dk_1 \\ &- 2\int_k^\infty k_1(k_1 - k)(n_k n_2 - n_1 n_k - n_1 n_2) \, dk_1 = -F_k \; . \end{aligned} \tag{3.4.10a}$$

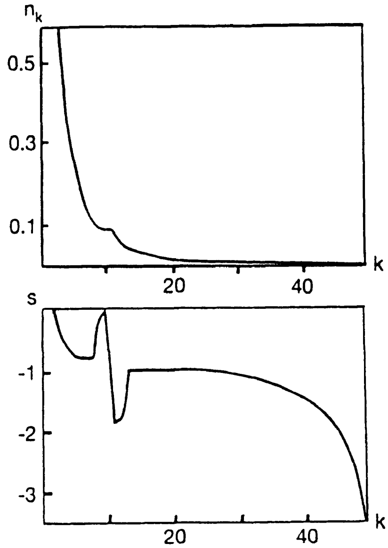

Fig. 3.12. Steady state distribution and its current index for optical turbulence excited by spectrally narrow pumping

Here $n_1 = n(k_1)$, $n_2 = n(|k - k_1|)$. Using the substitution $f(k) = kn(k)$, this equation can be represented as

$$-4f(k)\int_0^\infty f(k_1)\,dk_1 + \int_{-\infty}^\infty f(k_1)f(|k-k_1|)\,dk_1 = -F(k)' . \qquad (3.4.10b)$$

If we determine $f(k)$ and $F(k)$ at $k < 0$ with the help of $F(-k) = f(k)$ and $F(-k) = F(k)$, then the second term on the left-hand-side of (3.4.10b) is a convolution integral. Consequently, (3.4.10b) may be solved by Fourier transformation. Indeed, the Fourier transformations

$$f(\sigma) = \int_{-\infty}^\infty f(k)e^{ik\sigma}\,dk, \quad F(\sigma) = \int_{-\infty}^\infty F(k)e^{ik\sigma}\,,dk$$

reduce (3.4.10) to the algebraic equation

$$-2f(\sigma)f(0) + f^2(\sigma) + F(\sigma) = 0 .$$

From this we find $f(\sigma)$ and, using the reverse Fourier transform, we obtain the stationary distribution in the form

$$\begin{aligned} n_k &= \frac{1}{k}\int_{-\infty}^\infty e^{-ik\sigma}\left[\sqrt{F(0)} - \sqrt{F(0) - F(\sigma)}\right] d\sigma \\ &= \frac{\sqrt{2}}{k}\int_{-\infty}^{\infty} e^{-ik\sigma} \\ &\quad \times \left[\sqrt{\int_0^\infty F(k)\,dk} - \sqrt{\int_0^\infty \left(1 - e^{ik\sigma}\right)F(k)\,dk}\right] d\sigma . \end{aligned} \qquad (3.4.11)$$

In particular, setting $F_k = F\left[\delta(k - k_0) + \delta(k + k_0)\right]$, we get a solution in the form of a chain of peaks of decreasing intensity

$$n_k = \frac{\sqrt{2F}}{k} \sum_{j=1}^{\infty} \frac{\delta(k - jk_0)}{4j^2 - 1}\,, \tag{3.4.12}$$

It is seen that at $j \gg 1$, the number of waves in the jth peak N_j decreases with the growth of the number j according to the Kolmogorov law $N_j \propto j^{-3}$ [for two-dimensional acoustic turbulence the Kolmogorov solution (3.2.5) equals $n_k \propto k^{-3}$].

Let us now deal with the excitation of two-dimensional sound by the narrow increment Γ_k. We shall assume Γ_k to be nonzero in the small neighborhood Δk around k_0 ($\Delta k \ll k_0$). If there exists a solution in the form of a chain of sharp peaks against a low background (which will be checked by numerical experiments, see below Figs. 12-14), then the properties of such a distribution may be found using perturbation theory with the small parameter $\frac{\Delta k}{k_0}$.

As the zeroth approximation, we shall consider the interaction of peaks between each other. For the value

$$N_j = \int_{jk_0-\Delta k}^{jk_0+\Delta k} n_k dk$$

presenting the number of waves in the jth peak, we obtain

$$\begin{aligned} I_j = &\sum_{l=1}^{j} l(j-l)[N_j N_{j-1} - n_j(N_l N_{j-l}] \\ &- 2\sum_{l=j}^{\infty} l(l-j)[N_j N_{l-j} - N_l(N_j + N_{l-j})] = -\Gamma_0 N_1 \Delta_{j1}\,. \end{aligned} \tag{3.4.13}$$

We have assumed here the first peak to be narrower than the source (which will also be justified below by numeric simulations), Γ_0 is the value of the maximum of the increment. The solution of (3.4.13) evidently follows from (3.4.12) at $F = \Gamma_0 N_1$

$$N_j = \frac{2\Gamma_0}{3j(4j^2-1)}\,. \tag{3.4.14}$$

Certainly, within the full equation $I(k) + \Gamma_k n_k = \partial n_k/\partial t$ such solution does not have external stability, waves outside the peaks (with $k \neq jk_0$) should be excited. The distribution of these waves provides a background whose form may be found from the approximation following (3.4.13) where for the waves with $k \neq jk_0$ we shall take into account only their interaction with the chain of peaks:

$$\begin{aligned} -4n(k)\sum_{j=1}^{\infty} k_0 k j N_j &+ 2\sum_{j=1}^{i} jk_0(k-k_0)N_j n(k-jk_0) \\ &+ 2\sum_{j=1}^{\infty} k_0 j(jk_0 - k)N_j n(jk_0 - k) \\ &+ 2\sum_{j=1}^{\infty} jk_0(jk_0 + k)N_j n(jk_0 + k) = 0\,. \end{aligned}$$

Here $i = [k/k_0]$ is the integral part of the k/k_0-ratio. This equation has the solution

$$n(k) = g(k)k^{-1} , \tag{3.4.15}$$

where $g(k)$ is a periodic function (with period k_0) symmetric relative to the points $jk_0/2$. Comparing (3.4.14) and (3.4.15), we see that the background of the spectrum should with increasing k decrease slower than the peaks. Thus, the peaks may exist only in an intermediate range $k_0 < k < k_*$, where k_* is determined by the ratio of the amplitude of the first peak to that of the background:

$$k_*^2 \simeq \frac{2N_1}{\Delta k_1 n(k_0/2)} = \frac{4\Gamma_0}{9\Delta k_1 n(k_0/2)} .$$

Here Δk_1 is the width of the first peak. For $k > k_*$, an ordinary monotonically deccreasing Kolmogorov spectrum $n_k \propto k^{-3}$ should be observed.

The spectral background gives rise to additional damping for waves with $k = jk_0$, with the decrement

$$4jk_*\bar{g} = 4jk_* \int_0^{k_0} g(k)\, dk ,$$

due to which, with growth of j, the chain of peaks should more and more deviate from the Kolmogorov law, and decrease steeper with N_j.

Considering further orders of the perturbation theory and taking into account the terms quadratic in the amplitude of the background, we can obtain other properties of the above solution (deviations from the Kolmogorov law $N_j \propto j^{-3}$, the fine structure of peaks, etc).

The above consideration of the structure of the spectrum generated by a narrow source were verified by numerical simulations due to *Falkovich* and *Shafarenko* [3.36] who studied the kinetic equation

$$\frac{\partial n(k,t)}{\partial t} = I(k) + \Gamma_k n(k,t) \tag{3.4.16}$$

with a collision integral (3.4.10) and an increment of the form

$$\Gamma_k = \Gamma_0 \exp\left[-\left(\frac{k-k_0}{\Delta k}\right)^2\right] . \tag{3.4.17}$$

In order to provide in numerical simulations an efficient energy sink in the region of large k's one usually sets $n_k \equiv 0$ at $k > k_d$ [3.35–36]. As a result, the discretized collision integral becomes

$$\begin{aligned} I(k) = &\sum_{l=1}^{k} l(k-l)\left[n(l)n(k-l) - n(k)n(l) - n(k)n(k-l)\right] \\ &- 2\sum_{l=k}^{k_d} l(l-k)\left[n(k)n(l-k) - n(l)n(l-k) - n(l)n(k)\right] \\ &- 2n(k)\sum_{l=k_d}^{k_d+k} l(l-k)n(l-k) . \end{aligned} \tag{3.4.18}$$

The last term in (3.4.18) plays the role of nonlinear damping. It is due to the transfer of waves to the region $k > k_d$ at the expense of confluence processes.

In numerical simulations it has been found that, irrespective of the form of the initial condition, the solution of (3.4.16) evolved into the stationary distribution presenting a chain of peaks, which goes with the growth of k over into a monotonically decreasing function. Figure 3.13 illustrates such a steady distribution for $\Gamma_0 = 100$, $k_0 = 20$, $\Delta k = 4$, $k_d = 400$.

The dotted straight line in the figure has a slope of -3, i.e., it corresponds to the Kolmogorov power spectrum. The amplitude of the first peak exceeds that of the background at $k \approx k_0/2$ by two orders of magnitude. The half-width of the first peak at a height which is e times as small as the maximal one is approximately equal to $\Delta k_1 \simeq 1$, i.e., it makes up for a quarter of the half-width of the source. The next peaks are broadened according to the law $\Delta k_j \propto j$. The number of waves in the j-th period

$$N_j = \sum_{k=jk_0-k_0/2}^{jk_0+k_0/2} n_k$$

up to $j = 6$ decreases approximately by the Kolmogorov law $N_j \propto j^{-3}$ (for details see [3.36]). As a consequence of these two features ($\Delta k_j \propto j$, $N_j \propto j^{-3}$), the peaks' amplitudes decay according to the law

$$n(k_j) \approx \frac{N_j}{\Delta k_j} \propto j^{-4} ,$$

whose index differs from the Kolmogorov law by unity. The background of the spectrum decreases in conformity with (3.4.15) by the law $n(k) \propto k^{-1}$.

Such pre-Kolmogorov asymptotics, in which the relay transfer of energy is effected by the waves concentrated in narrow spectral intervals, is not an exclusive property of two-dimensional acoustic turbulence. Figure 3.14 shows the stationary distribution generated by the source (3.4.17) for three-dimensional sound at $\Gamma_0 = 100$, $k_d = 200$, $k_0 = 20$, $\Delta k = 4$.

The dotted straight line in Fig. 3.14 has the slope $-11/2$. The background of the distributions decays with the growth of k slower than the peaks.

Figure 3.15 shows the stationary distribution for capillary waves on a deep fluid with $\Gamma_0 = 100$, ω_d, $\omega_0 = 10$, $\Delta\omega = 2$ (the picture illustrates the situation in the space of eigenfrequencies $\omega_k \propto k^{3/2}$).

In both cases we have a chain of peaks whose amplitudes decrease by a power law with indices differing from the Kolmogorov index by unity. Their widths grow according to a linear law (in frequency space) which is easily understood. Indeed, let the waves exist in the range $(\omega_1 - \Delta\omega_1, \omega_1 + \Delta\omega_1)$. Due to the confluence process $2 \to 1$ the waves appear in the range $(2\omega_1 - 2\Delta\omega_1, 2\omega_1 + 2\Delta\omega_1)$ etc.

Evidently, distributions of this type should in the general case of wave turbulence with a scale-invariant decay dispersion law be generated by a spectrally narrow source.

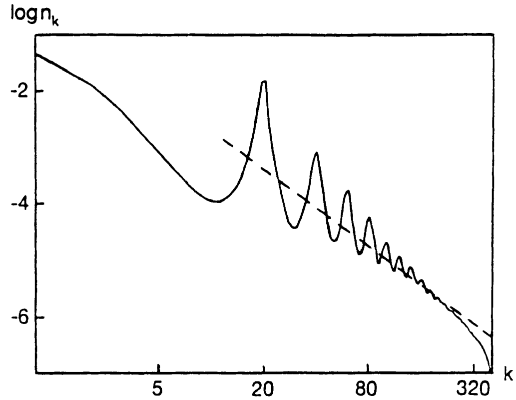

Fig. 3.13. Steady state distribution of two-dimensional sound excited by spectrally narrow pumping [3.36]

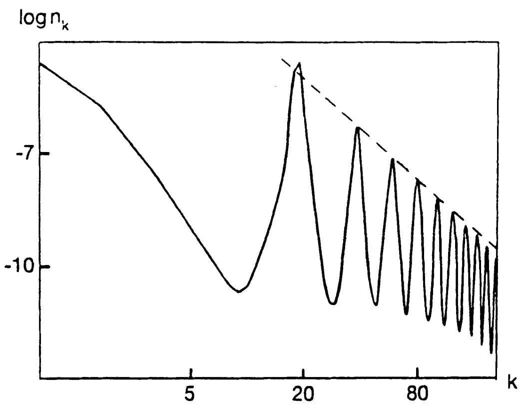

Fig. 3.14. Steady state distribution of three-dimensional sound excited by spectrally narrow pumping [3.36]

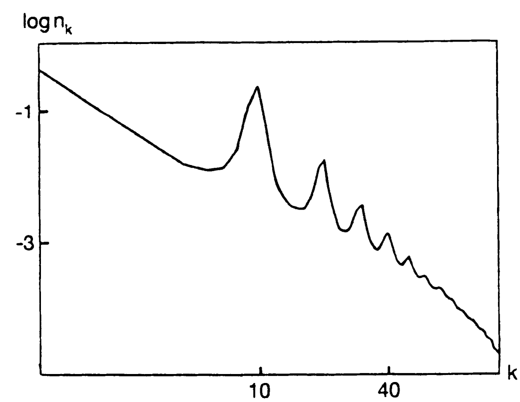

Fig. 3.15. Steady state distribution of capillary waves excited by narrow pumping [3.36]

Having obtained an idea about the structure of the distribution, we return now to discuss the energy flux absorbed by a wave system emitted by a narrow source. Due to narrowness of the first peak, (3.4.6) is replaced by

$$P = \Gamma_0 \omega_0 N_1 k_0^{d-1} \ . \tag{3.4.19a}$$

We shall estimate the number of waves in the first peak of this equation using the Kolmogorov law for N_j

$$N_j \simeq \lambda P^{1/2} k_0 (j k_0)^{-m-d}, \quad N_1 \simeq \lambda P^{1/2} k_0^{1-m-d} \ . \tag{3.4.19b}$$

Substituting (3.4.19b) into (3.4.19a), we again obtain (3.4.8), which is true irrespective of the width of the source.

The results of numerical modeling reported in detail in [3.36] directly support the dependence $P \propto \Gamma_0^2 k_0^{2h}$. For two-dimensional sound P does not depend on k_0; for three-dimensional sound the dependence is almost the inverse one $P \propto k_0^{-1}$, ($i.e., h = -1/2$); for capillary waves a reverse dependence of the flux on the source frequency is observed $P \propto \omega_0^{-1}$ ($h = -3/4$, $\alpha = 3/2$, $k_0^{2h} = \omega_0^{-1}$).

The flux is indeed proportional to the square of the maximal increment Γ_0. The flux depends only weakly on the strength of the integral over the source $\Gamma = \int \Gamma_k \, dk$. For example, for two-dimensional sound and constant amplitude of Γ_0 the flux slowly decreases with the integral strength of the source: at $\Delta k / k_0 = 1/10$ $P = 1534$, at $\Delta k / k_0 = 1/5$ $P = 1442$ and at $\Delta k / k_0 = 2/5$ $P = 1362$. This decrease seems to be due to a smearing out of the first peak with the broadening of the pump region.

3.4.2 Influence of Dissipation

In considering dissipation, one naturally encounters three questions:

1. What kind of function Γ_k describing the behavior of the wave damping decrement is required for the existence of the stationary distribution?
2. Which distortions of the Kolmogorov distribution in the inertial interval are caused by the presence of a remote dissipative region?
3. What is the structure of the stationary spectrum of wave turbulence in the region of strong dissipation?

We have started the discussion of the first question in Sect. 2.3.1. We have proved there the necessary condition for the existence of the nonequilibrium stationary distribution to be the presence of energy damping ($\Gamma_k < 0$) for large k. Now we shall clarify the further conditions which the asymptotics of the function Γ_k must satisfy at $k \to \infty$ [in addition to negativity] to ensure the existence of a Kolmogorov distribution in the inertial interval. In the dissipative region, the function Γ_k absorbes part of the flux transmitted by the Kolmogorov spectrum through the inertial interval. Let us substitute the Kolmogorov solution $n_k \propto k^{-s_0}$ into the stationary kinetic equation (2.2.19)

$$\frac{dP_k}{dk} = \Gamma_k E_k = \Gamma_k \pi (2k)^{d-1} \omega_k n_k \ .$$

It is seen that if the dissipative function Γ_k at $k \to \infty$ increases slower than

$$k^{\alpha+d-s_0} = k^h \ ,$$

then the dissipation is not able to absorb the whole flux. According to (3.4.2), the index h specifies the location of the energy-containing region of the spectrum. In the next section [see (3.4.23)], we shall see that the characteristic time of nonlinear wave interaction on the background of a Kolmogorov distribution as a function of k behaves as

$$t_{\mathrm{NL}} \propto k^{-h} \ .$$

This expression is understood by the following considerations: Let at some k there be a deviation of the distribution from the Kolmogorov one. Then, with k changed by a factor of A, the characteristic evolution time of this perturbation varies by a factor of A^{-h}.

Thus there are two physical explanations for the requirement that the function Γ_k should for $k \to \infty$ increase faster than k^h to ensure the existence of the stationary Kolmogorov distribution. On the one hand, this condition implies that the main portion of the energy should be absorbed in the short-wave region. On the other hand, it accounts for the quicker growth (or slower decrease) of the wave damping decrement compared to the inverse time of nonlinear interaction. It is only in the case of an infinite number of modes in a system that the requirement of a vanishing flux P at $k \to \infty$ does make sense. If there exists a maximal wave number k_{m} and the corresponding maximal frequency ω_{m}, then $n(\omega_{\mathrm{m}})$ is not bound to become zero. Indeed, having determined the flux in the case of the finite and discrete ω-space $\omega_l = l\omega_0$, $l = 1, \dots, L$ by

$$P_l = \sum_{i=1}^{l} i I_i \ ,$$

we see that $P_1 = P_L = 0$ holds due to the energy conservation law. Therefore, in this case, the Γ_l-function may be rather arbitrary in the whole ω-space. The only requirement which remains is the condition of entropy withdrawal (2.2.16) or, in the discrete form,

$$\sum_{i=1}^{L} \Gamma_i \le 0 \ .$$

Here the summation is to be carried out over all modes.

In the case of a model system with the ω-space consisting of three points (three spherical harmonics)

$$\frac{dn_1}{dt} = \Gamma_1 n_1 - 2V_1(n_1n_2 - n_1n_3 - n_2n_3) - 2V_2(n_1^2 - 2n_1n_2)$$
$$\frac{dn_2}{dt} = \Gamma_2 n_2 - 2V_1(n_1n_2 - n_1n_3 - n_2n_3) + V_2(n_1^2 - 2n_1n_3) \quad ,$$
$$\frac{dn_3}{dt} = \Gamma_3 n_3 + 2V_1(n_1n_2 - n_1n_3 - n_2n_3)$$

one can prove that the inequality $\Gamma_1 + \Gamma_2 + \Gamma_3 < 0$ is a necessary and sufficient condition for the existence of at least one (there may be more than one) steady state with positive n_1, n_2, n_3. We leave the derivation to the reader as a small algebraic exercise.

For an arbitrary number of modes in the system, it is only in the limit $\sum \Gamma_l \to -0$ that condition (2.2.16) may be proved to be sufficient. In this case, though separate Γ_l may be rather large, the system is almost in equilibrium. Really, considering (2.2.10), the equation for the rate of entropy variation becomes in the discrete case

$$\frac{dS}{dt} = \sum_{i,l} U(i,l)\frac{[n_i n_l - n_{i+l}(n_i + n_l)]^2}{n_i n_l n_{i+l}} + \sum_l \Gamma_l \, ,$$

where $U(i,l)$ is a positive function expressed via the square of the interaction coefficient and wave frequency. Hence it appears that at $\sum \Gamma_l \to -0$, each of the square brackets in the first sum should tend to zero. This is only possible for a Rayleigh-Jeans distribution $n_l = A/l$; here l is the coordinate in the ω-space; n_l the wave density in the k-space taken to be a function of the frequency. We consider now the isotropic case. The stationary distribution may be constructed using perturbation theory: $n_l = A/l + \psi_l + \ldots$. Substituting such a distribution into the discrete analog of the kinetic equation, one can show that $\psi_l \propto \Gamma_l$, $A \propto (\sum \Gamma_l^2)/(\sum \Gamma_l)$, the small expansion parameter being $(\sum \Gamma_l)^2 / \sum \Gamma_l^2$. Thus, at $\sum \Gamma_l \to -0$, the effective "temperature" of the stationary distribution tends to infinity implying that the characteristic saturation time should also increase.

Numerical simulation of the formation of a steady-state, given a source and the dissipation being distributed in the ω-space, was carried out by *Falkovich* and *Ryzhenkova* [3.37]. Following their paper, we shall consider at first the discrete kinetic equation describing capillary waves on deep water. In this case $\alpha = 3/2$, $m = 9/4$, $d = 2$, and the kinetic equation may be written in the form

$$\frac{\partial n_k}{\partial t} = \sum_{l=1}^{k-1} U(k,l)[n_l n_{k-l} - n_k(n_l + n_{k-l})]$$
$$+ \sum_{l=k+1}^{L} U(l,k)[n_k n_{l-k} - n_l(n_k + n_{l-k})] + \Gamma_k n_k \, .$$

Here

$$U(k,l) = k^{8/3}\frac{x^{2/3}(1-x)^{2/3}}{\sqrt{4x^{4/3}(1-x)^{4/3}-[1-x^{4/3}-(1-x)^{4/3}]^2}} \times \left\{\frac{(1-x^{2/3})^2}{(1-x)^{1/3}}+\frac{[1-(1-x)^{2/3}]^2}{x^{1/3}}-[x^{2/3}-(1-x)^{2/3}]^2\right\}^2$$

and $x = k/l$. The Γ_k-function was chosen in the form of $\Gamma_k = B\Delta_{k1} - \sqrt{k}$.

Figure 3.16 illustrates the time dependence of the total energy of a distribution for $L = 100$, $B = 100$, $\sum \Gamma_k = -571.4$. It is seen that at $t \gtrsim 3.8$ the evolution becomes exponential $E(t) = E_0 - E_1 \exp(-t/\Delta t)$. Determining the slope of the curve, one can find the characteristic saturation time Δt. It is interesting to follow the variation in Δt with an increasing number of modes L. The Γ_k-function grows with k slower than $k^{-h} = k^{3/4}$, therefore there cannot be a stationary distribution in an infinite system. In a finite system, however, the saturation time Δt falls off with the growth of L, which is due to the growth of the modulus $|\sum \Gamma_k|$, see Fig. 3.17. It is seen that at $\sum \Gamma_k \to 0$, $\Delta t^{-1} \propto \sum \Gamma_k$.

The two other wave systems with a decay dispersion law which we have considered above, the gravitational-capillary waves on shallow water (two-dimensional sound) and three-dimensional sound, exhibit in numerical simulations a similar behavior.

Let us now answer the second question posed at the beginning of this subsection. We consider the k-space and assume that starting from some k_d, there is a strong wave damping ($\Gamma_k < 0$), leading to a quick reduction of the occupation numbers n_k at $k \gtrsim k_\mathrm{d}$. In the region $k_0 \ll k \ll k_\mathrm{d}$, where k_0 gives the scale of the source, the stationary distribution should be close to the Kolmogorov distribution $n_k \propto k^{-m-d}$ – provided the condition for the locality of the interaction is satisfied, i.e., if the collision integral converges on the Kolmogorov solution. In Sect. 3.1 we have shown that the collision integral converges on power solutions $n_k \propto k^{-s}$ if the index s is a number out of the locality interval

$$m_1 + d - 1 + 2\alpha \equiv s_1 > s > s_2 \equiv 2m - m_1 + d + 1 - 2\alpha$$

and if for $k_1 \ll k$ the asymptotics of the interaction coefficient has the form

$$|V(k, k_1, k_2)|^2 = V^2 k_1^{m_1} k^{2m-m_1} \; . \tag{3.4.20}$$

The locality interval exists if

$$m_1 - m - 1 + 2\alpha > 0 \; .$$

If this condition is satisfied, the deviation of the stationary distribution from a power distribution, induced by the effect of a remote sink, is small and may be found with the help of perturbation theory in the small parameter k/k_d.

According to (3.1.11), the angle-averaged three-wave collision integral equals [except for a constant factor]

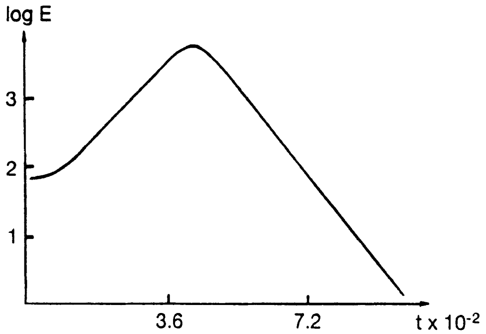

Fig. 3.16. The logarithm of E is given as a function of time t

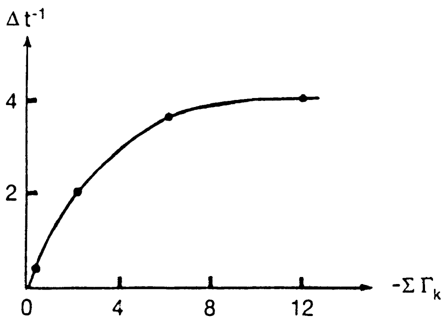

Fig. 3.17. The saturation time as a function of the entropy production rate

$$I(k) \propto \int_0^\infty \int_0^\infty [R(k, k_1, k_2) - R(k_1, k, k_2) - R(k_2, k, k_1)]\, dk_1 dk_2,$$
$$R(k, k_1, k_2) = |V_{k12}|^2 (k_1 k_2)^{d-1} \Delta_d^{-1} \delta(k^\alpha - k_1^\alpha - k_2^\alpha) \times \Theta(k - k_1)(n_1 n_2 - n_k n_1 - n_k n_2) \; . \qquad (3.4.21)$$

If, in an infinite interval $k \in (0, \infty)$, the Kolmogorov power solution $n_k = Dk^{-m-d}$ is realized, the collision integral is identically equal to zero $I(k) = 0$. The absence of waves at $k > k_\mathrm{d}$ (we shall set $n_k \equiv 0$ at $k > k_\mathrm{d}$) leads to a slight deviation of the collision integral from zero for $k \ll k_\mathrm{d}$

$$\delta I_1 = 2D^2 \int_{k_\mathrm{d}}^\infty |V(k_1, k, k_0)|^2 k_1^{d-1} \Delta_d^{-1}(k_1, k, k_0) k_0^{d-\alpha} \times \left[(kk_0)^{-m-d} - (kk_1)^{-m-d} - (k_1 k_0)^{-m-d}\right] dk_1 \; . \qquad (3.4.22)$$

Here $k_0^\alpha = k_1^\alpha - k^\alpha$. This additional contribution δI_1 owes its origin to the finite character of the inertial interval. For the distribution n_k to be stationary, δI_1 must be compensated for by a contribution δI_2, due to the small deviation of the solution from a pure power law ($n_k = n_k^0 + \delta n_k$, $\delta n_k \ll n_k^0$ at $k \ll k_\mathrm{d}$)

$$
\begin{aligned}
\delta I_2 &= \hat{L}_k \delta n_k \\
&= 2 \int_0^\infty \int_0^\infty (k_1 k_2)^{d-1} \Delta_d^{-1} \Big\{ |V_{k12}|^2 \delta(k^\alpha - k_1^\alpha - k_2^\alpha) \Theta(k - k_1) \\
&\quad \times \left[\delta n_1 \left(n_2^0 - n_k^0 \right) - \delta n_k n_1^0 \right] \\
&\quad - |V_{1k2}|^2 \delta(k_1^\alpha - k^\alpha - k_2^\alpha) \Theta(k_1 - k) \\
&\quad \times \left[\delta n_k \left(n_2^0 - n_1^0 \right) + \delta n_1 \left(n_k^0 + n_2^0 \right) + \delta n_2 \left(n_k^0 - n_1^0 \right) \right] \Big\} \, dk_1 dk_2 \, .
\end{aligned}
\tag{3.4.23}
$$

Here $\hat{L}_k$ is the operator of the kinetic equation linearized with respect to δn_k. This integral operator is scale-homogeneous $\hat{L}_{\lambda k} = \lambda^{m-\alpha} \hat{L}_k$ with the index $m - \alpha = -h$, see Sect. 4.2 for details.

Since the linearized collision integral $\hat{L}_k$ also determines the nonstationary behavior of small perturbations $\partial \delta n(k,t)/\partial t = \hat{L}_k \delta n(k,t)$ [see below (4.1.1)], the characteristic evolution time of such perturbations is proportional to k^h, as indicated in the preceding subsection.

Thus, in order to determine δn_k, we should solve the linear inhomogeneous integral equation

$$
\delta I_1 + \delta I_2 = \delta I_1 + \hat{L}_k \delta n_k = 0 \, . \tag{3.4.24}
$$

At first we calculate δI_1. Since in (3.4.22) $k_1 > k_{\mathrm{d}} \gg k$, we shall make use of the asymptotics (3.4.20) of the interaction coefficient and expand the square bracket in (3.4.22) up to the first nonvanishing terms in $(k/k_1)^\alpha$ to obtain

$$
\delta I_1 = \frac{2D^2(m+d)V^2}{\alpha(m_1 + 2\alpha - 1 - m)} k_{\mathrm{d}}^{m+1-2\alpha-m_1} k^{m_1 - m + \alpha - d - 1} \, . \tag{3.4.25}
$$

In conformity with the locality condition (3.1.12c) we have $m_1 + 2\alpha - 1 - m > 0$. Due to the homogeneity of the $\hat{L}_k$ operator the equation

$$
\hat{L}_k \delta n_k = k^{-x} \tag{3.4.26}
$$

has the power solution

$$
\delta n_k = k^{h-x} \left[W(x-h) D V^2 \right]^{-1} \tag{3.4.27}
$$

where $W(s)$ is a dimensionless integral obtained by substituting $\delta n_k = k^{-s}$ into (3.4.23) and factoring out $DV^2 k^{-h-s}$. The solution of (3.4.26) has the form of (3.4.27) only if the index $s = x - h$ is an element of the locality interval of the collision integral. It is easy to see that the locality interval (s_1, s_2) is the same for complete (3.1.11) and linearized collision integral (3.4.23). In our case, i.e., with (3.4.25) substituted into (3.4.24) we have $x = m + d + 1 - m_1 - \alpha$ and the quantity $x - h = 2m - m_1 + d + 1 - 2\alpha$ coincides with the lower bound of the locality interval s_2 [see (3.1.12a)]. Neglecting the slow logarithmic dependence in the integration, we obtain

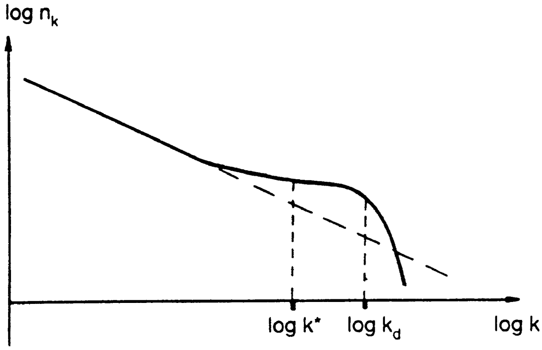

Fig. 3.18. Distortion of the Kolmogorov spectrum caused by damping

$$\delta n_k = Dk^{-m-d} \frac{2(m+d)\ln^{-1}\left(\frac{k_d}{k}\right)}{\alpha(m_1 - m + 2\alpha - 1)} \left(\frac{k}{k_d}\right)^{m_1 - m + 2\alpha - 1} . \tag{3.4.28}$$

Formula (3.4.28) is valid at $k \ll k_d$ and shows that the finiteness of the sink scale leads to increased occupation numbers in the inertial interval, since $\delta n_k > 0$. The δn_k-value grows with k, i.e., the distribution has a somewhat smoother slope. Of course, at $k \simeq k_d$, a sharp fall-off of n_k should take place, which now cannot be described in terms of perturbation theory. Thus, the stationary distribution should show the features depicted in Fig. 3.18, where the dashed line corresponds to the Kolmogorov power solution. The dependence of $\lg n_k$ on $\lg k$ should have an inflection point (marked in Fig. 3.18 as k^*), where the index of the solution for the current $s(k) = d \lg n_k / d \lg k$ passes through a minimum. Accumulation of waves at $k \lesssim k^*$ seems to be induced by the "bottle-neck" effect, arising due to a reduction of the flux at $k \gtrsim k^*$ because of a decrease in the occupation numbers at $k \lesssim k_d$. This picture is confirmed by numerical experiments carried out for capillary waves on the surface of a deep fluid and for sound [3.37]. Thus, Fig. 3.19a (top) represents the wave vector dependence of the current index of the stationary solution as obtained in the numerical simulation of two-dimensional acoustic turbulence. One observes a pronounced minimum $s(k)$ at $k^* = 77$. Using different k_d in the numerical experiments, one can verify that this effect is associated with the finiteness of the sink-scale; the position of the minimum of the index is proportional to $k_d : k^* = Bk_d$; for two-dimensional sound and damping appearing as a jump at $k = k_d$, we have $B \approx 1/3$. Fig. 3.19b (middle) illustrates the behavior of the index $s(k)$ for three-dimensional sound in a similar situation.

Finally, let us turn to the last question under consideration. We shall discuss the behavior of the stationary turbulence spectrum in the dissipative region at $k \gg k_d$. We take the damping decrement Γ_k to increase with k faster (or to decrease slower) than the inverse time of wave interaction in the inertial interval (i.e., the k^{-h} function). The implications are that the occupation numbers should

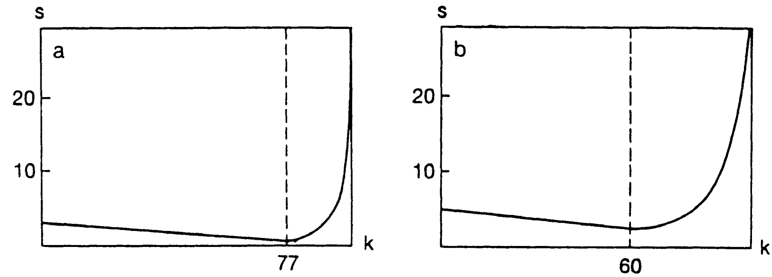

Fig. 3.19. Current index of the stationary spectra of acoustic turbulence for a) $d = 2$ and b) $d = 3$

in the dissipative region with growing k fall off faster than the Kolmogorov law. The character of this fall off is determined by the type of wave interaction in the dissipative region: they may predominantly interact with each other or with waves from the inertial interval. For waves with very different wave numbers, the dependence of their interaction time on k may be found by substituting into the collision integral (3.4.21) the asymptotic expression (3.4.20) for the interaction coefficient at $k_1 \ll k$

$$t_1^{-1}(k) \propto k^{2m-m_1+1-\alpha} . \tag{3.4.29}$$

If the damping decrement increases with k faster than t_1^{-1}, then the asymptotic of the distribution at $k \to \infty$ is determined by the interaction of waves in the dissipative region with each other. In this case, the exponential "quasi-Planck" spectrum $n_k = D\omega_k^{-b}\exp(-\omega_k/\omega_d)$ is formed. (Similar exponential asymptotics appear in the short-wave region with a free evolution of the distribution – see Sect. 3.4 below.) Really, assuming that the damping decrement Γ_k grows by the power law $\Gamma_k = -Gk^a$ (e.g., for viscosity $a = 2$) and that the main contribution to the collision integral stems from the integration over the region $k_1 \ll k$, we obtain the stationary kinetic equation in the form

$$Gk^a n_k = Dk^{2m-m_1+1-\alpha} n_k \int_{k_d}^{k} k_1^{d-1-\alpha+m_1-b} \times \left[1-\exp\left(-k_1^{\alpha}/k_d^{\alpha}\right)\right]^2 dk_1 . \tag{3.4.30}$$

As the integral in the right-hand-side of (3.4.30) is a nondecreasing function of k, a solution with such a form may only exist if the inequality

$$a > 2m - m_1 + 1 - \alpha \tag{3.4.31}$$

is satisfied. For all three systems under discussion and for viscous damping, this inequality is satisfied. Indeed, the numerical simulation [3.37] shows that for

$$\Gamma_k = G_1\Delta_{k1} - G_2k^2$$

in the strong dissipation region, the occupation numbers decrease according to an exponential law. For two-dimensional sound and capillary waves on the surface

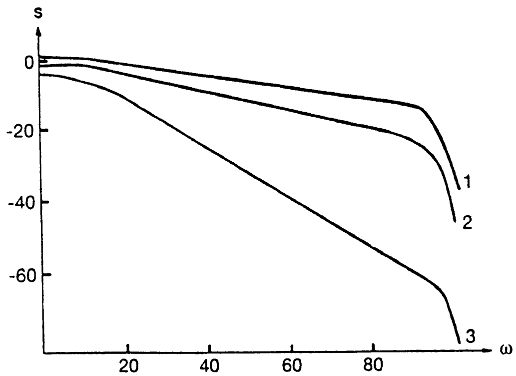

Fig. 3.20. Current index of a stationary spectrum in the damping region

of a deep fluid [3.37] the dependence of the steady distribution vs. the frequency is shown in Fig. 3.20. The section with a linear decrease of $s(\omega)$ corresponds to the exponential reduction of $n(\omega)$.

4. The Stability Problem and Kolmogorov Spectra

In this chapter we examine those of the solutions obtained in Chap. 3 which are suitable for modeling reality. Obviously, one can expect to observe only Kolmogorov distributions that are stable with regard to perturbations. Sections 4.1 and 4.2 deal with the behavior of distributions slightly differing from Kolmogorov solutions. The reason for the difference may be either a small variation in the boundary conditions (i.e., in the source and in the sink), or immediate modulation of the occupation numbers of the waves. Small perturbations are studied in terms of linear stability theory where the main object is the kinetic equation linearized with respect to the deviation of the resulting distribution from a Kolmogorov one. In Sect. 4.1, the basic properties of the linearized collision integral are considered and the neutrally stable modes, i.e., small steady modulations of the Kolmogorov distributions are obtained. Section 4.2 presents a mathematically correct linear stability theory of the Kolmogorov solutions, formulates the stability criterion and exemplifies instabilities. The last section of this chapter discusses the evolution of distributions which are initially far from Kolmogorov distributions.

4.1 The Linearized Kinetic Equation and Neutrally Stable Modes

4.1.1 The Linearized Collision Term

We shall start with the decay case. Assuming $n(\boldsymbol{k},t) = n(k)+\delta n(\boldsymbol{k},t)$, $\delta n(\boldsymbol{k},t) \ll n(k)$ and linearizing (2.1.12) with respect to the deviation $\delta n(\boldsymbol{k},t)$ we obtain

$$\begin{aligned}\frac{\partial \delta n(\boldsymbol{k},t)}{\partial t} = \int & d\boldsymbol{k}_1 d\boldsymbol{k}_2 \big\{ \mid V(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) \mid^2 \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2) \\ & \times \delta(\omega_k - \omega_1 - \omega_2)\big[[n(k_2)-n(k)]\delta n(\boldsymbol{k}_1,t) \\ & + [n(k_1)-n(k)]\delta n(\boldsymbol{k}_2,t) - [n(k_1)+n(k_2)]\delta n(\boldsymbol{k},t)\big] \\ & - 2 \mid V(\boldsymbol{k}_1,\boldsymbol{k},\boldsymbol{k}_2) \mid^2 \delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2)\delta(\omega_1-\omega_k-\omega_2) \\ & \times \big[[n(k)-n(k_1)]\delta n(\boldsymbol{k}_2,t) + [n(k)+n(k_2)]\delta n(\boldsymbol{k}_1,t) \\ & + [n(k_2)-n(k_1)]\delta n(\boldsymbol{k},t)\big]\big\} \\ = \hat{L}_k \delta n(\boldsymbol{k},t) = \int & L(\boldsymbol{k},\boldsymbol{k}_1)\delta n(\boldsymbol{k}_1,t)\, d\boldsymbol{k}_1 \ .\end{aligned} \qquad (4.1.1)$$

At the end of the preceding section we have already encountered the angle-averaged operator $\hat{L}$, see (3.4.23). If $n(k)$ is a stationary equilibrium distribution $n(k) = T/\omega(k)$, the kernel $L(\boldsymbol{k}, \boldsymbol{k}_1)$ of the integral operator has the important property of being symmetric [4.1]. Namely, one can normalize the function affected by the operator $\delta n(\boldsymbol{k}, t) = f(k)\varphi(\boldsymbol{k}, t)$ (using the function $f(k)$ to be defined below), so that the resulting kernel $M(\boldsymbol{k}, \boldsymbol{k}_1) = L(\boldsymbol{k}, \boldsymbol{k}_1)f(k_1)$ is symmetric

$$M(\boldsymbol{k}, \boldsymbol{k}_1) = M(\boldsymbol{k}_1, \boldsymbol{k}), \tag{4.1.2}$$

and the corresponding operator $\hat{M}$ hermitian. For a proof it is convenient to write the kernel of the renormalized operator (4.1.1) in the form

$$\begin{aligned} M(\boldsymbol{k}, \boldsymbol{k}_1) = \int d\boldsymbol{k}_2 \{ & 2U(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)[n(k_2) - n(k)] f(k_1) \\ & - U(\boldsymbol{k}, \boldsymbol{k}_2, \boldsymbol{k} - \boldsymbol{k}_2)[n(k_2) + n(|\boldsymbol{k} - \boldsymbol{k}_2|)] f(k)\delta(\boldsymbol{k} - \boldsymbol{k}_1) \\ & + 2U(\boldsymbol{k}_2, \boldsymbol{k}, \boldsymbol{k}_2 - \boldsymbol{k})[n(k_2) - n(|\boldsymbol{k}_2 - \boldsymbol{k}|)] f(k)\delta(\boldsymbol{k} - \boldsymbol{k}_1) \\ & + 2U(\boldsymbol{k}_1, \boldsymbol{k}, \boldsymbol{k}_2)[n(k) + n(k_2)] f(k_1) \\ & + 2U(\boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k})[n(k_2) - n(k)] f(k_1) \} \end{aligned} \tag{4.1.3}$$

where the function

$$U(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) = | V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) |^2 \, \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\delta(k^\alpha - k_1^\alpha - k_2^\alpha)$$

is invariant with regard to rearrangements of the second and third arguments. It should be noted that the terms in (4.1.3) containing $\delta(\boldsymbol{k} - \boldsymbol{k}_1)$ are, apparently, symmetric. Let us consider the last term in (4.1.3). If $n(k) \propto \omega^{-1}(k)$, then

$$U(\boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k}) f(k_1) \frac{\omega_k - \omega_2}{\omega_k \omega_2} f(k_1) = -U(\boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k}) \frac{f(k_1)\omega_1}{\omega_k \omega_2}.$$

Hence, this expression will also be symmetric if we choose $f(k_1) = \omega^{-2}(k_1)$. The first and fourth terms in (4.1.3) go upon the substitution $k \leftrightarrow k_1$ over into each other.

The symmetric character of the linearized four-wave collision term is proved in a similar way:

$$\begin{aligned} \frac{\partial \delta n(\boldsymbol{k}, t)}{\partial t} = \int & |T_{k123}|^2 \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\delta(\omega_k + \omega_1 - \omega_2 - \omega_3) \\ & \times \{[n(k_1)n(k_2) + n(k)n(k_2) - n(k)n(k_1)]\delta n(\boldsymbol{k}_3, t) \\ & + [n(k)n(k_3) + n(k_1)n(k_3) - n(k)n(k_1)]\delta n(\boldsymbol{k}_2, t) \\ & + [n(k_2)n(k_3) - n(k)n(k_2) - n(k)n(k_3)]\delta n(\boldsymbol{k}_1, t) \\ & + [n(k_2)n(k_3) - n(k_1)n(k_2) - n(k_1)n(k_3)]\delta n(\boldsymbol{k}, t)\} \\ & \times d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \\ = \hat{W}_k \delta n(\boldsymbol{k}, t) = \int & W(\boldsymbol{k}, \boldsymbol{k}')\delta n(\boldsymbol{k}', t) d\boldsymbol{k}' \, . \end{aligned} \tag{4.1.4}$$

For the case $n(k) = T/\omega_k$ we should choose again $f(k) = \omega^{-2}(k)$ and the kernel $Q(\boldsymbol{k}, \boldsymbol{k}') = W(\boldsymbol{k}, \boldsymbol{k}')\omega^{-2}(k')$ is also symmetric $Q(\boldsymbol{k}, \boldsymbol{k}') = Q(\boldsymbol{k}', \boldsymbol{k})$. Indeed, because of the symmetry of the interaction coefficients $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = T(\boldsymbol{k}_1, \boldsymbol{k}, \boldsymbol{k}_2, \boldsymbol{k}_3)$ with regard to the substitution $k \leftrightarrow k'$, it transforms each of the four terms in (4.1.4) into itself.

The hermitian character of the operators $\hat{M}_k$ and $\hat{Q}_k$ implies that the eigenvalues are real, i.e., there are no oscillations of the wave system around the equilibrium state. In this context it is useful to recall the theorem proved in Sect. 2.2, it states that distributions evolving according to the kinetic equations are associated with a growing entropy which has its maximum in the equilibrium state. In the next section we shall see that this theorem in liaison with the hermitian character of the linearized collision integral ensures that there is no absolute instability of equilibrium distributions. This prevents us from finding a perturbation which would grow with time in all points of the $\boldsymbol{k}$-space.

If the stationary solution $n(k)$ is not in thermodynamical equilibrium, the operator of the linearized collision integral is nonhermitian as may be directly verified. Therefore, deviations from nonequilibrium stationary distributions may behave absolutely differently. As we shall see in the next section, we may not just observe oscillations of the occupation numbers around the stationary values, but also various instabilities of the Kolmogorov solutions.

There is one more property of the operators (4.1.1, 4) which is common to equilibrium and isotropic Kolmogorov distributions $n(k)$. Owing to the parity of the stationary solution $n(-k) = n(k)$ and the δ-functions, the operators $\hat{L}$ and $\hat{W}$ conserve the parity of the function of k on which they act. This means that application of the operator on the even (odd) function $\delta n(k)$ results in an even (odd) function, respectively.

4.1.2 General Stationary Solutions and Neutrally Stable Modes

As we shall now show, deformations of stationary solutions may neither grow nor decrease. Such stationary additions are called *neutrally stable modes*. They owe their existence to the fact that the general stationary (equilibrium or nonequilibrium) solution depends on several parameters. As a consequence, the neutrally stable modes, i.e., the stationary solutions of the linearized equations (4.1.1, 4) are readily obtained from dimensional analysis.

Let us start with the equilibrium case. For waves with the decay dispersion law, the general equilibrium solution (2.2.13) depends on both integrals of motion of the system (i.e., the ones for energy and momentum)

$$n(\boldsymbol{k}, T, \boldsymbol{u}) = \frac{T}{\omega(k) - (\boldsymbol{k}\boldsymbol{u})} .$$

Such a solution is called *the drift equilibrium distribution.* At small momentum of the system ($u \to 0$), this expression may for $(\boldsymbol{k}\boldsymbol{u}) \ll \omega(k)$ be expanded into a series

$$n(\boldsymbol{k}, T, \boldsymbol{u}) = \frac{T}{\omega(k)} - \frac{(\boldsymbol{k}\boldsymbol{u})T}{\omega^2(k)} = n_0(k) + \delta n_0(\boldsymbol{k}) \; .$$

Hence, $\delta n_0(\boldsymbol{k}) \propto (\boldsymbol{k}\boldsymbol{u})/\omega^2(k)$ is a stationary solution of (4.1.1). For waves with a nondecay dispersion law there are three integrals of motion (energy, momentum, and action), the general equilibrium solution (2.2.14) has the form

$$n(\boldsymbol{k}, T, \boldsymbol{u}, \mu) = \frac{T}{\omega(k) - (\boldsymbol{k}\boldsymbol{u}) - \mu} \; ,$$

and the neutrally stable modes are obtained in a similar way

$$\delta n_0(\boldsymbol{k}) \propto (\boldsymbol{k}\boldsymbol{u})/\omega^2(k), \quad \delta n_1(k) \propto \omega^{-2}(k) \; .$$

The general nonequilibrium stationary solution should depend on all fluxes of the integrals of motion. A dimensional analysis shows that the stationary solution for the three-wave kinetic equation may be written in the form

$$\begin{aligned} n(\boldsymbol{k}, P, \boldsymbol{R}) &= \lambda P^{1/2} k^{-m-d} f(\xi), \\ \xi &= \frac{(\boldsymbol{R}\boldsymbol{k})\omega(k)}{P k^2} . \end{aligned} \tag{4.1.5}$$

Here $P, \boldsymbol{R}$ are the fluxes of energy and momentum, respectively, and λ is the dimensional Kolmogorov constant. Since the medium is assumed to be isotropic the solution depends on the scalar product $(\boldsymbol{R}\boldsymbol{k})$. The form of the dimensionless function $f(\xi)$ has so far only been established for sound with positive dispersion (see Sect. 5.1 below). In the general case, one can only indicate the asymptotics $f(\xi)$ at $\xi \to 0$ where the solution (4.1.5) should go over to the isotropic Kolmogorov distribution $n_0(k) = \lambda P^{1/2} k^{-m-d}$, therefore at $\xi \to 0$ we have $f(\xi) \to 1$. Assuming $f(\xi)$ to be analytical at zero and expanding (4.1.5) we obtain a stationary anisotropic correction to the isotropic solution

$$\begin{aligned} n(\boldsymbol{k}, P, \boldsymbol{R}) &\approx \lambda P^{1/2} k^{-m-d} + f'(0) k^{-m-d} (\boldsymbol{R}\boldsymbol{k}) \omega(k) P^{-1/2} k^{-2} \\ &= n_0(k) + \delta n(\boldsymbol{k}) \; . \end{aligned} \tag{4.1.6}$$

The solution (4.1.6) was first found by *Kats* and *Kontorovich* [4.2] and is called *the drift Kolmogorov solution.* It should be emphasized that, in contrast to the drift equilibrium distributions, it cannot be derived from the isotropic solution via the "Galilean" substitution $\omega \to \omega - (\boldsymbol{k}\boldsymbol{u})$ [4.2]. We used the quotation marks because this term refers in the given context not to a transition to a moving reference system [in such a transition, wave amplitudes transform according to $c(\boldsymbol{k}, t) \to c(\boldsymbol{k}, t) \exp[-i(\boldsymbol{k}\boldsymbol{u})t]$ while the simultaneous pair correlators $n(\boldsymbol{k}, t)$ do not change at all]. The equilibrium solutions are invariant with regard to the substitution $\omega(k) \to \omega(k) - (\boldsymbol{k}\boldsymbol{u})$ since the integrals of motion enter the entropy extremum condition additively. This invariance has nothing in common with the Galilean invariance. The lack of such an invariance in the nonequilibrium case implies that the Kolmogorov solution does possibly not correspond to an extremum of a functional.

In the nondecay case there are three integrals of motion and it should be possible to give the general nonequilibrium stationary solution as a function of two dimensionless variables

$$\begin{aligned} n(\boldsymbol{k}, P, Q, \boldsymbol{R}) &= \lambda_1 P^{1/3} k^{-d-2m/3} F[\omega_k Q/P,\ \omega_k(\boldsymbol{R}\boldsymbol{k})/Pk^2] \\ &= \lambda_1 P^{1/3} k^{-d-2m/3} F(\eta, \xi) \ . \end{aligned} \tag{4.1.7}$$

Assuming the function $F(\eta, \xi)$ to be analytical in both variables, we can obtain from (4.1.7) a small stationary correction to the solution with an energy flux $n_0(k) = \lambda_1 P^{1/3} k^{-d-2m/3}$. For for the drift solution we thus obtain

$$\begin{aligned} n(\boldsymbol{k}, P, \boldsymbol{R}) &\approx n_0(k) + \delta n_0(\boldsymbol{k}) \\ &= \lambda_1 P^{1/3} k^{-d-2m/3} + k^{-d-2m/3-2}(\boldsymbol{R}\boldsymbol{k}) P^{-2/3} \omega_k \left(\frac{\partial F}{\partial \xi}\right)_{\xi=\eta=0} \ . \end{aligned} \tag{4.1.8}$$

It is seen that for the drift Kolmogorov corrections to the solution with an energy flux the general formula

$$\frac{\delta n_0(k)}{n_0(k)} \propto \xi = \frac{(\boldsymbol{R}\boldsymbol{k})\omega_k}{Pk^2} \tag{4.1.9}$$

holds in the decay and nondecay cases [4.3]. If the system does not possess a momentum flux the general solution (4.1.7) goes over into the solution (3.2.25) of the isotropic stationary equation (3.1.23):

$$n(k, P, Q) = \lambda_1 P^{1/3} k^{-d-2m/3} F(\eta), \quad F(\eta) \equiv F(\eta, 0) \ . \tag{4.1.10}$$

The asymptotics of $F(\eta)$ may be found from the following considerations: at $Q = 0$ solution (4.1.10) should be transformed to a solution specified only by the energy flux and at $P = 0$ by the wave action flux. Thus, at $\eta = \omega Q/P \to 0$, $F(\eta) \to 1$, and at $\eta \to \infty$, $F(\eta) \to a\eta^{1/3}$, where a is some dimensionless constant. Physically such a solution corresponds to two well separated sources in ω-space. It describes the behavior of the Kolmogorov-like distribution between them with its energy flux in the small-frequency region which for large frequencies goes over to a solution with constant wave action flux. Expanding (4.1.10) for small η, we obtain a stationary addition to the solution with the energy flux. This addition carries the small action flux

$$\begin{aligned} n(k, P, Q) &= \lambda_1 P^{1/3} k^{-d-2m/3} + F'(0) Q P^{-2/3} \omega_k k^{-d-2m/3} \\ &= n_0(k) + \delta n_1(k) \ . \end{aligned} \tag{4.1.11}$$

The solution which transfers the small energy flux against the background of the main distribution with the wave action flux can also be obtained from (4.1.10) by expanding $F(\eta)$ in the $1/\eta$ parameter at $\eta \gg 1$. Another possibility would be to write the general solution (4.1.7) in the form

$$\begin{aligned} n(\boldsymbol{k}, P, Q, \boldsymbol{R}) &= \lambda_2 Q^{1/3} k^{-d-2m/3+\alpha/3} G[P/Q\omega_k, (\boldsymbol{R}, \boldsymbol{k})/Qk^2] \\ &= \lambda_2 Q^{1/3} k^{-d-2m/3+\alpha/3} G(\varsigma, \vartheta) \ . \end{aligned} \tag{4.1.12}$$

Expanding $G(\varsigma, \vartheta)$ at $\varsigma, \vartheta \to 0$, we obtain then the neutrally stable modes with small energy

$$\frac{\delta n_2(k)}{n_0(k)} \propto \varsigma \propto \omega^{-1}(k) \tag{4.1.13}$$

and momentum

$$\frac{\delta n_3(\boldsymbol{k})}{n_0(k)} \propto \vartheta \propto \frac{(\boldsymbol{R}, \boldsymbol{k})}{k^2} \tag{4.1.14}$$

fluxes against the background of the solution with the wave action flux $n_0(k) \propto k^{-d-2m/3+\alpha/3}$.

With the help of such dimensional and analytical analysis one can try to construct "hybrid" solutions depending on temperature and fluxes. For example, for the three-wave kinetic equation, the general spherically symmetric stationary solution may be written in the form

$$n(k, T, P) = \frac{T}{\omega(k)} g \left[P \left(\frac{\lambda \omega(k)}{T k^{m+d}} \right)^2 \right] . \tag{4.1.15}$$

At $P = 0$, the solution (4.1.15) should go over to the Rayleigh-Jeans distribution, hence $g(0) = 1$. At $T \to 0$ the Kolmogorov distribution $\lambda P^{1/2} k^{-m-d}$ should be obtained, therefore $g(x) \to x^{1/2}$ at $x \to \infty$. Since the dimensionless parameter $x = P(\lambda \omega_k / T k^{m+d})^2$ decreases with ω_k the solution (4.1.15) is close to the Kolmogorov solution at low frequencies. The high-frequency part of the distribution is an equilibrium one. Apparently, one often observes a strong interaction of the high-frequency part of the wave system with an external thermostat. The thermostat is supposed to have an infinite thermal capacity and appears in the form of a sink. Hence, the question about "the temperature of a turbulent medium" (see Sect. 3.1.2) is now answered. Such a "temperature" may be supposed to equal the mean energy of high-frequency motions.

Expanding $g(x)$ in a series up to the first-order terms, for $x \to 0$ we obtain $g(x) = 1 + g_0 x + \ldots$ and

$$n(k, T, P) = \frac{T}{\omega(k)} + \omega(k) k^{-2(m+d)} \frac{g_0 \lambda^2 P}{T} . \tag{4.1.16}$$

For Kolmogorov-like distributions we expand $g(x)$ at $x \to \infty$ in the asymptotic series $g(x) = x^{1/2}(1 + c/x + \ldots)$ to get, at $T \to 0$,

$$n(k, T, P) = \lambda P^{1/2} k^{-m-d} + c \lambda^{-1} T^2 P^{-1/2} \omega_k^{-2} k^{m+d} . \tag{4.1.17}$$

In the same way, one can obtain a nonisotropic correction to to the equilibrium distribution $n_0(k, T) = T/\omega(k)$. This correction carries the small momentum flux

$$\frac{\delta n(k)}{n_0(k, T)} \propto \frac{(\boldsymbol{R}\boldsymbol{k})\omega_k^3}{k^{-2(m+d+1)} T^{-2}} . \tag{4.1.18}$$

In the nondecay case, we write, similarly to (4.1.15),

$$n(k,T,P) = \frac{T}{\omega(k)} H\left[P\left(\frac{\lambda_1\omega_k}{Tk^{d+2m/3}}\right)^3\right] \tag{4.1.19}$$

and setting $H(y) = 1 + H_0 y$ at $y \to 0$ and $H(y) = y^{1/3}(1 + c_1/y)$ at $y \to \infty$, we obtain corrections to the equilibrium solution

$$n(k,T,P) = \frac{T}{\omega(k)} + \omega^2(k)k^{-3d-2m}\frac{H_0\lambda_1^3 P}{T^2} \tag{4.1.20}$$

and to the Kolmogorov solution

$$n(k,T,P) = \lambda_1 P^{1/3}k^{-d-2m/3} + c_1\lambda_1^{-2}P^{-2/3}T^3\omega^{-3}(k)k^{2d+4m/3}\,. \tag{4.1.21}$$

For the solution with temperature and action flux, we have

$$n(k,T,Q) = \frac{T}{\omega(k)} K\left[Q\left(\frac{\lambda_2\omega_k}{Tk^{d+2m/3-\alpha/3}}\right)^3\right]\,. \tag{4.1.22}$$

$$n(k,T,Q) = \frac{T}{\omega(k)} + \omega^2(k)k^{-3d-2m+\alpha}\frac{K_0\lambda_2^3 Q}{T^2}\,, \tag{4.1.23}$$

$$\begin{aligned} n(k,T,Q) = {} & \lambda_2 Q^{1/3}k^{-d-2m/3+\alpha/3} \\ & + c_2\lambda_2^{-2}Q^{-2/3}T^3\omega^{-3}(k)k^{2d+4m/3-2\alpha/3}\,. \end{aligned} \tag{4.1.24}$$

It is important to realize that, as opposed to the case of equilibrium solution, one should directly verify that the modes constructed, see (4.1.6, 16–18) and (4.1.8, 11, 13, 14, 20, 21, 23, 24), are actually the stationary solutions of (4.1.1) and (4.1.4), respectively. The point is that we know in the equilibrium case the explicit form of the general equilibrium solution, whereas in the nonequilibrium case neither the form of the functions f, F, g, G, H, K nor their analyticity properties are known.

It is easy to verify for isotropic corrections (4.1.11, 13, 20, 21, 23, 24). We set $n(k) \propto k^{-s}$, $\delta n(k) \propto k^{-p}$ and, having split up the integral (4.1.4) into four equal parts, we subject three of them to Zakharov transformations similar to (3.1.18)

$$\begin{aligned} k_3' &= k^2/k_3, \quad k_2' = k_1k/k_3, \quad k_1' = k_2k/k_3, \quad k = k_3k/k_3; \\ k_2' &= k^2/k_2, \quad k_1' = k_3k/k_2, \quad k_3' = k_1k/k_2, \quad k = k_2k/k_2; \\ k_1' &= k^2/k_1, \quad k_3' = k_2k/k_1, \quad k_2' = k_3k/k_1, \quad k = k_1k/k_1. \end{aligned} \tag{4.1.25}$$

As a result, the linearized collision integral (4.1.4) becomes:

$$\frac{\partial \delta n(\boldsymbol{k},t)}{\partial t} = \frac{k^\nu}{4}\int |T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_3)|^2 \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)$$
$$\times\, \delta(\omega_k+\omega_1-\omega_2-\omega_3)$$
$$\times \Big\{ \Big[(k_1k_2)^{-s}+(kk_2)^{-s}-(kk_1)^{-s}\Big]k_3^{-p}$$
$$+ \Big[(kk_3)^{-s}+(k_1k_3)^{-s}-(kk_1)^{-s}\Big]k_2^{-p} \tag{4.1.26}$$
$$+ \Big[(k_2k_3)^{-s}-(kk_2)^{-s}-(kk_3)^{-s}\Big]k_1^{-p}$$
$$+ \Big[(k_2k_3)^{-s}-(k_1k_2)^{-s}-(k_1k_3)^{-s}\Big]k^{-p}\Big\}$$
$$\times \left(k^{-\nu}+k_1^{-\nu}-k_2^{-\nu}-k_3^{-\nu}\right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 ,$$

where $\nu = 2m + 3d - \alpha - 2s - p$.

The braced term in (4.1.26) vanishes only for $s = \alpha$ and $p = 2\alpha, \alpha$, which corresponds to the equilibrium corrections to the equilibrium solution. To reduce the (rounded) last brackets to zero, one should choose $\nu = 0$ or $\nu = -\alpha$ (bearing in mind that $\omega_k \simeq k^\alpha$. For corrections to the equilibrium solution with $s = \alpha$, the choice of $\nu = 0$ gives $p = 3\alpha - 2m - 3d$ and $\nu = -\alpha$, $p = 2\alpha - 2m - 3d$, which coincides with (4.1.23) and (4.1.20), respectively. As in the case of stationary additions to the solution with the energy flux for which $s = d + 2m/3$, there are also two stationary indices $p = s = d + 2m/3$ and $p = \alpha + d + 2m/3$. The former coincides with the index of the main solution and refers to a mode with a small change of the energy flux

$$n(k,P) = n_0(k) + \delta n(k) = \lambda P^{1/2} k^{-d-2m/3} + \lambda \frac{\delta P}{2\sqrt{P}} k^{-d-2m/3} \tag{4.1.27}$$

(such modes corresponding to a simple change of the constant exist for all power solutions). The latter incex p corresponds to the correction (4.1.11) with a small wave action flux. The correction with the small temperature is not a stationary solution of (4.1.4). Similarly, for the solution with the wave action flux ($s = d + 2m/3 - \alpha/3$) we get $p = s$ and $p = d + 2m/3 + 2\alpha/3$ which coincides with (4.1.13) and the mode (4.1.24) does not satisfy the equation.

Likewise, using Zakharov's transformations, one can show that the mode (4.1.16) with a small energy flux is a stationary solution of (4.1.1), while the one with a small temperature (4.1.17) is not. In other words, there are two types of neutrally stable additions to the equilibrium solution, the respective variations of the equilibrium and nonequilibrium parameters. For the Kolmogorov solutions, the universal stationary corrections may only be obtained by varying the fluxes. From the mathematical viewpoint, the fact that the modes (4.1.17, 21, 24) are not stationary solutions of linearized equations, probably indicates nonanalyticity of the functions $g(x)$, $H(x)$, and $K(x)$ at $x \to \infty$. For example, that functions may allow for the expansion of T in noninteger powers for $x \to \infty$. Such powers are be nonuniversal, i.e., they are defined by specific properties of ω_k and V_{k12}.

It remains to check the stationary character of the drift Kolmogorov solutions (4.1.6, 8, 14, 18). They are anisotropic, i.e., they depend not only on the modulus of the wave vector but also on the angles in $\boldsymbol{k}$-space. Therefore Zakharov transformations affecting only the frequencies ω_k do not allow factorization of the collision integral. Elegant transformations which enable one to transform different terms in the linearized collision integrals with $\delta n(\boldsymbol{k}) \propto (\boldsymbol{Rk})$ into each other have been suggested by *Kats* and *Kontorovich* [4.4]. Following [4.2] we shall elaborate on these transformations for the three-wave equation (4.1.1). If

$$n(\boldsymbol{k}) = k^a[1 + k^b(\boldsymbol{\kappa R})] ,$$

where $\boldsymbol{\kappa} = \boldsymbol{k}/k$, then the linearized collision integral may be represented as $\hat{L}_k \delta n(\boldsymbol{k}) = (\boldsymbol{\kappa R})I(\boldsymbol{k})$, where

$$\begin{aligned} I(\boldsymbol{k}) = \int d\boldsymbol{k}_1 d\boldsymbol{k}_2 \big([&U(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\boldsymbol{f}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) \\ &- U(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k})\boldsymbol{f}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}) \\ &- U(\boldsymbol{k}_2,\boldsymbol{k},\boldsymbol{k}_1)\boldsymbol{f}(\boldsymbol{k}_2,\boldsymbol{k},\boldsymbol{k}_1)]\boldsymbol{\kappa}\big), \\ U(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) = {}& |V(k,k_1,k_2)|^2 \delta(\omega_k - \omega_1 - \omega_2)\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2), \\ \boldsymbol{f}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) = {}& -\boldsymbol{\kappa} k^{a+b}(k_1^a + k_2^b) + \boldsymbol{\kappa}_1 k_1^{a+b}(k_2^a - k^a) \\ &+ \boldsymbol{\kappa}_2 k_2^{a+b}(k_1^a - k^a) . \end{aligned} \tag{4.1.28}$$

The wanted transformations should convert the second and third terms of the integral (4.1.28) into the first one (possibly with additional factors). Consequently, it is necessary to transform into each other the surfaces on which the conservation laws given by the δ-functions are valid. This is achieved by applying similarity transformations (involving dilatations and rotations) to the triangles representing the momentum conservation laws. As an example, we show how to transform the third term. Let us provisionally denote the integration variables by k_1', k_2'. Figures 4.1a and 4.1c show similar triangles representing the momentum conservation laws of the first and the third integrals, respectively. The vector k is common to both triangles.

The transformation $\hat{G}_1$ changing triangle "c" into "a" involves two operations:

1) the rotation $\hat{g}_1^{-1}$ of the triangle $\boldsymbol{k}_1'\boldsymbol{k}_2'\boldsymbol{k}$, such that the vector $\hat{g}_1^{-1}\boldsymbol{k}_2'$ is directed along $\boldsymbol{k}$ (see Fig. 4.1b);

2) the dilatation $\hat{\lambda}$ with the coefficient $\lambda_1 = k/k_1$ such that

$$\hat{G}_1^{-1}\boldsymbol{k}_2' = (\hat{\lambda}\hat{g}_1)^{-1}\boldsymbol{k}_2' = \boldsymbol{k} .$$

In the integral (4.1.28) this transformation corresponds to the following substitution of variables:

$$\hat{G}_1 : \ \boldsymbol{k}_2' = (\lambda_1\hat{g}_1)^2\boldsymbol{k}_1, \quad \boldsymbol{k}_1' = \lambda_1\hat{g}_1\boldsymbol{k}_2, \quad \boldsymbol{k} = \lambda_1\hat{g}_1\boldsymbol{k}_1 . \tag{4.1.29a}$$

A similar transformation of the second term (where we also denote the old integration variables by $\boldsymbol{k}_1'$, $\boldsymbol{k}_2'$ and the new ones by $\boldsymbol{k}_1$, $\boldsymbol{k}_2$) has the form

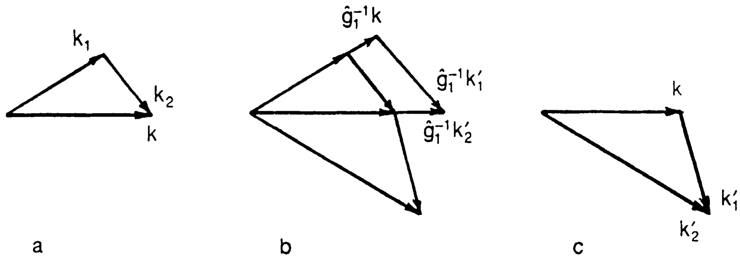

Fig. 4.1. The transformation converting the triangles a and c into each other is illustrated. The triangles represent the conservation laws of energy and momentum

$$\hat{G}_2: \ \boldsymbol{k}_1' = (\lambda_2 \hat{g}_2)^2 \boldsymbol{k}_2, \ \boldsymbol{k}_2' = \lambda_2 \hat{g}_2 \boldsymbol{k}_1, \ \lambda_2 = k/k_2 \ , \tag{4.1.29b}$$

where $\hat{G}_2 = \lambda_2 \hat{g}_2$ transforms $\boldsymbol{k}_2$ to $\boldsymbol{k}$.

Since the transformations (4.1.29) contain dilatations, the second and the third terms in (4.1.28) will acquire factors at the expense of the Jacobian of the transformation and the self-similarity of the functions U and f. As a result, the collision integral will become

$$I(\boldsymbol{k}) = \int d\boldsymbol{k}_1 d\boldsymbol{k}_2 U(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)[(\boldsymbol{\kappa} \boldsymbol{f}) - \lambda_1^r(\boldsymbol{\kappa} \boldsymbol{f}_1) - \lambda_2^r(\boldsymbol{\kappa} \boldsymbol{f}_2)]. \tag{4.1.30}$$

Here $r = 2d + 2m - \alpha$ and

$$\begin{aligned}
\boldsymbol{\kappa} \boldsymbol{f} &= \boldsymbol{\kappa} \boldsymbol{f}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2), \\
\boldsymbol{\kappa} \boldsymbol{f}_1 &\equiv \boldsymbol{\kappa} \boldsymbol{f}(\hat{G}_1 \boldsymbol{k}, \hat{G}_1 \boldsymbol{k}_1, \hat{G}_1 \boldsymbol{k}_2) \\
&= \lambda_1^{2a+b} (\hat{g}_1 \boldsymbol{\kappa}_1) \boldsymbol{f}(k \hat{g}_1 \boldsymbol{\kappa}, k_1 \hat{g}_1 \boldsymbol{\kappa}_1, k_2 \hat{g}_1 \boldsymbol{\kappa}_2) \\
&= \lambda_1^{2a+b} \boldsymbol{\kappa}_1 \boldsymbol{f}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) \ .
\end{aligned}$$

The last equality follows from the linearity of f with regard to the wave vectors $\boldsymbol{\kappa}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2$, and the definition of the rotation $\hat{g}_1$ ($\hat{g}_1 \boldsymbol{\kappa}_1 = \boldsymbol{\kappa}$), so that $(\boldsymbol{\kappa}, \hat{g}_1 \boldsymbol{\kappa}) = (\hat{g}_1 \boldsymbol{\kappa}_1, \hat{g}_1 \boldsymbol{\kappa}) = (\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)$. Similarly,

$$\boldsymbol{\kappa} \boldsymbol{f}(\hat{G}_2 \boldsymbol{k}, \hat{G}_2 \boldsymbol{k}_1, \hat{G}_2 \boldsymbol{k}_2) = \lambda_2^{2a+b} \boldsymbol{\kappa}_2 \boldsymbol{f}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) \ .$$

This gives the integral $I(\boldsymbol{k})$ in factorized form

$$\begin{aligned}
I(\boldsymbol{k}) = \int & d\boldsymbol{k}_1 d\boldsymbol{k}_2 U(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) \boldsymbol{f}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) \\
& \times [\boldsymbol{\kappa} - \boldsymbol{\kappa}_1 (k/k_1)^{r+2a+b} - \boldsymbol{\kappa}_2 (k/k_2)^{r+2a+b}] \ .
\end{aligned} \tag{4.1.31}$$

The products of the vectors in these expressions are obviously scalar products. From (4.1.31) it is seen that the integral $I(\boldsymbol{k})$ vanishes for the choice

$$r + 2a + b \equiv 2d + 2m - \alpha + 2a + b = -1 \ . \tag{4.1.32}$$

As in the case of the solution with the momentum flux, the vanishing of the integral is due to the δ-function in the wave vectors. Thus, the Kats-Kontorovich transformations factorize the linearized collision integral for a perturbation $\delta n(\boldsymbol{k})$ proportional to the cosine of an angle between $\boldsymbol{R}$ and $\boldsymbol{k}$. It is readily verified that the corrections found for the decay case (4.1.6, 18) satisfy condition (4.1.32). To conclude that the drift Kolmogorov solutions are neutrally stable modes, we have to verify their locality, i.e., the convergence of the integral (1.1.31). This is done in analogy to the isotropic case (3.1.12). The only particularity consists in the fact that the divergences are reduced by the power of k rather than ω_k [which compresses the locality strip by $2(\alpha - 1)$]. We leave it for the reader to check that for capillary waves on the surface the drift mode is local on deep water (3.1.15b) and not on shallow water (3.1.15a).

The momentum flux direction (towards large or small k) is given by the sign of the derivative of the collision integral with respect to the index of the solution

$$\operatorname{sign} R = -\operatorname{sign}(\partial I/\partial b) . \tag{4.1.33}$$

The derivative is calculated at the value of b which satisfies (4.1.32).

The fact that the drift corrections (4.1.8, 14) are the stationary solutions of (4.1.4) is proved in the same way. For factorization of the collision integral, one should use, apart from the dilatations (4.1.25) also rotations transforming similar quadrangles of wave vectors into each other.

Let us note that stationary corrections can also be obtained for the anisotropic spectra introduced in Sect. 3.3. Expanding the general solution (3.3.17) at $\xi \to 0$ one can obtain a stationary drift correction to the spectrum

$$n(p,q,P,R) = P^{1/2}|p|^{-1-u}q^{-2-v} + RP^{-1/2}|p|^{a-u-2}q^{b-v-2}$$

supporting an energy flux. In the opposite limit $\xi \to \infty$, a neutrally stable mode with a small energy flux can be obtained against a background of the spectrum

$$\begin{aligned} n(p,q,R,P) = {} & R^{1/2}|p|^{-u+(a-3)/2}q^{-v-2+b/2} \\ & + PR^{-1/2}|p|^{-u-(a+1)/2}q^{-v-2-b/2} \end{aligned}$$

with a momentum flux. These corrections can be shown to satisfy the linearized kinetic equation. Moreover, the nonlinear kinetic equation (2.5.2) has a stationary solution in the form of a sum of power functions

$$n(p,q) = \sum_{i=1}^{M} c_i p^{-x_i} q^{-y_i}$$

with $M \le 4$. Such a solution may exist even when different power functions are of the same order [4.5].

Concluding this section, it should be pointed out that the additional contribution of the anisotropic drift correction (4.1.9) to the energy flux distribution

$$\frac{\delta n_0(\boldsymbol{k})}{n_0(k)} \propto \frac{\omega_k}{k} \propto k^{\alpha-1}$$

in the decay case ($\alpha > 1$) grows with k, i.e., while going deeper into the inertial interval. Similarly, in the nondecay case, the contribution of the drift mode (4.1.14) grows from source to sink with respect to the wave action flux distribution

$$\frac{\delta n_3(\boldsymbol{k})}{n(k)} \propto k^{-1} .$$

Thus the drift Kolmogorov solutions (4.1.9, 14) imply a kind of structural instability of the isotropic Kolmogorov spectrum: even a small anisotropy of the wave source will lead to an essentially anisotropic distribution in the inertial interval. However, as we shall see in the next section, none of the drift solutions will be established in the case of anisotropic modulation of a wave source. Indeed, apart from the stationary solutions (4.1.9, 14) of the homogeneous equation $\hat{L}_k \delta n_k = 0$ and $\hat{W}_k \delta n_k = 0$ there may also exist solutions of inhomogeneous equations $\hat{L}_k \delta n(\boldsymbol{k}) = \delta\gamma(\boldsymbol{k}) n_0(k)$ and $\hat{W}_k \delta n(\boldsymbol{k}) = \delta\gamma(\boldsymbol{k}) n_0(k)$ (here $\delta\gamma$ is the anisotropic part of the source assumed to be small), which decrease while going further into the inertial interval $[\delta n(\boldsymbol{k})/n_0(k) \to 0]$. In order to clarify which distribution is generated by a weakly anisotropic source, it is necessary to solve the initial value problem. As will be shown in the next section, only those drift solutions may be observed in the inertial interval that transfer momentum flux into the same direction as the flux of the main integral of motion (of energy or action) – *the Falkovich criterion* [4.6]. It should be noted that this criterion is the natural generalization of the Frisch and Fournier criterion for isotropic solutions (see Sect. 3.1).

4.2 Stability Problem for Kolmogorov Spectra of Weak Turbulence

The proponents of the Kolmogorov spectrum concept in the hydrodynamics of incompressible fluids supposed this spectrum to be stable. An equivalent statement known as the "hypothesis of local isotropy of turbulence" asserts that in the step-by-step transfer of energy over scales, the turbulence spectrum becomes isotropic. In other words, it is usually supposed that the anisotropic spectrum, being determined by external anisotropic pumping in the region of small wave numbers, is in the inertial interval replaced by the isotropic Kolmogorov spectrum (see e.g. [4.7]). The concept of local isotropy for the small scales was introduced by *Taylor* [4.8]. In this section we shall show that an opposite situation may arise, at least for weak turbulence. Namely, the degree of anisotropy of the distribution may be small close to the source and increase further away in the inertial interval. This phenomenon is, in effect, one of the variants of a "self-organization process" or of the emergence of structures in nonlinear systems and may be considered as a kind of structural instability of the isotropic Kolmogorov spectrum. The possibility of such a kind of instabilities was first indicated by *L'vov* and *Falkovich* [4.9]. A general stability theory (including analyses of different types of instabilities and of various physical systems) was developed by *Balk* and *Zakharov* [4.10] and *Falkovich* and *Shafarenko* [4.11] observed the phenomenon in numerical simulations. We view this instability to be of an "interval" type. This name is associated with the fact that this instability owes its existence to the large inertial interval and has thus an asymptotic character: perturbations increase the more dramatically the larger the inertial interval. With interval instability, the perturbations grow by a power law while k goes into the inertial interval, so that the turbulent medium generates a universal (i.e., determined only by the properties of the medium itself), ordered structure in the region of large or small scales (in the remaining range, a Kolmogorov-like spectrum is realized).

Empirical Approach. Before delving into rigorous mathematical theory, let us try to guess a criterion for the structural stability of the Kolmogorov spectra on the basis of plausible physical reasoning. We consider a perturbation in the form of an angular l-harmonic Y_l. If the linearized kinetic equation has a stationary solution (neutrally stable mode)

$$\frac{\delta n(\boldsymbol{k})}{n(k)} = Y_l(\Omega)k^{-p} ,$$

then it also conserves the integral of motion

$$I_l = \int Y_l(\Omega)k^{p+h-1}\frac{\delta n(\boldsymbol{k})}{n(k)}\, dk d\Omega ,$$

whose constant flux is transported by that mode.

Let the spectrum in question carry a positive flux. Then the inertial interval is in the small-scale region. The harmonic may affect the stability of the spectrum in the inertial interval if $p < 0$. It is natural to assume that the harmonic can be generated by external anisotropic pumping if the flux of I_l is directed towards the damping region. The sign of the flux is defined by the derivative (4.1.33) of the linearized collision integral with respect to the exponent of the solution. Introducing the dimensionless collision integral $W_l(s)$ calculated with $\delta n(\boldsymbol{k}) = Y_l k^{-s} n(k)$ we thus obtain the instability criterion $W_l'(p_l) > 0$. Similarly one should in the case of negative main flux require $p_l > 0$ and $W_l'(p_l) < 0$ as a criterion for the existence of instability. Thus we get the simple physical criterion of structural instability connected with neutrally stable modes [4.6]:

$$-\operatorname{sign} p_l = \operatorname{sign} W_0'(0) = \operatorname{sign} W_l'(p_l) . \tag{4.2.1a}$$

Here $W_0'(0) = I'(\nu)$ is the derivative of the linearized collision integral with $\delta n(k) = n(k)k^{-s}$ which for $s = 0$ coincides with the derivative of the complete collision integral $\partial I/\partial x$ at x being equal to Kolmogorov index ν. Thus $W_0'(0)$ defines the flux of the main integral.

Equally simple we can obtain an instability criterion for the free evolution of a perturbation of the spectrum (without any additional pumping). Requiring the integral of motion I_l to be conserved while an l-harmonic perturbation evolves in k-space, we obtain

$$\frac{\delta n(\boldsymbol{k})}{n(k)} \propto k^{p+h} . \tag{4.2.1b}$$

In this case the role of p is played by the quantity $p + h$. A necessary condition for the existence of an instability is $\operatorname{sign}(p_l + h) = -\operatorname{sign} W_0'(0)$. Indeed, the ultraviolet instability (for a positive main flux) occurs for $p_l + h > 0$, etc.

Introduction to Stability Theory. Instability studies are usually reduced to deriving a complete set of eigenfunctions of the linearized operator and to investigating the eigenvalues determining the time evolution of the eigenfunctions. In our case, the operators $\hat{L}_k$ and $\hat{W}_k$ are scale-invariant, since the functions $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$, $\omega(k)$ and $n(k)$ possess this property. A natural set of functions consists of power functions k^s with different s. In this connection we encounter the first mathematical difficulty: such functions grow either at $k \to 0$ or at $k \to \infty$ giving rise to divergences of different terms in the collision integral. For the sake of convenience we will in this section use the variable $x = \ln k$ rather than k. The scale invariance of the operators $\hat{L}_k, \hat{W}_k$ will make it possible to use the more customary Fourier representation with eigenfunctions in the form of exponents $k^s \to \exp(sx)$. Unfortunately, our integral equation is not of the convolution type, since the operators $\hat{L}_k$, $\hat{W}_k$ have nonzero h-indices. So the linearized kinetic equation in the variables x may be written in the form

$$\frac{\partial}{\partial t}\delta n(x,t) = e^{-hx} \int_{-\infty}^{\infty} U(x - x')\delta n(x',t)\, dx' ,$$

which after Fourier transformation turns into an equation of the Carleman type (and not into an algebraic one)

$$\lambda \Psi(s + h) = W(s)\Psi(s) .$$

A rather developed theory of Carleman-type equations has been formulated by the mathematician *Cherskii* [4.12–13] for kernels of the form

$$U(x) = U_0\delta(x) + u(x) ,$$

where U_0 is a constant and $u(x)$ an ordinary integrable function. Unfortunately, it is in general not possible to represent the kernels of linearized collision integrals in such a form. This is due to the fact that in the kinetic equation separate integrals diverge, and the regularity of the whole expression is an effect of mutual cancellation of divergences. Indeed, the constant N equal to the sum of integrals appearing in (4.1.1) or (4.1.4) as factors at $\delta n(k)$ may, e.g., prove to be infinite. Thus, another mathematical difficulty is the singularity of the kernel of the integral operator. To overcome it and to obtain rigorous results the method of generalized functions should be applied carefully.

Finally, a last difficulty is associated with the fact that physical considerations are insufficient for choosing the boundary conditions to be imposed on perturbations. As we shall see, the correct conditions are obtained by considering the Cauchy initial value problem and the transition to the limit at $t \to \infty$ and by demanding this problem to be correctly posed for the chosen class of functions. Thus, we will not only have to analyze the eigenvalues and stationary solutions of the linearized equation, but we will also have to solve the associated initial value problem. We shall follow a similar way as Landau in the solution of the description of Langmuir wave damping in plasmas [4.14].

The mentioned mathematical difficulties explain the somewhat higher level of mathematical sophistication adopted in this section. However, as a reward quite beautiful results are obtained. In this section we shall fully classify possible types of behavior of a weakly turbulent medium in the vicinity of the Kolmogorov spectrum, show how one can efficiently identify the type of the system, describe the asymptotics of its behavior for large times ($t \to \infty$) and in the limits of small ($k \to \infty$) and large ($k \to 0$) scales.

The central result of the stability theory is a verifiable stability criterion for Kolmogorov spectra in the case of weak turbulence. This criterion reduces the examination of the stability of a Kolmogorov spectrum to calculating several integers (rotations of certain analytical functions around the imaginary axes; explicit formulas will be given). For the stability of the Kolmogorov spectrum it is necessary and sufficient that all these integers be equal to zero.

In this section we shall also try to reach a more profound understanding of locality of the Kolmogorov spectra. It appeared that, despite the locality of the Kolmogorov spectrum in the above sense, i.e., the convergence of the collision integral on the spectrum, the evolution of the distribution $n(\boldsymbol{k}, t)$ weakly deviating from the Kolmogorov spectrum may possibly not be determined by the interaction of waves only of scales of the same order and may considerably depend on the conditions at the ends of the inertia interval. This phenomenon was called the evolution nonlocality of Kolmogorov spectra; we shall describe the necessary and sufficient conditions of the evolution locality.

The general results of this chapter will be applied in the next one to the analysis of the turbulence of capillary waves, gravitational waves, Langmuir turbulence in plasmas and acoustic turbulence.

Our studies into the stability of Kolmogorov solutions were strongly stimulated by a desire to explain the experimentally observed anomalous angle narrowness of the spectrum of wind-stimulated undulation on the ocean surface.

The available technique enables one to study the stability of isotropic turbulence spectra of gravitational waves in the framework of the kinetic equation and to find that the spectra are stable. This probably suggests that for the description of wind-induced undulation it is insufficient to take into account only the interaction of gravitational waves with each other.

In the case of turbulent capillary waves, the Kolmogorov spectrum proved to be unstable with respect to anisotropic perturbations having only a first angular harmonic. In acoustic turbulence, the number of angular harmonics, with respect to which the Kolmogorov spectrum is unstable, is inversely proportional to the value $\sqrt{\varepsilon} = \sqrt{\alpha - 1}$ (see the Sect. 5.1). In both cases the instability proves to be of the hard interval type and shows itself in small scales.

The results formulated in this section for the Kolmogorov spectra of weak turbulence, which are the exact solutions of the kinetic equations for waves, may be extended to studies of the Kolmogorov spectra which are the exact solutions of other kinetic equations (Boltzmann equation, polymerization equation etc).

4.2.1 Perturbations of the Kolmogorov Spectrum

Statement of the Stability Problem. Inside the inertial interval there may be external effects or dissipation which is small by the very definition of the inertial interval (as compared to the values of source and sink forming the Kolmogorov spectrum). Consideration of these factors leads to an additional term of the form $\gamma(\boldsymbol{k}, t)n_{\boldsymbol{k}}^0$ on the right-hand-side of (4.1.1, 4). Assuming the initial solution to be of power type

$$n_{\boldsymbol{k}}^0 = Rk^{-\nu}, \tag{4.2.2}$$

we get, for the relative part of the perturbation,

$$A(\boldsymbol{k}, t) = \delta n(\boldsymbol{k}, t)/n_{\boldsymbol{k}}^0 , \tag{4.2.3}$$

an equation of the form

$$\frac{\partial A(\boldsymbol{k}, t)}{\partial t} = \hat{L}_{\boldsymbol{k}} A(\boldsymbol{k}, t) + \gamma(\boldsymbol{k}, t) . \tag{4.2.4}$$

One can study the stability of the Kolmogorov spectrum with respect to initial perturbations in terms of this equation (one should then set $\gamma = 0$) and external action ($\gamma \neq 0$), respectively.

The Evolution Equation and its Reduction to the Carleman Equation. Let us expand the function A in the Fourier series into an orthonormal system of angular harmonics $Y_l(\zeta)$ (ζ is a point on the sphere $\Omega = \{\zeta \in R^d / |\zeta = 1\}$):

$$A(\boldsymbol{k}, t) = \sum_l A_l(k, t) Y_l(\zeta), \quad A_l(k, t) = \int_\Omega A(\boldsymbol{k}, t) Y_l^*(\zeta)\, D\zeta .$$

Here $D\zeta$ is a surface element on the sphere, $d\boldsymbol{k} = k^{d-1} dk D\zeta$. For two-dimensional media ($d = 2$): $Y_l = 2\pi^{-1/2} e^{il\varphi}$, $l = 0, \pm 1, \pm 2, \ldots$, $D\zeta = d\varphi$. In the case of three-dimensional media ($d = 3$), the $Y_l(\zeta)$-functions are the normalized ordinary spheric functions $Y_l^j(\theta, \varphi)$, $l = 0, 1, 2, \ldots$, $j = 0, \pm 1, \pm 2, \ldots, \pm l$, $D\zeta = \sin\theta\, d\theta\, d\varphi$. For different functions A_l we have uncoupled equations of the form (the l-index is omitted):

$$\frac{\partial A}{\partial t} = \hat{L}[A] + \chi \, , \tag{4.2.5}$$

where $\chi = \chi(k,t) = \int_\Omega \gamma(\boldsymbol{k},t)Y^*(\zeta)\,D\zeta$, the linear operator $\hat{L}$ (assuming scale invariance of the medium) is homogeneous with a certain power $(-h)$:

$$\hat{L}[f \cdot \varepsilon](k) = \varepsilon^{-h}\hat{L}[f](\varepsilon k)$$

[$f = f(k)$ is an arbitrary function and ε is an arbitrary positive number]. It is convenient to go over from the variable k to the variable $x = \ln k$ by introducing the notation

$$F(x,t) = A(k,t), \quad \phi(x,t) = \chi(k,t) \quad (k = e^x) \; .$$

Then (4.2.5) reads

$$\frac{\partial F(x,t)}{\partial t} = e^{-hx}[U(x) * F(x,t)] + \phi(x,t) \tag{4.2.6}$$

where $U(x)$ is a generalized function determined by the form of the operator $\hat{L}$ and $*$ denotes the convolution

$$U(x) * F(x,t) = \int_{-\infty}^{+\infty} U(x - x')F(x',t)\,dx' \; .$$

The operator $\hat{L}$ is given by an integral of a sum of several expressions containing the function $A(k,t)$ with different arguments k, see (4.1.1) or (4.1.4). If this integral could be divided into the sum of integrals over these expressions then the generalized function $U(x)$ would have a form

$$U(x) = U_0\delta(x) + u(x) \tag{4.2.7}$$

where U_0 is a constant and $u(x)$ an ordinary integrable function. In the general case, when the integral specifying the operator $\hat{L}$ may not be divided into the sum of several integrals with the function A occurring only once in each of the integrals (the indivivual integrals diverge but the total integral converges due to a cancellation of the divergences of the different terms), then the generalized function $U(x)$ are regularizations of rather diverse singular functions, see [4.15].

In physically interesting situations, the kernel $U(x)$ exponentially tends to zero at $|x| \to \infty$:

$$U(x) = \begin{cases} O\left(e^{-ax}\right), & x \to -\infty, \\ O\left(e^{-bx}\right), & x \to +\infty. \end{cases} \tag{4.2.8}$$

Here $a < b$.

We shall study the Cauchy problem of the evolution equation (4.2.6) with the initial condition

$$F(x,0) = \phi_0(x) \; . \tag{4.2.9}$$

The solution of (4.2.6) should be sought in such a (possibly wider) class of functions that the convolution in (4.2.6) is determined. Such a class is constituted by the functions $f(x)$ which at $x \to +\infty$ grow slower than $\exp(-ax)$ and at $x \to -\infty$ slower than $\exp(-bx)$

$$f(x) = \begin{cases} O\left(e^{-\sigma_1 x}\right), & x \to +\infty \quad \sigma_1 > a \ ; \\ O\left(e^{-\sigma_2 x}\right), & x \to -\infty \quad \sigma_2 < b \ . \end{cases} \tag{4.2.10}$$

Let us denote this class of functions by $\mathcal{L}(a,b)$. We shall require that the solutions $F(x,t)$ of the evolution equation (4.2.6) are elements of the space $\mathcal{L}(a,b)$ at every fixed t. The class $\mathcal{L}(a,b)$ is analogous to the class of solutions treated by the classic Wiener-Hopf theory, see [4.15–16]. The exponential decrease of $F(x)$ corresponds to a power decrease of $\delta n(k)$. The quantities a, b coincide for zero spherical harmonics in the decay case with the boundaries of the locality interval s_1, s_2 determined in Sect. 3.1. For the remaining harmonics we have $a = s_2 + \alpha - 1$, $b = s_1 + 1 - \alpha$. Subjecting the evolution equation (4.2.6) to the Laplace time transformation

$$F(x) = F_\lambda(x) = \int_0^\infty F(x,t)e^{-\lambda t}\, dt$$

leads to

$$\lambda F(x) = e^{-hx}[U(x) * F(x)] + \Phi(x) \tag{4.2.11}$$

where

$$\Phi(x) = \Phi_\lambda(x) = \phi_0(x) + \int_0^\infty \phi(x,t)e^{-\lambda t}\, dt \ .$$

The functions $\phi_0(x)$ and $\phi(x,t)$ and consequently also $\Phi(x)$ may be considered to be finite functions of the variables x. The solution of (4.2.11), like those of the evolution equation (4.2.6), should be regarded in the class $\mathcal{L}(a,b)$.

If one formally Fourier transforms (4.2.11)

$$G(s) = \int_{-\infty}^{+\infty} F(x)e^{sx}\, dx \tag{4.2.12}$$

(whereby the convolution is converted into a product and multiplication by the exponential function into a translation) one obtains an equation of the Carleman type (see [4.12–13])

$$\lambda G(s+h) = W(s)G(s) + \Psi(s+h), \tag{4.2.13}$$

where $W(s)$ and $\Psi(s)$ are the Fourier images of the functions $U(x)$ and $\Phi(x)$, respectively.

For $h = 0$ it is easy to solve (4.2.13) and, hence, (4.2.6). We shall consider this case in Sect. 5.1.2 for two-dimensional acoustic turbulence. In this section we shall everywhere use $h \neq 0$.

The Fourier transformation in terms of the variable x corresponds to the Mellin transformation in the initial variable k

$$G(s) = \int_0^\infty F(k) k^{s-1}\, dk.$$

The function

$$W(s) = \int_{-\infty}^{+\infty} U(x) e^{sx}\, dx, \quad a < \operatorname{Re} s < b,$$

will play a key role in our further considerations. It is the image of an operator $\hat{L}_0 = \exp(hx)\hat{L}$ in the Mellin transformation and will be called the Mellin function. It is convenient to derive explicit expressions for Mellin functions by using

$$\begin{aligned} W(s)\delta(r-s) &= \frac{1}{2\pi}\int_0^\infty k^{r+h}\hat{L}[(k')^{-s}](k)\, dk/k \\ &= \frac{1}{2\pi}\int_0^\infty k^{r+h} Y^*(\zeta)\hat{L}[(k')^{-s}Y(\zeta')](k,\zeta)\, D\zeta\, dk/k\ . \end{aligned} \tag{4.2.14}$$

If we write in (4.2.14) the operator $\hat{L}$ in an explicit form, we obtain an integral in which the integration is to be performed over the variables $k,\ k_1,\ k_2$ in the decay case or $k,\ k_1,\ k_2,\ k_3$ in the nondecay case. This allows for a proper symmetrization of the integrand. Having chosen the variable ξ over which to integrate from zero to ∞ (for example, $\xi = k$), we can represent the integral (4.2.14) in the form

$$\frac{1}{2\pi}\int_0^\infty M(r,s,\xi)\xi^{r-s}\, d\xi/\xi, \tag{4.2.15}$$

where $M(r,s,\xi)\xi^{r-s}$ is the result of the remaining integrations. $M(r,s,\xi)$ is homogeneous and of zeroth power in ξ and does not explicitly depend on ξ. Taking

$$\frac{1}{2\pi}\int_0^\infty \xi^{r-s} d\xi/\xi = \delta(r-s)$$

into account we obtain that $W(s) = M(s,s,\xi) = M(s,s,1)$. Formula (4.2.14) should be used to obtain the Mellin functions for any kinetic equation. But when kinetic equations for waves are considered, this formula with the variable $\xi = k$ [see (4.2.15)] leads to the following expressions:

(i) in the decay case to

$$\begin{aligned} W(s) = R\int & 2\pi |V(k,k_1,k_2)|^2 \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\delta(\omega_k-\omega_1-\omega_2) \\ & \times (kk_1k_2)^{d-\nu}\{[(k^{-s}Y + k_1^{-s}Y_1 + k_2^{-s}Y_2)(k^\nu - k_1^\nu - k_2^\nu) \\ & - (k^{\nu-s}Y - k_1^{\nu-s}Y_1 - k_2^{\nu-s}Y_2)](k^{\mu+s}Y^* - k_1^{\mu+s}Y_1^* \\ & - k_2^{\mu+s}Y_2^*)\}\, D\zeta D\zeta_1 D\zeta_2 \frac{dk_1 dk_2}{k_1 k_2}, \end{aligned} \tag{4.2.16a}$$

(ii) in the nondecay case to

$$W(s) = R^2 \int |T_{k123}|^2 \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\delta(\omega_k + \omega_1 - \omega_2 - \omega_3) \times (kk_1k_2k_3)^{d-\nu}\{[(k^{-s}Y + k_1^{-s}Y_1 + k_2^{-s}Y_2 + k_3^{-s}Y_3) \times (k^\nu + k_1^\nu - k_2^\nu - k_3^\nu) - (k^{\nu-s}Y + k_1^{\nu-s}Y_1 - k_2^{\nu-s}Y_2 - k_3^{\nu-s}Y_3)](k^{\mu+s}Y^* + k_1^{\mu+s}Y_1^* + k_2^{\mu+s}Y_2^* - k_3^{\mu+s}Y_3^*)\}\pi D\zeta D\zeta_1 D\zeta_2 D\zeta_3 \frac{dk_1dk_2dk_3}{k_1k_2k_3}. \tag{4.2.16b}$$

Here $\mu = h + \nu - d$.

In the nondecay case we can use the representation

$$\delta(\omega + \omega_1 - \omega_2 - \omega_3) = \int_0^\infty \delta(y^\alpha - \omega - \omega_1)\delta(y^\alpha - \omega_2 - \omega_3)\alpha y^\alpha\, dy/y$$

to obtain a more symmetric expression for the Mellin function. Substituting this expression into (4.2.14) and specifying the variable $\xi = y$ [see (4.2.15)] we obtain

$$W(s) = R^2\alpha \int \pi |T_{k123}|^2 \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\delta(1 - \omega - \omega_1)\delta(1 - \omega_2 - \omega_3) \times \{[(k^{-s}Y + k_1^{-s}Y_1 + k_2^{-s}Y_2 + k_3^{-s}Y_3)(k^\nu + k_1^\nu - k_2^\nu - k_3^\nu) - (k^{\nu-s}Y + k_1^{\nu-s}Y_1 - k_2^{\nu-s}Y_2 - k_3^{\nu-s}Y_3)] \times (k^{\mu+s}Y^* + k_1^{\mu+s}Y_1^* - k_2^{\mu+s}Y_2 - k_3^{\mu+s}Y_3)\}(kk_1k_2k_3)^{d-\nu} \times D\zeta D\zeta_1 D\zeta_2 D\zeta_3 \frac{dk dk_1dk_2dk_3}{kk_1k_2k_3}. \tag{4.2.16c}$$

The integral (4.2.16) contains [to a linear approximation] all information about the behavior of the perturbations of the Kolmogorov spectra. That is, each of the Mellin functions $W(s)$ determines the behavior of the particular perturbation having the form of the angular harmonic contained in the integral (4.2.16) defining this Mellin function. The integrals (4.2.16a, b) are independent of the k-value [so that it may be chosen arbitrarily, for example, $k = 1$]. They are homogeneous functions of the variable k [of the zeroth order]. Thus is is possible to find the constant h (determining the scaling index of the $\hat{L}$-operator):

(i) in the decay case [see (4.1.1)],

$$h = \alpha - 2m - d + \nu, \tag{4.2.17a}$$

(ii) in the nondecay case [see (4.1.4)],

$$h = \alpha - 2m - 2d + 2\nu. \tag{4.2.17b}$$

It is much more convenient to handle the analytical functions $W(s)$ [for which explicit symmetrical integral representations (4.2.16) are available] than to deal with the generalized functions $U(x)$ or the $\hat{L}$-operators. Therefore, all conditions and statements referring to the evolution equation (4.2.5, 6) will be formulated in terms of the Mellin function $W(s)$.

To facilitate the further treatment we shall specify the following condition. The strip $\{s \in \mathbb{C}/\mathrm{Re}\, s \in I\}$, where $\mathbb{C}$ is the space of complex numbers and I is some interval or section, will be denoted by ΠI. We shall say that some function $g(s)$ in the strip ΠI is polynomially bounded on infinity if for any interval $K \in I$ there is such a number $j(K)$ that $g(s) = O(|s|^j)$, $s =\!\!\longrightarrow \infty$, $s \in \Pi K$.

The Mellin functions $W(s)$ (arising when considering the kinetic equations) have the following three properties.

1) The function $W(s)$ is analytical in some strip $\Pi(a,b)$ (on the straight lines $\mathrm{Re}\, s = a$ and $\mathrm{Re}\, s = b$ it has singularities).
2) The $W(s)$ and $1/W(s)$-functions in the $\Pi(a,b)$ strip are polynomially bounded on the infinity.
3) The value of the $W(s)$-function at $|\mathrm{Im}\, s| \to \infty$ becomes asymptotically real negative; to be more exact, for any interval $K \subset (a,b)$:

$$\arg[-W(s)] = O(1/s), \quad s \to \infty, \quad s \in \Pi K\ , \tag{4.2.18}$$

that is

$$\frac{\mathrm{Im}\, W(s)}{\mathrm{Re}\, W(s)} = O\left(\frac{1}{s}\right), \quad \mathrm{Re}\, W(s) < 0, \quad s \to \infty, \quad s \in \Pi K\ .$$

The first property and the polynomial constraint of the function $W(s)$ imply that the generalized function $U(x)$ satisfies the condition (4.2.8). If the function $U(x)$ were regular, the Mellin function $W(s)$ would have tended to zero at $|\mathrm{Im}\, s| \to \infty$, but since the generalized function $U(x)$ is a regularized singular function, the Mellin function may grow without bound at $|\mathrm{Im}\, s| \to \infty$, but not faster than the polynomial. The third property and the polynomial constraint of the $1/W(s)$-function are explained in the following way. The expression in the braces in (4.2.16) may be represented as a sum of two expressions, one of which is independent of s, and another (depending on s) is fast oscillating at large $|\mathrm{Im}\, s|$ because of the presence of the functions k^s; upon integration of (4.2.16) this expression will give values with different signs which will "quench" each other. Provided that the μ, ν indices correspond to the thermodynamic or Kolmogorov spectrum, the remaining, nonoscillating part in the braces in (4.2.16) is on the resonant manifold [specified by the δ-functions in (4.2.16)] transformed into the expression

$$-\left(k^{\mu+\nu} + k_1^{\mu+\nu} + k_2^{\mu+\nu} + k_3^{\mu+\nu}\right) \tag{4.2.19}$$

(in the decay case $k_3 \equiv 0$) which is negatively determined.

For the function $W(s)$ satisfying the conditions 1-3, the *function of rotation* $\kappa(\sigma)$ may be specified, which is important for all further treatment and has the properties mentioned below. We shall define rotations of the function $W(s)$ around a straight line $\mathrm{Re}\, s = \sigma$ [$\sigma \in (a,b)$] as a complete increment of the argument of a complex value $W(s)$ (with s moving from $\sigma - i\infty$ to $\sigma + i\infty$ along the straight line $\mathrm{Re}\, s = \sigma$) divided by 2π and denote it as $\kappa(\sigma)$. The $\kappa(\sigma)$-function is defined on the whole interval (a,b), except the points which are the real parts of zeros of the $W(s)$-function; it takes only integer values on and does not monotonically decrease in (a, b); it assumes each of its values in the whole interval rather and not just at a single point; if $\sigma_1, \sigma_2\, (\sigma_2 > \sigma_1)$ are points of its definition domain, then the difference $\kappa(\sigma_2) - \kappa(\sigma_1)$ is equal to the number of zeros of the Mellin function $W(s)$ in the strip $\Pi(\sigma_1, \sigma_2)$. We shall see below that the rotation $\kappa(\sigma)$ is the basic characteristic of the evolution equations (4.2.5, 6).

To be specific we shall consider below only $h > 0$. It is readily seen that this not really a limitation: the case with $h < 0$ reduces to the one with $h > 0$ if one performs in (4.2.5) or (4.2.6) the substitution $k \to 1/k$ or $x \to -x$, respectively; i.e., the function $W(s)$ is replaced by $W(-s)$.

Lemma. Having formally carried out the Fourier transformation in (4.2.11), we arrived at (4.2.13). The solutions of the latter are readily seen to have a structure with:

1. The general solution of the nonhomogeneous equation (4.2.13) is a sum of the particular solution of this equation and the general solution of the homogeneous equation

$$\lambda G(s + h) = W(s)G(s). \tag{4.2.20}$$

2. If $G_0(s)$ is a particular solution of the homogeneous equation (4.2.20), then the general solution of that equation has the form

$$G(s) = G_0(s)M(s), \tag{4.2.21}$$

where $M(s)$ is an arbitrary periodic function with period h, i.e., $M(s+h) = M(s)$.

Whence it is seen that (4.2.13) has many "extra" solutions. Besides, it is clear that the Fourier transformation is, generally speaking, not applicable to functions in the space $\mathcal{L}(a, b)$ [in which (4.2.11) is formulated], since they may simultaneously grow exponentially at $x \to +\infty$ and at $x \to -\infty$. In this section we shall verify the validity of the Fourier transformation in (4.2.11) and indicate the class of functions for which (4.2.13) should be considered to be equivalent to (4.2.11) in the class $\mathcal{L}(a, b)$.

First of all we would like to mention the following property of (4.2.11). Let $F(x)$ be a solution of (4.2.11) with $\lambda \neq 0$. Now, if the function $F(x)$ grows at $x \to +\infty$ not faster than $\exp(-\sigma x)$ with $\sigma \in (a, b)$, then this function grows at $x \to +\infty$ not faster than $\exp[-(\sigma + h)x]$. More precisely we have

$$F(x) = O\left(e^{-\sigma x}\right) \Longrightarrow F(x) = O\left(e^{-(\sigma+h)x}\right), \quad (x \to +\infty,\ a < \sigma < b) \tag{4.2.22}$$

Indeed, using (4.2.8), we obtain from the first estimate (4.2.22)

$$U * F = \int U(x - x')F(x')\, dx' = O\left(e^{-\sigma x}\right), \quad x \to +\infty, \tag{4.2.23}$$

and, consequently, owing to (4.2.11) at $\lambda \neq 0$ we have another estimate (4.2.22).

The property (4.2.22) makes it possible to "improve", by virtue of (4.2.11), the *a priori* characteristics of the solutions of this equation. Let $F(x)$ be a solution of (4.2.11) with $\lambda \neq 0$. Because of $F \in \mathcal{L}(a,b)$ there are numbers $\sigma_1, \sigma_2 \in (a,b)$ with

$$F(x) = \begin{cases} O\left(e^{-\sigma_1 x}\right) & \text{at } x \to +\infty, \\ O\left(e^{-\sigma_2 x}\right) & \text{at } x \to -\infty \end{cases}$$

see (4.2.10). Hence, we have by virtue of (4.2.22) $F(x) = O\left[\exp[-(\sigma_1 + h)x]\right]$, $x \to +\infty$. Now, for $\sigma_1 + h < b$ we can again make use of the property (4.2.22) to obtain $F(x) = O\left[\exp[-(\sigma_1 + 2h)x]\right]$ for $x \to +\infty$. Via repeated application of such a procedure one can show that $F(x) = O\left[\exp[-(b+h)x]\right]$, $x \to +\infty$. Thus, for $a < \varrho < b$

$$F(x) = \begin{cases} O\left(e^{-\varrho x}\right) & \text{at } x \to -\infty, \\ O\left(e^{-(b+h)x}\right) & \text{at } x \to +\infty \end{cases}$$

with $\varrho = \sigma_2$. From (4.2.23) it follows that the Fourier transformation (4.2.12) is applicable to $F(x)$ at $s \in \Pi(\varrho, b+h)$. The Fourier transformation $G(s)$ of this function is analytical and polynomially bounded in the strip $\Pi(\varrho, b+h)$. It satisfies (4.2.13) in the strip $\Pi(\varrho, b)$.

For a finite $\Phi(x)$ the function $\Psi(s)$ is an integer function and is polynomially bounded in the strip $\Pi(-\infty, +\infty)$.

The function $G(s)$ may be redefined in the strip $\Pi(a, b+h)$; for that purpose (4.2.13) must be rewritten in the form

$$G(s) = \frac{\lambda G(s+h) - \Psi(s+h)}{W(s)}. \tag{4.2.24}$$

Knowing the values of $G(s)$ in the strip $\Pi(\varrho, b+h)$, we can calculate by the aid of (4.2.24) its values in the strip $\Pi(\varrho - h, b)$, then in the strip $\Pi(\varrho - 2h,\ b-h)$, etc. Owing to the properties 1)–2) of the Mellin function $W(s)$, $G(s)$ thus redefined in the strip $\Pi(a, b+h)$ is meromorphic and polynomially bounded at infinity; it satisfies (4.2.13) in the strip $\Pi(a,b)$.

Let $\mathcal{M}I$ (where I is some interval) denote the space of such functions of a complex variable which i) are meromorphic and polynomially bounded at infinity in the strip $\Pi(a, b+h)$ and ii) are analytical in the strip ΠI.

We have shown that only those solutions $G(s)$ of (4.2.13) should be considered which belong to the space $\mathcal{M}(\varrho, b+h)$ at some $\varrho \in (a,b)$. These solutions satisfy (4.2.13) in the whole strip $\Pi(a,b)$. The quantity ϱ determines the rate of decrease (or growth) of the corresponding solutions $F(x)$ of (4.2.11) at $x \to -\infty$. The fact that the solution $G(s)$ belongs to the class $\mathcal{M}(\varrho, b+h)$ implies that the corresponding solution $F(x)$ satisfies the condition (4.2.23).

Remark 4.1. The arguments given in this section are not mathematically rigorous. If, for example, $U(x) = \delta''(x)$ then we have $U * F = F''$ and the property (4.2.23) turns out to be invalid in the form formulated here. The situation is similar when the generalized function $U(x)$ is a regularization of a singular function. However, all the foregoing and subsequent arguments may be made absolutely rigorous by using the method of generalized functions. For example, condition (4.2.8) is rigorously formulated in the following way: $U(x)\exp(\sigma x) \in S'$ for any $\sigma \in (a, b)$; S' is a space of generalized slow-growing functions (see [4.15]). It should be noted that properties 1) to 3) of the Mellin functions were formulated precisely; they go well with the generalized functions method.

Analysis of the Cauchy Problem for the Evolution Equation. It may be shown that for (4.2.13), the following alternative exists. Let the parameter λ in this equation be different from a negative number or zero and let ϱ be some number out of the interval (a, b). Then

A. If $\kappa(\varrho + 0) = 0$, then (4.2.13) has always a (unique) solution in the class $\mathcal{M}(\varrho, b + h)$.
B. If $\kappa(\varrho+0) < 0$, then for the solution of (4.2.13) to be in the class $\mathcal{M}(\varrho, b+h)$ it is necessary and sufficient that the function $\Psi(s)$ should satisfy some conditions (of the type of equations). The number of the conditions is equal to $|\kappa(\varrho + 0)|$ and these conditions are different for different λ.
C. If $\kappa(\varrho + 0) > 0$, the solution of (4.2.13) in the class $\mathcal{M}(\varrho, b + h)$ depends on arbitrary constants whose number is equal to $|\kappa(\varrho + 0)|$.

The properties 1)–3) of the Mellin functions cover all situations of note thus allowing for a complete investigation of the Carleman equation (4.2.13) and the proof of the above-formulated alternative similar to that described in [4.12–13].

Below the given alternative is proved in two steps. At first, a "basic" function is constructed which is a partial solution of the homogeneous equation, see (4.2.27). Then the inhomogeneous equation (4.2.13) is examined using the basic function. This will be done below [see (4.2.27) and later on]. We shall solve (4.2.13) in the space $\mathcal{M}(\varrho, b + h)$ at $\kappa(\varrho + 0) = 0$ and thus prove statement A. To avoid extensive mathemtical complexities, we shall not give the proof of the complete alternative.

This ABC-alternative allows one to analyze the Cauchy problem for (4.2.6) with the initial condition (4.2.9).

If the rotation function $\kappa(\sigma)$ is negative in the whole interval (a, b), the Cauchy problem (4.2.6, 9) either has no solution at all or its solutions grow "too quickly" with time, so that the Laplace transformation in time is inapplicable. [Equation (4.2.6), for example, might have solutions which become infinite within a finite time.] Indeed, if (4.2.11) had solutions in the space $\mathcal{L}(a, b)$, then (4.2.13) would be solvable in the class $\mathcal{M}(\varrho, b + h)$ for $\varrho \in (a, b)$. But since $\kappa(\varrho) < 0$, it is necessary that the $|\kappa(\varrho)|$ conditions (of the type of equalities) for the function $\Phi(x)$ or ultimately, the functions ϕ_0, ϕ should be satisfied. These conditions must be satisfied at all λ with sufficiently large real parts and are therefore rather rigorous (besides, these conditions are rather specific in their form, and it is difficult to assign a physical meaning to them). These conditions will be violated for practically all functions ϕ_0, ϕ.

If only in one point $\varrho \in (a, b)$ the rotation function assumes a positive value, the solution of the Cauchy problem (4.2.6, 9) is not unique. Using the inverse Fourier and Laplace transformations to solve the homogeneous equation (4.2.20) in the class $\mathcal{M}(\varrho, b+h)$, one can construct a nontrivial solution (not identically zero) $F_0(x,t)$ of the Cauchy problem (4.2.6, 9) with the functions $\phi_0 \equiv 0$, $\phi \equiv 0$ satisfying the condition:

$$F_0(x,t) = \begin{cases} O\left(e^{-\varrho x}\right) & \text{at } x \to -\infty, \\ O\left(e^{-(b+h)x}\right) & \text{at } x \to +\infty. \end{cases} \tag{4.2.25}$$

The mentioned nonuniqueness of the Cauchy problem (4.2.6, 9) suggests one should impose boundary conditions on the evolution equation (4.2.6) to consider (4.2.9) in a narrower space [belonging to the space $\mathcal{L}(a, b)$] where the Cauchy problem for this equation will have a unique solution. The choice of suitable boundary conditions is ambiguous: physical considerations only are insufficient for such a choice so that additional mathematical arguments are required.

At first glance it seems natural from the physical viewpoint to demand that the solution of (4.2.6) at every fixed t should be bounded on the whole straight line $-\infty < x < +\infty$. It appears, however, that this is not a good choice; given such boundary conditions the Cauchy problem (4.2.6, 9) may have many solutions or may have no solutions at all depending on the particulars of the situation.

We shall proceed in a different way. Since, from the physical viewpoint, we are concerned with the solutions of the Cauchy problem where the initial conditions are given by finite functions $\phi_0(x)$ we shall impose boundary conditions according to the following rule. The solution $F(x,t)$ should be chosen from a set of solutions of the Cauchy problem (4.2.6, 9) and the solution $F(x)$ from a set of solutions of (4.2.11) whose magnitude tends to zero as quickly as possible (or grows as slowly as possible) at $|x| \to \infty$. The solution $G(s)$ of (4.2.13) corresponding to this choice is an element of the space $\mathcal{M}(\beta, \gamma)$ with the widest interval (β, γ).

It is clear that this boundary condition leads to a unique solution of (4.2.11), provided the rotation function $\kappa(\sigma)$ tends to zero in the interval (a, b). Indeed, let (σ_-, σ_+) be a zero rotation interval:

$$\kappa(\sigma) \begin{cases} < 0 & \text{at } a < \sigma < \sigma_-, \\ = 0 & \text{at } \sigma_- < \sigma < \sigma_+, \\ > 0 & \text{at } \sigma_+ < \sigma < b\,. \end{cases}$$

Then (in line with the alternative formulated at the beginning of this subsection) (4.2.13) has a unique solution $G(s)$ belonging to the space $\mathcal{M}(\varrho, b+h)$ at any $\varrho \in (\sigma_-, \sigma_+)$, and all the remaining solutions of this equation belong to the space $\mathcal{M}(\varrho, b+h)$ at $\varrho \geq \sigma_+$. Consequently, (4.2.11) has a unique solution $F(x)$ satisfying the condition (4.2.23) at $\varrho \in (\sigma_-, \sigma_+)$; this solution tends to zero faster than all other solutions (or grows slower than the rest ones) at $x \to -\infty$.

Let $\mathcal{L}_\varrho$ denote the space of functions $f(x)$ for which $f \exp(\varrho x)$ is limited by a constant both at $x \to -\infty$ and $x \to +\infty$. Conversely, $\mathcal{L}_\varrho$ is a "space of functions with a weight $\exp(-\varrho x)$". It is readily understood that at $\varrho \in (a, b)$, the condition (4.2.23) for the solution $F(x)$ of (4.2.11) is equivalent to the condition $F \in \mathcal{L}_\varrho$.

Thus we have shown that the suggested boundary condition is equivalent to considering (4.2.6, 11) in the space $\mathcal{L}_\varrho$ at some ϱ from the zero rotation interval (σ_-, σ_+). In this space the solution of the Cauchy problem (4.2.6, 9) always exists, is unique and (as will be seen later on) stable against perturbations of the initial data and external disturbances: if the functions $|\phi_0 \exp(sx)|$ and $|\phi \exp(sx)|$ are small, the function $F \exp(sx)$ will also be small at $t > 0$. Whence follows the correctness of the Cauchy problem (4.2.6, 9) in the space $\mathcal{L}_\varrho$.

There are three alternatives for realizing a nonvanishing rotation function $\kappa(\varrho)$ in the interval (a, b).

First, the function κ may be negative in the whole interval (a, b) as discussed above.

Secondly, the function κ may be positive in the whole interval (a, b). In this case the Cauchy problem (4.2.6, 9) has a nonunique solution in the space $\mathcal{L}_\varrho$ at any $\varrho \in (a, b)$, see (4.2.25).

Finally, the function κ may be different from zero and take on both negative and positive values:

$$\kappa \begin{cases} < 0 & \text{at } a < \sigma < \sigma_0 \,, \\ > 0 & \text{at } \sigma_0 < \sigma < b \end{cases} \tag{4.2.26}$$

where σ_0 is a number out of the interval (a, b). On the straight line $\operatorname{Re} s = \sigma_0$ there must be at least two zeros of the Mellin function $W(s)$.

One can show that in all cases where there is no zero rotation, the adopted boundary condition does not allow to obtain the correct formulation of the Cauchy problem for (4.2.6) and it is not possible to find a physically sensible space in which the Cauchy problem (4.2.6, 9) is correct. In the next subsection 4.2.2 we shall clarify the physical meaning of this incorrectness and find the physical pictures corresponding to the above three cases.

Basic Function. The special solution $B(s)$ of the auxiliary homogeneous equation

$$-B(s + h) = W(s)B(s) \tag{4.2.27}$$

is of great importance for deriving a solution of the Carleman equation (4.2.13) and the evolution equation (4.2.9). We will call this solution the *basic function.* Let there be an interval of zero rotation (σ_-, σ_+). We shall define the basic function $B(s)$ as a solution of (4.2.27) having the following properties:

i) it is meromorphic in the strip $\Pi(\sigma_-, \sigma_+ + h)$,
ii) it has neither zeros nor poles in the strip $\Pi(a, b + h)$,
iii) the functions $B(s)$ and $1/B(s)$ are polynomially bounded on the infinity in the strip $\Pi(a, b + h)$.

Later on we shall only need the fact the basic function exists.

For an arbitrary Mellin function $W(s)$ the basic function $B(s)$ at $s \in \Pi(\sigma_-, \sigma_+ + h)$ is defined by the following formulas:

$$w(s) = \ln[-W(s)] \; ; \tag{4.2.28a}$$

$$\mathcal{R}(s) = \frac{\pi}{2ih^2} \int\limits_{\sigma-\varepsilon-i\infty}^{\sigma-\varepsilon+i\infty} \frac{w(r)}{\sin^2[\pi(s-r)/h]} dr \; , \tag{4.2.28b}$$

$$\sigma = \operatorname{Re} s, \; \sigma_< \sigma - \varepsilon < \sigma_+ + h, \; 0 < \varepsilon < h \; ;$$

$$\mathcal{P}(s) = \int \mathcal{R}(s) ds \; ; \tag{4.2.28c}$$

$$B(s) = \exp \mathcal{P}(s) \; . \tag{4.2.28d}$$

In (4.2.28a), that continuous branch of the logarithm is chosen for which $\operatorname{Im} w(s) \to 0$ at $\operatorname{Im} s \to \pm\infty$. In the strip $\Pi(a, b+h)$, the function $B(s)$ is by virtue of (4.2.27) through its values in the strip $\Pi(\sigma_-, \sigma_+ + h)$.

Starting from (4.2.28) and taking the logarithm of (4.2.27), we obtain:

$$\mathcal{P}(s+h) - \mathcal{P}(s) = w(s), \quad s \in \Pi(\sigma_-, \sigma_+) \; . \tag{4.2.29}$$

The integral (4.2.28b) does not depend on the choice of ε and determines the $\mathcal{R}(s)$-function which is analytical in the strip $\Pi(\sigma_-, \sigma_+ + h)$ (for every s from this strip one can select such an ε that $0 < \varepsilon < h$, $\sigma_- < \operatorname{Re} s - \varepsilon < \sigma_+ + h$). By direct substitution it can be verified that the function (4.2.28b) satisfies

$$\mathcal{R}(s+h) - \mathcal{R}(s) = w'(s), \quad s \in \Pi(\sigma_-, \sigma_+) \; . \tag{4.2.30}$$

Comparing (4.2.30) and (4.2.29), we see that as a solution $\mathcal{P}(s)$ of (4.2.29) one can take the integral (4.2.28c) of the function (4.2.28b). Then (4.2.28d) satisfies (4.2.27) in the strip $\Pi(\sigma_-, \sigma_+)$, is analytical, and different from zero in the strip $\Pi(\sigma_-, \sigma_+ + h)$. The last thing that is left to be done is to show that the function $B(s)$ and $1/B(s)$ are polynomially bounded in the strip $\Pi(\sigma_-, \sigma_+ + h)$. We do not give the proof here since it is very tedious. We shall only remark that it relies heavily on the properties 1)–3) of the Mellin functions, in particular, on the estimate (4.2.18).

Solution of the Cauchy Problem. Let there exist a zero rotation interval (σ_-, σ_+). We shall derive a solution of the Cauchy problem (4.2.6, 9) which satisfies the suggested boundary condition, i.e., which belongs to the space $\mathcal{L}_\varrho$ for $\varrho \in (\sigma_-, \sigma_+)$.

The solution $F(x)$ of (4.2.11) in the space $\mathcal{L}_\varrho$ corresponds to the solutions $G(s)$ of (4.2.13) in the space $\mathcal{M}(\varrho, b+h)$. According to (4.2.24), if $G \in \mathcal{M}(\varrho, b+h)$ and $\sigma_- < \varrho < \sigma_+$, then $G \in \mathcal{M}(\sigma_-, b+h)$. Let us seek the solution of (4.2.13) in the form

$$G(s) = B(s) g(s) \; .$$

For the g-function we have a difference equation with constant coefficients:

$$\lambda g(s) + g(s-h) = Q(s) = \frac{\Psi(s)}{B(s)}, \; s \in \Pi(a, b+h) \; , \tag{4.2.31}$$

which may be solved using the Fourier transformation. Owing to the properties of the basic function $B(s)$we have (i) $Q \in \mathcal{M}(\sigma_-, \sigma_+ + h)$ and (ii) (4.2.13) in the class $\mathcal{M}(\sigma_-, b + h)$ is equivalent to (4.2.31) in the class $\mathcal{M}(\sigma_-, \sigma_+ + h)$. Whence follows the existence and uniqueness of functions $P(x)$, $f(x)$ ensuring

$$Q(s) = \int_{-\infty}^{+\infty} P(x)e^{sx}\,dx,\ g(s) = \int_{-\infty}^{+\infty} f(x)e^{sx}\,dx,\ s \in \Pi(\sigma_-, \sigma_+ + h)\ .$$

Substituting these expressions into (4.2.31), we obtain

$$\lambda f(x) + f(x)e^{-hx} = P(x)$$

leading to

$$f(x) = \frac{P(x)}{\lambda + e^{-hx}}\ .$$

Consequently (at $\lambda > 0$), the solution $F(x)$ of (4.2.11) in the space $\mathcal{L}_\varrho$ ($\sigma_- < \varrho < \sigma_+$) exists, is unique and defined by

$$\begin{aligned}
&\Psi(s) = \int_{-\infty}^{+\infty} \Phi(x)e^{-sx}\,dx,\ Q(s) = \frac{\Psi(s)}{B(s)}\ ,\\
&P(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} Q(s)e^{-sx}\,dx,\\
&f(x) = \frac{P(x)}{\lambda + e^{-hx}},\ g(s) = \int_{-\infty}^{+\infty} f(x)e^{sx}\,dx\ ,\\
&G(s) = B(s)g(s),\ F(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} G(s)e^{-sx}\,ds\ ,
\end{aligned} \tag{4.2.32}$$

where $\sigma = \operatorname{Re} s \in (\sigma_-, \sigma_+ + h)$. The formulas (4.2.32) may be written in the form

$$F = \mathbb{Z}^{-1}[\lambda + \exp(-hx)]^{-1}\mathbb{Z}[\Phi]\ , \tag{4.2.33a}$$

where $\mathbb{Z}$, $\mathbb{Z}^{-1}$ are the mutually inverse convolution operators with generalized functions $z_1(x)$, $z_2(x)$ whose Fourier images are $1/B(s)$ and $B(s)$, respectively,

$$\begin{aligned}
&\mathbb{Z}f = z_1 * f,\ \frac{1}{B(s)} = \int_{-\infty}^{+\infty} z_1(x)e^{sx}\,dx,\\
&\mathbb{Z}^{-1}f = z_2 * f,\ B(s) = \int_{-\infty}^{+\infty} z_2(x)e^{sx}\,dx\ ,
\end{aligned} \tag{4.2.33b}$$

where $s \in \Pi(\sigma_-, \sigma_+ + h)$; evidently, $z_1 * z_2 = \delta(x)$. The function $1/[\lambda + \exp(-hx)]$ is understood in (4.2.33) as an multiplication operator acting on this function.

Making in (4.2.33a) the inverse Laplace transformation in the variable λ, we get a solution of the Cauchy problem (4.2.6, 9). If there is no external action ($\phi = 0$), the function $\Phi = \phi_0$ does not depend on λ and

$$F(x,t) = T(t)\phi_0(x) \quad (\phi \equiv 0),$$

where

$$T(t) = \mathbb{Z}^{-1} \exp(-te-hx)\,\mathbb{Z} \quad \text{for } t \geq 0. \tag{4.2.34}$$

Here we took advantage of the fact that the operator $\mathbb{Z}$ is independent of λ, and

$$\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{e^{\lambda t}}{\lambda + e^{-hx}}\, d\lambda = \exp[-t\exp(-hx)]$$

at any $\delta > 0$.

For an arbitrary external action, the solution of the Cauchy problem (4.2.6, 9) has the form

$$F(x,t) = T(t)\phi_0(x) + \int_0^t T(t-\tau)\phi(x,\tau)d\tau. \tag{4.2.35}$$

Remark 4.2. In the study of the evolution equation (4.2.6) for an incorrect Cauchy problem of this equation, it is of interest to establish mathematically rigorously the correctness of this Cauchy problem, provided there exists a zero rotation interval. This may be done using the method of generalized functions. Let S'_ϱ be a space of such generalized functions $f(x)$ that $f(x)\exp(\varrho x)$ is a slowly growing generalized function (cf. the definition of the space $\mathcal{L}_\varrho$). The correctness of the Cauchy problem (4.2.6), (4.2.9) and the fact that its solutions are actually determined by (4.2.34–35) are ensured by the following.

Theorem. A family of operators (4.2.34) forms a semigroup [4.17] in the space S'_ϱ at any $\varrho \in (\sigma_-, \sigma_+ + h)$; the operator

$$f \longmapsto e^{-hx}[U * f] \tag{4.2.36}$$

is an infinitesimal generating operator of this semigroup in the space S'_ϱ at any $\varrho \in (\sigma_-, \sigma_+ + h)$. (For the relationship between correctness and semigroups see, e.g., [4.15]). An evident consequence of this theorem is the stability of solutions of (4.2.6) (relative to initial perturbations) in the topology of space S'_ϱ at any $\varrho \in (\sigma_-, \sigma_+ + h)$.

4.2.2 Behavior of Kolmogorov-Like Turbulent Distributions. Stability Criterion

Based on the results of the preceding subsection, we shall examine the behavior of the solutions of the evolution equation (4.2.6) which describes the evolution of perturbations of the Kolmogorov spectrum having the form of an arbitrary angular harmonic. The character of this behavior depends mainly on the details of the rotation function κ. Let us first consider a "regular" situation in which there exists a zero rotation interval (σ_-, σ_+). In this case the solution $F(x,t)$ of the Cauchy problem (4.2.6, 9) is given by (4.2.34–35) whose form is determined by the basic function $B(s)$, see (4.2.33b).

Zeros and Poles of the Basic Function. The asymptotics of solutions of the Cauchy problem (4.2.6, 9) (at $t \to \infty$ or $|x| \to \infty$) are determined by the zeros and poles of the basic function $B(s)$. $B(s)$ has neither zeros nor poles in the strip $\Pi(\sigma_-, \sigma_+ + h)$. Hence, it follows from (4.2.27) that its zeros and poles in the strip $\Pi(a, b+h)$ are characterized as follows: if p is a zero of the Mellin function $W(s)$ lying on the right of the strip $\Pi(\sigma_-, \sigma_+)$ [i.e., $p \in \Pi(\sigma_+, b)$], then the $B(s)$-function has zeros in all points of the form

$$p+h,\ p+2h,\ p+3h, \ldots \tag{4.2.37}$$

located in the strip $\Pi(a, b+h)$. All zeros of a sequence like (4.2.37) have the same multiplicity equal to that of the zero p of the function $W(s)$. If the function $W(s)$ has a zero q on the left of the strip $\Pi(\sigma_-, \sigma_+)$ [i.e., $q \in \Pi(a, \sigma_-)$], then the function $B(s)$ has zeros in all points

$$q,\ q-h,\ q-2h, \ldots\ , \tag{4.2.38}$$

on the strip $\Pi(a, b+h)$, with all poles of such a sequence (4.2.38) having the same multiplicity equal to the one of the zero at q. The function $B(s)$ has no other zeros and poles in the strip $\Pi(a, b+h)$, in particular, no zeros in the strip $\Pi(a, \sigma_+ + h)$ and no poles in the strip $\Pi(\sigma_-, b+h)$

For the sake of simplicity of the form of the asymptotic expansion, we shall later on consider all zeros and poles of the $W(s)$, $B(s)$-functions to be of first order.

Asymptotics at $|x| \to \infty$. Using (4.2.24), it is easy to see that the solution $G(s)$ of (4.2.13) in the class $\mathcal{M}(\varrho, b+h)$ $(\sigma_- < \varrho < \sigma_+)$ may have poles in the strip $\Pi(a, b+h)$ only in points of the form (4.2.37) in which the basic function $B(s)$ possesses poles. The poles of the Fourier image $G(s)$ of the solution $F(x,t)$ of (4.2.6) have the same points as $B(s)$. Therefore, the asymptotic behavior of the solution $F(x,t)$ at $|x| \to \infty$ is:

$$F(x,t) \approx \begin{cases} \sum_q K_q(t) e^{-qx} & \text{at } x \to -\infty\ , \\ O\left(e^{-(b+h)x}\right) & \text{at } x \to +\infty\ , \end{cases} \tag{4.2.39}$$

$$K_q(t) = \operatorname{res} G_t(q)\ ,$$

where the summation extends over the set of poles q of the basic function $B(s)$ of the series (4.2.38). The main terms in the sum (4.2.39) are those in which the values of q have the largest real part. Therefore

$$F(x,t) \sim \sum_{\operatorname{Re} q = \sigma_-} K_q(t) e^{-qx}, \quad x \to -\infty\ , \tag{4.2.40}$$

where the summation includes the set of zeros q of the Mellin function $W(s)$ lying on the line $\mathrm{Re}\, s = \sigma_-$.

Free Evolution of Initial Perturbations. Let us describe at first the behavior of the solution $F(x,t)$ of the evolution equation (4.2.6) without external action ($\phi \equiv 0$). These solutions are determined by

$$F(x,z) = Z^{-1}\left[\Theta_t(x)P(x)\right] \tag{4.2.41}$$

where $P(x) = \mathbb{Z}\phi_0(x)$ is a time-independent function and

$$\Theta_t(x) = \exp[-t\exp(-hx)]\ , \tag{4.2.42}$$

see (4.2.34–35). The function (4.2.42) has the property

$$\Theta_{t\tau}(x) = \Theta_\tau\left(x - \frac{\ln t}{h}\right)\ . \tag{4.2.43}$$

The Fourier transforms $\psi_0(s)$, $Q(s)$ of the respective functions $\phi_0(x)$, $P(x)$ are related by $Q(s) = \psi_0(s)/B(S)$. Since $\psi_0(S)$ is an integer function, the poles of $Q(s)$ are determined by the zeros of the basic function $B(s)$. Therefore

$$P(x) \approx \begin{cases} O\left(e^{-ax}\right) & \text{at } x \to -\infty\ , \\ \sum_p -\psi_0(p)/B'(p) & \text{at } x \to +\infty\ , \end{cases} \tag{4.2.44}$$

where summation is effected over the set of zeros p of the function $B(S)$ being a combination of the sequences (4.2.37). Since the function (4.2.42) is "practically equal to zero" at $x \ll \ln t/h$, we have at sufficiently large t in conformity with (4.2.41, 44)

$$F(x,t) \approx \sum_p -\frac{\psi_0(p)}{B'(p)} Z^{-1}\left[\Theta_t(x)e^{-px}\right],\ (t \to \infty) \tag{4.2.45}$$

where the summation is performed over the set of the zeros p of the basic function $B(s)$ being a combination of the sequences (4.2.37). The main terms in the sum (4.2.45) are those in which the zeros p have the smallest real parts. Hence,

$$F(x,t) \sim \sum_{\mathrm{Re}\, p=\sigma_+} -\frac{\psi_0(p+h)}{B'(p+h)} Z^{-1}\left[\Theta_t(x)e^{-(p+h)x}\right],\quad (t \to \infty)\ , \tag{4.2.46}$$

where the summation is performed over the set of zeros p of the Mellin function $W(s)$ lying on the straight line $\mathrm{Re}\, s = \sigma_+$.

Since the function $\Theta_t(s)$ satisfies (4.2.43) we know that according to (4.2.46), the perturbation $F(x,t)$ at large t represents a superposition of several "running waves"

$$F_p(x,t) = -\frac{\psi_0(p+h)}{B'(p+h)} Z^{-1}\left[\Theta_t(x)e^{-(p+h)x}\right] \tag{4.2.47}$$

possessing the self-similarity

$$F_p(x, t\tau) = F_p\left(x - \frac{\ln t}{h}, \tau\right) t^{-(p+h)/h}, \quad (t > 0, \tau > 0) . \tag{4.2.48}$$

The form of any such "wave" is universal, it does not depend on the initial conditions but is entirely determined by the characteristics of the medium; the initial conditions affect only the "amplitudes" of these "waves". From (4.2.48) we see that the "wave" (4.2.47) travels in the positive direction of the x axis according to the law const $+ \ln t/h$ (at $h < 0$ the "wave" moves into the negative direction). If the function (4.2.47) assumes at the moment τ in the point ξ a certain value $C_p = F_p(\xi, \tau)$, then at the moment $t > \tau$ at the point $x = \xi + (1/h)\ln(t/\tau)$ it will have the value

$$F_p(x, t) = C_p\left(\frac{t}{\tau}\right)^{-(p+h)/h} = C_p e^{-(p+h)x} .$$

Therefore, one can say that the "wave amplitude" changes according to the law $t^{-(p+h)/h}$ and that the "wave" has the "envelope" const $\cdot \exp[-(\sigma_+ + h)x]$ with $\operatorname{Re} p = \sigma_+$). If the zero p is a real number ($p = \sigma_+$), the "wave amplitude" changes monotonically; if the zero p contains a nonzero imaginary part, the "wave amplitude" oscillates with time. The sum (4.2.46) of the "waves" represents a "wave" evolving according to the logarithmic law const$+\ln t/h$ with the "envelope" const $\cdot \exp[-(\sigma_+ + h)x]$. However, the amplitude of the "complete wave" may show a much more complex behavior in time [the point is that zeros p of the function $W(s)$ located on the straight line $\operatorname{Re} s = \sigma_+$ may have imaginary parts whose values are rather different]. At every fixed moment of time t the behavior of the perturbation $F(x, t)$ at $|x| \to \infty$ is determined by the asymptotics (4.2.39–40). Figures 4.2 a,b,c ($t_1 < t_2 < t_3$) schematically show the evolution of the perturbation $F(x, t)$ at different positions of the interval $(\sigma_-, \sigma_+ + h)$ relative to the point $\sigma = 0$.

If $\sigma_- < 0 < \sigma_+ + h$, then any small initial perturbation will give rise to deviations from the Kolmogorov distribution that remain small throughout the whole spectrum and tend to zero as time progresses (Fig. 4.2). Thus the Kolmogorov spectrum proves to be stable against small perturbations. (A rigorous confirmation of this statement follows from the theorem given in Remark 4.2.) If $\sigma_+ + h < 0$, such a perturbation of the Kolmogorov spectrum increases for large x (Fig. 4.2b); the turbulent medium develops the above-mentioned "running structure", see (4.2.45–48). Such an instability is usually referred to as a convective instability. If $\sigma_- > 0$, a small initial perturbation leads to a large deviation from the Kolmogorov spectrum at large negative x (Fig. 4.2c); the form of this deviation is also universal, see (4.2.39–40).

Thus if the interval $(\sigma_-, \sigma_+ + h)$ does not contain the point $\sigma = 0$, the Kolmogorov spectrum is unstable with regard to initial perturbations. But the character of this instability is unusual. Indeed, since in real systems the inertial interval (k_0, k_d) is always finite ($0 < k_0 < k_d < \infty$), the perturbations of the Kolmogorov

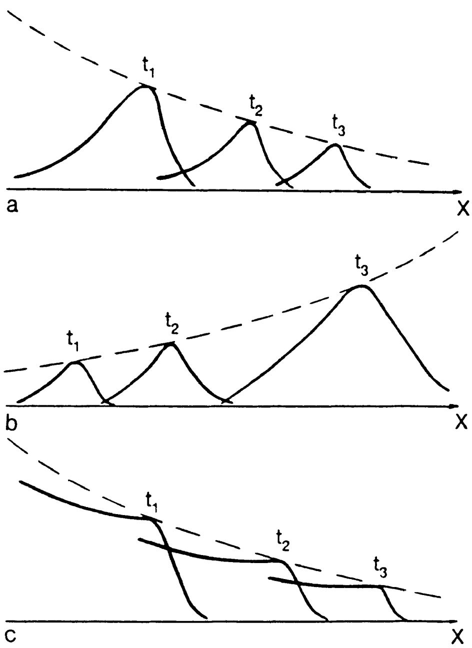

Fig. 4.2. The behavior ($t_1 < t_2 < t_3$) of the perturbation for different positions of the zero rotation interval: **a)** $\sigma_- < 0, \sigma_+ + h$, **b)** $\sigma_+ + h < 0$, **c)** $\sigma_- > 0$. The left slope is proportional to $\exp(-\sigma_- x)$, the right one to $\exp[-(b+h)x]$. The dotted line corresponds to $\exp[-(\sigma_+ + h)x]$

spectrum cannot increase infinitely as a result of this instability (see Fig. 4.2b, c) and, consequently, the Kolmogorov spectrum is in fact stable with respect to infinitely small perturbations. The above instability of the Kolmogorov spectrum is of asymptotic character: the perturbation of the Kolmogorov spectrum may increase arbitrarily strongly if the inertial interval is sufficiently large ($k_d/k_0 \gg 1$). Such an instability having an asymptotic meaning and originating from the existence of a large inertial interval will be referred to as *interval instability*.

The two qualitatively different interval instabilities are characterized by (i) $\sigma_+ + h < 0$ and (ii) $\sigma_- > 0$. In the first case the Kolmogorov spectrum perturbations grow gradually [at large t the perturbation value grows by a power law proportion to $t^{-(\sigma_+ + h)/h}$, see Fig. 4.2b], and in the second case the small initial perturbation leads almost instantly to a strong deviation from the Kolmogorov spectrum [in the range of small wave numbers] (see Fig. 4.2c). But in reality this "instancy" is also the result of the infinity of the inertial interval. Within a finite interval, all interaction times are finite. In the former case we shall call the instability *soft interval instability* and in the latter the *hard interval instability*.

Evolution of Perturbations Under External Action. Under the influence of a constant external action $\phi = \phi(x)$ in the system (4.26) the solution of the Cauchy problem (4.2.6, 9) is according to (4.2.34, 35) determined by

$$F(x,t) = Z^{-1}\Theta_t(x)Z[\varrho_0] + Z^{-1}\big(1 - \Theta_t(x)\big)e^{hx}Z[\phi] \,. \tag{4.2.49}$$

At $t \to \infty$ this solution tends to

$$F_\infty(x) = Z^{-1} e^{hx} Z[\phi] = \frac{1}{2\pi i} \int_{\varrho - i\infty}^{\varrho + i\infty} -\frac{\psi(s+h)}{W(s)}\, ds\,, \qquad \sigma_- < \varrho < \sigma_+\,, \tag{4.2.50}$$

where $\psi(s)$ is the Fourier image of $\phi(x)$. The function (4.2.50) is, evidently, the stationary solution of (4.2.6). In general (4.2.6) has many stationary solutions. In particular, further stationary solutions of this equation may be obtained with the help of the integral from (4.2.50) provided the parameter ϱ is chosen not to be from the (σ_-, σ_+) interval. If the interval (a, b) contains the point $\sigma = 0$ then there always exists a stationary solution of (4.2.6), which is bounded on the whole line $-\infty < x < +\infty$. However, only the solution (4.2.50) may be a bounded solution of the Cauchy problem for (4.2.9); all other stationary solutions have no relation to the evolution equation (4.2.6).

The asymptotic behavior of the solution (4.2.50) at large $|x|$ is

$$F_\infty(x) \approx \begin{cases} \sum_q -[\psi(q+h)/W'(q)]e^{-qx} & \text{at } x \to -\infty\,, \\ \sum_p -[\psi(p+h)/W'(p)]e^{-px} & \text{at } x \to +\infty\,, \end{cases} \tag{4.2.51}$$

where q (or p) goes through many zeros of the Mellin function $W(s)$ situated on the left (or right, respectively) of the band $\Pi(\sigma_-, \sigma_+)$. In the upper sum of (4.2.51), the main terms are those in which $\operatorname{Re} q = \sigma_-$, in the lower one the ones with $\operatorname{Re} p = \sigma_+$.

The character of the evolution of the solution (4.2.49) to the limiting stationary solution (4.2.50) is schematically indicated in Figs. 4.3a, b, c ($t_1 < t_2 < t_3$) for different positions of the zero rotation interval (σ_-, σ_+) relative to the point $\sigma = 0$. The external action feeds in the perturbation of the Kolmogorov spectrum with the perturbation front expanding according to the logarithmic law $x_b = \text{const} + \ln t/h$.

If $\sigma_- < 0 < \sigma_+$, the Kolmogorov spectrum is stable against weak external actions: at all wave numbers the turbulence spectrum differs only slightly from the Kolmogorov spectrum (Fig. 4.3a). When the zero rotational interval (σ_-, σ_+) does not contain the point $\sigma = 0$, there is an interval instability of the Kolmogorov spectrum against external effects. As a result of this instability, a stationary distribution is formed which strongly differs from the Kolmogorov spectrum either at $x \to +\infty$ (if $\sigma_+ < 0$, see Fig. 4.2b) or at $x \to -\infty$ (if $\sigma_- > h$, see Fig. 4.3c); in the remaining part of the inertial interval the state of the turbulent medium is Kolmogorov-like. The form of the resulting large deviation from the Kolmogorov spectrum is universal and according to (4.2.15) it is determined by the zeros of the Mellin function $W(s)$. Thus, a turbulent medium generates an ordered stationary structure. Since the zeros of the function $W(s)$ located on the straight lines $\operatorname{Re} s = \sigma_\pm$ may have nonzero imaginary parts, this structure may have rather a complex (nonmonotonic) form. When $\sigma_+ < 0$, the structure appears gradually, covering an increasingly larger region [at $t \to \infty$, the magnitude of the structure grows by a power law proportional to $t^{-\sigma_+/h}$, see Fig. 4.3b]

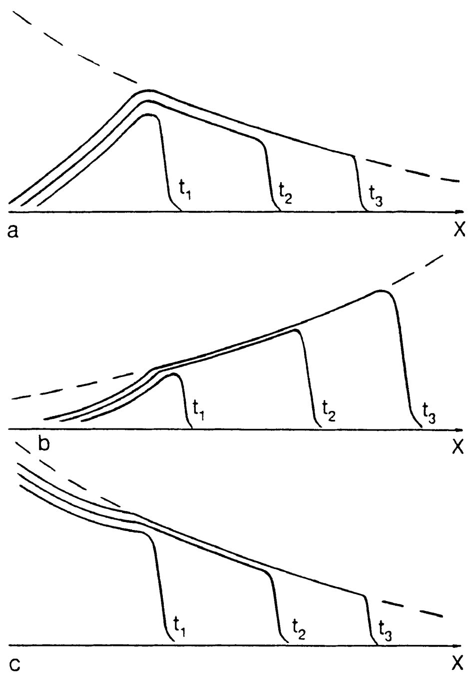

Fig. 4.3. Temporal behavior ($t_1 < t_2 < t_3$) of the perturbation under external pumping; **a)–c)** illustrate different positions of the zero rotation interval: **a)** $\sigma_- < 0 < \sigma_+$, **b)** $\sigma_+ < 0$, **c)** $\sigma_- > 0$. The dotted line depicts $\exp(-\sigma_+ x)$

and, consequently, the interval instability is soft. This instability results in the *soft generation of a stationary structure.* When $\sigma_- > 0$ a stationary structure [exponentially growing at $x \to -\infty$, see Fig. 4.3c] is formed within a finite time. We have a hard interval instability leading to the *hard generation of the stationary structure.*

As seen from comparison of (4.2.47) with (4.2.50) and of Fig. 4.2 with Fig. 4.3, the stationary solution stability condition with regard to external actions is more strict than the stability condition with regard to initial perturbations. In the former case, a necessary condition for stability is that the point $\sigma = 0$ corresponding to the Kolmogorov solution index falls within the zero rotational interval (σ_-, σ_+). In the latter case, it is sufficient that the point $\sigma = 0$ falls beyond a wider interval $(\sigma_-, \sigma_+ + h)$ at $h > 0$ or $(\sigma_- + h, \sigma_+)$ at $h < 0$. The physical reason for this difference is, of course, the fact that external actions constantly generate perturbations of the distribution.

The difference in the behavior of perturbations exhibited in Figs. 4.2, 3 may also be explained with the help of conservation laws. Indeed, the boundaries of the zero rotation interval σ_-, σ_+ are specified by the zeros of the Mellin function $W_l(s)$. But every zero $W_l(p) = 0$ implies the presence of an integral of motion of the form

$$I_l = \int Y_l(\Omega) k^{p+h-1} A(\boldsymbol{k})\, dk\, d\Omega \tag{4.2.52}$$

(we shall take p to be a real quantity) in the linearized kinetic equation. Besides, the presence of a zero of the Mellin function implies the presence of a stationary solution of the form

$$A_l(\boldsymbol{k}) = Y_l(\Omega)k^{-p},$$

transferring the constant flux of the integral I_l (examples of such solutions are the neutrally stable modes derived in Sect. 4.1 for $l = 0, 1$). It is readily understood that the behavior of the perturbations shown in Fig. 4.2 corresponds to conservation of the integral of motion I_l, while Fig. 4.3 illustrates the formation of the power asymptotics with a constant flux of this integral.

Remark 4.3. One can have the impression that the exponentially growing asymptotics at $|x| \to \infty$ [occurring for $\kappa(0) \neq 0$, see (4.2.51, 40)] imply the inapplicability of the linear approximation (4.2.4). As a matter of fact, the use of the linear approximation in such situations is based on the fact that in real systems the inertial interval (k_0, k_d) is finite $(0 < k_0 < k_\mathrm{d} < \infty)$. We suppose that real systems do not have a spectrum like $n(\boldsymbol{k}, t) = n_k^0[1 + A(\boldsymbol{k}, t)]$ as determined by (4.2.4–6), but to have a distribution which is close to this spectrum within the inertial interval [and is very different from it for $k \ll k_0$ and $k \gg k_\mathrm{d}$] – in a similar way as supposed for the Kolmogorov spectrum itself (4.2.2). Then applicability of the linear approximation requires $|F| \ll 1$ only inside the inertial interval.

Evolution Locality and Nonlocality of the Kolmogorov Spectra. We shall discuss the case that the locality interval contains the zero rotation interval. In this case, all integrals in (4.2.5, 6) converge, i.e., the evolution of perturbations having the form of the respective harmonics is determined only by the interaction of waves with scales of the same order. It seems natural to call this property the *evolution locality of the spectrum* (as opposed to the locality for which the collision integral converges on the Kolmogorov spectrum, see Sect. 3.1).

The nonexistence of a zero rotation interval within the locality interval may be shown to imply nonlocality of the evolution. In other words, the dependence of the behavior of the perturbation on conditions at the ends of the inertia interval. To understand this feature, it is convenient to discuss the continuous transition from systems with the interval (σ_-, σ_+) within (a, b) to systems having no zero rotation interval at all. Such a transition may be realized in three ways: by contraction of the interval (σ_-, σ_+) to the left or right end or to an inner point of the interval (a, b).

Let, for example, the interval (σ_-, σ_+) be contracted to the right limit of the locality interval $\sigma_\pm \to b$ then the rotation function is negative for most values within (a, b). At any moment of time $t > 0$ the perturbation $F(x, t)$ grows at $x \to -\infty$ proportionally to $\exp(-\sigma_- x)$, see Fig. 4.3. Consequently, the convolution $U * F$ is "located" on the boundary of the divergence so that the dominant role is played by the interaction with the left end of the inertia interval, i.e., with small k. Thus, if the rotation function is negative over the whole locality interval, the behavior of the perturbation arising at the moment

$t = 0$ will immediately (at all $t > 0$) depend strongly on the conditions at the left end of the inertial interval. We may consider this as a hard evolution nonlocality.

Let us now discuss the transition to a rotation function that is positive over the whole interval (a, b). If $\sigma_\pm \to a$, the external action should give rise to a perturbation growing at $x \to +\infty$ proportional to $\exp(-\sigma_+ x)$, see Fig. 4.3b. The perturbation front expands according to the law x_{fr} = const + $(\ln t)/h$. Consequently, if (k_0, k_d) is the inertial interval and the quantity k_1 characterizes the scale of the external action ($k_0 \ll k_1 \ll k_d$), then after a time of the order of $(k_d/k_1)^h$ the behavior of the perturbation will considerably depend on the conditions at the right end of the inertial interval.

Let us recall that above we set $h > 0$. If $h < 0$, in both cases described above, small and large scales exchange roles [i.e., the behavior of the perturbations depends on the conditions at the right end of the inertial interval for the negative rotation function, etc.].

If there is no convergence strip for integral (4.2.16) then the system has no evolution locality at all. An example for such a situation is given by shallow-water capillary waves [see (1.2.40)]: in the isotropic case the width of the locality interval equals $s_1 - s_2 = 2$ while it is for even angular harmonics given by $s_1 - s_2 - 2(\alpha - 1) = 0$. Thus, the evolution of the perturbation in the form of an even angular harmonic depends on the conditions at the ends of the inertial interval.

Strong Instability of Kolmogorov Spectra. Let us finally consider the third case in which there is no zero rotation. It is defined by (4.2.26) and is not related to nonlocality. It may even occur when the carrier of the function $U(x)$ is concentrated in a single point $x = 0$ [the convolution with the function $U(x)$ yields in this case the differential operator].

It is clear that with the help of an arbitrarily small perturbation, one can go over from the function $U(x)$ to $U_\varepsilon(x)$ in which the Mellin function $W_\varepsilon(s)$ has a zero rotation interval $(\sigma_-^\varepsilon, \sigma_+^\varepsilon)$ such that $\varepsilon \to 0$, $\sigma_\pm^\varepsilon \to \sigma_0$, $W_\varepsilon \to W_0$. Equation (4.2.6) with the function $U_\varepsilon(x)$ should be considered in the space $\mathcal{L}_\varrho$ at $\varrho \in (\sigma_-^\varepsilon, \sigma_-^\varepsilon)$ in which the Cauchy problem for this equation always has one and only one solution $F_\varepsilon(x, t)$.

The incorrectness of the (unperturbed) Cauchy problem (4.2.6, 9) in the case under discussion is easily understood from the fact that the limit of the function $F_\varepsilon(x, t)$ at $\varepsilon \to 0$ largely depends on the family $\{U_\varepsilon(x),\ \varepsilon \to 0\}$ within which we approach $U(x)$; that holds in particular for the asymptotic of the limiting function at $x \to -\infty$.

Since $\sigma_\pm \to \sigma_0$ at $\varepsilon \to 0$, it would be natural to consider (4.2.6) in the space $\mathcal{L}_{\sigma 0}$. However, the set of eigenvalues of the operator (4.2.36) in that space covers the entire complex plane. Therefore, we can consider the Kolmogorov spectrum to be strongly unstable (to perturbations having the form of the angular harmonic in question). In contrast to interval instability, the perturbations of the Kolmogorov spectrum grow in this case with time throughout the whole inertial interval. Under the influence of constant external actions, no stationary solution is formed and an essentially nonstationary regime, "secondary turbulence", may result. It should be noted that a more consistent formulation of the problem of the stability of the Kolmogorov spectrum should be as follows. First of all, the kinetic equation should be supplemented by terms describing the isotropic pumping and damping regions and a stationary solution of this equation should be found that is close to the Kolmogorov spectrum in an interval (k_1, k_2); outside this interval the solution may strongly differ from the Kolmogorov spectrum. Then the kinetic equation must be linearized in the vicinity of this stationary solution and expanded in angular harmonics. For a particular angular harmonic the expansion leads to an evolution equation of the form (4.2.5) for which the Cauchy problem always has one and only one solution; the operator $\hat{L}$ is no longer homogeneous. Having examined the behavior of the solutions of this equation, one should clarify the changes that occur when the ranges of the source and sink in k-space go to zero or to infinity and examine the behavior of the perturbations established in the interval (k_1, k_2). Finally, one should analyze in which situations this behavior is independent of the specific type of the source and sink.

This program for examining the stability of Kolmogorov spectra turns out to be too complex. Currently there exists no strict proof of the fact that in general the kinetic equation with a source and sink has a stationary solution close to the Kolmogorov spectrum in some interval.

As seen above, the strongly unstable equation (4.2.6) may become stable as the result of an arbitrary small variation in the medium characteristics (the interval instability is not an absolute instability). Such a sharp transition from strong instability to stability occurs only in the limit when both the range of a source and the range of a sink tend to zero or to infinity and the inertial interval becomes infinitely large ($k_0 \to 0$, $k_d \to \infty$). With a finite interval the Kolmogorov spectrum may also be unstable in the case of a rather small zero rotation interval (σ_-, σ_+) with $(\sigma_+ - \sigma_- \approx 0)$; the perturbations of the Kolmogorov spectrum will exponentially grow with time in the whole inertia interval. In the case of a small zero rotation interval the stability of the Kolmogorov spectrum established above has the following asymptotic meaning: no matter how small the value of $\sigma_+ - \sigma_-$ is, it is always possible to find a sufficiently large inertial interval in which the Kolmogorov spectrum is stable (with regard to perturbations having the form of a respective angular harmonic). If there exists no zero rotation interval then the increment of the instability of the Kolmogorov spectrum is finite within a finite inertial interval (k_0, k_d). The increment tends to infinity at $k_0 \to 0$, $k_d \to \infty$.

The Kolmogorov Spectrum. Stability Criterion. The treatment of the preceding subsections allows us to clarify the conditions under which the state of a turbulent medium is in the whole inertial interval under various perturbations and at any moment of time close to the Kolmogorov distribution. Thus we arrive at the stability criterion for the Kolmogorov spectrum obtained by *Balk* and *Zakharov* [3.7]:

The Kolmogorov spectrum is stable against disturbances having the form of the angular harmonic $Y(\zeta)$ if and only if $\kappa(0) = 0$, i.e., if the rotation of the Mellin function $W(s)$ corresponding on the imaginary axis to the harmonic $Y(\zeta)$ is defined and equal to zero. (If the zero rotation interval exists and the point $\sigma = 0$ is its boundary, the Kolmogorov spectrum is indifferently stable against perturbations of the form of the respective angular harmonic.)

When $\kappa(0) > 0$, the instability of the Kolmogorov spectrum is strongest for large k and when $\kappa(0) < 0$, for small k. The interval instability and evolution nonlocality are soft if the quantity $h\kappa(0)$ is positive and hard if it is negative.

It should be noted that for different angular harmonics the behavior of the perturbations may be of different types and may have different asymptotics, so that the overall perturbation of the Kolmogorov spectrum may be rather diverse since it is a superposition of perturbations corresponding to all angular harmonics.

It is readily seen that for the order of angular harmonic $Y(\zeta)$ tending to infinity, the value of the integral (4.2.16) specifying the Mellin functions becomes real and negative, just as at $|\mathrm{Im}\, s| \to \infty$, see (4.2.18). Consequently, for angular harmonics of sufficiently high order l, the quantity $\kappa(0)$ is always zero, with the zero rotation interval extending at $l \to \infty$ over the whole interval (a, b). Hence, the Kolmogorov spectrum is always stable with regard to perturbatins in form of higher angular harmonics. The first few angular harmonics determine the behavior of a turbulent medium near the Kolmogorov spectrum.

Examination of the instability of the Kolmogorov spectrum involves verification of a finite number of conditions of the form $\kappa(0) = 0$ which may be conveniently checked using a computer. Since the calculation of the value of $\kappa(0)$ should necessarily yield an integer, the use of a computer including an estimate of the error would even yield a rigorous mathematical proof of the stability status of the Kolmogorov spectrum.

It is worthwhile to draw attention to the following three general symmetries of the Mellin functions.

The fact that the kinetic equation is real implies that the Mellin functions W_Y and W_{Y^*} corresponding to the angular harmonics Y and Y^*, respectively, satisfy

$$W_Y(s) = W^*_{Y^*}(s^*)$$

and that their rotation functions coincide

$$\kappa_Y(0) \equiv \kappa_{Y^*}(0) .$$

For three-dimensional media the Mellin functions $W_l^j(s)$ corresponding to angular harmonics $Y_l^j(\zeta)$, $j = -l, \dots, l$ of the same order l are identically equal

$$W_l^j(s) = \frac{1}{2l+1} \sum_{n=-l}^{l} W_l^n(s) . \tag{4.2.53}$$

If a continuous isotropic medium is also mirror-symmetric then the interaction coefficient is invariant with regard to reflections in k-space and the Mellin functions satisfy

$$W(s^*) = W^*(s) . \tag{4.2.54}$$

This equation is always satisfied in the decay three-dimensional case, because in three dimensions it is always possible to accomplish reflections of three vectors $\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2$ by the aid of rotations.

From (4.2.54) it follows that the zeros of the function $W(s)$ are either real or form pairs of complex conjugate numbers. Hence, the third case for the nonexistence of zero rotation (4.2.26) may be the case of general position (on the line $\operatorname{Re} s = \sigma_0$, there may be a pair of complex conjugated zeros of the Mellin function).

When the function $W(s)$ obeys (4.2.54), one can formulate a rather simple sufficient condition of the instability of the Kolmogorov spectrum

$$W(0) > 0 . \tag{4.2.55}$$

Indeed, it follows from (4.2.54–55) that the rotation $\kappa(0)$ is inevitably odd.

The criterion for the instability of the Kolmogorov solution with regard to isotropic perturbations is formulated in a different way. For the zero harmonic, $W_0(0)$ is always zero and the rotation function $\kappa_0(0)$ is not defined. To obtain the wanted criterion, one should slightly shift the vertical axis on which κ is calculated to the right or to the left depending on the region in which the Kolmogorov solution is realized, i.e., for small or large k, respectively. For example, a sufficient condition for the instability of the short-wave spectrum with regard to isotropic perturbations is the inequality $W_0(\varepsilon) > 0$ for the small negative ε. It is readily seen that this is equivalent to the condition $W_0'(0) < 0$. Thus we arrive again at the Fournier-Frisch criterion described in Sect. 3.1: solutions with the "wrong" sign of the flux are unstable.

4.2.3 Physical Examples

As we have seen in the preceding subsection, the behavior of perturbations is determined by position of the zeros of the Mellin functions $W_l(s)$. However, every zero $W_l(p) = 0$, for example, for real p corresponds to the stationary power-type solution $A(\boldsymbol{k}) = \delta n(\boldsymbol{k})/n_{\boldsymbol{k}}^0 = Y_l k^{-p}$ of the linearized kinetic equation. For zeroth and first angular harmonics, the stationary power solutions [the neutrally stable modes (4.1.9, 11, 13, 14, 16, 18, 20, 23)] were derived in Section 4.1. These solutions are universal, i.e., they do not depend on the particular form of the interaction coefficient, but are entirely determined by the indices. All these modes correspond to small fluxes of the integrals of motion (4.2.52). The locality of a neutrally stable mode implies that p is an element of the analyticity strip of the corresponding function $W_l(s)$.

The zeros corresponding to universal modes, like all other zeros of Mellin functions, determine the terms of the asymptotic expansions of the Kolmogorov spectrum, see (4.2.39, 45, 51). The role of an individual term depends considerably on the position of the corresponding zero relative to the zero rotation strip $\Pi(\sigma_-, \sigma_+)$: it matters whether the location is on the right or left of this strip, on its boundary or far away from it. In particular, in the case of the instability of the Kolmogorov spectrum, the main and fastest growing correction to the spectrum is entirely determined by the particular zero p of $W(s)$ for which either $\operatorname{Re} p = \sigma_- > 0$ holds or $\operatorname{Re} p = \sigma_+ > 0$.

Let us consider at first the stability problem of Rayleigh-Jeans spectra that are in thermodynamic equilibrium. If the power solution (4.2.2) is a thermodynamic spectrum, then it may easily be seen from (4.2.16) that the Mellin functions $W(s)$ have the properties

$$W(r+s) = W(r-s), \quad W(r+i\omega) < 0, \quad \text{where} \quad r = \frac{\nu-\mu}{2} = \frac{d-h}{2}\,.$$

That condition just presents the H-theorem for a given harmonic of the linearized kinetic equation. It follows that the analyticity strip $\Pi(a,b)$ of the Mellin function $W(s)$ is symmetric with regard to the line $\operatorname{Re} s = r$; on this line the rotation of the function $W(s)$ is zero. Consequently, there exists a zero rotation interval symmetric relative to the point r. However, this interval does not necessarily include the point $s = 0$. Thus, the H-theorem does not ensure the stability of the equilibrium distribution: the initially small perturbation can grow in the process of evolution. The existence of a zero rotation interval implies that the instability of thermodynamic spectra is not strong, i.e., the perturbations cannot grow in the whole k-space. If interval instability exists, it will inevitably occur at large scales if $r > 0$ or at small scales if $r < 0$. Interval instability will take place, for example, when the zero p corresponding to a neutrally stable mode is located between the points $s = 0$ and $s = r$. Let us consider, for example, the decay case. For physically interesting media, the index of the Kolmogorov spectrum is generally larger than that of the thermodynamic spectrum. Therefore the quantity $r = d + m - \alpha$ is positive and, consequently, the interval instability can manifest itself only at large scales. In the decay case it is also easy to show that the zero $p_0 = 2(m+d-\alpha) = 2r$ corresponding to the isotropic mode (4.1.16) is not located between the points $s = 0$ and $s = r$ and cannot lead to interval instability. For the first angular harmonic the situation is different: since $\alpha > 1$ we know that out of the two zeros $p_1 = \alpha - 1$, $p_2 = 2r + 1 - \alpha$ symmetric with regard to the point r, one is always located in the interval $(0, r)$. This is in general an equilibrium zero p_1 (in all cases considered we have $p_1 < p_2$). Hence, the spectrum T/ω_k will show interval instability in the region of small k, provided that this zero falls also within the analyticity strip. The physical meaning of this instability is rather simple. An external source generates a perturbation in the form of the first angular harmonic having a nonzero momentum. The instability corresponds to the rearrangement process of the distribution T/ω_k to $T/[\omega_k - (\boldsymbol{k}\boldsymbol{u})]$ with the nonzero momentum. For example, for deep-water capillary waves ($m = 9/4$, $d = 2$ $\alpha = 3/2$), we have $0 < p_1 < r$ and $h = -7/2 < 0$, i.e., the perturbations of the equilibrium spectrum are shifted to small k, with the part of the perturbation due to the first harmonic growing in magnitude. Thus we have a soft interval instability. Since $p_1 + h < 0$, the spectrum under consideration is stable with regard to initial perturbations.

We can discuss the nondecay case in a similar way. For the solution $T/\omega_k \propto k^\alpha$ with zero chemical potential, the presence of the mode (4.1.20) with a small energy flux P cannot lead to instability, because $p_0(P) = 2m + 3d - 3\alpha = 2r$. However, there are two more isotropic, neutrally stable modes whose indices are symmetric relative to the point r: the equilibrium one with $p_0(\mu) = \alpha$ and the nonequilibrium mode (4.1.23) with $p_0(Q) = 2r - \alpha$. At $r > \alpha$, the instability may be associated with the former and at $\alpha > r > 0$ with the latter. At $r < 0$, the Rayleigh-Jeans spectrum is stable with regard to isotropic perturbations, but we may have interval instability with respect to perturbations in the form of the first angular harmonic, as one of the zeros $p_1 = \alpha - 1$, $p_2 = 2r + 1 - \alpha$ will inevitably be found in the interval $(0, r)$. As we see, all these instabilities are associated with the conservation laws and correspond to structural rearrangements of the distributions after adding to them the initially nonexistent integral of motion.

Let us discuss now possible instabilities of the Kolmogorov spectrum that are associated with the universal neutrally stable modes. In particular, we shall consider formation of these modes under perturbations of a source.

As usual, we shall start from the decay case. In an isotropic general system, there are no other integrals of motion besides the one of energy; therefore, the isotropic neutrally stable modes are not formed (except for the trivial one $\delta n_k/n_k^0$ = const corresponding to energy flux variations). In all the examples the Mellin function $W_0(s)$ has a single zero $W_0(0) = 0$ in the locality strip, and the Kolmogorov spectra are indifferently stable with respect to isotropic perturbations. We shall note, however, that this statement has not been proved in the general case.

With regard to the first angular harmonic, there is a drift mode (4.1.9) corresponding to constant momentum flux. Its index $p_1 = 1 - \alpha < 0$ gives the zero of the Mellin function $W_1(p_1) = 0$ located on the left of the point $s = 0$. Consequently, the instability associated with this zero may be observed in the region of large k, i.e., just in the inertial interval [we suppose as usual $\nu = m + d > \alpha$ which, according to (3.1.13), corresponds to a positive energy flux and to a source at small k]. If the perturbations of the source have the form of the first angular harmonic, a drift mode may be formed (i.e., determine the spectrum perturbation asymptotics at $k \to \infty$), if p_1 is the zero closest to the point $s = 0$. Besides, in conformity with (4.2.55), the condition $W_1(0) > 0$ should be satisfied. From this follows the necessity of the condition

$$W_1'(p_1) = W_1'(1 - \alpha) > 0 \, . \tag{4.2.56}$$

But, as we have seen in Sects. 3.1.3 and 4.1, the derivative of the collision integral with regard to the index of the stationary-state solution, specifies the sign of the corresponding flux. Thus, the condition (4.2.56) implies that in the decay case the drift mode is formed only for positive momentum flux, i.e., it has the same direction as the energy flux of the main solution. Physically this condition seems to be quite natural, as fluxes should be directed towards the damping region.

This criterion (first formulated by *Falkovich* [4.6]) is also valid for any of the universal steady-state modes (4.1.9, 11, 13, 14, 16, 18, 20, 23): a neutrally stable mode is formed and leads to structural instability of the Kolmogorov spectrum only when the flux of the integral of motion transferred by it has the same direction as the flux of the main integral of motion. Indeed, in mirror-symmetric media, for the mode (4.1.11) dominating at large k, a sufficient instability condition $W_0(0) > 0$ is provided by the inequality $W_0'(-\alpha) > 0$, i.e., by the positive character of the small-wave action flux. On the other hand, the modes (4.1.13–14) can lead to structural instability of a spectrum with an action flux if the fluxes transferred by those modes are directed towards small k: $W_0'(\alpha) < 0$, $W_1'(1) < 0$.

Let us consider some examples. The integrals (4.2.16) determining the Mellin function are rather complex and cannot be calculated analytically. However, they may be calculated on a computer. Since it is sufficient to find only the Mellin function rotations (being integers), these computations can be rather inaccurate. In [4.10] one can find a transformation of the integrals (4.2.16) to a form suitable for machine computations.

Let us start with the turbulence of gravitational waves on the surface of a deep incompressible fluid. The appropriate dispersion law is given by (1.1.42) and the four-wave interaction coefficient, by (1.2.42), so that $\alpha = 1/2$, $m = 3$, $d = 2$. There are two Kolmogorov solutions, one with action flux towards the region of large scales (3.1.27) and the other one with energy flux towards the region of small scales (3.1.28).

Calculations with formula (A.4.10) show that for both spectra the values of all Mellin functions $W_l(s)$ on the imaginary axis ($s = i\omega$, $-\infty < \omega < \infty$) are located in the left half-plane. [The calculations were performed for $l = 0, 1, \dots, 29$; with growing l, the values of the function $W_l(i\omega)$ for $-\infty < \omega < \infty$ are displaced further into the left half-plane.] Consequently, rotations of all Mellin functions around the imaginary axis vanish and the Kolmogorov spectra (3.1.27–28) are stable. One can also directly verify that in this case the neutrally stable modes (4.1.11, 13–14) transfer backwards fluxes of action, energy and momentum, respectively, that are small compared to the fluxes of the main integrals of motion. Consequently those nodes could not be realized. In real situations at very small k ($k \simeq k_0$) and at very large k ($k \simeq k_d$), the medium gives rise to damping (sometimes the damping may be due to the nonsteady state of the spectrum in the region of small or large k); pumping is observed in an intermediate range $k \simeq k_1$. If the pumping and damping regions have sufficiently strong differences in their scales $k_0 \ll k_1 \ll k_d$, then in a stationary state all the energy pumped into the system should be transferred to the region of large k and all the wave action pumped in, to the region of small k, see (3.1.26). Thus (at least for weak anisotropic perturbations), the Kolmogorov spectrum (3.1.28) supporting an energy flux should be formed in the inertial interval $k_1 \ll k \ll k_d$ and the Kolmogorov spectrum (3.1.27) supporting a wave action flux in the inertial interval $k_0 \ll k \ll k_1$.

A typical example for a system with a decay law and weak turbulence are turbulent capillary waves (on the surface of a deep incompressible fluid). The dispersion law of these waves and the interaction coefficient are determined by (1.2.40). Thus we have $\alpha = 3/2$, $m = 9/4$, $d = 2$. Numerical computation shows that the rotations $\kappa_l(0)$ of the Mellin functions $W_l(s)$ are zero for all $m \neq \pm 1$ and $\kappa_{\pm 1}(0) = 1$. Hence, the spectrum (3.1.15b) is unstable with regard to perturbations having the form of the first angular harmonic. Numerical computation also yields $\kappa_{\pm 1}(-2/3) = 0$. This leads to the following conclusions. First, the instability of the first angular harmonic is of the interval type and is hard since $h = -3/4 < 0$; this instability manifests itself in the region of large k. Second, in the strip $\Pi(-2/3, 0)$ the function $W_1(s)$ has a single zero which occurs at $s = 1 - \alpha = -1/2$ and corresponds to the universal mode (4.1.9). In this case the direction of the (small) momentum flux coincides with the one of the energy flux. Consequently, in a system of capillary waves under weak constant anisotropic action, a spectrum of the form (4.2.51) should be established:

$$n(\boldsymbol{k}) \simeq \lambda P^{1/2} k^{-17/4} \begin{cases} 1, & k \to 0\,, \\ \mathrm{const}\sqrt{k}\cos\theta, & k \to \infty\,. \end{cases}$$

Thus, the anisotropy extends into the region of large k and the hypothesis about local isotropy does not hold for capillary wave turbulence; small fluctuations at large scales will lead to large fluctuations at small scales.

It is interesting to have a look at the structure of the stationary turbulence spectrum in the region of k-space in which the anisotropy is no longer small. The structural instability of the isotropic spectrum is associated with the stationary mode (4.1.9) transferring the momentum flux. Therefore it is natural to suppose that in the short-wave region a universal stationary distribution should be given by the fluxes of both conserved values P and R and have the form (4.1.5)

$$n(\boldsymbol{k}, P, \boldsymbol{R}) = \lambda P^{1/2} k^{-m-d} f(\xi), \quad \xi = \frac{(\boldsymbol{R}\boldsymbol{k})\omega_k}{Pk^2} .$$

In Sect. 4.1 we discussed the properties of such a solution at $\xi \to 0$, which is almost isotropic with $f(0) = 1$. One can make the hypothesis that at $\xi \to +\infty$ (i.e., at $k \to \infty$ and $\cos\theta > 0$) the distribution should be determined only by the momentum flux. A necessary for this is $f(\xi) \propto \sqrt{\xi}$. Finally, we assume $f(\xi) \to 0$ at $\xi \to -\infty$ to hold. Let us now describe in brief the properties of such a hypothetical solution. In the direction of the vector $\boldsymbol{R}$, i.e., the pumping has a maximum (i.e., at $\theta = 0$), the occupation numbers should decrease slower than for the isotropic Kolmogorov solution $n(\boldsymbol{k}) \propto k^{-15/4}$. In transversal directions $\theta = \pm\pi/2$, the decrease of $n(\boldsymbol{k})$ with increasing k coincides with the behavior of the isotropic case (3.1.15b): $n(\boldsymbol{k}) \propto k^{-17/4}$. Most of the waves are found in the right part of the hemisphere with $|\theta| < \pi/2$. It should be noted that in the only case in which $f(\xi)$ was determined unambiguously [for sound turbulence, see below (5.1.12, 14)], its properties proved to be identical to the above case: $f(\xi) \propto \sqrt{\xi}$ at $\xi \to +\infty$ and $f(\xi) \to 0$ at $\xi \to -\infty$.

It should be remarked that such an instability does not occur for all media with a decay dispersion law. In general, the vaules of $\kappa_l(0)$ can for $l = 0, 1, \ldots$ assume rather diverse sets of values. Thus we may consider a model with the same values of the parameters α, m, d as for capillary waves where we have $\kappa_1(0) = 1$, $\kappa_7(0) = 3$ and for l other than 1 and 7 the value $\kappa_l(0) = 0$.

An important example of nondecay weak turbulence is the turbulence of Langmuir waves in plasmas. The dispersion law of these waves (1.3.3) may be considered to be scale-invariant with $\alpha = 2$. If the main nonlinear process of plasmon interaction is the exchange of virtual ion sound oscillations, the interaction coefficient is given by (1.3.14) and has the scaling index $m = 0$.

In this situation, the only local power spectrum is a Kolmogorov spectrum with the wave action flux Q. In two dimensions, the computer calculation has shown this Kolmogorov spectrum to be unstable with regard to perturbations having the form of angular harmonics with $l = 0, \pm 1$; the other harmonics are stable. As we see, in this case the Kolmogorov solution is unstable with respect to isotropic perturbations.

If the dynamic equations (1.1.14) depict the nonlinear Schrödinger equation (1.4.24) written in momentum representation, then $\alpha = 2$, $m = 0$ and the interaction coefficient is a constant. In this case the computations show the only local Kolmogorov spectrum (with wave action flux) at $d = 2$ to be unstable with regard to the same angular harmonics as in the case of Langmuir turbulence.

Both these instabilities are associated with universal modes (4.1.13–14). Curiously, for both modes the (energy and momentum) fluxes are positive, i.e., are directed towards large k. However, for the initial spectra the wave action flux is also positive since $n_0(k) \propto \omega^{-2/3}$ and according to (3.1.22) we have $\operatorname{sign} Q > 0$. It is the "wrong" direction of the flux Q that gives in these two cases rise to the spectrum instability of isotropic media according to the Frisch-Fournier criterion (see Sect. 3.1.3).

In all cases considered, the instabilities of the Kolmogorov spectrum occur only for the zero and first angular harmonic and are associated with universal modes carrying small fluxes of the integrals of motion. These structural instabilities obviously describe the rearrangement processes of single-flux distributions to multi-flux distributions as for equilibrium systems. In this case, the asymptotics of stationary turbulent distributions at $k \to 0, \infty$ are determined by the directions of the fluxes of the integrals of motion. One can also assume that in general cases, structural instabilities of single-flux spectra may be associated only with the universal modes derived in Sect. 4.1 which carry the fluxes of energy, momentum or wave action. We thus come to the more sophisticated form of the universality hypothesis: in the inertial interval, a stationary spectrum should be defined by those fluxes of the integrals of motion which are directed from the source to the sink. Such spectra are universal since they depend on the fluxes only, being independent of the fine structure of the source. However, a universal spectrum is not necessarily an isotropic one. If the momentum flux is directed from the pumping to the damping region, then the stationary spectrum is anisotropic and the isotropy hypothesis is incorrect while the universality hypothesis may still hold.

In degenerate cases (for example, with additional integrals of motion) the structural instabilities of Kolmogorov spectra could be connected with angular harmonics larger than zero or unity. In the Sect. 5.1 we shall discuss structural instabilities of isotropic turbulence spectra for small-dispersion waves. Such wave systems are close to a degenerate system of nondispersive waves obeying a linear dispersion law that is intermediate between the decay and nondecay cases.

4.3 Nonstationary Processes and the Formation of Kolmogorov Spectra

Of all happenings of Nature,
explosion is the last to be deemed unexpected.

M. Tscvetaeva

In this section we shall discuss the nonstationary behavior of weakly turbulent distributions. We shall be concerned with both free evolution regimes (in the absence of external sources) and the formation of stable Kolmogorov distributions after pumping is switched on. As we shall see now, the character of the evolution dramatically depends on the sign of the index $h = \alpha+d-s_0$ dealt with in Sect. 3.4–4.2. Indeed, the index of the collision integral linearized against the background of the Kolmogorov solution is equal to $-h$. This means that if at some k the distribution deviates from a stationary one then the typical time for variations of the occupation numbers is

$$t_{NL}^{-1} \propto k^{-h} . \tag{4.3.1}$$

If there are no external effects to be considered then the energy should be conserved so that the characteristic time of the system may be evaluated from the kinetic equation

$$t_{NL}^{-1} \propto k^{-2h} E \tag{4.3.2}$$

where $E = \int \omega_k n_k d\boldsymbol{k}$ is the total energy of the distribution. Thus, in both cases the process of wave transfer, e.g., to large k is accelerated or slowed down depending on the sign of h. It is appropriate to draw attention to the point that the quantity h shows also at which end of the Kolmogorov distribution transporting an energy flux, the major part of the energy of the turbulence is concentrated (see Sect. 3.4.1)

$$E = \int \omega_k n_k^0 d\boldsymbol{k} \propto k^h . \tag{4.3.3}$$

For example, for $h > 0$ most of the energy of the Kolmogorov spectrum is confined to the region of large k. As we can see from (4.3.1, 2), the motion of the distribution slows in this case down as it moves towards larger wave numbers while its evolution into the opposite direction (containing little or no energy) is an accelerated process.

In Sect. 4.3.1 we shall first discuss the nonstationary behavior of weakly turbulent distributions of waves with a decay dispersion law and $s_0 = m + d$ and $h = \alpha - m$, see (3.4.3). Since h is equal to the difference between the frequency index and the index of the interaction coefficient, its sign indicates which coefficient of the Hamiltonian grows quicker with k, the one responsible for linear phenomena or for the interaction. The dimensional analysis of Sect 4.3.1 will show that distributions initially localized in the long-wave region evolve on their way towards large k in a self-similar manner. After a long-wave source has been switched on, the stationary Kolmogorov distribution is formed with the help of a self-similar relaxation front.

Wave systems with the opposite sign of h, i.e., with $h < 0$ are characterized by an evolution of the front of spectrum formation according to an explosion law, i.e., it approaches infinity within a finite time.

A strict analytical proof of the explosive character of pumping is given in Sect. 4.3.2 for the particular case of weak three-dimensional sound turbulence ($h = -1/2$). The idea of the proof is to consider the dynamics of the moments of the distribution function in k-space. Proceeding from the kinetic equation, one can prove that if initially some moments were finite (i.e., the distribution was decreasing fast) then they will become infinite within a finite time, which corresponds to the formation of the power asymptotics at $k \to \infty$. For the intermediate case $h = 0$ of two-dimensional acoustic turbulence, the same section gives analytical proofs of the nonexplosive character of the evolution. Section 4.3.2 gives the results of numerical simulations which vividly support the ideas displayed in Sects. 4.3.1, 2.

4.3.1 Analysis of Self-Similar Substitutions

We start with the three-wave kinetic equation (2.1.12). For the sake of simplicity we assume the distributions to be isotropic. In this case, there is only one integral of motion, the energy. As usual we shall consider $m+d > \alpha$, i.e., after its formation, the Kolmogorov distribution transfers energy to the short-wave region, see (3.1.13). We shall discuss two physically different statements of the problem: 1) extension of the Kolmogorov distribution into the region of large k; 2) decaying turbulence: the free evolution of the initially long-wave packet which should in an attempt to arrive at a equilibrium distribution be spread out over the entire k-space.

It would be natural to assume that far from the source or from the initial localization site of a wave packet, the evolution will after some time become self-similar. Let us discuss possible self-similar substitutions for the three-wave kinetic equation. We shall seek the solution of the nonstationary equation (2.1.12) in the form

$$n(k,t) = t^{-q} f(kt^{-p}) = t^{-q} f(\xi) \ . \tag{4.3.4}$$

We assume the variable ξ to be dimensionless, measure k in units of k_0 (where k_0 is the initial location of the source or packet) and t in units of $t_N = V_0^2 k_0^{2m+d} n^2(k_0)/\omega(k_0)$. In here t_N is the characteristic time of nonlinear wave interaction in the region $k \simeq k_0$ and V_0 the dimensional constant of the interaction coefficient, see (3.1.7c).

Substituting (4.3.4) into the three-wave kinetic equation

$$\frac{\partial n(k,t)}{\partial t} = I(k) , \tag{4.3.5}$$

we obtain

$$-(gf + p\xi f') = I(\xi)t^{p(2m-\alpha+d)-q+1} .$$

It follows that a solution of the form (4.3.4) may only exist if the condition

$$p(2m - \alpha + d) - q + 1 = 0 \tag{4.3.6}$$

is satisfied.

To obtain another relationship between the parameters p and q, we should consider cases 1) and 2) separately:

1) In the region between the source and the relaxation front, the quasi-stationary Kolmogorov distribution $n_k^0 \propto k^{-m-d} = k^{-s_0}$ should be formed. This means that at $\xi \to 0$ we should have $f(\xi) \propto \xi^{-m-d}$ and since we are dealing with a stationary state,

$$p(m + d) = q . \tag{4.3.7}$$

From (4.3.6, 7), we find $q = (m + d)/(\alpha - m) = s_0/h$ and get

$$p = (\alpha - m)^{-1} = 1/h , \tag{4.3.8}$$

which is consistent with the estimate (4.3.1). The boundary of the Kolmogorov distribution corresponds to $\xi \simeq 1$ and moves in the k-space according to the law $k_b \propto t^p = t^{1/h}$. We encountered this law already when considering the motion of small perturbations against the background of the stationary spectrum. It is obvious that the solution (4.3.4) describes in this case the evolution of the Kolmogorov spectrum towards large k only for $h > 0$. The same conclusion will be arrived at by considering the kinetic equation in the self-similar variables

$$-(qf + p\xi f') = I(\xi) \tag{4.3.9}$$

where $I(\xi)$ is the three-wave collision integral $I(k)$ (3.1.11) in which k has been replaced by ξ . Substituting $f(\xi) \propto \xi^{-m-d}$ into (4.3.9), we see that $I(\xi)/f(\xi) \propto \xi^{-h}$, i.e., the $I(\xi)$ term prevails in the region $\xi \ll 1$ [and $f(\xi)$ has the Kolmogorov asymptotics there] at $h > 0$. The front velocity measured on the logarithmical scale

$$\frac{d \ln[k_b(t)/k_b(0)]}{dt} \propto (ht)^{-1}$$

decreases with time, in line with the notion that the expansion process of the distributions with the expansion of the energy-containing region is a slowing-down process.

At $h \to 0$, the front velocity dramatically increases to be infinite at $h = 0$. This indicates that at $h < 0$ the formation rate of the Kolmogorov spectrum is so high (to be more exact, it increases with time so quickly) that the relaxation front reaches infinity within a finite time. To obtain at $h < 0$ a self-similar relaxation front moving towards large k, we replace the t in (4.3.4) by $\tau = t_0 - t$:

$$n(k,t) = \tau^{-q} f(k\tau^{-p}) \ . \tag{4.3.10}$$

Equations (4.3.6–8) for p and q will regain their previous form, but now the right boundary of the Kolmogorov distribution with $k_b\tau^{-p} \simeq 1$ evolves according to the explosion law $k_b \propto \tau^{1/h}$ and reaches infinity within the finite time t_0 determined by the initial distribution (see below). Since for a system with $h < 0$ the energy of the Kolmogorov distribution is localized in the long-wave region, the relaxation of the stationary spectrum in the interval from a finite k to ∞ demands redistribution of the finite energy and takes finite time.

It is of interest to clarify the behavior of the energy accumulated in the self-similar part of the distribution. For the solutions (4.3.4) and (4.3.10) we have

$$E(t) = t^{2ph-1} \int_0^\infty f(\xi)\xi^{\alpha+d-1} d\xi \ , \tag{4.3.11a}$$

$$E(t) = \tau^{2ph-1} \int_0^\infty f(\xi)\xi^{\alpha+d-1} d\xi \ , \tag{4.3.11b}$$

respectively. $p = 1/h$ leads to $E \propto t$ at $h > 0$ and $E \propto \tau = t_0 - t$ at $h < 0$. The linear growth of the energy in systems with $h > 0$ means, that a self-similar solution is formed when the occupation numbers of the waves associated with the source do not change any longer and a constant energy flux has been established. In this case, the main portion of the energy is concentrated in the self-similar region. However, at $h < 0$ the portion of energy contained in the solution (4.3.10) decreases as the self-similar wave moves towards the short-wave region.

Thus, very different time scales are realized for the universal relaxation regimes of the Kolmogorov spectra with $h > 0$ or $h < 0$. At $h > 0$, the self-similar wave (4.3.4) is formed for a time much larger than the typical stabilization time for the occupation numbers in the pumping region. For $h < 0$, the self-similar wave is realized for a small period of time ($t_0 - t \ll t_0$), which is too small for the occupation numbers of long waves to undergo any essential changes.

2) Now we consider the free evolution starting from the system with $h < 0$. Since the wave (4.3.10) is self-accelerating

$$\frac{dk_b/dt}{k_b} \propto 1/\tau \ ,$$

the behavior of the short-wave part of the distribution should be insensitive to the presence or absence of a long-wave source that changes the occupation numbers not faster than exponentially. Hence, for times smaller than t_0, the free expansion of a turbulent distribution towards large k proceeds in the same way as in the case of the source: the self-similar wave $n(k,t) = \tau^{-q} f(k\tau^{-p}) = \tau^{-q} f(\xi)$ with $q = s_0/h, p = 1/h$ moves according to an "explosion" or power law, leaving the Kolmogorov distribution behind, $f(\xi) \propto \xi^{-s_0}$ at $\xi \ll 1$. Ahead of the relaxation front, the occupation numbers should rapidly decrease with growing ξ (and, accordingly, k). In [4.18] it has been assumed that if waves in the short-wave region of the distribution interact mainly with each other, then the (quasi-Planck) asymptotics are exponential $n_k \propto \exp[-(\omega_k/\omega_b)] = \exp[-(k/k_b)^\alpha]$. Let us look for the conditions under which this is possible. Going over to the variable $\eta = \xi^\alpha$, we write the kinetic equation (4.3.9) in the form

$$\begin{aligned} -[qf(\eta) + \alpha p\eta f\prime(\eta)] = \int_0^\eta & [\eta_1(\eta - \eta_1)]^{d/\alpha - 1}\eta^{2m/\alpha} f_1^2(\eta/\eta_1) \\ & \times \Delta_d^{-1}[f(\eta_1)f(\eta - \eta_1) - f(\eta)f(\eta_1) \\ & - f(\eta)f(\eta - \eta_1)]d\eta_1 \\ & - 2\int_\eta^\infty [\eta_1(\eta_1 - \eta)]^{d/\alpha - 1}\eta_1^{2m/\alpha} f_1^2(\eta/\eta_1)\Delta_d^{-1} \\ & \times [f(\eta)f(\eta_1 - \eta) - f(\eta_1)f(\eta_1 - \eta) \\ & - f(\eta_1)f(\eta)]\, d\eta_1 \, . \end{aligned} \tag{4.3.12}$$

Here f_1 is the structural function of the interaction coefficient (3.1.7c). Expressed in terms of the frequency ratio $x = \omega_1/\omega$ it has the properties $f_1(x) = f_1(1-x)$ and $f_1(x) \propto x^{m_1/\alpha}$ at $x \to 0$. The quantity Δ_d^{-1} is a result of angle averaging of the δ-function of wave vectors, see its definition in Sect. 3.1. Let us consider (4.3.12) at $\eta \gg 1$ and set $q = s_0/h$, $p = 1/h$, $f(\eta) = \eta^{-b}\exp(-\eta)$. Using the asymptotics of the function $f_1(x)$ at $x \to 0$, we obtain from (4.3.12)

$$\begin{aligned} \alpha\eta^{1-b}e^{-\eta} = & -2he^{-\eta}\eta^{(1-\alpha+2m-2m_1)/\alpha - b} \\ & \times \int_1^\eta \eta_1^{(d-1-\alpha+2m_1)/\alpha - b}(1 - e^{-\eta_1})^2 d\eta_1 \, . \end{aligned}$$

Hence, such asymptotics can only exist if the inequality

$$m_1 - m + \alpha \geq \frac{1}{2} \tag{4.3.13}$$

is satisfied. For example, for capillary waves on a deep fluid this condition is satisfied ($h = \frac{3}{4}$, $\alpha = \frac{3}{2}$, $m = \frac{9}{4}$, $m_1 = \frac{7}{2}$) while it is violated for three-dimensional sound ($h = -\frac{1}{2}$, $\alpha = 1$, $m = \frac{3}{2}$, $m_1 = \frac{1}{2}$). The question with regard to the possibility of an analytical construction of the self-similar asymptotics in the region ($\xi \geq 1$) if condition (4.3.13) is violated remains open. Numerical simulations carried out for three-dimensional sound (see Fig. 4.4 below) show that here again n_k is ahead of the wave front quickly diminished with the growth of k (i.e., quicker than by the Kolmogorov law).

The total energy is conserved, since for $t < t_0$ the asymptotics of the distribution at $k \to \infty$ fall off steeper than the Kolmogorov one. What happens at $t \geq t_0$, when the right boundary of the Kolmogorov distribution will reach infinity? The energy should start to decrease, because then $n_k \propto k^{-s_0}$ at $k \to \infty$ and the energy flux at $k = \infty$ is nonzero. Naturally, any realistic system has a sink at finite k_m so that the energy will start to decrease somewhat earlier than at t_0, namely, when the relaxation front reaches the sink. It would be natural to assume that after a large enough period of time ($t \gg t_0$), the decrease of the distribution would start to proceed according to the second self-similar regime with the energy decaying according to a power law. Such a behavior should be described by a self-similar solution of the form (4.3.4). The relationship between q and p is given by (4.3.6) and the energy decreases by the law (4.3.11a). To determine the index p, we have to solve the nonlinear eigenvalue problem (4.3.9) with the additional requirement $f(\xi) \geq 0$ at $0 < \xi < \infty$. Starting from the condition ensuring the decrease of the energy, one can impose on p the limitation

$$p > (2h)^{-1} .$$

Since we consider negative h, this inequality allows p to be positive or negative. The solutions with positive p describe a distribution moving to the right and those with negative p move to the left. We can, however, *suppose* that, since the sink (be it at $k = k_m$ or $k = \infty$) is stationary, the distribution on the whole should be diminished without being in motion, i.e., $p = 0$ and $E \propto p^{-1}$. The law depicting the energy decrease may also be obtained from a sequence of estimates based on the same assumption as above to yield $dE/dt \propto P \propto E^2 \Rightarrow E \propto t^{-1}$. The results of numerical simulations (see below Fig. 4.9) are consistent with our supposition. Thus, at $h < 0$ the behavior is bi-self-similar.

As time increases, the free evolution of systems with $h > 0$ should for large k approach a self-similar solution of the type (4.3.4). But now the energy should be conserved, therefore the index p is determined unambiguously by $p = (2h)^{-1}$ which is consistent with the estimate (4.3.2). The positiveness of p implies that the distribution moves towards the short-wave region. Such a solution has no Kolmogorov asymptotics at all, since the energy flux vanishes at $k \to 0$ and $k \to \infty$.

The intermediate case with $h = 0$ is degenerate since it has zero measure in the space of conceivable wave systems. We shall study it now to obtain a more complete general picture and to account for two systems observed in nature, gravitational-capillary (1.2.39a), (3.1.3) and capillary (1.2.39b, c) waves on shallow water. As a rule, there is a different type of self-similarity in the degenerate points separating regions of deviating behavior ($h < 0$ and $h > 0$). Indeed, according to (4.3.1, 2) the condition $h = 0$ implies the independence of the characteristic interaction time from the wave number. For this reason, the velocity of the relaxation front measured on a logarithmical scale, should be constant which implies a self-similarity of the exponential rather than of the power type.

$$n(k,t) = \exp[-qt]f(k\exp[-pt]) = \exp[-qt]f(\xi) \tag{4.3.14}$$

For (4.3.14) to be a solution of (4.3.5), one should have $q = p(2m + d - \alpha) = p(m + d) = ps_0$. In this case the energy

$$E = \exp[(ps_0 - q)t] \int_0^\infty f(\xi)\xi^{\alpha+d-1}d\xi = \int_0^\infty f(\xi)\xi^{\alpha+d-1}d\xi$$

remains unchanged, therefore such a self-similar solution corresponds to the free evolution regime. Having reached the sink, the evolution of the self-similar front (4.3.14) may switch over to the regime (4.3.11) in which the energy is diminished inversely proportional to time [see (4.3.11a) and Fig. 4.10 below].

According to the character of their evolution, wave systems with a decay dispersion law are divided into two classes: one with $\alpha > m$ $(h > 0)$ and the other with $\alpha < m$ $(h < 0)$.

In the nondecay case, the classification also includes two Kolmogorov solutions: $n_1(k) \propto k^{-s_1}$, $s_1 = d + 2m/3$, with the energy flux towards the region of large k and $n_2(k) \propto k^{-s_2}$, $s_2 = d + 2m/3 - \alpha/3$, with the wave action flux towards the long-wave region. Accordingly, there are two significant indices, h_1 and h_2, determining the position of the energy-containing region of the n_1 solution and the region where the main part of wave action of the n_2 solution is concentrated:

$$E = \int \omega_k n_1(k)dk \propto k^{\alpha+d-s_1} = k^{\alpha-2m/3} \equiv k^{h_1}\,, \tag{4.3.15a}$$

$$N = \int n_2(k)dk \propto k^{d-s_2} = k^{(\alpha-2m)/3} \equiv k^{h_2}\,. \tag{4.3.15b}$$

Since we consider, as a rule, systems with $\alpha > 0$, we usually have $h_1 > h_2$. Consequently, in the space of systems there are three regions of parameters corresponding to evolutions of weakly turbulent distributions with diverse characters: a) $h_1, h_2 > 0$; b) $h_1, h_2 < 0$; c) $h_1 > 0 > h_2$.

In the presence of a source, the Kolmogorov solution $n_1(k)$ at $k \to \infty$ is established by a self-similar front of the form (4.3.4) for $h_1 > 0$ and an explosion front (4.3.10) for $h_1 < 0$, while the $n_2(k)$ solution at $k \to 0$ is established by the wave (4.3.4) for $h_2 < 0$ and by (4.3.10) for $h_2 > 0$. In all these cases we have $p = 1/h_i$, $i = 1, 2$.

We shall now briefly outline the different cases of free evolution of turbulent distributions. In case a), the long-wave Kolmogorov asymptotics corresponding to a constant action flux is accomplished via an explosive power law: $n(k,t) = (t_0 - t)^q f[k(t_0 - t)^{-p}]$, $f(x) \propto x^{-s_2}$ at $x \gg 1$, $p = 1/h_2$, $q = ps_2$. Then, at $t > t_0$, energy conservation leads to a self-similarity regime with (4.3.4) and

$q = p(\alpha + d)$, $p = (3h_1)^{-1}$. A time progresses, this distribution is shifted towards the short-wave region. Specifying the position of the energy-containing region in ω-space by $\omega_E = E/N$, energy is seen to be pumped over towards large ω due to the fact that N is decreasing while E is conserved. In case b), the short-wave Kolmogorov asymptotics corresponding to a constant energy flux arise according to the explosion law (4.3.10). Then we observe the gradual formation of a wave which propagates into the long-wave region according to the self-similar law (4.3.4) with parameters $q = pd$, $p = (3h_2)^{-1}$ corresponding to conservation of the wave action integral. The maximum of the energy distribution ω_E decreases with time. Finally, in the case c) both integrals, E and N, should conserve, precluding the existence of two-parameter self-similar solutions (4.3.4) or (4.3.10), therefore the evolution is non-self-similar.

For example, in the case of gravitational waves on deep water we have $\alpha = 1/2$, $m = 3$, i.e., we are dealing with case b), since $h_1 = -3/2 < 0$, $h_2 = -11/6 < 0$. Thus, in the short-wave region a Kolmogorov spectrum with an energy flux is formed according to the explosion law $n(k,t) = (t_0 - t)^{8/3} f[k(t_0 - t)^{2/3})$. The long-wave spectrum with an action flux is formed by the decelerating relaxation front $n(k,t) = t^{23/11} f(kt^{6/11})$. The boundary frequency of such a spectrum moves according to $\omega \propto t^{-3/11}$. What about the attenuation of waving after the wind (or pumping) abates? The distribution has Kolmogorov asymptotics with constant energy flux at $k \to \infty$, so the energy should not be conserved. The evolution of decaying turbulence should reach the self-similar regime $n(k,t) = t^{4/11} f(kt^{2/11})$. Thus, in the isotropic case, the mean waving frequency decreases according to $\omega_E \propto t^{-1/11}$.

4.3.2 Method of Moments

The treatment of the previous section was based on plausible arguments rather than rigorous proofs. Indeed, we left not only the transition of arbitrary distributions to the regime (4.3.4) or (4.3.10) without substantiation, but did not even provide a proof for the existence of self-similar solutions of such form.

However, it turns out that the most interesting property of evolution, the explosive character of the formation of the Kolmogorov asymptotics, may be strictly established for a particular case of weak sound turbulence. The idea of the proof is quite simple and was suggested by *Falkovich* [4.19]. It is based on consideration of the dynamics of the moments of the distribution $n(k,t)$

$$M_i(t) = \int k^i n(k,t) d\boldsymbol{k} \ .$$

Indeed, if at $k \to \infty$ the (power) asymptotics $n(k) \propto k^{-s}$ are established during a finite time, then the moments M_i with $i > s - d$ should become infinite. (A similar train of thought has also been used in the discussion of approximate models of hydrodynamic turbulence [4.20]).

The evolution of three-dimensional isotropic distributions of weak sound turbulence is described by (3.2.3). Going over to dimensionless variables, it assumes the form

$$\frac{\partial n(k,t)}{\partial t} = 4\pi \int_0^k (k-k_1)^2[n(k_1)n(k-k_1) - n(k)n(k_1) - n(k)n(k-k_1)]k_1^2 dk_1 - 8\pi \int_k^\infty (k_1-k)^2[n(k)n(k_1-k) - n(k_1)n(k_1-k) - n(k_1)n(k)]k_1^2\, dk_1 \ . \tag{4.3.16}$$

In the weak dispersion limit when $\omega_k \approx k$, the energy is the first moment of the distribution function

$$E = M_1 = \int kn(k,t)d\boldsymbol{k} = 4\pi \int_0^\infty k^3 n(k,t)\, dk$$

It is conserved if, at $k \to \infty$, the function $n(k,t)$ decreases faster than by the Kolmogorov law (3.2.5) $n(k) \propto k^{-9/2}$. In our case of weak sound turbulence we have $\alpha = 1$, $m = \frac{3}{2}$, $h = -\frac{1}{2}$.

Let us consider the behavior of other moments of the distribution function. For the zero moment, the total number of waves

$$N(t) = 4\pi \int_0^\infty n(k,t)k^2\, dk$$

is readily calculated from (4.3.16)

$$\begin{aligned}\frac{dN}{dt} &= -(4\pi)^2 \int_0^\infty k^2 dk \int_k^\infty k_1^2(k-k_1)^2[n(k)n(k_1-k) - n(k_1)n(k) - n(k_1)n(k-k_1)]dk_1 \\ &= (4\pi)^2 \int_0^\infty k^2 n(k)\, dk \int_0^\infty k_1^2 n(k_1)\left[(k-k_1)^2 - (k+k_1)^2\right] dk_1 \\ &= -4E^2 \ .\end{aligned} \tag{4.3.17}$$

In deriving this equation, we rearranged the integration limits, which is only correct if the integral $\int_0^\infty k^3 n(k)dk$ converge. Thus, (4.3.17) is valid, if at $k \to \infty$ the quantity $n(k)$ diminishes quicker than k^4. We shall note here that we assume that there are no singularities of $n(k)$ at $k = 0$.

We see from (4.3.17) that an initial distribution that rapidly decreases at $k \to \infty$ and that has finite and nonzero N and E, cannot remain unchanged for arbitrary long times.

The existence of the equality (4.3.17) prompts the following picture of evolution: during the time $t_0 = N(0)/4E^2$, the initial long-wave packet will spread out over the k-space, N will vanish and the energy-containing scale $k_E = E/N$ will become infinite. This scenario implies that all the energy is pumped over to infinity (to large k) within a finite period of time.

However, the evolution is actually different. Let us consider the behavior of the second moment $M_2 = 4\pi \int_0^\infty k^2 n(k) k^2 dk$:

$$\frac{dM_2}{dt} = 6M_2^2 + 8\pi E \int_0^\infty k^3 n(k) k^2 dk + 32\pi^2 \int_0^\infty u^2 n(u) du \int_0^u v^4 (u-v)^2 n(v) dv \geq 8M_2^2 . \tag{4.3.18}$$

Here we have used the simple inequality

$$\int_0^\infty k^5 n(k) dk \int_0^\infty k^3 n(k) dk \geq \left(\int_0^\infty k^4 n(k) dk \right)^2 .$$

The inequality (4.3.18) is valid if the integral $M \propto \int_0^\infty k^4 n(k) dk$ converges, i.e., if $n(k)$ decays faster than k^{-5}.

The equation $dx/dt = 8x^2$ has an explosive solution $x(t) = x(0)/\big(1 - 8x(0)t\big)$. Consequently, it follows from (4.3.18) that M_2 should become infinite during the time $t \leq t_1 = [8M_2(0)]^{-1}$. It is easy to show $t_1^{-1} = 8M_2(0) \geq 2t_0^{-1} = 8E^2/N(0)$ to hold. In other words, the "explosion" occuring when the second moment M_2 becomes infinite, takes place at least twice as quickly as the N vanish. We shall see below (see Figs. 4.4–6) that as M_2 increases, the Kolmogorov asymptotic $k^{-9/2}$ is explosively formed in the region of large k.

On the distribution $n(k) \propto k^{-9/2}$, the energy is contained in the region of small k: $\varepsilon \propto k^{-3/2}$. Therefore, during the relaxation time of the Kolmogorov asymptotics, only a small part of the initial energy of the turbulence is located in the region of large k. The effect of explosive formation of the Kolmogorov distribution in the short-wave region may be compared to a weak collapse (see [4.21]) when the value of the integral of motion captured into a singularity region (in our case, into $k \to \infty$) tends to zero.

Thus, the evolution of a weakly turbulent acoustic distribution should exhibit two stages. During the first, "explosive" stage energy is conserved, the number of waves decreases by a linear law and the second moment increases explosively. As a result, a finite energy flux will set on at $k \to \infty$, the total energy will start to diminish. Then at $t \gg t_1$, the self-similar regime of the type (4.3.4) is established with $p = 0$ – $n(k,t) = t^{-1} f(k)$ and with the energy decreasing by the power law $E \propto t^{-1}$. The short-wave asymptotic in this case is of the Kolmogorov type $n(k) \propto k^{-9/2}$, so that (4.3.17) remains valid, i.e., the total number of waves decreases monotonically (though this time not by the linear law, see Fig. 3.9 below).

We also note that, as the averaged interaction coefficient in (4.3.16) is proportional to k^2, consideration of moments above the second one will not change this picture: their derivatives may be expressed in terms of lower-order moments.

From (4.3.16) we see that the inverse of the time of nonlinear interaction $t_{NL}^{-1} \propto kE$ [see (4.3.2)] grows with k slower than the dispersion correction to the frequency $\delta\omega = a^2k^3$. This means that the applicability criterion of the weakly turbulent approximation, $\delta\omega t_{NL} \gg 1$, will be satisfied increasingly better as the distributions move towards large k. Consequently, the formation of the power-type Kolmogorov asymptotics $n(k) \propto k^{-9/2}$ in an infinite k-space will proceed up to the absorption region or to $k \simeq a^{-1}$ where the dispersion will cease to be small and (4.3.16) will become inapplicable.

It should also be noted that the presence of an external wave source, i.e., the addition of positive terms F_k (external force) or $\gamma_k n_k$ (the instability increment) to the right-hand side of (4.3.16) does not violate the inequality (4.3.18). Therefore, the stabilization of the stationary Kolmogorov spectrum should terminate at a finite time.

4.3.3 Numerical Simulations

Any realistic experiment, be it with the real or a model system or numerical, deals with a finite number of modes. This is due either to finiteness of the system and the discreteness of the medium or to strong damping of higher harmonics. With these considerations in mind, *Falkovich* and *Shafarenko* [4.22] carried out numerical simulations of (4.3.16) in two variants:

(i) for a closed system of L modes,

$$\begin{aligned}\frac{\partial n(k,t)}{\partial t} &= \sum_{l=1}^{k} l^2(k-l)^2\{n(l)n(k-l) - n(k)[n(l)+n(k-l)]\}\\ &\quad - 2\sum_{l=k}^{L} l^2(l-k)^2\{n(k)n(l-k)\\ &\quad - n(l)[n(k)+n(l-k)]\} = W_k,\end{aligned} \tag{4.3.19a}$$

(ii) for an open system with wave drift [we set $n(k) \equiv 0$ at $k > L$],

$$\frac{\partial n(k,t)}{\partial t} = W_k - 2n(k)\sum_{l=L}^{L+k} l^2(l-k)^2 n(l-k)\ . \tag{4.3.19b}$$

The last term in (4.3.19b) implies that due to confluence processes waves drift into the region with $k > L$. It plays the role of nonlinear damping and provides an efficient energy sink, see (3.4.18).

The evolution of the closed system should lead to the equilibrium distribution $[n(k) = T/k$ at $k \gg 1]$ with (4.3.19a) retaining the total energy

$$E = \sum_{k=1}^{L} k^3 n(k) .$$

For the total number of waves $N = \sum_{k=1}^{L} k^2 n(k)$ we have from (4.3.19a):

$$\frac{\partial N}{\partial t} = -4E^2 + \sum_{k=1}^{L} k^2 n(k) \sum_{l=L-k}^{L} l^2 (k+l)^2 n(l) .$$

The last term is an "overlap integral" covering the regions $(1, L/2)$, $(L/2, L)$. Consequently, the law describing the decrease of N will deviate more and more from a linear rule as the waves are distributed over the whole interval,

In the open system (4.3.19b) the energy monotonically decreases

$$\frac{\partial E}{\partial t} = -2 \sum_{k=1}^{L} k^3 n(k) \sum_{L-k}^{L} l^2 (k-l)^2 n(l)$$

due to similar "overlap integrals".

Thus, if a wave packet was initially concentrated in the region $k \ll L$, its evolution will be the same for (4.3.19a) and for (4.3.19b), respectively, until the finiteness of k-space manifests itself. In a closed system, the waves will be accumulated near the right; in an open one the occupation numbers will decrease (as compared to the evolution in an infinite system) because of nonlinear damping. For those moments of time and regions of k-space for which the solutions (4.3.19a) and (4.3.19b) are close to each other, one can expect numerical experiments to be a good simulation of the behavior of waves in an infinite medium. In these simulations, the initial distribution was chosen to be localized in the region of small k

$$n(k, 0) = \exp[-k^2/k_0^2] = \exp[-k^2/10]$$

and the equations (4.3.19) were solved numerically. The time derivative was approximated by the first difference

$$\frac{\partial n(k,t)}{\partial t} \approx \frac{n(k, t+\Delta t) - n(k,t)}{\Delta t}$$

where Δt was made sufficiently small $(5 \cdot 10^{-5} \div 2 \cdot 10^{-8})$ to ensure stability of the numerical procedure. The number of modes was chosen to be 200, 400, and 1000.

Figure 4.4 presents the evolution of $n(k, t)$ up to the time when the solutions (4.3.19a) and (4.3.19b) start to deviate from each other. At the moment $t^* = 5.3 \cdot 10^{-4}$ the Kolmogorov distribution transmitting the constant energy flux to

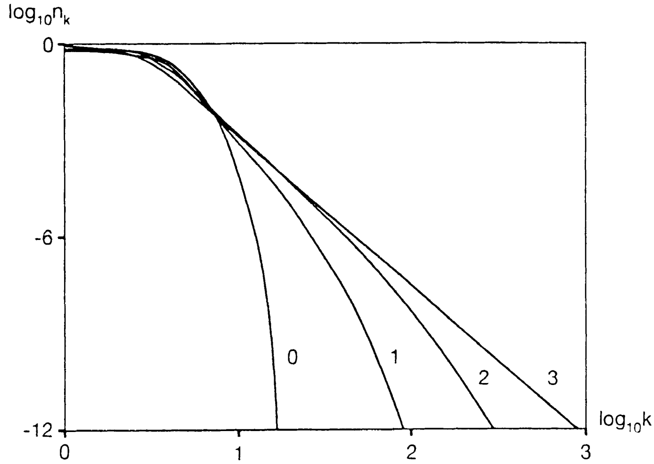

Fig. 4.4. The evolution of $n(k, t)$ is illustrated. The curves 0 to 4 represent ln $n(k)$ at the moments of time 0, 10^{-4}, $3 \cdot 10^{-4}$ and $5.3 \cdot 10^{-4}$, respectively. The dotted line depicts the Kolmogorov power law $n(k)^0 = Ak^{-9/2}$

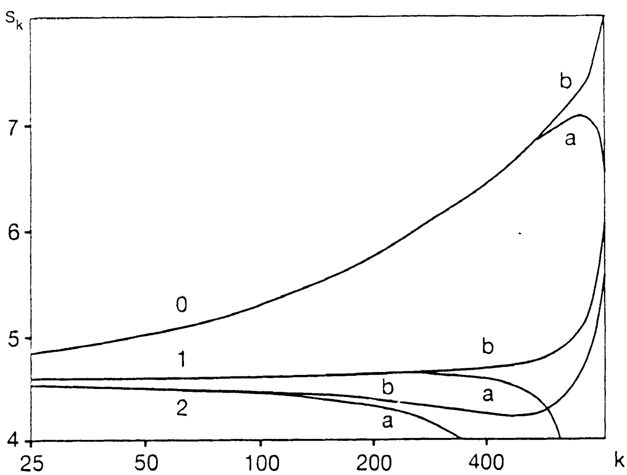

Fig. 4.5. The dependence of the local index of the spectrum on k is shown for different moments of time. The curves labeled 0, 1, and 2 correspond to the times $t = 4 \cdot 10^{-3}$, $t = 5.3 \cdot 10^{-4}$, and $t = 6 \cdot 10^{-4}$, respectively

large k is established almost over the whole interval of 1000 points. In detail this is illustrated in Fig. 4.5.

$$s(k) \equiv -\frac{\ln\big(n(k)/n(k+1)\big)}{\ln(k/k+1)} \ .$$

The labels "a" and "b" indicate that the corresponding curve are based on (4.3.19a) and (4.3.19b), respectively. An appreciable difference is noted between the behaviors of open and closed system when starting the evolution from $t \simeq t^*$. In the closed system, the distribution quickly becomes smoother [$s(k)$ decreases] and starting from the right end of the interval an equilibrium spectrum with $s(k) \equiv 1$ is formed.

Figure 4.6 shows the time dependence of all the three moments of the distribution function.

One can see that there are two well-separated evolution stages. Approximating the dependence $M_2^{-1}(t)$ by a dashed line we get $t^* = 5.32 \cdot 10^{-4} \approx [8.7M(0)]^{-1}$. The fact that $M_2^{-1}(t)$ does not go down to zero and the energy starts to decrease earlier than the time $t = t^*$ is a result of the finiteness of the system, see also Fig. 4.7.

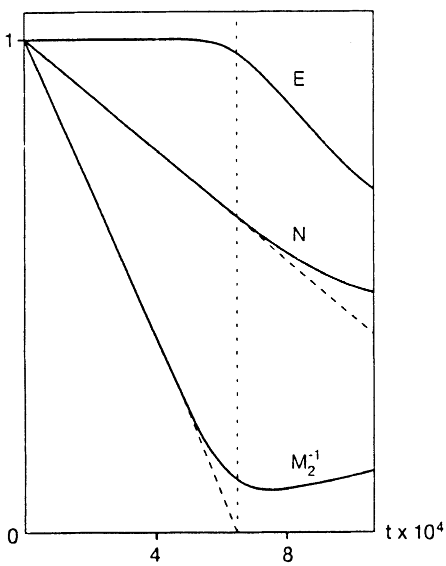

Fig. 4.6. The three moments of the distribution function are shown for (4.3.19b), $L = 1000$

Let us now verify the supposition formulated in Sect. 4.3.1, it indicates that the self-accelerated formation of Kolmogorov asymptotics involves the region $k \gg k_0$ at $\tau = t^* - t \ll t^*$. Equation (4.3.16) allows for a self-similar explosive-type substitution

$$n(k,t) = \tau^{-5p-1} f(k\tau^{-p}) = \tau^{-5p-1} f(z) \tag{4.3.20}$$

One can expect the formation of self-similar asymptotics, provided that in the region $k \gg k_0$ the occupation numbers change much quicker than at $k \simeq k_0$. Then the self-similar part of the solution can be matched with the energy-containing region $k \simeq k_0$ via the quasi-stationary intermediate asymptotics. For these asymptotics to be of the Kolmogorov type, it is necessary that at $z \ll 1$ we have $f \propto z^{-9/2}$. From the steady state condition we get $p = -2$. Thus, the boundary of the region containing the Kolmogorov distribution should be specified by the condition $z \simeq 1$ and should become infinite during the finite time $k_b \propto (t^* - t)^{-2}$. The numerical simulations give a qualitative confirmation of such a behavior [4.22].

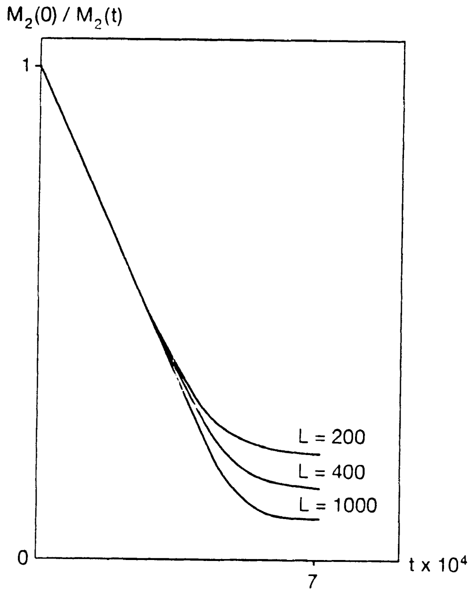

Fig. 4.7. The behavior of $M_2^{-1}(t)$ is illustrated for different L-modes

At $t \gg t^*$, the energy $E(t)$ approaches the power law $E \propto t^{-\beta}$ as illustrated in Fig. 4.8 showing the time dependence of

$$\beta = \frac{\partial \ln E}{\partial \ln t} .$$

According to our presumption, β should tend to unity with time which turns out to be consistent with Fig. 4.8.

Concluding this section, we shall consider two-dimensional weak sound turbulence [(3.2.3) at $d = 2$] as an example of a system with $h = 0$. The isotropic kinetic equation has a kernel which is polynomial in k, k_1

$$\begin{aligned} \frac{\partial n(k,t)}{\partial t} = 2\pi \int_0^k & (k - k_1)[n(k_1)n(k - k_1) \\ & - n(k)n(k_1) - n(k)n(k - k_1]k_1\, dk_1 \\ - 4\pi \int_k^\infty & (k_1 - k)[n(k)n(k_1 - k) \\ & - n(k_1)n(k_1 - k) - n(k_1)n(k)]k_1\, dk_1 . \end{aligned} \tag{4.3.21}$$

Since the averaged interaction coefficient is proportional to the first power of k, it is reasonable to consider the behavior of the zeroth and first moments of the distribution function $n(k)$. The first moment $\int_0^\infty kn(k)kdk = E$ is the energy which is conserved if there is no external damping. For the zeroth moment (the total number of waves), we can obtain from (4.3.21) [4.19] the expression

$$\frac{dN}{dt} = 2\int_0^\infty kn(k)dk \int_0^\infty k'(k+k')n(k+k')dk' - 2NE$$
$$= -\int_0^\infty dk\left(\int_k^\infty k'n(k')dk'\right)^2 .$$

Now we see that $dN/dt \leq 0$ holds, i.e., the energy-containing scale $k_E = E/N$ of an arbitrary distribution is monotonically shifted to large k. On the other hand, $dN/dt \geq -2NE$, i.e., $N(t)$ does not decrease faster than by an exponential law. *Falkovich* and *Shafarenko* [4.22] examined the evolution of $n(k,t)$ in terms of a discrete open system (3.4.18) corresponding to (4.3.21) up to times when the energy is decreased by almost a factor of three with regard to the initial value. From Fig. 4.9 (with $L = 400$) one can see that there are two stages of evolution. During the first stage the energy is conserved and the Kolmogorov distribution forms until the absorption region is reached [in the given case $n(k) \propto k^{-3}$, see (3.2.5)].

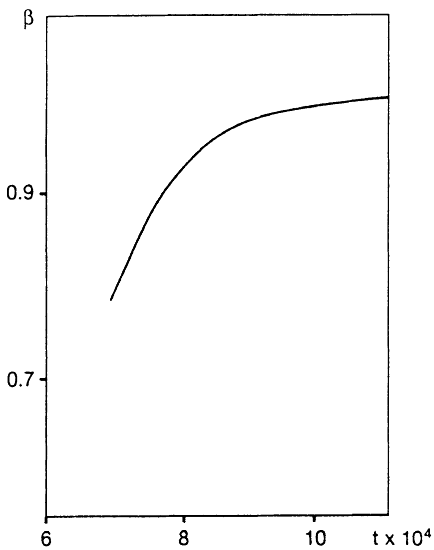

Fig. 4.8. The time dependence of β is illustrated

The time t_1 required to form a constant energy dissipation rate, grows with the size of the system approximately logarithmically (L = 50, $t_1 \simeq 5.70$; L = 100, $t_1 \simeq 8.05$; $L = 200$, $t_1 \simeq 10.70$; $L = 400$, $t_1 \simeq 13.49$), which is in agreement with (4.3.14). At $t \gg t_1$ the self-similar regime (4.3.4) is established and the energy decreases like t^{-1}. Such a regime persists until the energy concentrated in the region without external damping (approximately $k \leq L/2$) becomes comparable to the energy in the interval $k \geq L/2$. At this stage, the energy dissipation rate starts to decrease.

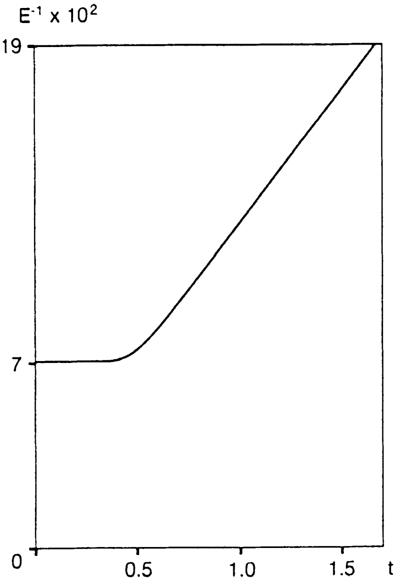

Fig. 4.9. Different stages of the evolution are depicted

5. Physical Applications

... a time to gather stones together

Ecclesiastes

Now it is time for our theory to bear fruit and for us to harvest. In this chapter we shall recollect the results concerning specific physical systems and shall obtain some new results by specializing general approaches from the previous chapters. We shall try to describe most hitherto known facts with regard to the spectra of developed weak turbulence. The general model of wave turbulence adopted in this book is based on the detailed consideration of the wave-wave interaction. The interaction with the external environment was described by the function $\Gamma(\boldsymbol{k})$ specifiying the decrement of wave attenuation or the growth-rate of wave instability. Nature is naturally more complex (for example, the interaction of water waves with wind and currents or wave-particle interactions in plasmas). Nevertheless, we believe the following fomulas and interpretations to provide a good basis for the further development of the theory of wave turbulence in various systems each of which might require a separate monograph for an adequate detailed description.

5.1 Weak Acoustic Turbulence

In this section we shall discuss wave turbulence with a near-sound dispersion law. Plenty of physical systems belong to this type. In spatially homogeneous media according to the Goldsone theorem, the wave frequency $\omega(k)$ should vanish together with the wave number k. In most cases the frequency expansion at small k starts from the first (i.e., linear) term. So the large-scale perturbations produce acoustic waves in solids, fluids, gases, and plasmas. The dispersion is supposed to be weak but sufficiently large to justify applicability of the kinetic equation and to be greater than dissipation. The magnitude of the nonlinear interaction coefficient and the sign of the dispersive frequency addition are different for various media. As repeatedly mentioned above, the properties of acoustic turbulence strongly depend on the sign of dispersion and the dimension of the space. Indeed, for positive dispersion three-wave interactions are allowed, while for negative dispersion one should take four-wave processes into account. Different kinetic equations are to be used in these cases. As far as the space dimension is concerned, we shall mention here only one fact to demonstrate the large difference between two- and three-dimensional cases. Considering acoustic turbulence to be close to scale invariant turbulence and supposing the index of the Kolmogorov spectrum to be close to the general formula $s_0 = m + d$, we obtain the correct value for the three-dimensional case only (see Sect. 3.2). For the two-dimensional case it is not possible to use the scale-invariant approximation of the dispersion law $\omega(k) = ck^{1+\epsilon}$ instead of the physically better justified expression (1.2.22)

$$\omega(k) = ck(1 + a^2k^2) . \tag{5.1.1}$$

Even for the index of the Komogorov solution one gets different answers. Two-dimensional sound turbulence may be referred to as "nonanalytical" with regard to the dispersion parameter, i.e., the solution depends on the way which the dispersion law tends to a linear one. We shall treat the case with positive dispersion in Sect. 5.1.1 ($d = 3$) and Sect. 5.1.2 ($d = 2$) and the one with negative dispersion in the first part of the Sect. 5.1.3.

It should be noted that the dispersion law can be close to a linear one not only for long waves. There exist short acoustic-like waves in some systems. For example, the dispersion law of spin waves in an antiferromagnet (1.4.18) coincides with that of relativistic particle and has the form

$$\omega^2(k) = \omega_0^2 + (ck)^2 . \tag{5.1.2a}$$

In the "ultrarelativistic" limit $vk \gg \omega_0$ the frequency is approximately equal to

$$\omega(k) = ck + \omega_0^2/(2vk) . \tag{5.1.2b}$$

The second term on the right-hand-side describes small dispersion. With the help of a figure like Fig. 1.1 it is easy to understand that both dispersion laws (5.1.2a, b) are of the nondecay type. Therefore in the case of short waves, the positive additional term leads to the four-wave kinetic equation as opposed to the long wave case. Generally speaking, the decay criterion for the almost acoustic dispersion law

$$\omega(k) = ck + \Omega(k) , \qquad \Omega(k) \ll ck \tag{5.1.3}$$

can be formulated as follows

$$\mathrm{sign}\left[\Omega(k)\frac{\partial}{\partial k}\frac{\Omega(k)}{ck}\right] > 0 . \tag{5.1.4}$$

The dispersion law (5.1.2) is also valid for atmospheric inertio-gravity waves with lengths ($\simeq$100–1000 km) shorter than the Rossby radius. Such waves are two-dimensional. The second part of Sect. 5.1.3 deals with short-wave acoustic turbulence.

5.1.1 Three-Dimensional Acoustics with Positive Dispersion: Magnetic Sound and Phonons in Helium

This subsection is devoted to long three-dimensional sound waves with positive dispersion thus allowing for three-wave interactions. In magnetized media, a positive dispersion term proportional to powers of k that are greater than unity is frequently oberved. The interaction between acoustic and spin-wave subsystems in crystals gives rise to the both dispersion and nonlinear interaction of sound, since the spin subsystem is usually strongly dispersive and nonlinear unlike the acoustic one. This interaction is especially strong in antiferromagnets [5.1]. Magnetic sound in plasmas possesses also positive dispersion [5.2]. In the presence of a magnetic field the dispersion law is usually nonisotropic. For example, in plasmas, the dispersive addition to the linear term depends on the angle and even changes sign at some angles. However, in the case of small dispersion only waves with close propagation directions interact with each other. Thus, the angular dependence can often be ignored. Considering locally isotropic and slightly anisotropic distributions seems thus reasonable. Within some interval of the values of the pressure, phonons in helium provide another example for systems with positive dispersion [5.3].

Thus, we consider three-dimensional waves with the dispersion law (5.1.1). The coefficient of the three-wave interaction can be represented in the extremely simple form

$$|V(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)|^2 = bkk_1k_2 \ , \tag{5.1.5}$$

defined by a single dimensional constant b. Indeed, the scaling index m of the interaction coefficient of long acoustic waves can be obtained from a dimensional analysis: $m = 3/2$ (see Sect. 1.1.4). The powers of k_1 and k_2 should be equal because of the symmetry properties of the system. The power of k can be obtained by the following simple argument. Let us consider the case $k \ll k_1, k_2$ with $k_1 \approx k_2$. The long k-wave can be regarded as giving rise to density variations thus varying also the frequency. Therefore, the interaction Hamiltonian

$$\int V(k, k_1, k_2)a(k)a(k_1)a^*(k_1)\, d\boldsymbol{k}d\boldsymbol{k}_1d\boldsymbol{k}_2$$

may be written in the form

$$\int \delta\omega(k_1)|a(k_1)|^2\, d\boldsymbol{k}_1 \ .$$

Consequently, $V(k, k_1, k_1)$ varies with k_1 like

$$\lim_{k \ll k_1} V(k, k_1, k_1) \propto k_1 k^{1/2} \ . \tag{5.1.6}$$

If we add the requirement that the interaction coefficient has to vanish with every wave number k, k_1, or k_2 (since there should be no interaction with a homogeneous shift) then we arrive at (5.1.5) with an accuracy of a dimensionless function of angles, see also (1.1.38). But the angular dependence of the interaction coefficient can be neglected, because the space-time synchronization condition $\omega(\boldsymbol{k}) = \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k} - \boldsymbol{k}_1)$ allows in the case of weak dispersion for interactions between waves with almost collinear wave vectors, see (3.2.2).

Thus, we can write the kinetic equation for three-dimensional sound in the form

$$\begin{aligned}\frac{\partial n(\boldsymbol{k},t)}{b\partial t} = \int & dk_1 d\theta_1 d\varphi_1 k k_1^3 (k-k_1)\Big\{\delta\big(3ca^2 k k_1 (k-k_1) - \theta_1^2 k k_1/2\big) \\ & \times \left[n(\boldsymbol{k}_1) n(\boldsymbol{k}-\boldsymbol{k}_1) - n(\boldsymbol{k}) n(\boldsymbol{k}_1) - n(\boldsymbol{k}) n(\boldsymbol{k}-\boldsymbol{k}_1)\right] \\ & - 2\delta\big(3ca^2 k k_1 (k_1 - k) - \theta_1^2 k k_1/2\big) \\ & \times \left[n(\boldsymbol{k}) n(\boldsymbol{k}_1 - k) - n(\boldsymbol{k}_1) n(\boldsymbol{k}) - n(\boldsymbol{k}_1) n(\boldsymbol{k}_1 - \boldsymbol{k})\right]\Big\}\end{aligned} \tag{5.1.7}$$

where we retained the small dispersion parameter a only in the arguments of the δ-functions. The applicability criterion for this equation will be derived in the second volume of this book and is given by

$$bk^2 N \ll c \ln(ak)^{-2} .$$

For isotropic distributions, (5.1.7) can be integrated over the angles to arrive at (3.2.3):

$$\begin{aligned}\frac{\partial n(k,t)}{4\pi b\partial t} = \int_0^k & k_1^2 (k-k_1)^2 \{n(k_1) n(k-k_1) \\ & - n(k)\left[n(k_1) + n(k-k_1)\right]\} \, dk_1 \\ & - \int_k^\infty k_1^2 (k_1 - k)^2 \{n(k) n(k_1 - k) \\ & - n(k_1)\left[n(k) + n(k_1 - k)\right]\} \, dk_1 \, .\end{aligned} \tag{5.1.8}$$

Thus, the small dispersion parameter a is eliminated from the isotropic kinetic equation of the zeroth order.

There exists only one isotropic Kolmogorov solution (2.4.5)

$$n(k) = \lambda P^{1/2} k^{-9/2} , \qquad \lambda \approx (5/b)^{-1/2} (4\pi)^{-1} \tag{5.1.9}$$

carrying the energy flux P. It was obtained by *Zakharov* [5.4] as the first example of a weakly turbulent Kolmogorov-like spectrum. Spectrally narrow pumping should generate at intermediate values of k a stationary distribution in the form of a chain of sharp peaks with the amplitudes dropping down like $n(k_j) \propto k_j^{-11/2}$ with $k_j = jk_0$ (see Fig. 3.14) [5.5]. With the growth of k such a pre-Kolmogorov solution goes over into a Kolmogorov solution.

According to the general formula (3.4.8), the flux P can be expressed in terms of the pumping characteristics. If an external environment generates waves with wave numbers of the order of k_0 with the growth-rate Γ_0, then the energy flux (i.e., the energy dissipation rate) can be evaluated by

$$P \simeq (\Gamma_0 c\lambda)^2 k_0^{-1} \ .$$

We see that in this case the index h equals $\alpha - m = 1 - 3/2 = -1/2$. So the long-wave part of the spectrum (5.1.9) is the one containing most of the energy

$$E = 4\pi \int \omega(k) n(k) k^2 \, dk \propto k_0^{-1/2} \ .$$

Given a stationary pumping, the spectrum in the short-wave region is formed according to an "explosive" power law. It can be described in terms of the self-similar solution (4.3.10)

$$n(k,t) \propto (t_0 - t)^9 f\big(k(t_0 - t)^2\big) \ .$$

For small $k(t_0 - t)^2$ the distribution is quasi-stationary and almost a Kolmogorov one. The right tail of the Kolmogorov spectrum evolves according to the explosive power law $k_b \propto (t_0 - t)^{-2}$ and reaches infinity within a finite time. In the case of a free decay of acoustic turbulence, the Kolmogorov short-wave asymptotics (right up to the damping region) also have an explosive evolution. Once the right boundary of the Kolmogorov asymptotics has reached infinity, the total energy starts to decrease (see Sect. 4.3 for details). It should be noted that the distribution also expands to the long-wave region (to $k < k_0$), but this is a self-decelerating process. The relaxation to the distribution (5.1.9) was first observed in the numerical simulations by *Zakharov* and *Musher* [5.6]. The nonstationary behavior was studied by *Falkovich* and *Shafarenko* [5.7] (see also Sect. 4.3.3).

For the Kolmogorov spectrum the applicability parameter (3.1.33) of weak turbulence has in this case the form

$$\xi^{-1} = a^2 k^3 t_{NL} \propto k^{5/2} \ .$$

Thus the applicability criterion $\xi^{-1} \gg 1$ will be satisfied increasingly better, as the distributions move towards large k.

Let us discuss the stability of the isotropic spectrum with regard to anisotropic perturbations. As we shall now see, the stability problem for such systems is nontrivial because of its closeness to the degenerate case. Indeed, in the limiting case of a linear dispersion law $\omega_k = ck$, the conditions of spatio-temporal synchronization $\omega(\boldsymbol{k}_1 + \boldsymbol{k}_2) = \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2)$ allow only interactions of waves propagating along a line. Waves traveling at different angles do not interact. Therefor, besides the energy integral

$$E = \int k n_{\boldsymbol{k}} \, d\boldsymbol{k}$$

and components of the momentum

$$\boldsymbol{\Pi}_i = \int \cos\theta_i \, k n_k \, d\boldsymbol{k} \; ,$$

the kinetic equation has (in the limit) $\omega_k \to ck$ an infinite set of integrals of motion

$$\int f(\zeta) k n_k \, d\boldsymbol{k} \; ,$$

where $f(\zeta)$ is an arbitrary function of the angular variables: $\zeta = (\theta, \phi)$. Expanding $f(\zeta)$ in angular harmonics, one can thus obtain an infinite series of integrals

$$I_l^0 = \int Y_l(\zeta) k n_k \, d\boldsymbol{k} \; . \tag{5.1.10}$$

It is interesting to clarify what happens to these integrals under the action of small dispersion of the wave velocity. To use the stability theory developed in Sect. 4.2, we shall approximate the dispersion law by the scale-invariant expression $\omega_k \propto k^{1+\varepsilon}$, $\varepsilon \ll 1$. Due to the dispersion, the interaction of waves with noncollinear wave vectors becomes possible. However, it is easy to see from the analysis of the synchronization conditions that waves with wave numbers of the same order essentially interact only within a narrow cone of angles $\theta_{\text{int}} \simeq \varepsilon^{1/2}$ [see (5.1.24) below]. One can prove that for angular harmonics changing only a little bit on the scale of the interaction angle θ_{int} (i.e., for $l \ll \varepsilon^{-1/2}$), there exist integrals of motion of the linearized kinetic equation that are similar to (5.1.10)

$$I_l = \int Y_l(\zeta) k^{1+p_l+\varepsilon} n_k \, d\boldsymbol{k} \; . \tag{5.1.11}$$

Accordingly, there are power corrections to the isotropic Kolmogorov spectrum that are the stationary solutions of the linearized equation and transfer the constant fluxes of the integrals (5.1.11). These fluxes have the same direction as the energy flux of the isotropic solution. The latter is thus structurally unstable with respect to the excitation of angular harmonics with $l \ll \varepsilon^{-1/2}$. In this section we shall give a rigorous proof of the instability of the isotropic spectrum of three-dimensional sound turbulence and we shall analytically derive the anisotropic Kolmogorov solution supporting two fluxes, namely, of energy and momentum. Such an analytical derivation is possible due to the presence of the small parameter ε.

To apply the criteria obtained in Sect. 4.2 to the analysis of sound turbulence, we thus model the near-sound dispersion law by the scale-invariant expression

$$\omega_k = c k^{1+\varepsilon} \; . \tag{5.1.12}$$

The use of this dispersion law instead of (5.1.3) is sensible only in the three-dimensional case when in the acoustic limit the use of different dispersion laws leads to the same Kolmogorov solutions.

Let us substitute ω_k and V_{k12} specified by (5.1.12) and (5.1.5) into the three-wave kinetic equation (2.1.12)

$$\begin{aligned}\frac{\partial n_k}{\partial t} &= \int \big[U_{k12}(n_1 n_2 - n_1 n_k - n_2 n_k) \\ &\quad - 2U_{1k2}(n_k n_2 - n_1 n_2 - n_1 n_k)\big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2, \\ U_{k12} &= \frac{B}{c}(q + \cos\theta_{k1} + \cos\theta_{12} + \cos\theta_{k2})^2 \\ &\quad \times \delta\big(k^{1+\varepsilon} - k_1^{1+\varepsilon} - k_2^{1+\varepsilon}\big)\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\ .\end{aligned} \tag{5.1.13}$$

We shall restrict ourselves to the axially symmetric case when $n(\boldsymbol{k}) = n(k, \theta)$. Let us integrate (5.1.13) over $d\boldsymbol{k}_2$ using $\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)$ and over $d\cos\theta_1$ using $\delta\big(k^{1+\varepsilon} - k_1^{1+\varepsilon} - k_2^{1+\varepsilon}\big)$, to obtain

$$\begin{aligned}\frac{1}{B}\frac{\partial n_k}{\partial t} &= \int_0^k (q + \cos\theta_{k1} + \cos\theta_{k2} + \cos\theta_{12})^2 \big(k^{1+\varepsilon} - k_1^{1+\varepsilon}\big)^{(2-\varepsilon)/(1+\varepsilon)} \\ &\quad \times k_1^2 (n_1 n_2 - n_k n_1 - n_k n_2)\, dk_1\, d\phi \\ &\quad - 2\int_k^\infty (q + \cos\theta_{k1} + \cos\theta_{k2} + \cos\theta_{12})^2 \\ &\quad \times \big(k_1^{1+\varepsilon} - k^{1+\varepsilon}\big)^{(2-\varepsilon)/(1+\varepsilon)} k_1^2 (n_2 n_k - n_1 n_k - n_1 n_2)\, dk_1\, d\phi\ .\end{aligned} \tag{5.1.14}$$

Here $\phi = \phi_1 - \phi_k$, and the arguments of $n_1 \equiv n(k_1, \theta_1, \phi)$ should be taken, in accordance with δ-functions from (5.1.13), on the surface given by

$$\begin{aligned}\frac{(\boldsymbol{k}\boldsymbol{k}_1)}{kk_1} \equiv \cos\theta_{k1} &= \frac{k^2 - k_1^2 - \big(k^{1+\varepsilon} - k_1^{1+\varepsilon}\big)^{2/(1+\varepsilon)}}{2kk_1} \\ &\approx 1 + \varepsilon\frac{k - k_1}{kk_1}\left[k_1 \ln\frac{k_1}{k} + (k - k_1)\ln\frac{k - k_1}{k}\right]\end{aligned} \tag{5.1.15a}$$

in the first integral, and

$$\begin{aligned}\cos\theta_{k1} &= \frac{\big(k_1^{1+\varepsilon} - k^{1+\varepsilon}\big)^{2/(1+\varepsilon)} - k^2 - k_1^2}{2kk_1} \\ &\approx 1 - \varepsilon\frac{k_1 - k}{kk_1}\left[k_1 \ln\frac{k_1}{k} - (k_1 - k)\ln\frac{k_1 - k}{k}\right]\end{aligned} \tag{5.1.15b}$$

in the second integral. For

$$n_2 = n\left(\left|k^{1+\varepsilon} - k_1^{1+\varepsilon}\right|^{1/(1+\varepsilon)}, \theta_2, \phi\right),$$

one should use

$$\sin\theta_{k2} = \sin\theta_{k1}\frac{k_1}{\left|k^{1+\varepsilon} - k_1^{1+\varepsilon}\right|^{1/(1+\varepsilon)}}\ .$$

Similar expressions with the substitution $k_1 \leftrightarrow k$ are valid for the second integral. Due to the smallness of the dispersion ($\varepsilon \ll 1$), the angles between interacting waves are small, except for narrow regions in the k-space which are close to the integration limits $k_1 = 0, k$ and $k_1 \to \infty$. However, these regions do not contribute to the collision integral because of the locality of the interaction, i.e., because of the convergence of the integrals with all the solutions obtained, see (5.1.18, 24). Thus, one can approximate all cosines by unity and use $\varepsilon = 0$ everywhere except for calculations involving the angular arguments of the occupation numbers n_1, n_2.

Let us obtain the stationary solutions of the kinetic equation (5.1.5) linearized with regard to a small deviation from the isotropic Kolmogorov solution. Let

$$n_k = k^{-9/2}[1 + A(k,\theta)] = k^{-9/2}[1 - k^{-p} f(\cos\theta)] \ .$$

Making use of the smallness of the interaction angle, we employ the differential approximation for the angular variables and use in (5.1.14) the expansion

$$f(\cos\theta_1) = f(\cos\theta) - \left(f' \cos\theta - f'' \sin^2\theta \cos^2\phi\right)\frac{\theta_{k1}^2}{2} \ . \tag{5.1.16}$$

Thus we obtain from (5.1.14) an equation for $f(\cos\theta) = f(z)$ and the parameter p. This equation is a factor multiplied by the converging integral (which we met in Sect. 3.2.1, after the substitution $x \to 1/x$ it coincides with $I_3 \approx 0.2$)

$$\left(\varepsilon z \frac{df}{dz} - \varepsilon \frac{1-z^2}{2}\frac{d^2 f}{dz^2} + pf\right) \int_0^1 x^2(1-x)^2[x \ln x + (1-x)\ln(1-x)]$$
$$\times \left[x^{-9/2} + (1-x)^{-9/2} - x^{-9/2}(1-x)^{-9/2}\right] dx = 0 \ .$$

In the parenthesis in front of the integral we have the Legendre equation possessing a regular solution of the form

$$f_l = P_l(\cos\theta), \qquad p_l = -\varepsilon\frac{l(l+1)}{2} \ , \tag{5.1.17}$$

where the P_l are Legendre polynomials. Thus, for not too high harmonics satisfying the differential approximation ($\varepsilon l^2 \ll 1$) we obtain a set of neutrally stable modes of the form

$$A(k,\theta) = k^{\varepsilon l(l+1)/2} P_l(\cos\theta) \ . \tag{5.1.18}$$

For $l = 1$ the solution (5.1.18) coincides with the Kats-Kontorovich drift mode. The formula (5.1.17) has been obtained by *L'vov* and *Falkovich* [5.8].

The existence of the set of stationary solutions (5.1.18) implies that in this approximation the Mellin functions $W_l(s)$ have (with an accuracy to terms of the order of $\varepsilon^2 l^4$) zeros in the points $s = p_l$. Does the exact expression (4.2.16a) for $W_l(s,\varepsilon)$ have a zero? At small ε it does due to the implicit function theorem [5.9]: if $W_l(s,0)$ has the zero $W_l(0,0) = 0$ and $[\partial W_l(s,\varepsilon)/\partial s]_{0,0} \neq 0$, then $W_l(s,\varepsilon)$ also has a zero for sufficiently small ε. The derivatives of the Mellin functions with respect to s may be calculated directly [5.11] to verify that for $l \ll \varepsilon^{-1/2}$ we have

$$\left(\frac{\partial W_l(s,\varepsilon)}{\partial s}\right)_{s=\varepsilon=0} > 0 ,$$

i.e., the implicit function theorem is applicable to answer our question affirmative. It is also easy to prove that in the zeros $s = p_l$ of the $W_l(s)$ functions their derivatives are also positive. As we already know, this implies that the fluxes of the integrals of motion (5.1.11) transferred by the modes (5.1.18) are also positive. Since the initial isotropic solution also transfers a positive energy flux, the modes (5.1.18) should be formed in the presence of an anisotropic source. Indeed, one can directly calculate that

$$W_l(0) = l(l+1)I_3 > 0 ,$$

i.e., criterion (4.2.55) is satisfied.

It is also easy to check if the function $W_l(s)$ has a zero rotation interval. It is convenient to start from the zeroth spherical harmonic for which the Mellin function can be calculated directly and is expressed via the gamma functions [5.10, 11]

$$W_0(s) = -\frac{8}{3} + \frac{1}{(s^2-1/4)(s+3/2)} + \frac{\Gamma(-3/2)\Gamma(4+s)}{4\Gamma(s+5/2)} \frac{1+\cos\pi s+\sin\pi s}{\cos\pi s} . \tag{5.1.19}$$

This function has the analyticity strip $\Pi(-1/2, 1/2)$ and the zero rotation interval $\sigma_-^0 = -1/2$, $\sigma_+^0 = 0$. At $\varepsilon l^2 \ll 1$ we have $W_l(s) \approx W_0(s)$ so that the functions $W_l(s)$ also have zero rotation intervals (σ_-^l, σ_+^l) close to (σ_-^0, σ_+^0). Thus, the conditions for structural instability are satisfied. This was first proved by *Balk* and *Zakharov* [5.10].

For $p_l < 0$, the zeros of the Mellin functions are located on the left of $s = 0$. Hence, the discussed instability is found in the region of large k, deep in the inertial interval. For three-dimensional sound, $h = \alpha - m = -1/2 < 0$, i.e., the instability is of the hard interval type. Thus, a weakly anisotropic source should generate in the inertial interval a distribution of the form

$$n(k,\theta) = \lambda P^{1/2} k^{-9/2} \left[1 + \sum_{l=1}^{L} c_l P_l(\cos\theta) k^{-\varepsilon l(l+1)/2}\right] . \tag{5.1.20}$$

As seen from (5.1.20), the higher the number of the angular harmonic, the quicker is the increase (with growing k) of its contribuiton to the stationary spectrum. Thus, a small anisotropy of a source located in the region of small k, leads to an essentially anisotropic spectrum in the short-wave region. This phenomenon was first predicted by *L'vov* and *Falkovich* [5.8].

It is interesting to have a closer look at the form of the stationary spectrum in the region of large k where the anisotropic part of the solution is no longer small and the linear approximation is inapplicable. At $l > 1$ it is only in the linear approximation that the quantities (5.1.11) are integrals of motion. One can come up with the hypothesis that in the inertial interval an essentially anisotropic spectrum should be determined by the fluxes of the first two integrals (of energy ($l = 0$) and momentum ($l = 1$)) which are also conserved quantities within the nonlinear kinetic equation (5.1.13). Such a two-flux universal solution generalizing the Kolmogorov solution has been analytically constructed by *L'vov* and *Falkovich* [5.12]. According to the dimensional relation $Pk \propto R\omega_k$, the two-flux spectrum should have the form (4.1.5)

$$n_k = P^{1/2}k^{-9/2}F(y), \quad y = \frac{(\boldsymbol{R}\boldsymbol{k})\omega(k)}{Pk^2} \, . \tag{5.1.21}$$

Here $P, \boldsymbol{R}$ are the energy and momentum fluxes, respectively.

The form of the dimensionless function $F(y)$ may in this case be found using the smallness of the dispersion and employing the differential approximation in the variable y. Indeed, for k and k_1 we obtain from (5.1.14)

$$\begin{aligned} y_1 &= \frac{(\boldsymbol{R}\boldsymbol{k})\omega(k_1)}{Pk_1^2} \equiv \cos\theta_1\left(\frac{k_1}{k_a}\right)^{\varepsilon} \\ &\approx \left[\cos\theta + \theta_{k1}\sin\theta\,\cos\phi + \left(\varepsilon\ln\frac{k_1}{k} - \frac{\theta_{k1}^2}{2}\right)\cos\theta\right]\left(\frac{k}{k_a}\right)^{\varepsilon}, \end{aligned} \tag{5.1.22}$$

i.e., $|y_1 - y| \ll y$. Now we expand in (5.1.14) the functions $F(y_1)$ and $F(y_2)$ up to terms of the order of ε. Then, we split the second integral up into two identical terms and make in one of them the substitution $k_1 \to k^2/k_1$ and in the other, $k_1 \to kk_1/(k-k_1)$ (Zakharov transformations, see Sects. 3.1, 2). Recalling then that $n_k \propto k^{-9/2}$ is an exact solution, we obtain to the first order in ε the equation

$$\left[\left(\frac{\partial F}{\partial y}\right)^2 + F\frac{\partial^2 F}{\partial y^2}\right]\left(\frac{k}{k_a}\right)^{\varepsilon}\sin^2\theta\, I_3 = 0 \, .$$

The equation for $F(y)$ is given by a factor multiplied by the converging integral I_3. The solution is

$$F(y) = \begin{cases} \sqrt{C_1 y + C_2} & \text{for } y > -C_2/C_1 \\ 0 & \text{for } y < -C_2/C_1 \, . \end{cases} \tag{5.1.23}$$

According to (5.1.21) and (5.1.23), the distribution n_k should vanish on a surface in the k-space. As a matter of fact, at $y \to -C_2/C_1$ the derivatives of $F(y)$ increase sharply and in the close vicinity of the surface (at $y + C_2/C_1 \lesssim \sqrt{\varepsilon}$) the applicability conditions of the differential approximation are violated. This tells us that it is not sufficient to consider only the first and second derivatives. The solution of the initial equation (5.1.14) should lead to a smooth, but (on the scale of the characteristic angle of interaction, i.e., $\sqrt{\varepsilon}$) rapidly decreasing function $F(y)$. At $y \to -\infty$ the function $F(y)$ should tend to zero. The integration constants C_1 and C_2 may be included into the definition of the fluxes R and P. If that is intended, then the constant C_1 must be considered to be positive, because the substitution $C_1 \to -C_1$ implies a simple rotation of the coordinate system $\theta \to \pi - \theta$. The two signs of C_2 specify two different families of solutions

$$n_k = k^{-9/2}\left(\frac{R\omega_k \cos\theta}{k} + P\right)^{1/2}, \tag{5.1.24a}$$

$$n_k = k^{-9/2}\left(\frac{R\omega_k \cos\theta}{k} - P\right)^{1/2}. \tag{5.1.24b}$$

The first of these, (5.1.24a), corresponds to spectrum narrowing with the growth of k. In particular, it should describe a stationary distribution which is generated by a weakly anisotropic source located at $k = k_0$ and supports a small momentum flux $[R\omega(k_0) \ll Pk_0]$. Expansion of (5.1.24) in $R\omega(k)/(Pk)$ yields then at small k: in the zeroth order, the isotropic Kolmogorov solution; in the first order, the Kats-Kontorovich drift correction; in higher orders, the higher harmonics (5.1.17) whose contributions to the spectrum increase with k. At large k practically all waves are concentrated in the right hemisphere.

The solution (5.1.24) describes an expanding spectrum. Its width $\Delta\theta(k)$ increases with k according to the law

$$\cos\Delta\theta(k) = \frac{Pk}{R\omega_k}.$$

If on the boundary of the inertial interval (at $k = k_0$) $R\omega(k_0) \approx Pk_0$, then the initial width of the spectrum $\Delta\theta(k_0)$ may be very small. From below the quantity $\Delta\theta(k_0)$ is limited only by the interaction angle $\sqrt{\varepsilon}$, because at such a width the differential approximation used to obtain the solution (5.1.24) becomes invalid. Thus, the solution (5.1.24b) should presumably be generated by narrow sources with widths $\sqrt{\varepsilon} < \Delta\theta \ll \pi/2$. It is essential that in the limit of large k and at $-\pi/2 < \theta < \pi/2$ the solutions (5.1.24a) and (5.1.24b) coincide. The spectrum is entirely determined by the momentum flux and presents a wide jet whose angular form does not depend on the form of the boundary conditions

$$n_k \to k^{-(9+\epsilon)/2}(R\cos\theta)^{1/2}.$$

As we see, the family of solutions (5.1.24) allows for a wide range of boundary conditions at small k (in the vicinity of a source): from isotropic to extremely narrow with a width of the order of the interaction angle. If such a solution is really generated by arbitrary pumping, then the universality hypothesis is still alive in spite of the structural instability of the isotropic Kolmogorov spectrum. But the hypothesis should be reformulated in a more sophisticated form: a steady spectrum in the inertial interval should be defined by the influxes of all motion integrals whose fluxes are directed from source to sink.

5.1.2 Two-Dimensional Acoustics with Positive Dispersion: Gravity-Capillary Waves on Shallow Water and Waves in Flaky Media

This section is devoted to the two-dimensional turbulence of sound with positive dispersion, see the dispersion law (5.1.1). We met it first in Sect. 1.2.5 when we studied short waves ($k \gg \sqrt{\varrho g/\sigma}$ with ϱ, σ, g are the density, surface tension coefficient and gravity acceleration, respectively) on shallow water ($kh \ll 1$), see (1.2.39). Such a dispersion also describes the evolution of sound in flaky media with a weak interaction between the layers. Since we again consider the long-wave case, the interaction coefficient is given by (5.1.5). We can obtain the kinetic equation in the same way as in the case of (5.1.7) and it has also a rather similar form

$$
\begin{aligned}
\frac{\partial n(\boldsymbol{k},t)}{b\partial t} = \int & dk_1 d\theta_1 k k_1^2 (k-k_1)\Big\{\delta\big(3ca^2 k k_1(k-k_1) - \theta_1^2 k k_1/2\big) \\
& \times \big[n(\boldsymbol{k}_1)n(\boldsymbol{k}-\boldsymbol{k}_1) - n(\boldsymbol{k})n(\boldsymbol{k}_1) - n(\boldsymbol{k})n(\boldsymbol{k}-\boldsymbol{k}_1)\big] \\
& - 2\delta\big(3ca^2 k k_1(k_1-k) - \theta_1^2 k k_1/2\big) \\
& \times \big[n(\boldsymbol{k})n(\boldsymbol{k}_1-\boldsymbol{k}) - n(\boldsymbol{k}_1)n(\boldsymbol{k}) - n(\boldsymbol{k}_1)n(\boldsymbol{k}_1-\boldsymbol{k})\big]\Big\}\ .
\end{aligned} \tag{5.1.25}
$$

As usual the applicability criterion for this equation has the form (2.1.25)

$$bk^2 N \ll c(ak)^2\ .$$

For isotropic distributions we can integrate (5.1.25) over angles to obtain (3.2.3)

$$
\begin{aligned}
\frac{\partial n(k,t)}{\partial t} &= \frac{2\pi b}{\sqrt{6}a} \\
\times \Bigg\{ & \int_0^k k_1(k-k_1)\big\{n(k_1)n(k-k_1) - n(k)\big[n(k_1)+n(k-k_1)\big]\big\}\, dk_1 \\
& - \int_k^\infty k_1(k_1-k)\big\{n(k)n(k_1-k) - n(k_1)\big[n(k)+n(k_1-k)\big]\big\}\, dk_1 \Bigg\}.
\end{aligned} \tag{5.1.26}
$$

Its isotropic Kolmogorov solution (2.4.5)

$$n(k) = \lambda P^{1/2} k^{-3}\ , \qquad \lambda \approx (2b)^{-1/2}(2\pi)^{-1}\ , \tag{5.1.27}$$

carries the energy flux P. It was obtained by *Falkovich* [5.13] and *Musher* [5.14]. Spectrally narrow pumping at intermediate k should generate the stationary distribution (3.4.14–15) consisting of a chain of sharp peaks with the amplitudes obeying the law $n(k_j) \propto k_j^{-4}$, $k_j = jk_0$, see Fig. 3.13 [5.6]. With growing k such a pre-Kolmogorov solution goes over into a Kolmogorov solution.

According to the general formula (3.4.8), the flux P can be expressed in terms of the pumping characteristics (3.4.8). If an external environment generates waves with wave numbers of the order k_0 with the growth-rate Γ_0, then the energy flux (i.e., energy dissipation rate) can be evaluated according to

$$P \simeq (\Gamma_0 c\lambda)^2 ,$$

i.e., it is indepedendent of k_0. This is related to the fact that $h = 0$ in this case. Thus the energy is uniformly distributed over all scales of the inertial interval. The spectrum is formed in a "nonexplosive" process. The free decay takes place in two stages – see Fig. 4.10. The first corresponds to energy conservation and can be described in terms of the exponential self-similar solution (4.3.14). When the boundary of the Kolmogorov spectrum has reached the damping region, the energy starts to decrease like $E(t) \propto t^{-1}$. When going over towards large k, the applicability criterion for weak turbulence improves since in this case $\xi(k) \propto k^{\varepsilon}$, see (3.1.33).

The stability of an isotropic spectrum of two-dimensional acoustic turbulence deserves special consideration for two reasons. First, in this case $h = m+d-s_0 = 1+2-3 = 0$, so that the stability criteria obtained in Sect. 4.2 are inapplicable. Second, in two dimensions the weak decay dispersion law cannot be modeled by the scale invariant formula $\omega_k \propto k^{1+\varepsilon}$; the true expression (5.1.1) has to be used. This gives rise to large mathematical difficulties in the treatment of the stability problem, because the kinetic equation has a scale-invariant limit at $a \to 0$ only in the isotropic case, see (5.1.26). In the description of anisotropic perturbations, the dispersion length a enters the angular dependences of the occupation numbers and may not be separated out as a factor from the collision integral as in (5.1.26). Due to this, we shall treat the stability problem of Kolmogorov solutions with regard to isotropic perturbations analytically, while numerical simulations will have to be used for the anisotropic case.

Let us follow *Falkovich* [5.13] in the treatment of the isotropic case. At $h = 0$, the operator of the linearized collision integral has the scaling index zero. The eigenfunctions of such an operator are obviously the power functions

$$\delta n_s(k,t) = k^{-s} e^{W(s)t} ,$$

where the Mellin function $W(s)$ is an eigenvalue.

Thus the stability problem reduces to the calculation of $W(s)$ and the study of the evolution of localized perturbations consisting of a superposition of eigenfunctions

$$\delta n(k,t) = \int_\gamma g(s) k^{-s} e^{W(s)t}\, ds , \qquad (5.1.28)$$

where $g(s)$ and the contour γ are such that at $k \to 0, \infty$ we have for the perturbation $\delta n(k,t) \to 0$.

Let us linearize (5.1.26) with regard to small deviations from the Kolmogorov solution $n_k^0 = Nk^{-3}$. Substituting $n(k,t) = n_k^0 + \delta n(k,t)$ and using $D = \pi b\sqrt{2/3} = \pi\sqrt{2/3}B/c$ we have

$$\frac{a}{2DN}\frac{\partial\delta n(k,t)}{\partial t} = -\int_0^\infty \left\{ \frac{\delta n(k,t)k}{k_1^2} + \delta n(k_1,t) \right. \\ \left. \times \left[(k-k_1)^{-2} + (k+k_1)^{-2} - 2k^{-2}\right] k_1\, dk_1 \right\} . \tag{5.1.29}$$

Equation (5.1.29) has fundamental solutions of the form $k^{-s}e^{W(s)t}$. The eigenvalues are determined by the integral

$$W(s) = \frac{2DN}{a}\int_0^\infty \left[\frac{x^{1-s}}{(x-1)^2} + \frac{x^{1-s}}{(x+1)^2} - 2x^{1-s} - 2x^{-2}\right] dx \\ = \frac{2DN}{a}(1-s)\cot\frac{\pi s}{2} , \tag{5.1.30}$$

which converges in the strip $2 < \operatorname{Re} s < 4$. This implies that among the perturbations having the asymptotics $\delta n_k \propto k^{-s}$, only those with $\operatorname{Re} s \in (2,4)$ are local. For them the main wave interaction takes place between wave vectors of the same order. The evolution of nonlocal perturbations should not be discussed in terms of (5.1.29), but rather by implicitly introducing into the integration a cut-off on the scale of source and sink, taking into account the distortions of the stationary solution that occur because of finiteness of the inertial interval. We shall restrict ourselves to the study of the evolution of local perturbations. We assume the evolution of nonlocal perturbations to proceed in two steps. The fast first process will – within times determined by the scales of the source k_0 or sink k_d – lead to a rearrangement of the perturbation (e.g., an initially narrow packet will broaden and the slowly diminishing tails will damp and start to diminish rapidly). Further on it will become local and evolve in the manner discussed below.

As seen from (5.1.30), we have $W(3) = 0$ which corresponds to an indifferent stability of the Kolmogorov solution with regard to variations in the energy flux. It should be noted that $W(s)$ is complex in the strip $\operatorname{Re} s \in (2,4)$. This is due to the fact that the $\hat{L}$ operator of the kinetic equation linearized with regard to small deviations from the Kolmogorov spectrum is in general nonhermitian, in contrast to the weakly nonequilibrium case, see Sect. 4.1.

Among the fundamental solutions of the form $k^{-s}e^{W(s)t}$ there are solutions for which $\operatorname{Re} W(s) > 0$, see (5.1.30). Since the eigenfunctions k^{-s} do not satisfy the boundary conditions, this does not imply instability of the Kolmogorov solution. It is more convenient to impose on the perturbations physical conditions in terms of $F(x,t) = \delta n(k,t)/n_k^0$, where $x = \ln(k/k_0)$ [in the variable x the representation (5.1.28) goes over to the Fourier transformation of the perturbation]. Demanding finiteness of the perturbation energy

$$\delta E = \int_0^\infty ck^2\delta n(k,t)\, dk \le M < \infty \tag{5.1.31}$$

leads to the condition $F(x) \to 0$ at $x \to \pm\infty$, with $F(x) \propto \exp(-\varepsilon_1 x)$ at $x \to +\infty$ and $F(x) \propto \exp(\varepsilon_2 x)$ at $x \to -\infty$ with $\varepsilon_1, \varepsilon_2 > 0$. It is also convenient to "translate" the analyticity strip by going over from s to $z = (s-3)/2$:

$$\frac{a}{2DN} W(z) = 2(1+z) \tan \pi z \ . \tag{5.1.32}$$

As we shall show now, the function $W(s)$ (5.1.30, 32) has such a structure that any perturbation localized in the k-space is swept over towards large k. Let us first consider hump-shaped perturbations given locally by $F(x,t) = F_0 \exp[-2zx + W(z)t]$ where z is a slow real function of x $\left(|dz/dx| \ll |z/x|\right)$, such that on the right of the maximum $0 \leq z < 1/2$ and on the left $-1/2 < z \leq 0$. As we are discussing local perturbations $(-1/2 < z < 1/2)$, the evolution of each section is entirely determined by wave vectors of the same order, i.e., by the quantity $W\{z(x)\}$. From (5.1.20) we see that on the real axis $\operatorname{sign} W(z) = \operatorname{sign} z$ holds at $z \in (-1/2, 1/2)$. This implies that a perturbation of the above type develops in the following way: the right slope increases, the left one decreases and the maximum value remains constant $[W(z=0) = 0]$. This is readily understood to correspond to a packet traveling to the right. Its top moves according to the law $k(t) = k_0 \exp[W'(0)t] = k_0 \exp(2DNt/a)$. The exponential character of the motion is readily appreciated if it is borne in mind that the condition $h = 0$ implies the independence of the characteristic nonlinear interaction time on k. Therefore, e.g., in its travel across the spectrum any wave doubles its wave vector during a period of time independent of k.

The arbitrary localized solution of (5.1.29) may be written in the form of (5.1.28) where the contour γ extends into the analyticity strip ranging from $-i\infty$ to $+i\infty$ and $g(z)$ is the Fourier transformation of the initial perturbation $F(x,0)$. Having aligned the contour γ with the axis $\operatorname{Re} s = 0$, $s = i\sigma$, we see that arbitrary localized perturbations are damped, since $a\operatorname{Re} W(z) = -2DN\sigma \tanh \sigma < 0$.

Now we discuss this in more detail. First we take the narrow Gaussian wave packet $g(z) = \exp\left[(z-z_0)^2/\Delta^2\right]$ with $(\Delta \ll |z_0| = |\kappa_0 + i\sigma_0|)$. We shall calculate the integral (5.1.28) by the saddle point method. The saddle point z_* is specified by

$$t \frac{\partial W(z_*)}{\partial z} + 2 \frac{z_* - z_0}{\Delta^2} - 2x = 0 \tag{5.1.33}$$

i.e., $z_*(x,t)$ depends on x and t. Let us discuss the behavior of the maximum of the envelope. We shall designate the coordinate of the maximum coordinate by $x_0(t)$. As may be easily seen from (5.1.32–33), $x_0(0) = -\kappa_0/\Delta^2$ holds and the saddle point $z_*(x_0,t) = z_*^0(t)$ with $x = x_0$ is located at the imaginary axis

$$z_*^0(0) = i\sigma_0 , \quad aW(i\sigma_0) = 2DN(2i \tanh \sigma_0 - 2\sigma_0 \tanh \sigma_0) \ .$$

At $t \ll \sigma_0 \cosh^2 \sigma_0$ (see below) we have

$$x_0(t) = -\frac{\kappa_0}{\Delta^2} + t\operatorname{Re} \frac{\partial W(z_*^0)}{\partial z} , \quad z_*^0(t) = -i\sigma_0 - it\Delta^2 \left(\tanh \sigma_0 + \frac{\sigma_0}{\cosh^2 \sigma_0} \right) \ .$$

The saddle point $z_*^0(t)$ moves along the imaginary axis towards zero. Since on the imaginary axis the velocity is $\operatorname{Re} W'(i\sigma) = 4DN/a \cosh^2 \sigma > 0$, the narrow Gaussian packet travels towards large x, with a decreasing maximum of the envelope $[\operatorname{Re} W(i\sigma) < 0]$. As $\operatorname{Im} W(i\sigma) \neq 0$, the damping of the packet is accompanied by oscillations. At $\sigma_0 \ll 1$, the oscillation period is much smaller than the damping time.

But any perturbation is eventually transferred to large x leaving behind a trail damping out with time. Indeed, let us consider $t \to \infty$ and $x/t = v = \text{const}$. For $x \to +\infty$, the function $g(z)$ is analytical at $\text{Re}\, z \geq 0$. Shifting the integration contour to the right (to the saddle point which at $v > W'(0)$ is located on the real axis and is given by $W'(\kappa) = v$) we get

$$F(x,t) \propto \exp\left[t\big(W(\kappa) - \kappa W'(\kappa)\big)\right] .$$

It is easy to show that for the function (5.1.32) we have at $\kappa > 0$ the relation $W(\kappa) - \kappa W'(\kappa) < 0$ which corresponds to a system without a convective instability. With regard to the trail left behind it has to be noted that at $t \to \infty$ the saddle point is for finite x ($ax \ll 2DNt$) determined by the condition $W'(z_*) = 0$ or $\sin 2\pi z_* = -2\pi(1 + z_*)$. This transcendental equation has two roots. They are located in the left half-plane (approximately $z_* \approx -1/4 \pm 2i/5$). Since $\text{Re}\, W(z_*) < 0$ we thus have

$$F(x,t) \propto \exp[x/2 + W(z_*)t] ,$$

showing that the trail is indeed decaying or damped as time progresses.

As we have seen above, the drift of the packet is determined by the quantity $W'(0)$. The same derivative is proportional to the energy flux of the Kolmogorov spectrum.

Summarizing, we conclude that small isotropic perturbations against the background of the Kolmogorov spectrum should be driven towards the damping region without growing in magnitude. If the perturbation have a structure oscillating in k-space [$\sigma_0 \neq 0$, $F \propto \exp(i\sigma_0 \ln k)$], the drift is accompanied by damping and oscillations in time.

Now we go over to discuss the stability of the isotropic spectrum of two-dimensional sound turbulence with regard to anisotropic perturbations. Again, like in the three-dimensional case, one can try to analytically derive the stationary anisotropic solutions of the linearized kinetic equation. The small parameter characterizing the approximation of the dispersion law to the linear one is in this case the quantity $(ak)^2$ rather than ε. As a consequence, the drift solution

$$A(k,\theta) = \frac{\omega_k(\boldsymbol{Rk})}{Pk^2} \propto (ak)^2 \cos\theta$$

has already for the first harmonic an index which is located outside the limits of the locality strip. The same holds for all subsequent harmonics; the linearized collision integral diverges for such solutions. The authors of Ref. 5.8 attempted an analytical derivation of the stationary anisotropic solutions by introducing a formal cut-off into the diverging integrals and requiring the most divergent terms to vanish. This lead to stationary solutions of the form

$$A_l(k,\theta) \propto (akl)^2 \cos(l\theta) . \tag{5.1.34}$$

However, these calculations fail to provide a rigorous proof. The structure of a stationary spectrum in the presence of an anisotropic source may be discussed with the help of computer simulations. Let us consider the two-dimensional kinetic equation (5.1.25) with an external source

$$\begin{aligned}\frac{\partial n(k,t)}{b\partial t} = \int \Big[& \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)(n_1 n_2 - n_k n_1 - n_k n_2) \\ & \times \delta\big(k - k_1 - k_2 + a^2(k^3 - k_1^3 - k_2^3)\big) - 2\delta(\boldsymbol{k}_1 - \boldsymbol{k} - \boldsymbol{k}_2) \\ & \times \delta\big(k_1 - k - k_2 + a^2(k_1^3 - k^3 - k_2^3)\big) \\ & \times (n_2 n_k - n_1 n_k - n_1 n_2)\Big] k k_1 k_2 \, d\boldsymbol{k}_1 d\boldsymbol{k}_2 + \Gamma_k n_k \, .\end{aligned} \tag{5.1.35}$$

As above, we have neglected the angular dependence in the interaction coefficient by using $\cos\theta_i \approx 1$. Let us integrate (5.1.35) over $d\boldsymbol{k}_2$ using the δ-function in the wave vectors and over $d\theta_1$ using the frequency δ-function. To first order in (ak) we obtain an equation formally coinciding with (5.1.26)

$$\begin{aligned}\frac{\partial n_k}{\partial t} = \frac{b 2\pi}{\sqrt{6}a} \Big[& \int_0^k k_1(k-k_1)(n_1 n_2 - n_k n_1 - n_k n_2)\, dk_1 \\ & - 2\int_k^\infty (k_1 - k) k_1 (n_k n_2 - n_1 n_2 - n_1 n_k)\, dk_1 \Big] + \Gamma_k n_k \, .\end{aligned}$$

In this equation we took $n_k = n(k,\theta,t)$ and assumed

$$n_1 = n\big(k_1, \theta \pm \sqrt{6}a(k-k_1), t\big), \quad n_2 = \big(k - k_1, \theta \mp \sqrt{6}ak_1, t\big)$$

in the first integral and

$$n_1 = n\big(k_1, \theta \pm \sqrt{6}a(k_1-k), t\big), \quad n_2 = n\big(k_1 - k, \theta \pm \sqrt{6}ak_1, t\big)$$

in the second one.

To remove the constant factor in front of the collision integral, we shall renormalize the occupation numbers via the substitution $n_k \to n_k\sqrt{6}a/b2\pi$. The resulting equation for the discrete case ($k = ik_0$, $\mu = \sqrt{6}ak_0$)

$$\begin{aligned}\frac{\partial n(i,\theta,t)}{\partial t} = & \sum_{j=1}^{i-1} i(i-j)\{n(j,\theta \pm \mu i \mp \mu j) n(i-j, \theta \mp \mu j) \\ & - n(i,\theta)[n(j,\theta \pm \mu i \mp \mu j) + n(i-j,\theta \mp \mu j)]\} \\ & - 2\sum_{j=i+1}^{L} j(j-i)\{n(i,\theta) n(j-i,\theta \pm \mu j) \\ & - n(j,\theta \pm \mu j \mp \mu i)[n(i,\theta) + n(j-i,\theta \pm \mu j)]\} \\ & + \Gamma_i n(i,\theta) - 2n(i,\theta)\sum_{j=L}^{L+i} j(j-i) n(j-i,\theta \pm \mu j)\end{aligned} \tag{5.1.36}$$

was numerically solved by *Falkovich* and *Shafarenko* [5.14] in θ, i coordinates on a rectangular grid. At $i > L$ the occupation numbers were assumed to be zero. As mentioned in Sect. 3.4, [see (3.4.18)], this assumption gives rise to the last term in (5.1.36) which plays the role of nonlinear damping. Periodic boundary conditions in the coordinate θ were imposed. The dimensions of the coordinate grid were chosen to be equal to 100 points in k and 32 points in θ. The summations in (5.1.36) were performed separately for the upper and lower signs before μ. The integration path in the plane θ_1, j is a line with the slope μ; at sufficiently small μ it passes through between the nodes of the grid. Therefore, the values $n(j, \theta \pm \mu i \mp \mu j)$ and $n(i-j, \theta \mp \mu j$ were calculated using a linear approximation inbetween the two closest grid nodes taking into account the periodicity in θ.

The evolution of the two-dimensional spectrum was modeled starting from the isotropic solution. A weak angle modulation was imposed on the source ($\varepsilon = .01$)

$$\Gamma(i, \theta) = 100\Delta_{i1}(1 + \varepsilon \cos m\theta) \ .$$

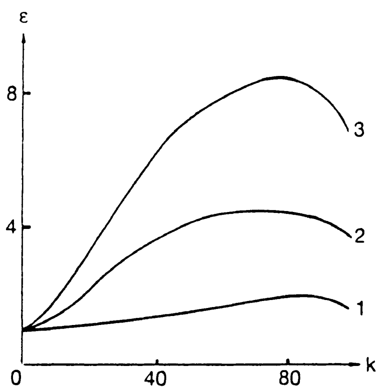

Fig. 5.1. Dependence of the relative angular modulation ε on k for different angular harmonics l and dispersion parameters μ: **1)** $l = 1$, $\mu = \pi/256$, **2)** $l = 2$, $\mu = \pi/256$, and **3)** $l = 1$, $\mu = \pi/128$

The results of the numerical simulations are represented in Fig. 5.1 showing the dependence of the relative depth of the modulation angle of the emerging spectrum $[n_{max} = \max_\theta n(k, \theta)]$

$$\varepsilon(k) = \frac{n_{max}(k) - n_{min}(k)}{n_{max}(k) + n_{min}(k)}$$

on the modules of the wave vector k. The quadratic dependence $\varepsilon = \varepsilon_0 + \nu k^2$ characterizes the initial section (for curve 3 up to $k \simeq 15$, $\varepsilon'''/\varepsilon'' \simeq 10^{-1}$), then the finiteness of $\mu i = \sqrt{6}ak$ starts to show itself. Comparison of curves 2 and 1 shows that under the same conditions the first harmonic increases by 1.02×10^{-2} and the second one, by 3.74×10^{-2}. This corresponds approximately to the relationship (5.1.34) $A_l \propto l^2$ [the difference by a factor of 3.7 rather than 4 seems to be due to the fact that for the second harmonic finiteness of μi begins to show itself at smaller $i = k/k_0$ and the growth of $\varepsilon(k)$ slows down]. The fact that the reduced growth of the $\varepsilon(k)$ is not only associated with influence of the right end of the inertial interval, but also with the finiteness of μi follows, e.g., from a comparison of curves 1 and 2. For curve 1, $\varepsilon(k)$ reaches its maximum at $k = 90$, and for 3 (corresponding to μ which is twice as high), at $k = 78$. The dependence of the occupation numbers on the angle is sinusoidal at all k. It reproduces the form of the source to an accuracy not worse than 10^{-4}. This is due to the smallness of the angular modulation ($\varepsilon \ll 1$) and to the fact that the operator of the kinetic equation linearized against the background of the isotropic solution is diagonal with regard to the number of the angular harmonics. Increasing ε, we go over into the nonlinear region where different angular harmonics interact with each other more strongly. Figure 5.2 corresponding to $\varepsilon = 1/5$ shows the relative contributions of the first three angular harmonics of the emerging spectrum for different values of k.

$$n_l(k) = \int n(k, \theta) a^{il\theta} \, d\theta \; .$$

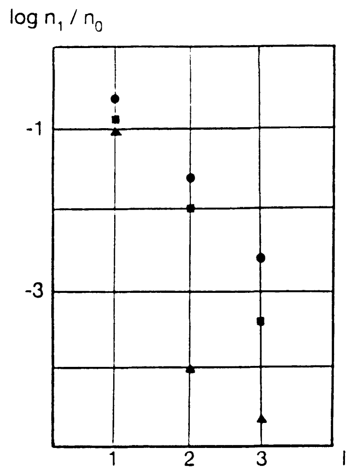

Fig. 5.2. Relative contributions of the first three harmonics: • $k = 2$, ■ $k = 40$, △ $k = 90$

The contributions from the higher harmonics are seen to increase with k. However, this does not imply that for large k the spectrum rapidly oscillates with the angle. Thus we have completed modeling the system at $\varepsilon = 1$, i.e., for a strongly anisotropic source. The dependence of the spectrum on θ is for all k smooth, see Fig. 5.3. The angular width of the spectrum $\Delta\theta$ decreases with the growth of the wave vector (Fig. 5.3).

It should be noted that up to $k = 50$, the quantity $d\Delta\theta/dk$ decreases with the growth of k. This possibly corresponds to $\Delta\theta$ approaching an asymptotically constant value. The increase of $d\Delta\theta/dk$ at $k > 50$ may be attributed to the influence of the right end of the inertial interval. Also up to $k \simeq 50$, the following effect is observed: in the direction of maximal values of the quantities $\Gamma(\theta)$ and $n_k(\theta)$, the spectrum decays slower than the isotropic one and the index $s(k) = \partial \ln[n(k,\theta)]/\partial \ln k$ decreases from $s(3) = 3.0$ to $s(50) = 2.4$ while in the direction $\theta = \pm\pi/2$ it increases: $s(3) = 3.0$ and $s(50) = 3.2$.

Such a behavior qualitatively corresponds to the properties of the two-flux solution (5.1.24a) obtained for three-dimensional sound.

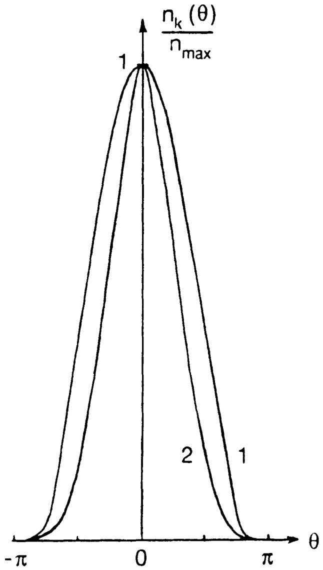

Fig. 5.3. Dependence of the stationary spectrum on the angle θ, 1) $k = 2$, 2) $k = 50$

It is of interest to note the nonmonotonic character of the relaxation of the anisotropic spectrum at large ε ($\varepsilon = 0.2, 1$). Thus, for $\varepsilon = 1$, the angular width $\Delta\theta$ for $k \leq 5$ diminishes at the final relaxation stage, for $k > 5$, it grows with time. For $\varepsilon = 0.2$ the value $\Delta\theta$ at first grows and is then reduced down to stationary values. It should be borne in mind that the initial distribution was isotropic. Now we would like to make a general remark. The effect of an increased anisotropy of the spectrum for growing k may be more pronounced at the relaxation stage than in the steady state (in which such an effect may not be observed at all). This is the result of a nonmonotonic evolution of occupation numbers during the formation stage and of the inversely proportional dependence of the relaxation time on the amplitude of the source. Indeed, let us consider as a model the limiting case of nondispersive sound $\omega(k) = ck$ where only the waves with collinear wave vectors interact and relaxation occurs along the direction of each beam independently of the other beams. Clearly, towards the directions of increasing source amplitude, the spectrum is closer to the formed one, i.e., with growing k it decays from a certain moment of time onwards slower than in the directions into which the relative angular modulation of the spectrum $\varepsilon(k)$ is an increasing function of k. On the other hand, it is evident that in the stationary state of such a model the quantity ε is independent of k and is equal to the relative modulation of the source. In a system with the dispersion relation $d^2\omega/dk^2$ allowing for the interaction of waves moving at different angles, such an "intermediate" anisotropy (at intermediate times) may be less pronounced or nonexistent.

Taking into account the terms of the next order $\sim (ak)^2$ in the $n(k)$ and $V(k, k_1, k_2)$ does not change the conclusions arrived at [5.15].

Thus, the isotropic spectrum of turbulence is structurally unstable for two-and three-dimensional weak sound turbulence. For large k the spectrum should be essentially anisotropic.

5.1.3 Nondecay Acoustic Turbulence: Ion Sound, Gravity Waves on Shallow Water and Inertio-Gravity Waves

Long-Wave Acoustic Turbulence. This section is devoted to acoustic waves with a nondecay dispersion law

$$\omega(k) = ck(1 - a^2k^2) . \tag{5.1.37}$$

The principal role in the interaction is played by four-wave scattering processes with the coefficient (1.1.29b)

$$\begin{aligned} T_p' = &- \frac{U_{-1-212}U_{-3-434}}{\omega_3 + \omega_4 + \omega_{3+4}} + \frac{V^*_{1+212}V_{3+434}}{\omega_1 + \omega_2 - \omega_{1+2}} \\ &- \frac{V^*_{131-3}V_{424-2}}{\omega_{4-2} + \omega_2 - \omega_4} - \frac{V^*_{242-4}V_{313-1}}{\omega_{3-1} + \omega_1 - \omega_3} \\ &- \frac{V^*_{232-3}V_{414-1}}{\omega_{4-1} + \omega_1 - \omega_4} - \frac{V^*_{141-4}V_{323-2}}{\omega_{3-2} + \omega_2 - \omega_3} . \end{aligned} \tag{5.1.38}$$

Here we used the notation $(j \pm i) = k_j \pm k_i$.

The stationary solutions can be found with the help of Kats-Kontorovich transformations the definition of which has been given in Sect. 3.2.1. In the three-dimensional case the stationary solutions are determined by (3.2.6):

$$n_1(k, P) = \lambda_1 P^{1/3} k^{-11/3}, \qquad n_2(k, Q) = \lambda_2 Q^{1/3} k^{-10/3} .$$

Both of these solutions are local [5.16] and correspond to the general formulas (2.3.10) with the renormalized index $m = 1$ of the interaction coefficient. Defining the indices $h_1 = 1/3$ and $h_2 = -1/3$ in the usual manner [see (4.3.15)], we see that both spectra are formed by decelerating waves like (4.3.4) and that the fluxes may be expressed in terms of the pumping amplitude Γ_0 and typical wave number k_0:

$$P \propto \Gamma_0^{3/2} k_0^{1/2}, \qquad Q \propto \Gamma_0^{3/2} k_0^{-1/2} .$$

For weak nonlinearity in the region of large k the applicability parameter for the solution n_1 should be evaluated according to (3.1.33b):

$$\xi^{-1}(k) \propto k^{3+h_1} = k^{10/3} .$$

Then the criterion for weak nonlinearity $\xi \ll 1$ is obeyed at large k. In the same way, one can see that for small k weak turbulence goes over into strong turbulence.

The two-dimensional case corresponds to waves on shallow water. It should be recalled that to the zeroth order in the small parameter ak (where a is proportional to the fluid depth) the interaction coefficient $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ vanishes on the resonance manifold $\omega(k)+\omega(k_1) = \omega(k_2)+\omega(k_3)$, $\boldsymbol{k}+\boldsymbol{k}_1 = \boldsymbol{k}_2+\boldsymbol{k}_3$, see the end of Sect. 3.2.1. Such a remarkable property is connected with the integrability of the Kadomtsev-Petviashvili equation (1.5.4). So, to first order in the (small) water depth, there are neither nonlinear interaction nor an evolution of the spectrum to a stationary state. Considering also the subsequent expansion terms we can obtain the Kolmogorov solutions [5.16]

$$n_3(k,P) = \lambda_3 P^{1/3} k^{-10/3}, \qquad n_4(k,Q) = \lambda_4 Q^{1/3} k^{-3}$$

with $h_1 = 2/3$ and $h_2 = -1$ so that

$$P \propto \Gamma_0^{3/2} k_0, \quad Q \propto \Gamma_0^{3/2} k_0^{-3/2}$$

are seen to hold.

The stability problem has not yet been solved. It is natural to expect a long-wave structural instability of the spectra n_2 and n_4 with regard to an additional perturbation carrying a momentum flux, since for such a perturbation we have

$$\delta n(\boldsymbol{k},\boldsymbol{R})/n(k,P) \propto 1/k \ .$$

Spectra carrying an energy flux should be stable, since

$$\delta n(\boldsymbol{k},\boldsymbol{R})/n(k,P) \propto \omega(k)/k$$

holds and the phase velocity decreases with k.

Short-Wave Acoustic Turbulence. Concluding this section let us briefly consider short-wave acoustic turbulence. In this case, the linear term in the dispersion law is supplemented by a positive term inversely proportional to k

$$\omega(k) = ck + \omega_0^2/2ck \ . \tag{5.1.39}$$

This dispersion relation is of the nondecay type. Stationary Kolmogorov solutions can be obtained by the general formulas (3.2.5) with $\beta = -1$. For $d = 3$ (which corresponds to spin waves in antiferromagnets and to ultrarelativistic particles)

$$n_1(k,P) \propto k^{-5}, \qquad n_2(k,Q) \propto k^{-14/3}$$

is obtained. So far neither locality nor stability of this solution has been checked.

The two-dimensional case is of special interest. The dispersion relation (5.1.39) is encountered in different physical situations. The frequency gap ω_0 could be connected either with a magnetic field (for plasmas) or with a rotation (for fluids or gases). The latter case is apparently realized in the Earth's atmosphere. If the wavelength is larger than the height of the atmosphere (10 km), then the wave may be treated as being two-dimensional. Such motions (like inertio-gravity waves in the atmosphere) is described in the framework of the shallow water equations [5.17]. If, in addition to the above, the wavelength is smaller than the Rossby radius ($\simeq$ 3000 km for medium latitudes) then it is reasonable to neglect the dependence of the Coriolis parameter $f = 2\Omega\cos\alpha$ (see Sect. 1.3.2) on the latitude α. In this case we have $\omega_0 = f$. The contribution of rotations to (5.1.39) is so small that the waves could be referred to as being "gravitational" rather than "inertial", in spite of the fact that rotation gives rise to the dispersive term supplementing the linear acoustic term.

Proceeding from (3.2.5), we get

$$n_1(k, P) \propto k^{-14/3}, \qquad n_2(k, Q) \propto k^{-13/3} .$$

According to *Falkovich* and *Medvedev* [5.18], only the second solution is local. It carries the flux of the wave action towards the large-scale region. In contrast to the long-wave limit, the interaction coefficient (5.1.38) does not vanish on the resonant manifold.

5.2 Wave Turbulence on Water Surfaces

From Sect. 1.2.5 we see that there are several cases of wave turbulence for different interrelations both between wave length and water depth and between surface tension and gravity. Prominent examples of shallow-water waves were considered in Sect. 5.1 since they belong to acoustic-type waves. Other cases of capillary waves on shallow water will be considered in Sect. 5.2.3.

5.2.1 Capillary Waves on Deep Water

As mentioned in Sect. 1.1.4, for sufficiently short water waves (with wavelengthes not exceeding a centimeter) the restoring force should be entirely determined by surface tension. The dispersion relation and interaction coefficient are

$$\omega(k) = \sqrt{\frac{\sigma}{\varrho}}\, k^{3/2} , \tag{5.2.1}$$

$$\begin{aligned} V(\boldsymbol{k}, 12) = \frac{1}{8\pi}\left(\frac{\sigma}{4\varrho^3}\right)^{1/4} &\left[(\boldsymbol{k}_1\boldsymbol{k}_2 + k_1k_2)\left(\frac{k_1k_2}{k}\right)^{1/4}\right. \\ &\left. + (\boldsymbol{k}_1\boldsymbol{k} - k_1k)\left(\frac{k_1k}{k_2}\right)^{1/4} + (\boldsymbol{k}\boldsymbol{k}_2 - kk_2)\left(\frac{kk_2}{k_1}\right)^{1/4}\right] , \end{aligned} \tag{5.2.2}$$

see also (1.2.41). This dispersion relation is of the decay type. There exists only a single isotropic stationary distribution (3.1.15b) as obtained by *Zakharov* and *Filonenko* [5.19]. It supports an energy flux

$$n(k) = \left(\frac{P}{a}\right)^{1/2} 8\pi\left(\frac{4\varrho^3}{\sigma}\right)^{1/4} k^{-17/4} . \tag{5.2.3}$$

Here a is the dimensionless constant from (3.1.13b) still to be evaluated. Determining the asymptotics of the interaction coefficient at $k_1 \ll k$ we obtain $m_1 = 7/2$ to see that the locality conditions (3.1.12) are satisfied. The index h is equal to $\alpha - m = 3/2 - 9/4 = -3/4$. Consequently, the energy flux is expressed in terms of the pumping characteristics [see (3.4.8)]:

$$P \propto \Gamma_0^2 \omega_0^{-1} \ .$$

Here ω_0 is the frequency of the waves generated and Γ_0 is the maximum growth-rate of linear instability. Knowledge of h also allows us to predict the character of nonstationary evolution. Since $h < 0$, short-wave Kolmogorov-like asymptotics (5.2.3) are reached in an "explosive" manner [5.20]

$$n(k,t) = (t_0 - t)^{17/3} f\left(k(t_0 - t)^{4/3}\right) \ .$$

Here f is a universal dimensionless function (see Sect. 4.3.1) and the dimensionless quantities k and t, respectively, are measured in units of an initial k_0 and of the typical nonlinear interaction time at $k = k_0$. At $k \to 0$ such a solution has quasi-stationary Kolmogorov-like asymptotics $f(x) \propto x^{-17/4}$. Before the front distribution falls exponentially: $f(x) \propto \exp(-x^{\alpha})$ (see Sect. 4.3.1).

The dimensionless parameter of the nonlinearity level (3.1.33) has the k-dependence

$$\xi^{-1}(k) = \omega_k t_{NL} \propto k^{3/4} \ ,$$

hence, the weak-turbulence approximation is violated for small k.

The isotropic distribution is structurally unstable as shown in Sect. 4.2. The relative contribution of the anisotropic part grows with k like $\delta n(\boldsymbol{k})/n(k) \propto \omega_k/k \propto \sqrt{k}$. In the short-wave region the stationary spectrum is thus substantially anisotropic. For positive angles θ_k between the wave vector and momentum flux, the stationary distribution of short waves should be defined by the momentum flux:

$$n(\boldsymbol{k}) = k^{-4}\sqrt{R\cos\theta_k} \ ,$$

compare with (5.1.24).

Experiments with capillary waves generated in a wind-wave tunnel are discussed in *Lleonart* and *Blackman* [5.21].

5.2.2 Gravity Waves on Deep Water

This is the kind of wave which is excited by wind blowing over the surfaces of seas and oceans. The corresponding kinetic equation was first obtained by *Hasselmann* [5.22]. The dispersion relation of gravity waves $\omega(k) = \sqrt{gk}$ is of the nondecay type. The interaction coefficients (1.2.43) are

$$\begin{aligned} U_{k,12} = V_{-k12} = \frac{1}{8\pi}\left(\frac{g}{4\varrho^2}\right)^{1/4} \Bigg[&(\boldsymbol{k}_1\boldsymbol{k}_2 + k_1k_2)\left(\frac{k}{k_1k_2}\right)^{1/4} \\ &+ (\boldsymbol{k}\boldsymbol{k}_1 + kk_1)\left(\frac{k_2}{kk_1}\right)^{1/4} + (\boldsymbol{k}\boldsymbol{k}_2 + kk_2)\left(\frac{k_1}{kk_2}\right)^{1/4}\Bigg] \ , \end{aligned} \qquad (5.2.4)$$

$$W(\boldsymbol{k}\mathbf{1},\mathbf{23}) = \frac{(kk_1k_2k_3)^{1/2}}{64\varrho\pi^2}[R(\boldsymbol{k}\mathbf{123}) + R(\boldsymbol{k}\mathbf{123}) \\ - R(\boldsymbol{k}\mathbf{213}) - R(\boldsymbol{k}\mathbf{312}) - R(\mathbf{12}\boldsymbol{k}\mathbf{3}) - R(\mathbf{13}\boldsymbol{k}\mathbf{2})] \, , \tag{5.2.5}$$

$$R(\boldsymbol{k}\mathbf{123}) = \left(\frac{kk_1}{k_2k_3}\right)^{1/4} [2(k+k_1) \\ - |\boldsymbol{k}-\boldsymbol{k}_2| - |\boldsymbol{k}-\boldsymbol{k}_3| - |\boldsymbol{k}_1-\boldsymbol{k}_2| - |\boldsymbol{k}_1-\boldsymbol{k}_3|] \, . \tag{5.2.6}$$

Eliminating three-wave processes, the resulting coefficient of the four-wave interaction has the form (1.1.30) and possesses the same homogeneity properties as $W(\boldsymbol{k}\mathbf{1},\mathbf{23})$, see also (1.1.43). Therefore we have in this case $\alpha = 1/2$, $d = 2$ and $m = 3$.

In the isotropic case there are two integrals of motion implying that there are also two isotropic Kolmogorov-like spectra (3.1.27–28). The first one was obtained by *Zakharov* and *Filonenko* [5.23]. It carries a constant energy flux to the short-wave region

$$n(k,P) = \varrho^{2/3}(P/a_1)^{1/3}k^{-4} \, . \tag{5.2.7}$$

Here a_1 is a dimensionless constant defined by the integral (3.1.13b) not yet evaluated for this particular case. The index $h_1 = \alpha - 2m/3 = -3/2$ [see Sect. 3.4.1 and (4.3.15a)] is negative and the spectrum (5.2.7) is thus formed "explosively"

$$n(k,t) = (t_0 - t)^{8/3} f_1\left(k(t_0-t)^{2/3}\right) \, .$$

Here f_1 is a universal dimensionless function (see Sect. 4.3.1 for details) of the dimensionless variables k (measured in the units of an initial k_0) and t (measured in the units of the typical nonlinear interaction time at $k = k_0$). The spectrum behind the front is (5.2.7) $f_1(x) \to x^{-4}$ at $x \to 0$. The right boundary of the Kolmogorov spectrum evolves according to the explosive law $k_b \propto (t_0 - t)^{-2/3}$. The value of t_0 is defined by the initial conditions.

Isotropic spectrum is stable with regard to angular modulations, since the contribution of the momentum-carrying additional term is proportional to the phase velocity $\omega(k)/k$ and thus decreases with k. With regard to the applicability condition of the weak turbulence approximation it is of note that the dimensionless nonlinearity parameter (3.1.33b) $\xi(k) \propto k^{-\alpha-h_1} = k$ grows with k. Thus the short-wave turbulence is a strong one which manifests itself in the observation of breakers and whitecaps.

The second stationary spectrum was obtained by *Zakharov* and *Zaslavskii* [5.24]. It carries a constant flux of wave action towards the long-wave region

$$n(k,Q) = g^{1/6}\varrho^{2/3}(Q/a_2)^{1/3}k^{-23/6} \, . \tag{5.2.8}$$

The dimensionless constant a_2 equals the integral (3.1.22b) and has not yet been computed. The locality of both spectra (5.2.7–8) could be proved using the following remarkable property of the interaction coefficient for $\boldsymbol{k}_1 \parallel \boldsymbol{k}$

$$T(k,k_1;k,k_1)=\begin{cases} k^2k_1 & \text{for } k<k_1\,, \\ kk_1^2 & \text{for } k>k_1\,. \end{cases}$$

The index $h_2 = (\alpha - 2m)/3 = -11/6$ [see Sect. 3.4.1 and (4.3.15b)] is negative and spectrum (5.2.8) is thus formed by a decelarating relaxation wave

$$n(k,t) = t^{23/11} f_2(kt^{6/11})\ .$$

Here f_2 is a universal dimensionless function. Behind the front there exists the spectrum (5.2.8) with $f(x) \to x^{-23/6}$ at $x \to \infty$. The right boundary of the Kolmogorov spectrum moves according to $k_b \propto t^{-6/11}$. Considering both spectra (5.2.7–8), one can see that most of the wave energy is contained in the long-wave region. So the mean frequency of the distribution approximately corresponds to the left edge of the spectrum (5.2.8) and thus decreases like $\omega_b \propto t^{-3/11}$.

The isotropic spectrum (5.2.8) has been proved to be stable with regard to angular modulations, see Sect. 4.3.3. With regard to the applicability condition of the weak-turbulence approximation we would like to not that the dimensionless nonlinearity parameter (3.1.33c) $\xi(k) \propto k^{-\alpha-h_2} = k^{4/3}$ grows with k. So the waving becomes weaker as the structure moves towards large k.

In what way is the attenuation of waving effected when the wind abates? The short-wave part of the distribution has the Kolmogorov asymptotics (5.2.7) with constant energy flux implying that the energy decreases. The evolution of decaying turbulence should approach the self-similar regime $n(k,t) = t^{4/11} f_3(kt^{2/11})$ which describes the propagation into the long-wave region. Thus, in the isotropic case, the mean frequency of waving decreases like $\omega_E \propto t^{-1/11}$.

Several field and laboratory observations are summarized by *Phillips* [5.25], see also *Forristall* [5.26]. The findings presented therein agree with both versions (5.2.7–8), but do not allow to distinguish between them.

5.2.3 Capillary Waves on Shallow Fluids

Such waves exist on very shallow fluids with $h_0 \ll \sqrt{\sigma/\varrho g}$. For intermediate wave numbers with $\sqrt{\varrho g/\sigma} \ll k \ll h_0^{-1}$, the dispersion relation and interaction coefficient have the extremely simple form (1.2.40)

$$\omega(k) = \sqrt{\frac{\sigma h_0}{\varrho}}\, k^2, \quad V_{k12} = \frac{k^2}{8\pi}\left(\frac{\sigma}{4\varrho h_0}\right)^{1/4}.$$

The dispersion law is of the decay type. For our purposes it is sufficient to consider only three-wave processes which are seen to correspond to $\alpha = 2$, $d = 2$, $m = 2$ $m_1 = 0$. A stationary Kolmogorov-like solution with constant energy flux was found by *Kats* and *Kontorovich* [5.27] to have the form

$$n(k) = P^{1/2} 8\sqrt{\pi}\left(\frac{\varrho h_0}{\sigma}\right)^{1/4} k^{-4}\ . \tag{5.2.9}$$

According to criterion (3.1.12) it is local and it has been proved to be stable with regard to isotropic perturbations [5.28]. In this case, the index h is given by $h = \alpha - m = 0$ so that the energy is uniformly distributed over the scales in the inertial range. The spectrum is formed in a "nonexplosive" process. The free decay takes place in two steps, see Fig. 4.13. The first step conserves energy and can be described in terms of the exponential self-similar solution (4.3.14). When the boundary of the Kolmogorov spectrum has reached the damping region, the energy starts to decrease $E(t) \propto t^{-1}$. The applicability criterion for weak turbulence is violated when going over to large k, since we have in this case $\xi(k) \propto k^{-2}$, see (3.1.33).

With regard to anisotropic perturbations it is found that a locality strip exists only for even harmonics. Odd harmonics with the numbers $l = 2j + 1$ evolve in a nonlocal manner, perturbations in the inertial range strongly interact with both left and right edges. The sign of the time derivative of the perturbation is seen to be proportional to $(-1)^j$. So the spectrum should be unstable with respect to harmonics with even j. The angular form of the spectrum should be substantially anisotropic in the inertial interval while index still be the same as in the isotropic case [5.28]. The transfer of energy is thus local while that of momentum is nonlocal in k-space,

5.3 Turbulence Spectra in Plasmas, Solids, and the Atmosphere

The media listed in the title are sometimes anisotropic due to electromagnetic fields (for plasmas and magnetics) or to rotation (for the atmosphere). However, in some situations it can be regarded as locally isotropic for wave turbulence. So, both isotropic and anisotropic Kolmogorov spectra may arise. Isotropic turbulence of ion sound waves was considered in Sect. 5.1. So we consider here an isotropic turbulence of plasmons.

5.3.1 Langmuir Turbulence in Isotropic Plasmas

For weak external fields, plasmas can be regarded as isotropic and we shall use (1.3.3–5)

$$\omega^2(k) = \omega_p^2(1 + 3k^2 r_D^2) \tag{5.3.1}$$

for the Hamiltonian coefficients. Here ω_p and r_D are, respectively, plasma frequency and the Debye length

$$\omega_p^2 = \frac{4\pi \varrho_0}{m^2}, \quad r_D^2 = \frac{T_e m}{4\pi \varrho_0 e^2} .$$

The coefficient of the three-wave interactions has the form

$$U_{k12} = V_{k12} = \frac{1}{8\sqrt{2}\pi^3 \varrho_0} \left[\left(\frac{\omega_1 \omega_2}{2\omega_k} \right)^{1/2} k \cos\theta_{12} + \left(\frac{\omega_k \omega_1}{2\omega_2} \right)^{1/2} k_2 \cos\theta_1 + \left(\frac{\omega_k \omega_2}{\omega_1} \right)^{1/2} k_1 \cos\theta_2 \right] .$$

However, the dispersion relation (5.3.1) holds only in the long-wave range $kr_D \ll 1$ and is of the nondecay type. Using the transformation (1.1.28), one can obtain an efficient Hamiltonian (1.1.29). In the range of $kr_D \ll 1$ the interaction coefficients U and V become scale-invariant with the scaling index unity and the efficient four-wave interaction coefficient (1.1.29b) has the scaling index two, since $\omega(k) \approx \omega_p$

$$T_{k123} = \frac{1}{\omega_p} \left[V_{k+1,k1} V_{2+3,23} - V_{-k-1,k1} V_{-2-3,23} - V_{k2,k-2} V_{133-1} - V_{k3,k-3} V_{122-1} - V_{k22-k} V_{131-3} - V_{k33-k} V_{122-1} \right] . \tag{5.3.2}$$

Such a nonlinear interaction has an electronic origin. Let us briefly explain the conditions under which this interaction is the main one. Considering plasma turbulence, the following dimensionless parameters are usually introduced: kr_D, which corresponds to the plasmon dispersion and E/nT which equals the ratio of the wave energy to the thermal one and is a measure of the nonlinearity level. So the typical time for turbulence evolution due to wave-wave interactions can be estimated from the kinetic equation to be roughly

$$\frac{1}{t_{NL}} \sim \omega_p (kr_D)^2 \left(\frac{E}{nT} \right)^2 .$$

It should be smaller than the time of induced scattering interactions with ions

$$\frac{1}{t_i} \sim \omega_p (kr_D)^3 \frac{E}{nT} .$$

Thus we obtain

$$\frac{E}{nT} > kr_D .$$

It is necessary to demand the validity of the weak turbulence approximation. In the case of plasma turbulence that means that there should be no Langmuir collapse the typical time of which contains another small parameter, the mass ratio

$$\frac{1}{t_c} \sim \sqrt{\frac{mE}{MnT}} .$$

Requiring $t_{NL} \ll t_c$ we obtain as a criterion for the weak turbulence approximation

$$\frac{E}{nT} \gg (kr_D)^{-4/3}\left(\frac{m}{M}\right)^{1/3} .$$

Under such conditions, the Kolmogorov spectra of Langmuir turbulence can be obtained [5.29], see (3.1.29–30). The first one corresponds to a constant energy flux P towards the short-wave region

$$n(k) = (P/a_1)^{1/3} n^{2/3} k^{-13/3} . \tag{5.3.3}$$

According to (4.3.15a), an "energy-containing" region is a short-wave region, since we have in that case $h_1 = \alpha - 2m/3 = 2 - 4/3 = 2/3 > 0$. So the spectrum (5.3.3) is formed by the decelerating wave (4.3.4) and the right boundary of the spectrum k_{rb} evolves according to $k_{rb}(t) \propto t^{3/2}$. The dimensionless parameter of the nonlinearity is according to (3.1.33b) given by $\xi \simeq (\omega t_{NL})^{-1} \propto k^{-8/3}$ implying a weak short-wave turbulence.

The second spectrum carries the flux of the wave action Q towards small k

$$n(k) = (Q/b_1)\omega_p^{1/3}(r_D n)^{2/3} k^{-11/3} . \tag{5.3.4}$$

In this case $h_2 = (\alpha - 2m)/3 = -2/3$ and the action-containing region is the same as the pumping region (i.e., the right edge of the spectrum). The left boundary of the spectrum (5.3.4) moves with $k_{lb} \propto t^{-3/2}$. According to the estimate (3.1.33c), the dimensionless nonlinearity parameter behaves as $\xi \propto k^{-4/3}$. The long-wave turbulence should be strong. According to *Zakharov* [5.29] both spectra (5.3.3–4) are local.

If the external pumping generates waves with some k_0 and a growth rate equal to Γ, then the fluxes are estimated to be

$$P \propto \Gamma^{3/2} k_0, \qquad Q \propto \Gamma^{3/2} k_0^{-1} .$$

The stability problem for such spectra is not yet solved. Both spectra (5.3.3–4) have indices which are larger than the one of dispersion $\alpha = 2$. According to the criterion (3.1.22), that means that both fluxes have correct directions: the spectrum (5.3.3) transfers energy towards the short-wave region while the spectrum (5.3.4) transfers action towards the long-wave end. Therefore, it is natural to expect that both spectra are stable in the isotropic case. As far as structural stability is concerned, they can be supposed to be unstable, since the contributions of the drift corrections (4.1.9,14) with constant momentum flux $\boldsymbol{R}$

$$\frac{\delta n(\boldsymbol{k})}{n(k,P)} \propto \frac{(\boldsymbol{Rk})\omega_k}{Pk^2} \propto k$$
$$\frac{\delta n(\boldsymbol{k})}{n(k,Q)} \propto \frac{(\boldsymbol{Rk})}{Qk^2} \propto k^{-1}$$

increase when going over from pumping to the inertial intervals. In this case further investigations are necessary to clarify all details of interest.

Another case of Langmuir turbulence can be observed in a nonisothermal plasma ($T_e \gg T_i$) provided the conditions

$$\frac{T_i}{T_e}(kr_D)^2 < \frac{E}{nT} < (kr_D)^2 < \frac{m}{M}$$

are satisfied. Then the plasmons interact via virtual ion sound waves. The coefficient of the four-wave interaction is given by (1.3.14)

$$T_{k123} = -\frac{\omega_p(\cos\theta_{k1}\cos\theta_{23} + \cos\theta_{k3}\cos\theta_{12})}{8\pi^3 nT} .$$

Its scaling index is zero. All features of such a turbulence are the same as for optical turbulence (which corresponds to T_{k123} = const). So we shall treat them together.

5.3.2 Optical Turbulence in Nonlinear Dielectrics and Turbulence of Envelopes

As stated in Sect. 1.5, the nonlinear Schrödinger equation describes the behavior of envelopes of quite general high-freguency quasi-monochromatic waves. We referred to such turbulence of envelopes as optical turbulence. It is characterized by waves with a quadratic dispersion law like (5.3.1)

$$\omega(k) = \omega_0 + \beta k^2 .$$

The constant ω_0 may be eliminated by the canonical transformation (1.4.22). The only remaining memory of it is that three-wave processes are prohibited. The coefficient of the four-wave interaction is supposed to be a constant. Therefore, the correspondent kinetic equation has the form

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = T\int \delta(k^2 + k_1^2 - k_2^2 - k_3^2)\delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) \times n_k n_1 n_2 n_3 \left(n_k^{-1} + n_1^{-1} - n_2^{-1} - n_k^{-1}\right) d\boldsymbol{k}d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 .$$

It is worthwhile to consider this equation for both two-dimensional media (which mainly corresponds to the envelopes of water waves) and three-dimensional ones. According to the general formula (3.1.9), its Kolmogorov solutions are equal to

$$\begin{aligned} n_1(k,P) &= \lambda_1 P^{1/3} k^{-d-2m/3} \propto k^{-d} , \\ n_2(k,Q) &= \lambda_2 Q^{1/3} k^{-d-2m/3+\alpha/3} \propto k^{-d+2/3} . \end{aligned} \tag{5.3.5a}$$

Substituting these solutions into the kinetic equation, one can see that the only local solution is n_2 for both $d = 2$ and $d = 3$. Since the nonlinearity parameter behaves in this case like $\xi \propto k^{-8/3}$, the turbulence becomes strong at the long-wave region. It corresponds to the collapse of envelopes (or self-focusing) which provides a sink for Kolmogorov spectra with action flux [5.30].

In the two-dimensional case, the only local solution $n_2(k,Q) \propto k^{-4/3} \propto \omega_k^{-2/3}$ has an index 2/3 smaller than unity. The implications are that for this solution the action flux is directed towards large k. From the Frisch-Fournier criterion in follows that this spectrum is unstable, even in the isotropic situation.

Rigorous stability theory demonstrates the presence of instabilities with regard to both the zero and the first angular harmonics (Sect. 4.2.3). What are the characteristics of the stationary turbulence spectrum in this case? The answer is quite unexpected [5.32]. An external pump heates the wave system and creates a steady state which is close to thermodynamical equilibrium with a small nonequilibrium fraction carrying action flux towards small k. According to Sect. 4.1, such a solution should have the form (4.1.23). In numerical simulations [5.30] it was found that the stationary turbulent distribution is close to the more general solution

$$n(\omega_k) = \frac{T}{\omega + \mu + aQ\omega^2/T^3} \tag{5.3.5b}$$

with temperature, chemical potential, and action flux. It should be noted that the collision integral logarithmically diverges in this case. Probably, it hints that the Kolmogorov-like addition should slightly deviate from a power form (e.g., with a logarithmic factor) to ensure convergence. For the small-scale part of the spectrum the formally defined Kolmogorov index $s = d = 2$ is just equal to the equilibrium index $s = \alpha = 2$. Hence, it is naturally to suppose the long-wave part of spectrum also to be close to the equilibrium distribution and to be slightly distorted either by a logarithmic factor (as in Sect. 3.2.2 where the coincidence index for plasmon-sound turbulence was considered) or by a small nonequlibrium additional contribution as for the long-scale part (5.3.5b).

The stability problem has not been solved for the three-dimensional case. The spectrum with an action flux carries a negative flux so it should be stable with regard to isotropic perturbations. In this case, $h_2 = (\alpha - 2m)/3 = 2/3$. In terms of the pumping characteristics the flux is given by

$$Q \simeq \Gamma^{3/2} k_0 \ .$$

Such a spectrum should be formed in an explosive way with the left boundary moving according to $k_{lb} \propto (t_0 - t)^{3/2}$. Its stability with regard to the first harmonics depends on the sign of the momentum flux carried by the drift Kolmogorov solution (4.1.14). If the momentum flux is negative, then the isotropic spectrum with an action flux is structurally unstable.

As far as a spectrum with energy flux is concerned, the collision integral diverges logarithmically for $d = 3$. It may be shown that the short-wave asymptotics of the stationary turbulent distribution slightly (e.g., logarithmically) differ from $k^{-3} \propto \omega_k^{-3/2}$, so the index might be close to 3/2. Numerical simulations [5.31] support such a hypothesis.

5.3.3 Spin Wave Turbulence in Magnetic Dielectrics

Spin waves in ferromagnetics and antiferromagnetics may have the dispersion law (1.4.9a, 21)

$$\omega(k) = \omega_0 + \beta k^2$$

hence, $\alpha = 2$. Spin waves in ferromagnets correspond to an interaction coefficient with $m = 2$. The Kolmogorov solutions (3.1.29c, 30c) coincide (up to a constant) with (5.3.3–4) for Langmuir waves with a nonlinearity induced by electrons. Thus everything said above about plasmons, holds also for magnons. For spin waves in antiferromagnets, such a case corresponds exactly to optical turbulence.

Now we shall discuss a somewhat detailed example of the Kolmogorov solution for spin waves whose dispersion is due to the exchange interaction (1.4.9a) and the interaction coefficient, to the magnetic dipole interaction (1.4.12). Such a "hybrid" is realized for an intermediate range of wave numbers k, which are (i) sufficiently large to allow it to neglect the gap ω_0 in the dispersion law (1.4.9a) and consider it to be of the decay type

$$\omega_k \propto k^2$$

and (ii) sufficiently small to neglect the four-wave exchange interaction (1.4.19b). The latter requirement, however, can always be satisfied assuming the turbulence level to be sufficiently small. The three-wave interaction coefficient (1.4.12) is in this case anisotropic

$$V_{k12} \propto \sin 2\theta_1 \, \exp(i\phi_1) + \sin 2\theta_2 \, \exp(i\phi_2). \tag{5.3.6}$$

Here θ_i, ϕ_i are, respectively, polar and azimuthal angle of $\boldsymbol{k}_i$ ($i = 1,2$) with the constant magnetization direction $\boldsymbol{M}$. The following fact found in [5.32] seemed a surprise: despite the angular dependence of the interaction coefficient, the three-wave kinetic equation (2.1.12) has an isotropic stationary solution. This is associated with the fact that on isotropic distributions $n(\boldsymbol{k}) \equiv n(\omega)$, the angular dependence is the same for all terms in the collision integral. Indeed, substituting (5.3.10) into (2.1.12), going over to the variable $\omega = k^2$ and integrating over angles, we get

$$\begin{aligned} \frac{\partial n(\omega,t)}{\partial t} &\propto (1 + 2\cos^2\theta - 3\cos^4\theta)\left\{ \int_0^\omega [n(\omega')n(\omega-\omega') \right. \\ &\quad - n(\omega)n(\omega-\omega') - n(\omega)n(\omega')]\,d\omega' - 2\int_0^\infty [n(\omega)n(\omega') \\ &\quad \left. - n(\omega+\omega)n(\omega') - n(\omega+\omega')n(\omega)]\,d\omega' \right\} \\ &= I(\omega,\theta) \ . \end{aligned} \tag{5.3.7}$$

As we see, the angular dependence in the collision integral is separated out. Using Zakharov transformations (2.3.14), we obtain from (3.3.27) the stationary solution $n(\omega) \propto \omega^{-3/2}$, in accordance with the general formula $n(\omega) \propto \omega^{-(m+d)/\alpha}$ (here $m = 0$, $d = 3$, $\alpha = 2$). The solution is local as may be verified using (3.1.12) or (3.3.27). Solving $\operatorname{div} P = -\omega I(\omega,\theta)$, we find that the only nonzero component is the radial component of the flux, which depends only on the angle θ : $P \propto 1 + 2\cos^2\theta - 3\cos^4\theta$. The flux is maximal at $\cos\theta = \pm 1/\sqrt{3}$.

This example shows that Kolmogorov distributions in anisotropic media may also be obtained analytically (when the frequency and the interaction coefficient are not bihomogeneous). However, it is obvious that the efficiency of an analytical treatment depends in such a case strongly on the particular forms of dispersion law and interaction coefficient.

5.3.4 Anisotropic Spectra in Plasmas

Before considering specific examples of spectra in anisotropic media, a general remark should be made. The theory of such spectra is far from being complete and well developed. The matching problem is not yet solved. As seen in Sect. 3.3, there exist families of power-like stationary spectra a good understanding of the physical meanings of which will require further studies. Hence, we present here briefly some known theoretical results concerning different anisotropic spectra.

If a plasma is strongly magnetized, then the dispersion laws and interaction coefficients of ion sound and Langmuir waves are both bihomogeneous functions of the compoents of the wave vectors, i.e., they satisfy equations (3.3.1) in some angular intervals.

Ion Sound Turbulence in Magnetized Plasmas. Let us start from the ion sound which we considered in Sect. 5.1.3 for an isotropic plasma. Here we follow *Kuznetsov* [5.33] in considering the anisotropic case.

For sound waves $\omega(k_z, k_\perp) = \omega(p, q)$ and V_{k12} are according to (1.3.20) given by

$$\omega \propto pq^{-2}, \quad V \propto (pp_1p_2)^{1/2}\Theta(p)\Theta(p_1)\Theta(p_2) \,. \tag{5.3.8}$$

Thus, $a = 1$, $b = 1$, $u = 3/2$, $v = 0$. The presence of Θ-functions in the interaction coefficient implies that only waves with $p > 0$ are considered. The Kolmogorov solution supporting an energy flux is

$$n(p, q) = \lambda_1 P^{1/2} p^{-5/2} q^{-2} \,, \tag{5.3.9a}$$

and with momentum flux

$$n(p, q) = \lambda_2 R^{1/2} p^{-5/2} q^{-1} \,. \tag{5.3.9b}$$

Now we should verify locality of the resulting distributions. We shall substitute (5.3.8) into the kinetic equation (3.3.2). Consider first the convergence of the integral at zero, i.e., at $q_1 \ll q$ and $p_1 \ll p$. Using the δ-functions we integrate now over the variables p_2, q_2. The expression (3.3.3) for Δ_2 will simplify to

$$\Delta_2(q_1, p_1) \approx \frac{q}{2}\sqrt{4q_1^2 - q^2(p_1/p)^2} \,,$$

and the integration will be only be performed for the region in which root in Δ_2 is positive. Let us gather all the terms that become infinite at $q_1 \to 0$, $p_1 \to 0$:

$$2\int dp_1 \int dq_1 \frac{qp^2 p_1}{\Delta_2(q_1,p_1)} n(p_1,q_1) \times [n(p-p_1, q+qp_1/2p) + n(p+p_1, q-qp_1/2p) - 2n(p,q)] \, .$$

This expression takes into account the presence of two identical singularities in the first term (3.3.2a), namely at $(p_1, q_1) \to 0$ and $(p_1, q_1) \to (p, q)$. We see that, like for the isotropic solutions, see Sect. 3.1, the singularities are twice reduced. Let us expand the expression in square brackets up to the first nonvanishing terms (quadratic in the small parameters q_1/q and p_1/p). Introducing the dimensionless variables $\zeta = q_1/q$, $\eta = p_1/p$ and setting $n(p,q) \propto p^{-x}q^{-y}$, we reduce the integral to

$$q^{3-y}p^{3-x} \int_0 d\zeta\, \zeta^{4-y-x} \int_0 d\eta\, \eta^{3-x}\sqrt{1-\eta^2} \, .$$

Whence, we obtain the convergence conditions $x < 4$ and $x + y < 5$. Both solutions (5.3.9), as well as the equilibrium solutions (3.3.9) (in the given case $n \propto p^{-1}q^{-2}, p^{-1}$) satisfy these conditions. In a similar way we obtain the convergence conditions for the integrals at $q_1 \to \infty$, $p_1 \to \infty$ with

$$\Delta_2(q_1,p_1) \approx \frac{q_1}{2}\sqrt{4q^2 - q_1^2(p/p_1)^2} \, .$$

The most dangerous terms are arranged into the combination

$$p^2 n(p,q) \int dp_1 \int dq_1 \frac{q_1 p_1}{\Delta_2(q_1,p_1)} \left[\frac{\partial n(p_1,q_1)}{\partial p_1} - \frac{\partial n(p_1,q_1)}{\partial q_1}\frac{q_1}{2p_1} \right] .$$

Going over to the dimensionless variables $\zeta = q/q_1$, $\eta = \zeta^{-1}p/p_1$ we obtain

$$(x-y)p^{2-x}q^{2-y} \int_0 d\zeta\, \zeta^{x+y-3} \int_0^1 d\eta \frac{\eta^{x-2}}{\sqrt{1-\eta^2}} \, .$$

The convergence of the integral is ensured for $x > 1$, $x+y > 2$. These conditions are satisfied for both distributions (5.3.9). By the way, in the given case we have for equilibrium distributions $x = -1, y = 0$; $x = -1$, and $y = -2$, i.e., the ultraviolet convergence conditions are not satisfied. This certainly does not mean divergence of the collision integral of which every term is identically zero in equilibrium. It implies simply that for any wave belonging to the Rayleigh-Jeans distribution, the principal role in the equilibrium state is played by the interaction with high-frequency waves. Since at $\omega \gtrsim T/\hbar$ the Rayleigh-Jeans decrease $n \propto \omega^{-1}$ goes over to the Planck decrease $n \propto \exp(-\hbar\omega/T$ [see (2.2.12)], the main interaction takes place with thermal waves (for which $\omega \simeq T/\hbar$).

Thus, for ion-sound turbulence in a magnetic field, both Kolmogorov distributions are local.

Langmuir Turbulence in Magnetized Plasmas. For the Langmuir waves propagating almost perpendicularly to a magnetic field, the dispersion relation has in the limiting cases of strong and weak fields, respectively, a similar dependence on p and q, see (1.3.22, 24)

$$\omega(p,q) \propto \frac{|p|}{q} .$$

We consider waves moving in the entire k-space, i.e., p varies from $-\infty$ to $+\infty$. The total momentum vanishes for the power solution $n \propto |p|^{-x} q^{-y}$ and (3.3.10d) should not exist. Indeed, the transformations (3.3.7) will cast the stationary kinetic equation into the form

$$\begin{aligned} &\int\int_{-\infty}^{\infty} dp_1 dp_2 \int\int_0^{\infty} dq_1 dq_2\, U(p,p_1,p_2,q,q_1,q_2)\delta(p-p_1-p_2) \\ &\times \delta(p^a q^b - p_1^a q_1^b - p_2^a q_2^b)\Big[|p_1 p_2|^{-x}(q_1 q_2)^{-y} - |pp_1|^{-x}(qq_1)^{-y} \\ &- |pp_2|^{-x}(qq_2)^{-y}\Big]\left[1 - \left|\frac{p}{p_1}\right|^{\tilde{x}}\left(\frac{q}{q_1}\right)^{\tilde{y}} - \left|\frac{p}{p_2}\right|^{\tilde{x}}\left(\frac{q}{q_2}\right)^{\tilde{y}}\right] = 0 , \end{aligned} \tag{5.3.10}$$

see also (3.3.8). If we set $\tilde{x} = -1$, $\tilde{y} = 0$ like in (3.3.10b), the left-hand side of (5.3.10) will not vanish because of the presence of $|p/p_1|$ and $|p/p_2|$ in the second square bracket.

First we shall consider strong magnetic fields. In the angular range limited by the inequality $|\cos\theta_k| \ll \omega_p/\omega_H$, the interaction coefficient is given by (1.3.23a)

$$V \propto (\boldsymbol{q}_1[\boldsymbol{h}\boldsymbol{q}_2])\sqrt{\frac{|p_1 p_2|q}{q_1 q_2 |p|}}\operatorname{sign} p\left[\frac{\operatorname{sign} p}{q}\left(\frac{q_1}{q_2} - \frac{q_2}{q_1}\right) + \frac{\operatorname{sign} p_1}{q_2} + \frac{\operatorname{sign} p_2}{q_1}\right] .$$

In this case $u = v = 1/2$ and

$$n(p,q) \propto P^{1/2}|p|^{-3/2}q^{-5/2} \tag{5.3.11}$$

is the Kolmogorov solution with a constant energy flux. We can direct verify that it is local [5.34]. For a weak magnetic field, (1.3.24) holds for the interaction coefficient

$$V \propto \sqrt{\frac{p_1 p_2 q}{q_1 q_2 p}}\frac{(\boldsymbol{q}_1[\boldsymbol{h}\boldsymbol{q}_2])}{q q_1 q_2}\left[q^2 + qq_2 \operatorname{sign}(pp_2) + qq_1 \operatorname{sign}(pp_1)\right] .$$

Its indices $u = v = 1/2$ and the solution coincide with (5.3.11). Locality is also verified directly.

The common turbulence of Langmuir and ion-sound waves has been described in Sect. 3.2.2 as an example for the interaction of high- and low-frequency waves.

One also knows the case in which simultaneously the properties of low- and high-frequency wave interactions (see Sect. 3.2.2) and of bihomogeneity are observed. An example is the interaction of high-frequency Alfven waves ($\omega_k = k_z v_A$) with low-frequency magnetic sound ($\Omega_k = k_z c_s$) in a magnetized plasma at low pressure ($c_s \ll v_A$) [5.35]. The kinetic equations are obtained from magnetohydrodynamic equations and have the form

$$\frac{\partial n(k,t)}{\partial t} = -\int U(k_2,k,k_1)T(k_2,k,k_1)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 ,$$

$$\frac{\partial N(k,t)}{\partial t} = \int [U(k,k_1,k_2)T(k,k_1,k_2) - U(k_1,k,k_2)T(k_1,k,k_2)]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 ,$$

with

$$T(k,k_1,k_2) = N(k_1)n(k_2) - N(k)n(k_2) - N(k)N(k_1) ,$$

$n(k)$, $N(k)$ are the densities of sound and Alfven waves, respectively, and

$$U = 2\pi |V_{k12}|^2 \delta(k - k_1 - k_2)$$

is a bihomogeneous function of the components of the wave vector

$$U(\lambda k_z, \mu k_\perp) = \lambda \mu^{-2} U(k_z, k_\perp) .$$

The resulting Kolmogorov solution is

$$n(k) = A_1 k_z^{-2} k_\perp^{-2}, \quad N(k) = B_1 k_z^{-2} k_\perp^{-2} ,$$

and corresponds to a constant flux of high-frequency waves. The Kolmogorov spectrum with a constant energy flux has the form

$$n(k) = A_2 k_z^{-5/2} k_\perp^{-2}, \quad N(k) = B_2 k_z^{-5/2} k_\perp^{-2} .$$

For both spectra one can obtain the estimate $A_i c_s \sim B_i v_A$, $(i = 1,2)$. Therefore, for stationary turbulence, the energies of Alfven and sound waves are of the same order.

Checking the locality, we find that the integral over k_z converges, while the integral over $k_\perp$ is just about divergent, i.e., it diverges logarithmically. To assert that such a turbulence is local, it is necessary to consider also dispersion (i.e., the next terms in ω_k, Ω_k) [5.35].

5.3.5 Rossby Waves

Let us now discuss the turbulence of the barotropic Rossby waves introduced in Sect. 1.3.2. Their k-space is two-dimensional, which does not allow to take over the results of the cases discussed above. Therefore we shall consider this example in more detail. We shall restrict ourselves to the region of small-scale ($k \gg k_0$) motions close to *zonal* ones (for which $k_x \equiv p \ll k_y \equiv q$). In this limit the dispersion relation (1.3.31) and the interaction coefficient (1.3.54) are bihomogeneous functions of the wave vector components [5.36–37]

$$\omega \propto pq^{-2} , \tag{5.3.12a}$$

$$V \propto (pp_1p_2)^{1/2}(q_1^{-1} + q_2^{-1} - q^{-1}) \tag{5.3.12b}$$

so that we have $a = 1$, $b = -2$, $u = 3/2$, and $v = -1$.

The kinetic equation for Rossby waves takes the form

$$\begin{aligned}\frac{\partial n}{\partial t} = \int\int_0^\infty dp_1 dp_2 \int\int_{-\infty}^{\infty} dq_1 dq_2 \\ \times \left[U(k, k_1, k_2)(n_1n_2 - nn_1 - nn_2)\right. \\ \left. - 2U(k_1, k, k_2)(nn_2 - nn_1 - n_1n_2)\right],\end{aligned} \tag{5.3.13}$$

where

$$U(k, k_1, k_2) = |V|^2\delta(p - p_1 - p_2)\delta(q - q_1 - q_2)\delta(pq^{-2} - p_1q_1^{-2} - p_2q_2^{-2}),$$
$$n = n(p, q, t), \quad n_i = n(p_i, q_i, t), \quad \boldsymbol{k} = (p, q) .$$

Equation (5.3.13) has three general integrals of motion: the energy

$$E = \int \omega n \, dpdq$$

and the *zonal* and *meridional* momentum components

$$\text{(zonal)} \quad \int pn \, dpdq \qquad \text{and} \quad \text{(meridional)} \quad \int qn \, dpdq ,$$

respectively. However, it would be wrong to suppose that this equation must have three stationary Kolmogorov solutions of the type $n \propto p^{-x}|q|^{-y}$ corresponding to the fluxes of the three conserved quantities. Because of its parity with regard to q, the meridional component of the total momentum is identically zero on power distributions so that there should be no corresponding Kolmogorov distribution. Indeed, let us use the transformations (3.3.7) to reduce the stationary kinetic equation to

$$\begin{aligned}I(p, q) = \int\int_0^\infty dp_1 dp_2 \int\int_{-\infty}^{\infty} dq_1 dq_2 \left(q_1^{-1} + q_2^{-1} - q^{-1}\right)^2 \\ \times \delta(p - p_1 - p_2)\delta(q - q_1 - q_2)\delta\left(\frac{p}{q^2} - \frac{p_1}{q_1^2} - \frac{p_2}{q_2^2}\right) \\ \times \left[(p_1p_2)^{-x}(q_1q_2)^{-y} - (pp_1)^{-x}(qq_1)^{-y} - (pp_2)^{-x}(qq_2)^{-y}\right] \\ \times \left\{1 - \left|\frac{q_1}{q}\right|^{2y-2}\left(\frac{p_1}{p}\right)^{2x-4} - \left|\frac{q_2}{q}\right|^{2y-2}\left(\frac{p_2}{p}\right)^{2x-4}\right\} pp_1p_2 = 0\end{aligned} \tag{5.3.14}$$

This equation may have two solutions if the expressions in braces coincide with the arguments of the first or third δ-function. The first solution corresponds to the choice $2x - 4 = 1$, $2y - 2 = 0$ and should transfer the constant flux R_x of the zonal component of momentum:

$$n(p, q) \propto R_x^{1/2} p^{-5/2} |q|^{-1} . \tag{5.3.15a}$$

The second solution corresponds to the constant energy flux P

$$n(p, q) \propto P^{1/2} p^{-5/2} \ . \tag{5.3.15b}$$

The second δ-function may not be utilized for obtaining an extra stationary solution, as the choice $2y - 2 = 1$, $x = 2$ reduces only the integral over the region $q_2/q > 0$ to zero, cf. (3.3.20).

It is interesting to mention the existence of an additional integral of motion recently obtained for Rossby waves [5.38]. It has the simple form $\int (\omega^3/q^2) n \, dp dq$. The corresponding Kolmogorov distribution $n(p, q) \propto p^{-7/2} q^3$ is nonlocal.

Substituting (5.3.15) into (5.3.13), it is easy to verify that the local distribution is only (5.3.15b). The energy flux in the k-space has two components proportional to the derivatives of the collision term (5.3.14) with respect to the solution indices. Calculating $\partial I/\partial x$ and $\partial I/\partial y$ at $x = 5/2$, $y = 0$ we see that $P_x > 0$, $P_y < 0$, i.e., the flux is directed towards larger p and smaller $|q|$. Considering the form of the dispersion law (5.3.12a), we conclude that the Kolmogorov distribution transfers the energy flux into the high-frequency region [5.39].

The treatment of baroclinic Rossby waves is complicated by the presence of various vertical harmonics, see (1.3.61). If only the waves corresponding to $l = 1$ are excited, then (1.3.61) coincides with (3.3.22) and the problem reduces to the solved one.

It is worthwhile to point out that anisotropic Kolmogorov spectra are more often found to be nonlocal than isotropic ones. In [5.40], a model of nonlocal turbulence of Rossby waves has been proposed. It suggests that the waves from the inertial interval interact mainly with low-frequency waves. Such an interaction is almost elastic and redistributes the high-frequency waves over the surface of the constant frequency $p - \omega(p, q) = \text{const}$. However. for the $\omega(p, q)$ of (5.3.12a), the corresponding surface is not closed. Thus, the local interaction may pump waves from the source to the region of sufficiently large q of the damping area. As a result, the evolution of the drift turbulence probably leads to the separation of the spectrum into two well-distinquished components; a small-scale component concentrated on a certain line in $\boldsymbol{k}$-space and a large-scale zonal flow with its level determined by the interaction with the short-wave turbulence. Further investigations (possibly involving computer simulations) are necessary to find out whether drift turbulence is local or nonlocal.

6. Conclusion

There may be a Moral, though some say not;
I think there's a moral, though I don't know what.

A. Milne "Now We Are Six"

Methodological Guide

The reader who has worked his way through the book up to this place should have found out that the Kolmogorov spectra of wave turbulence are fairly easy to handle. Indeed, detailed treatment of any new case will not require much further efforts. The answers to most questions, including rather subtle ones, may be obtained from dimensional analysis and from simple asymptotic estimates. Now, what is the procedure to be followed in such an analysis? Let us formulate the program to be carried out. This outline will simultaneously serve as a methodological guide for this volume:

1. If we don't even know the dispersion relation of the waves, we have to determine from a dimensional analysis the form of the dependence of the frequency ω on the wave vector k; in particular, the index of the dispersion relation α, see Sect. 1.1.4 (1.1.31). With the parameters of our problem we can now construct several different expressions having the dimension of a frequency. If these expressions depend differently on the wave vector, then scale-invariance and the universal Kolmogorov spectra of turbulence may only be observed in the region of k-space where one of the contributions to the frequency is by far larger than all others. The only exception occurs when the frequency consists of two terms, a constant $\omega(0)$ and the dispersion $\delta\omega(k)$. Then, in the nondecay case, the constant term $\omega(0)$ does not enter the resonance conditions for four-wave processes

$$\omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) = \omega(\boldsymbol{k}_3) + \omega(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3)$$

and Kolmogorov solutions exist when $\omega(0)$ and $\delta\omega$ are of the same order. However, such dispersion relations are frequently of the decay type in one region of k-space and of the nondecay type in another. Consequently they have to be treated differently. An example is the dispersion law $\omega(k) = \omega(0) + \beta k^2$ which goes for $\beta k^2 > 2\omega(0)$ over to the decay type.

2. From the form of the dispersion relation it is seen whether three- or four-wave interactions, see (1.1.24, 29), are dominant. For different interaction types there are different numbers of the integrals of motion (see Sect. 2.2.1) and one can

also expect a different number of Kolmogorov spectra (Sect. 3.1) for them. From a dimensional analysis we can now extract the scaling index m of the respective interaction coefficient, Sect. 1.1.4 (1.1.32). Knowing α, m and the space dimension d, we may utilize one of the formulas (3.1.4–6) to obtain the Kolmogorov spectrum.

3. In the next step we have to verify the locality of the interaction of the Kolmogorov spectrum (i.e., stationary locality). For example, for three-wave interactions the verification of locality by the aid of (3.1.12) requires the knowledge of the index m_1 determining the asymptotics of the interaction coefficient

$$|V(\boldsymbol{k}, \boldsymbol{k}-\boldsymbol{\kappa}, \boldsymbol{\kappa})|^2 \propto \kappa^{m_1} k^{2m-m_1} \quad \text{at} \quad \kappa \ll k \, . \tag{6.1}$$

From this step onwards it is therefore not sufficient to rely on dimensional analysis. It is necessary to carry out a real calculation. However, this calculation is a rather simple one since we do not need to obtain the exact expression for the interaction coefficient but only the index of its asymptotics.

In most cases the asymptotic index m_1 may also be found from a dimensional analysis. For example, if the frequency parameter ω_k depends on the medium parameter that oscillates for the wave a_κ with $\kappa \to 0$, then the interaction Hamiltonian

$$\int V(k, k-\kappa, \kappa) a(\kappa) a(k) a^*(k-\kappa) \approx \int V(k, k, \kappa) a(\kappa) a(k) a^*(k)$$

may be represented as

$$\int \delta\omega_k a_k a_k^* \, .$$

As a consequence, the k-dependence of $V(k, k, \kappa)$ should be similar to the one of the frequency, namely

$$V(k, k, \kappa) \propto k^\alpha \kappa^{m-\alpha} \, . \tag{6.2}$$

Comparing (6.2) and (6.1) we obtain

$$m_1 = 2(m-\alpha) = -2h \, . \tag{6.3}$$

For example, a sound wave with a small wave vector corresponds to a small density variation which is almost constant in space. Since, the frequency depends on the density we have $m_1 = 2(m-\alpha) = 2(3/2-1) = 1$, compare with (5.1.6). It should be noted that for two-dimensional sound in (6.3), one should substitute the index of the initial interaction coefficient 3/2, rather than the angle-averaged one 1. For shallow-water capillary waves, the waves alter the fluid height which in turn determines the frequency so that (6.3) is correct, see Sect. 3.1.2. But for deep-water capillary waves, the frequency does not depend on the depth so that (6.3) cannot be used. Indeed, according to (1.2.41b) we have in this case $m_1 = 7/2$ while (6.3) would lead to $m_1 = 3/2$.

4. Thus we have made sure that the turbulence is local. Consequently the corresponding Kolmogorov solution exists. Now, (4.2.17) yields the index h. Knowing h, we can immediately answer two important questions simultaneously. Namely, with regard to

(i) the type of the dependence of the flux absorbed by the turbulence on the pumping frequency (3.4.8);

(ii) the character of the nonstationary behavior of the wave system. In particular, it is in the isotropic case possible to describe the formation process of a Kolmogorov distribution, see Sect. 4.3.

5. Certainly, we can only speak about the formation of Kolmogorov spectra after having checked their stability (at least with regard to isotropic perturbations). For this purpose it is sufficient to compare the index of the Kolmogorov solution with the index of the respective equilibrium distribution. This allows to determine the sign of the flux, see (3.1.13b, 22). In line with the Fournier-Frisch criterion (see Sect. 3.1.3 and the end of Sect. 4.2), positive-flux spectra are stable in the short-wave region and negative-flux spectra in the long-wave region.

6. It is only when assessing the problem of structural stability of an isotropic spectrum that we will need to know the exact expression of the coefficient $V(k, k_1, k_2)$ or $T(k, k_1, k_2, k_3)$ of the interaction Hamiltonian. This expression must be substituted into (4.2.16) for the Mellin functions $W_l(s)$. It is sufficient to analyse the first few harmonics.

In general, the dispersion relation is nondegenerate (see the end of Sect. 2.2.1) and the interaction coefficient does not vanish on the resonance surface. So we cannot expect any anisotropic integrals of motion except the one for momentum. Therefore, it is in this case sufficient to consider only the first angular harmonic ($l = 1$) which is proportional to the cosine. Now we have to check the sign of the zero of the Mellin function. Positiveness of $W_l(0)$ implies according to (4.2.55) structural instability of the isotropic solution against perturbations of the form of the l-th harmonic. In this case determination of the growth rate of the anisotropic perturbation contribution with k makes it necessary to find that zero of the $W_l(s)$ function which is closest to the point $s = 0$. For the first harmonic it can be found from a dimensional analysis, in line with (4.1.9, 14). If $W_l(p) = 0$, the relative perturbation term behaves like $A_l(k) = \delta n(k)/n(k) \propto k^{-p}$.

Let us take $W_l(0) < 0$ to hold, a condition which does not ensure stability. We have to calculate the rotation $\kappa_l(0)$ of the Mellin function $W_l(i\sigma)$ around the imaginary axis (as σ moves from $-\infty$ to $+\infty$). According to the Balk-Zakharov criterion (see Sect. 4.2.2) the isotropic spectrum is stable with regard to the given harmonic, provided $\kappa_l(0) = 0$ holds.

In the general case, when we are concerned with the first harmonic only, we can proceed in a much simpler way. The zero of the $W_1(s)$ function (i.e., the index of the indifferently stable mode) may be found from a dimensional analysis following (4.1.9, 14). Whether this mode may be excited (leading to structural instability of the isotropic solution) is according to the Falkovich criterion (see Sect. 4.2) determined by the sign of the momentum flux $W_1'(p)$ it carries.

7. Finally, we have to determine the applicability conditions for the obtained results. For this purpose, we have to determine scale and amplitude ranges to

which the weak turbulence approximation is applicable. This may be done following a dimensional analysis according to (1.1.34, 35) and (2.1.14, 15, 21, 25, 27, 30).

With the recipe laid out in items 1. to 7. and the material provided in the preceding chapters of this volume the reader should have all tools at his disposition to investigate in details turbulence and Kolmogorov spectra.

A. Appendix

A.1 Variational Derivatives

Without giving a strict justification, we shall explain several simple rules for calculating variational derivatives. They follow from the fact that $\delta/\delta f(r)$ generalizes the notion of the partial derivative $\partial/\partial f(r_n)$ with discrete r_n to continuous variables.

1. The variational derivatives of a linear functional of the form $I = \int \phi(r')f(r')\,dr'$ are calculated by

$$\frac{\delta I}{\delta f(r)} = \int \phi(r')\frac{\delta f(r')}{\delta f(r)}\,dr' = \int \phi(r')\delta(r-r')\,dr' = \phi(r)\ . \qquad (A1.1)$$

To obtain this formula, one can mentally substitute $\delta/\delta f(r)$ by $\partial/\partial f(r_n)$, simultaneously replacing the integration by a summation to return after differentiation to the continuous version

$$\frac{\delta}{\delta f(r)}\int \phi(r')f(r')\,dr' \rightarrow \frac{\partial}{\partial f(r_n)}\sum_m \phi(r_m)f(r_m) = \phi(r_n) \rightarrow \phi(r)\ .$$

Symbollically this result may be represented by

$$\frac{\delta f(r')}{\delta f(r)} = \delta(r-r')\ . \qquad (A1.2)$$

2. If the function f in the functional is affected by differential operators, then application of the rule (A1.2) requires first to bring them to the left-hand-side and to integrate by parts. For example,

$$\frac{\delta}{\delta f(r)}\int \phi\nabla f\,dr' = -\frac{\delta}{\delta f(r)}\int f\nabla\phi\,dr' = -\nabla\phi\ . \qquad (A1.3)$$

Here we assumed that the product $f(r')\phi(r')$ vanishes on the boundary of the integration region.

The variational derivative of nonlinear functionals is calculated following a procedure similar to the one for the partial differentiation of a complex function:

$$\frac{\delta}{\delta f(r)}\int F[f(r')]\,dr' = \int \frac{\delta F}{\delta f(r')}\frac{\delta f(r')}{\delta f(r)}\,dr' = \frac{\delta F}{\delta f(r)}\ .$$

For example,

$$\frac{\delta}{\delta f(r)} \int f^n(r')\, dr' = n f^{n-1}(r), \quad \text{etc} . \tag{A1.4}$$

3. Variation of multi-dimensional integrals over the function describing the boundary of the integration domain is not so trivial. Thus, in deriving the Hamiltonian description of waves on a fluid surface (see Sect. 1.2), one should calculate the variational derivative $\delta/\delta\eta(r)$ of the following functional

$$J = \int d\boldsymbol{r} \int^{\eta(r)} A[\boldsymbol{r}, z; \eta(r)]\, dz. \tag{A1.5}$$

Here $\boldsymbol{r} = (x, y)$ is a two-dimensional vector, $A[\boldsymbol{r}, z, \eta(\boldsymbol{r})]$ depends not only on spatial variables but also on the form of the function $\eta(\boldsymbol{r})$. For example, the boundary condition on A is given on the surface $z = \eta(\boldsymbol{r})$. For the variation $\eta \to \eta + \delta\eta$, the variation in J consists of two terms

$$\delta J = \int d\boldsymbol{r}'\, A[\boldsymbol{r}', z; \eta(\boldsymbol{r}')]_{z=\eta} \delta\eta(\boldsymbol{r}') + \int d\boldsymbol{r}' \delta A[\boldsymbol{r}', z; \eta(r')]_{z=\eta}. \tag{A1.6}$$

The first term is due to the variation in the size of the integration domain; the second one, to the variation of the integrand, e.g. $\delta A = A(z + \delta z) - A(z) = \delta\eta \partial A/\partial z$, see (1.2.33).

A.2 Canonicity Conditions of Transformations

1. Let $a_j(r,t)$, $a_j^*(r,t)$ be the canonical variables, so that their equations of motion have the canonical form (1.1.6). Now we introduce new variables $b_l(r,t)$, $b_l^*(r,t)$ via transformations which are not explicit functions of time

$$b_l = f_l\{a_j, a_j^*\}, \quad b_l^* = f_l^*\{a_j, a_j^*\}. \tag{A2.1}$$

Here f_i is a functional. Let us obtain the conditions under which the equations of motion in the b, b^* variables retain the form (1.1.6):

$$\begin{aligned}
\frac{\partial b_l(r,t)}{\partial t} &= \sum_j \int dr' \left[-i \frac{\delta b_l(r)}{\delta a_j(r')} \frac{\delta \mathcal{H}}{\delta a_j * (r')} + i \frac{\delta b_l(r)}{\delta a_j^*(r')} \frac{\delta \mathcal{H}}{\delta a_j(r')} \right], \\
\frac{\delta \mathcal{H}}{\delta a_j(r')} &= \sum_m \int dr'' \left[\frac{\delta \mathcal{H}}{\delta b_m(r'')} \frac{\delta b_m(r'')}{\delta a_j(r')} + \frac{\delta \mathcal{H}}{\delta b_m^*(r'')} \frac{\delta b_m^*(r'')}{\delta a_j(r')} \right], \\
i\frac{\partial b_l}{\partial t} &= \sum_j \int dr' \left[\frac{\delta \mathcal{H}}{\delta b_m^*(r')} \{b_l(r) b_m^*(r')\}_{aa^*} \right. \\
&\quad \left. + \frac{\delta \mathcal{H}}{\delta b_m(r')} \{b_l(r) b_m(r')\}_{aa^*} \right] .
\end{aligned} \tag{A2.2}$$

Here we have made use of the definition

$$\{f(r)g(r')\}_{aa^*} = \sum_m \int dr'' \left[\frac{\delta f(r)}{\delta a_m(r'')} \frac{\delta g(r')}{\delta a_m^*(r'')} - \frac{\delta f(r)}{\delta a_m^*(r'')} \frac{\delta g(r')}{\delta a_m(r'')} \right] \qquad (A2.3)$$

of Poisson brackets. Equations (A2.2) are to have the canonical form (1.1.6) if the following conditions are satisfied

$$\{b_l, b_j\} = 0, \quad \{b_l(r) b_j^*(r')\} = \delta_{lj}\delta(r - r') , \qquad (A2.4)$$

which are classical analogues of commutation equations for Bose operators.

The canonicity conditions for transformations of Fourier images $b(k,t)$ and $b^*(k,t)$ have the same form. Thus, for the linear $u - v$ transformation (1.1.16) diagonalizing the quadratic part of the Hamiltonian

$$b_j(k) = \sum_l \left[u_{jl}(k) a_l(k) + v_{jl}(k) a_l^*(-k) \right],$$

we obtain from (A2.4) the canonicity conditions

$$\begin{aligned} \sum_l \left[u_{jl}(k) u_{ml}^*(k) - v_{jl}(k) v_{ml}^*(k) \right] &= \delta_{jm}, \\ \sum_l \left[u_{jl}(k) v_{ml}(-k) - v_{jl}(k) u_{ml}(-k) \right] &= 0. \end{aligned} \qquad (A2.5)$$

For quasi-linear transformations of the type

$$\begin{aligned} b(k) = a(k) + \int \big[& A(k, k_1, k_2) a(k_1) a(k_2) + B(k, k_1, k_2) a^*(k_1) a(k_2) \\ & + C(k, k_1, k_2) a^*(k_1) a^*(k_2) \big] \, dk_1 dk_2 \, , \end{aligned}$$

used in Sects. 1.1,2 to eliminate the nonresonant three-wave processes, the canonicity conditions are also obtained using the Poisson brackets (A2.4) with an accuracy to next order terms. They have the form [see also (A3.4)]:

$$\begin{aligned} B(k, k_1, k_2) &= B(k_1, k, k_2) = -2A^*(k_2, k_1, k) \, , \\ C(k, k_1, k_2) &= C(k_1, k, k_2) = C(k, k_2, k_1) \, . \end{aligned} \qquad (A2.6)$$

2. The canonical transformations may be formulated in terms of generating functionals, which are the continuous analogues of the generating functions of finite-dimensional systems. It is known that the Hamilton equations of motion may be obtained from an extremal action principle of the form

$$\delta \int dt \left[\int p(r,t) \frac{\partial q(r,t)}{\partial t} \, dr - \mathcal{H}\{p, q\} \right] = 0. \qquad (A2.7)$$

In this equation, all coordinates and momenta should be varied independently. The equation in the new variables P and Q has a canonical form if the principle

$$\delta \int dt \left[\int P(r,t) \frac{\partial Q(r,t)}{\partial t} dr - \mathcal{H}\{P,Q\} \right] = 0 \tag{A2.8}$$

is satisfied.

These two principles will be equivalent if the (sub)integrands differ only by a total variation (the analogue of the total differential) of an arbitrary functional Φ of coordinates, momenta and time

$$\delta\Phi = \int p(r)\delta q(r)\, dr - \int P(r)\delta Q(r)\, dr + (\mathcal{H}' - \mathcal{H})dt. \tag{A2.9}$$

Hence we obtain

$$p = \frac{\delta\Phi}{\delta q}, \quad P = -\frac{\delta\Phi}{\delta Q}, \quad \mathcal{H}' = \mathcal{H} + \frac{\partial\Phi}{\partial t}, \tag{A2.10}$$

specifying (at a given $\Phi\{q, Q, t\}$) the relation between the old and new variables and the new Hamiltonian.

In Sect. 1.2.4, we needed the generating functional in the Q, p-variables. To derive the transformation formulas for this case, one should subject (A2.9) to a Legendre transformation

$$\begin{aligned} \delta\left(\Phi + \int q(r)p(r)\, dr\right) = &\int q(r)\delta p(r)\, dr \\ &- \int P(r)\delta Q(r)\, dr + (\mathcal{H}' - \mathcal{H})\, dt\ . \end{aligned} \tag{A2.11}$$

The new generating functional is thus equal to $F(p, Q, t) = \Phi + pq$, and

$$q = \frac{\delta F}{\delta p}, \quad P = -\frac{\delta F}{\delta Q}, \quad \mathcal{H}' = \mathcal{H} + \frac{\partial F}{\partial t}\ . \tag{A2.12}$$

It should be noted that usually the condition (A2.9) or (A2.11) is taken as a definition of the canonicity of the transformation $(p, q) \to (P, Q)$. Yet, it should be borne in mind that the canonical form of the equations of motion is retained by a wider class of transformations, for example, those in which the Hamiltonian is multiplied by an arbitrary constant.

A.3 Elimination of Nonresonant Terms from the Interaction Hamiltonian

We shall consider the nondecay case to show how a transformation may be used to eliminate three-wave and nonresonant four-wave processes from the interaction Hamiltonian. Let us seek the transformation in the form of a power series. Since expansion of the Hamiltonian starts with a quadratic term and ends (for us) with fourth-order terms, the transformation should contain linear, quadratic and cubic terms.

Let us demonstrate a simple method for computing the coefficients of a power series transformation, which is by its derivation canonical. That method is based on the fact that a hamiltonian system possesses at all times hamiltonian properties. Therefore, the transformation $c(k,0) \to c(k,t)$ is canonical. Let us consider an auxiliary Hamiltonian in the standard form (1.1.24)

$$\begin{aligned}\tilde{\mathcal{H}} = & \frac{1}{2}\int \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\big(\tilde{V}_{123}c_1^*c_2c_3 + \text{c.c.}\big)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \\ & + \frac{1}{6}\int \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\big(\tilde{U}_{123}c_1^*c_2^*c_3^* + \text{c.c}\big)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \\ & + \frac{1}{4}\int \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\tilde{W}_{1234}c_1^*c_2^*c_3c_4\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 \\ & + \int \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\big(\tilde{G}_{1234}c_1c_2^*c_3^*c_4^* + \text{c.c.}\big)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 \\ & + \int \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 + \boldsymbol{k}_4)\big(\tilde{R}_{1234}c_1^*c_2^*c_3^*c_4^* + \text{c.c.}\big)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 .\end{aligned} \tag{A3.1}$$

Here c.c. denotes the complex conjugate of the preceding term.

Using a Taylor series we can express the old variables $b(k,t) = c(k,t)$ in terms of $c(k,0)$

$$b(k,t) = c(k,0) + t\left(\frac{\partial c(k,t)}{\partial t}\right)_{t=0} + \frac{t^2}{2}\left(\frac{\partial^2 c(k,t)}{\partial t^2}\right)_{t=0} + \dots \,. \tag{A3.2}$$

According to (A3.1):

$$\begin{aligned}\left(\frac{\partial c(k,t)}{\partial t}\right)_{t=0} &= -i\frac{\delta\mathcal{H}\{c(k,0), c^*(k,0)\}}{\delta c^*(k,0)} , \\ \left(\frac{\partial^2 c(k,t)}{\partial t^2}\right)_{t=0} &= -i\frac{\partial}{\partial t}\frac{\delta\mathcal{H}}{\delta c^*} .\end{aligned} \tag{A3.3}$$

Substituting for $\partial c/\partial t$ and $\partial^2 c/\partial t^2$ and setting, for example, $t = 1$ (the other choice of t just corresponds to the redefinition of the transformation coefficients) we get the general form of the canonical transformations in terms of a power series:

$$
\begin{aligned}
b(k) = c(k) &- \frac{i}{2}\int \Big[\tilde V_{k12}\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)c_1c_2 \\
&+ 2\tilde V^*_{1k2}\delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2)c_1c_2^* + \tilde U_{k12}\delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2)c_1^*c_2^*\Big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 \\
&+ \int \Big[\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)c_1c_2c_3\big(-\tilde G^*_{k123} - \frac{1}{4}\tilde V_{k1k-1}\tilde V_{2+323} \\
&+ \frac{1}{4}\tilde V^*_{1k1-k}\tilde U_{-2-323}\big) + \delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3)c_1^*c_2^*c_3\big(-3i\tilde G_{3k12} \\
&+ \frac{1}{4}\tilde V_{k3k-3}\tilde U_{-2-121} + \frac{1}{4}\tilde V^*_{3k3-k}\tilde V^*_{2+323} - \frac{1}{2}\tilde V^*_{323-2}\tilde V^*_{1k1-k} \qquad (A3.4)\\
&+ \frac{1}{2}\tilde V_{131-3}\tilde U_{-k-2k2}\big) + \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)c_1^*c_2c_3\big(-\frac{i}{2}\tilde W_{k123} \\
&- \frac{1}{2}\tilde V_{k2k-2}\tilde V^*_{313-1} - \frac{1}{4}\tilde V^*_{1+k1k}\tilde V_{2+323} + \frac{1}{2}\tilde V_{131-3}\tilde V^*_{2k2-k} \\
&+ \frac{1}{2}\tilde U_{-k-1k1}\tilde U_{-2-323}\big) + \delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)c_1^*c_2^*c_3^*\big(-4i\tilde R_{k123} \\
&- \frac{1}{4}\tilde V^*_{1+k1k}\tilde U_{-2-323} + \frac{1}{4}\tilde V^*_{2+323}\tilde U_{-k-1k1}\big)\Big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \ .
\end{aligned}
$$

In our shorthand notation we replace the k_j-arguments by the j-indices.

The transformation is seen to contain seven different terms made up of the five arbitrary functions

$$
\tilde V(k,k_1,k_2),\ \tilde U(k,k_1,k_2),\ \tilde G(k,k_1,k_2,k_3),\ \tilde R(k,k_1,k_2,k_3),\ \tilde W(k,k_1,k_2,k_3)\ .
$$

These functions have also to satisfy the usual symmetry conditions (1.1.25) for Hamiltonian coefficients. The canonicity condition (A2.6) is satisfied identically.

Let us now substitute (A3.4) into the Hamiltonian (1.1.24). The resulting Hamiltonian has the same form (1.1.24) but other coefficients. Demanding the coefficients of the cubic terms to be equal to zero, we obtain

$$
i\tilde V_{k12} = \frac{V_{k12}}{\omega_k-\omega_1-\omega_2}, \qquad i\tilde U_{k12} = -\frac{U_{k12}}{2(\omega_k+\omega_1+\omega_2)}. \qquad (A3.5)
$$

These are exactly the coefficients ($2A_1 = -i\tilde V$, $A_3 = -i\tilde U$) given in (1.1.28b) for $\omega_k = \omega_1 = \omega_2 = \omega_3 = \omega$.

The fact that the fourth-order terms with $c_1c_2c_3c_k$ and $c_k^*c_1c_2c_3$ vanish allows one to obtain two more transformation coefficients. After proper symmetrization, we find

$$
\begin{aligned}
i\tilde R_{k123} &= \frac{R_{k123} + \frac{i}{24}\big(U_{-k-iki}\tilde V_{j+ljl} + V_{k+iki}\tilde U_{-j-ljl}\big)}{\omega_k+\omega_1+\omega_2+\omega_3} \\
&\quad + \frac{1}{48}\big(\tilde V_{j+ljl}\tilde U_{-k-iki} - \tilde V_{k+iki}\tilde U_{-j-ljl}\big), \\
i\tilde G_{k123} &= \frac{G_{k123} + \frac{i}{6}\big(V_{kik-i}\tilde V_{j+ljl} + V_{iki-k}\tilde U_{-l-jlj}\big)}{\omega_k-\omega_1-\omega_2-\omega_3} \qquad (A3.6)\\
&\quad + \frac{1}{12}\big(\tilde V^*_{iki-k}\tilde U_{ljl-j} - \tilde V_{kik-i}\tilde V_{l+jlj}\big)\ .
\end{aligned}
$$

Here summation over the divergent values of i, j, l indices running over the numbers $1, 2, 3$ is implied.

Thus, since the denominators in (A3.5–6) do not vanish, the respective terms may be excluded from the Hamiltonian and the corresponding transformation coefficients may be obtained. In the new variables the rest of the interaction Hamiltonian has the form

$$\begin{aligned}\mathcal{H}_{int} = \mathcal{H}_4 = \frac{1}{4}\int \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4) c_1^* c_2^* c_3 c_4 \big[W_{1234} + T_{1234} \\ + (\omega_1 + \omega_2 - \omega_3 - \omega_4)\big(i\tilde{W}_{1234} + \frac{1}{4}\tilde{V}^*_{pip-i}\tilde{V}_{jqj-q} \\ - \frac{1}{4}\tilde{V}^*_{qjq-j}\tilde{V}_{ipi-p}\big)\big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 \ ,\end{aligned} \tag{A3.7}$$

where

$$\begin{aligned}2T_{1234} = &-\frac{U_{-3-434}U_{-1-212}}{\omega_3 + \omega_4 + \omega_{3+4}} - \frac{U_{-3-434}U_{-1-212}}{\omega_1 + \omega_2 + \omega_{1+2}} \\ &- \frac{V^*_{1+212}V_{3+434}}{\omega_{1+2} - \omega_1 - \omega_2} - \frac{V^*_{1+212}V_{3+434}}{\omega_{3+4} - \omega_3 - \omega_4} - \frac{V^*_{ipi-p}V_{qjq-j}}{\omega_{q-j} + \omega_j - \omega_q}\end{aligned}$$

and indices i, j run over the (divergent) numbers $1, 2$, indices p, q over the $3, 4$. The renormalized interaction coefficient satisfies the same symmetry conditions (1.1.25) as W_{1234} and $\tilde{W}_{1234}$.

Since we always can find $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_4$ satisfying

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4 \quad \text{and} \quad \omega_1 + \omega_2 = \omega_3 + \omega_4, \tag{A3.8}$$

it is impossible to eliminate the term (A3.7) in all $\boldsymbol{k}$-space. That particular Hamiltonian describes scattering processes allowed for all wave systems.

Formula (1.1.29b) suffices to consider weak turbulence of wave interaction with wave vectors lying only on the resonance surface (A3.8) [where (A3.7) coincides with (1.1.29)]. But if intrinsically nonlinear phenomena are discussed (for example, when using truncated equations for the description of water waves [A.1]), the use of (1.1.29b) at $\omega_1 + \omega_2 \neq \omega_3 + \omega_4$ implies besides other errors that the energy is not conserved by the equations [A.2].

The arbitrary function $\tilde{W}$ may be chosen as desired, it has only to satisfy the symmetry conditions (1.1.25). Varying $\tilde{W}$ we simultaneously vary c_k, leaving b_k constant.

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