Lagrangian Description of Transport in Turbulence

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Abstract

The description of scalar and vector fields transport by a random flow hinges on methods of statistical mechanics applied to the motion of fluid particles, i.e. to the Lagrangian dynamics. We first present the propagators describing evolving probability distributions of different configurations of fluid particles. We then use those propagators to describe growth, decay and steady states of different scalar and vector quantities transported by random flows. We discuss both practical questions like mixing and segregation and fundamental problems like symmetry breaking in turbulence.

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In the absence of viscosity ... the velocity field does not remain differentiable! The ordinary formulation of the laws of motion in terms of differential equations becomes inadequate and must be replaced by a more general description ...

L. Onsager, May 20, 1949

I. INTRODUCTION

This lecture course is an abridged and updated version of the review [1]. Here, we try to use intuitive arguments whenever possible and avoid most of the technicalities (but not all). Readers interested in a more rigorous and detailed presentation are encouraged to turn to the review [1].

The subject of the course is the combined effect of molecular diffusion and random flow on scalar and vector fields transported by a fluid. We want to understand first when there is mixing and when, on the contrary, inhomogeneities are created and enhanced. We want to distinguish between cases when flow create small-scale inhomogeneities of the transported fields which are then killed by molecular diffusion and cases when large-scale structures of the fields appear. Our goal is to describe both temporal and spatial statistical properties of transported fields.

A. Propagators

If we wish to describe the statistics of different fields transported by the flow we need a formalism to describe the probabilities of different flow trajectories. Consider the evolution of a passive scalar tracer $\theta(\mathbf{r},t)$ in a random flow. The mean value of the scalar tracer at a given point is an average over values brought by different trajectories:

$$\Big\langle \theta(\mathbf{r},s)\Big\rangle = \int \mathcal{P}(\mathbf{r},s;\mathbf{R},0)\,\theta(\mathbf{R},0)\,d\mathbf{R}\,, \tag{1}$$

Here, $\mathcal{P}(\mathbf{r},s;\mathbf{R},t)$ is the probability density function (PDF) to find the particle at time t at position $\mathbf{R}$ given its position $\mathbf{r}$ at time s. That PDF is called the propagator or the Green function. Multi-point correlation functions of the tracer

$$C_N(\underline{\mathbf{r}},s) \equiv \Big\langle \theta(\mathbf{r}_1,s)\,\ldots\,\theta(\mathbf{r}_N,s)\Big\rangle = \int \mathcal{P}_N(\underline{\mathbf{r}},s;\underline{\mathbf{R}},0)\;\theta(\mathbf{R}_1,0)\,\ldots\,\theta(\mathbf{R}_N,0)\;d\underline{\mathbf{R}} \tag{2}$$

are expressed via the multi-particle Green functions $\mathcal{P}_{_N}$ which are the joint PDF's of the equal-time positions $\underline{\mathbf{R}} = (\mathbf{R}_{_1}, \ldots, \mathbf{R}_{_N})$ of N fluid trajectories. The next Chapter is devoted to the analysis of the one-, two- and multi-particle Green functions. The results of Chapter II are used then in the subsequent Chapters III and IV for the description of the transported passive fields. Chapter V describes active tracers which influence the velocity that transports them.

The trajectory of the fluid particle that passes at time s through the point $\mathbf{r}$ is described by the vector $\mathbf{R}(t;\mathbf{r},s)$ which satisfies $\mathbf{R}(t;\mathbf{r},t) = \mathbf{r}$ and the stochastic equation [2]

$$\dot{\mathbf{R}} = \mathbf{v}(\mathbf{R},t) + \mathbf{u}(t) \ . \tag{3}$$

Here, $\mathbf{u}(t)$ describes the molecular Brownian motion, it has zero average and covariance function $\langle u^i(t)\, u^j(t')\rangle = 2\kappa\delta^{ij}\delta(t-t')$. We shall also consider the macroscopic velocity $\mathbf{v}$ as random with different statistical properties and different dependencies on space and time in different cases. The molecular diffusivity κ is of order $10^{-1}cm^2s^{-1}$ for gases in gases so it would take many hours for a smell to diffuse across the dinner table. Similarly, to diffuse salt a kilometer depth of the ocean molecular diffusion would take 10^7 years. It is the motion of fluids that provides large-scale transport and mixing in most cases. There is a clear scale separation between the macroscopic velocity $\mathbf{v}$ and the molecular diffusion $\mathbf{u}$ which allows one to treat them separately.

Using (3) one can write the Green function as an integral over paths that satisfy $\mathbf{q}(s) = \mathbf{r}$ and $\mathbf{q}(t) = \mathbf{R}$ (see, e.g. [1, 3]):

$$\begin{aligned}
\mathcal{P}(\mathbf{r},s;\mathbf{R},t) &= \left\langle \int \exp\left\{ -\int_s^t \imath\mathbf{p}(\tau)\cdot\left[\dot{\mathbf{q}}(\tau) - \mathbf{v}(\mathbf{q}(\tau),\tau) - \mathbf{u}(\tau)\right] d\tau \right\} \mathcal{D}\mathbf{p}\,\mathcal{D}\mathbf{q} \right\rangle_{\mathbf{v},\mathbf{u}} && (4)\\
&= \left\langle \int \exp\left\{ -\int_s^t \left[\imath\mathbf{p}(\tau)\cdot\Big(\dot{\mathbf{q}}(\tau) - \mathbf{v}(\mathbf{q}(\tau),\tau)\Big) + \kappa\mathbf{p}^2(\tau)\right] d\tau \right\} \mathcal{D}\mathbf{p}\,\mathcal{D}\mathbf{q} \right\rangle_{\mathbf{v}} && (5)\\
&= \left\langle \int \exp\left\{ -\frac{1}{4\kappa}\int_s^t \left[\dot{\mathbf{q}}(\tau) - \mathbf{v}(\mathbf{q}(\tau),\tau)\right]^2 d\tau \right\} \mathcal{D}\mathbf{q} \right\rangle_{\mathbf{v}} = \langle P(\mathbf{r},s;\mathbf{R},t|\mathbf{v})\rangle_{\mathbf{v}} \ . && (6)
\end{aligned}$$

The integration over the auxiliary field $\mathbf{p}$ in (4) enforces the delta function of (3). One passes from (4) to (5) by averaging over the Brownian noise, and from (5) to (6) by calculating the Gaussian integral over $\mathbf{p}$.

It is sometimes useful to consider only the partial average over the molecular diffusion, then the tracer satisfies the advection-diffusion equation:

$$\frac{\partial\theta}{\partial t} + (\mathbf{v}\cdot\nabla)\theta - \kappa\nabla^2\theta = 0 \ . \tag{7}$$

The solution can be expressed via the $\mathbf{v}$-dependent propagator $P(\mathbf{r}, s; \mathbf{R}, t \,|\, \mathbf{v})$ defined by (6). It satisfies the initial condition $P(\mathbf{r}, t; \mathbf{R}, t \,|\, \mathbf{v}) = \delta(\mathbf{R} - \mathbf{r})$ and the equation

$$\left[\partial_t - \nabla_{\mathbf{R}} \cdot \mathbf{v}(\mathbf{R}, t) - \kappa \nabla^2_{\mathbf{R}}\right] P(\mathbf{r}, s; \mathbf{R}, t|\mathbf{v}) \;=\; 0, \tag{8}$$

for $s > t$. For a regular velocity with deterministic trajectories, one has at $\kappa = 0$

$$P(\mathbf{r}, s; \mathbf{R}, t|\mathbf{v}) = \delta\Big(\mathbf{R} - \mathbf{R}(t; \mathbf{r}, s)\Big)\,. \tag{9}$$

We shall see below that, even when the velocity field is not regular and the notion of a single Lagrangian trajectory does not make sense, the propagators are well defined.

B. Kraichnan model

Generally, exact calculations are only possible for Gaussian random processes delta-correlated in time like in (5). The simplest case is the Brownian motion when the advection is absent. One then obtains from (6) the Gaussian PDF of the displacement,

$$\mathcal{P}(\mathbf{R}, t) = (4\pi\kappa t)^{-d/2} \mathrm{e}^{-R^2/(4\kappa t)}\,, \tag{10}$$

which satisfies the heat equation $(\partial_t - \kappa\nabla^2)\mathcal{P}(\mathbf{r}, t) = 0$. The short-correlated case is far from being an exotic exception but rather presents a long-time limit of an integral of any finite-correlated random function. Indeed, such an integral can be presented as a sum of many independent equally distributed random numbers, the statistics of such sums is a subject of the Central Limit Theorem. For the long-time description of the advection in finite-correlated flows, it is useful to consider the extreme case of random homogeneous and stationary velocities with a very short correlation time. This case may be regarded as describing the sped-up-film view of velocity fields with temporal decay of correlations or, more formally, as the scaling limit $\lim\limits_{\mu\to\infty} \mu^{\frac{1}{2}} \mathbf{v}(\mathbf{r}, \mu t)$. When $\mu \to \infty$ one gets a Gaussian velocity field with the 2-point function

$$\langle v_i(\mathbf{r}, t)\, v_j(\mathbf{r}', t')\rangle = 2\,\delta(t - t') D_{ij}(\mathbf{r} - \mathbf{r}'). \tag{11}$$

It is common to call the Gaussian ensemble with a white-noise 2-point function (11) the Kraichnan ensemble [4]. For the Kraichnan velocities $\mathbf{v}$, the Lagrangian velocity $\mathbf{v}(\mathbf{R}, t)$ has the same white noise temporal statistics as the Eulerian velocity $\mathbf{v}(\mathbf{r}, t)$ for fixed $\mathbf{r}$ and the displacement along a Lagrangian trajectory $\mathbf{R}(t)-\mathbf{R}(0)$ is a Brownian motion for all times. In exactly the same way as one derives (6,10) from (4) one gets

$$\mathcal{P}(\mathbf{R},t) = |\hat\beta|^{1/2}(4\pi t)^{-d/2}\mathrm{e}^{-\beta_{ij}R_iR_j/4t}\,, \tag{12}$$

where $(\hat\beta^{-1})_{ij} = D_{ij}(\mathbf{0}) + \kappa\delta_{ij}$. We shall see that the Kraichnan ensemble of velocities constitutes an important theoretical laboratory of the particle behavior in fluid turbulence.

C. Large Deviation Approach

One can move beyond the consideration of the previous section considering the correlation time finite (yet small comparing to the time of evolution). Such generalization is the subject of the Large Deviation Theory. Let us present here the basic idea which will be used extensively in this course. Consider some quantity X which is an integral of some random function over time t much larger than the correlation time τ. At $t \gg \tau$, X behaves as a sum of many independent equally distributed random numbers y_i: $X = \sum_1^N y_i$ with $N \propto t/\tau$. The generating function $\langle \mathrm{e}^{zX}\rangle$ of the moments of X is the product, $\langle \mathrm{e}^{zX}\rangle = \mathrm{e}^{NS(z)}$, where we have denoted $\langle \mathrm{e}^{zy}\rangle \equiv \mathrm{e}^{S(z)}$ (assuming that the generating function $\langle \mathrm{e}^{zy}\rangle$ exists for all complex z). The PDF $\mathcal{P}(X)$ is given by the inverse Laplace transform $(2\pi i)^{-1}\int \mathrm{e}^{-\,z\,X+NS(z)}\,dz$ with the integral over any axis parallel to the imaginary one. For $X \propto N$, the integral is dominated by the saddle point z_0 such that $S'(z_0) = X/N$ and

$$\mathcal{P}(X) \;\propto\; \mathrm{e}^{-NH(X/N-\langle y\rangle)}\,. \tag{13}$$

Here $H = -S(z_0) + z_0S'(z_0)$ is the function of the variable $X/N - \langle y\rangle$, it is called entropy function as it appears also in the thermodynamic limit in statistical physics [5]. A few important properties of H (also called rate or Cramér function) may be established independently of the distribution $\mathcal{P}(y)$. It is a convex function which takes its minimum at zero, i.e. for X equal to its mean value $\langle X\rangle = NS'(0) = N\langle y\rangle$ which grows linearly with N. The minimal value of H vanishes since $S(0) = 0$. The entropy is quadratic around its minimum with $H''(0) = \Delta^{-1}$, where $\Delta = S''(0)$ is the variance of y. The fluctuations $X - \langle X\rangle$ on the scale $\mathcal{O}(N^{1/2})$ are governed by the Central Limit Theorem that states that $(X - \langle X\rangle)/N^{1/2}$ becomes for large N a Gaussian random variable with variance $\langle y^2\rangle - \langle y\rangle^2 \equiv \Delta$ as in (10,12). Finally, its fluctuations on the larger scale $\mathcal{O}(N)$ are governed by the large deviation form (13). The possible non-Gaussianity of the y's leads to a non-quadratic behavior of H for (large) deviations from the mean, starting from $X - \langle X\rangle/N \simeq \Delta/S'''(0)$. Note that if y is Gaussian then X is Gaussian too for any t but the universal formula (13) with $H = (X - N\langle y\rangle)^2/2N\Delta$ is valid only for $t \gg \tau$.

II. PARTICLES IN FLUID TURBULENCE

As explained in the Introduction, understanding the properties of transported fields involves the analysis of the behavior of fluid particles. We present here the results on the time-dependent statistics of the Lagrangian trajectories $\mathbf{R}_n(t)$. In this Chapter, we sequentially increase the number of particles involved in the problem. We start from a single trajectory whose effective motion is a simple diffusion at times longer than the velocity correlation time in the Lagrangian frame (Sect. II A). We then move to two particles. The separation law of two close trajectories depends on the scaling properties of the velocity field $\mathbf{v}(\mathbf{r}, t)$. If the velocity is smooth, that is $|\mathbf{v}(\mathbf{R}_n) - \mathbf{v}(\mathbf{R}_m)| \propto |\mathbf{R}_n - \mathbf{R}_m|$, then the initial separation grows exponentially in time (Sect. II B). The smooth case can be analyzed in much detail using the large deviation arguments presented in Sect. I C. The reader mainly interested in applications to transported fields might wish to take the final results (23) and (28) for granted, skipping their derivation. If the velocity is non-smooth, that is $|\mathbf{v}(\mathbf{R}_n) - \mathbf{v}(\mathbf{R}_m)| \propto |\mathbf{R}_n - \mathbf{R}_m|^\alpha$ with $\alpha < 1$, then the separation distance between two trajectories grows as a power of time (Sect. II C). We discuss important implications of such a behavior for the nature of the Lagrangian dynamics. The difference between the incompressible flows, where the trajectories generally separate, and compressible ones, where they may cluster, is discussed in Sect. II D. Finally, in the consideration of three or more trajectories, the new issue of geometry appears. Statistical conservation laws come to light in two-particle problem and then feature prominently in the consideration of multi-particle configurations. Geometry and statistical conservation laws are the main subject of Sect. II E. Although we try to keep the discussion as general as possible, much of the insight into the trajectory dynamics for the non-smooth case is obtained by studying the Kraichnan model.

A. Single-particle diffusion

We now consider the single Lagrangian trajectory $\mathbf{R}(t)$. For the pure advection without noise, the displacement $\mathbf{R}(t) - \mathbf{R}(0) = \int_0^t \mathbf{V}(s)\, ds$, with $\mathbf{V}(t) = \mathbf{v}(\mathbf{R}(t), t)$ being the Lagrangian velocity. The properties of the displacement depend on the specific trajectory under consideration. We shall always work in the frame of reference with no mean flow: $\langle \mathbf{v} \rangle = 0$. We assume statistical homogeneity of the Eulerian velocities which implies that the stochastic process $\mathbf{V}(t)$ does not depend on the initial position $\mathbf{R}(0)$ of the trajectory. If, additionally, the Eulerian velocities are statistically stationary, then so is $\mathbf{V}(t)$. This follows by averaging the expectations involving $V(t+\tau)$ over the initial position $\mathbf{R}(0)$ (which they do not depend on) and the change of variables $\mathbf{R}(0) \mapsto \mathbf{R}(\tau)$ under the velocity ensemble average. Note that the Jacobian of the change of variables is supposed to be unity which requires incompressibility. For $\kappa = 0$, the mean square displacement satisfies the equation:

$$\frac{d}{dt} \langle [\mathbf{R}(t) - \mathbf{R}(0)]^2 \rangle = 2 \int_0^t \langle \mathbf{V}(0) \cdot \mathbf{V}(s) \rangle ds. \tag{14}$$

The behavior of the displacement is crucially dependent on the range of temporal correlations of the Lagrangian velocity. Let us define the correlation time τ of $\mathbf{V}(t)$ by

$$\int_0^\infty \langle \mathbf{V}(0) \cdot \mathbf{V}(s) \rangle ds = \langle \mathbf{v}^2 \rangle \tau. \tag{15}$$

The value of τ provides a measure of the Lagrangian velocity memory, its divergence being symptomatic of persistent correlations. No general relation between the Eulerian and the Lagrangian correlation times has been established, except for the case of short-correlated velocities. For times $t \ll \tau$, the 2-point function in (14) is approximately equal to $\langle \mathbf{V}(0)^2 \rangle = \langle \mathbf{v}^2 \rangle$. The fluid particle transport is then ballistic with $\langle [\mathbf{R}(t) - \mathbf{R}(0)]^2 \rangle \simeq \langle \mathbf{v}^2 \rangle t^2$ and the PDF $\mathcal{P}(\mathbf{R}, t)$ is determined by the whole single-time velocity PDF. When the correlation time of $\mathbf{V}(t)$ is finite (a generic situation in a turbulent flow where τ is of order of a large-scale turnover time) an effective diffusive regime is expected to arise for $t \gg \tau$ with $\langle (\mathbf{R}(t) - \mathbf{R}(0))^2 \rangle \simeq 2 \langle \mathbf{v}^2 \rangle \tau t$ [2]. Indeed, the particle displacements over time segments much larger than τ are almost independent. At long times, the displacement $\delta\mathbf{R}(t)$ behaves then as a sum of many independent variables and falls into the class of stationary processes treated in Sects. I B, I C. In other words, $\delta\mathbf{R}(t)$ for $t \gg \tau$ becomes a Brownian motion in d dimensions, normally distributed with $\left\langle \delta R^i(t) \delta R^j(t) \right\rangle \simeq D_e^{ij} t$, where the so-called eddy diffusivity tensor is as follows

$$D_e^{ij} = \frac{1}{2}\int_0^\infty \langle V_i(0)\, V_j(s) + V_j(0)\, V_i(s)\rangle\, ds\,. \tag{16}$$

The symmetric second order tensor D_e^{ij} is the only characteristics of the velocity which matters in this limit of $t \gg \tau$. The trace of the tensor is equal to $\langle \mathbf{v}^2\rangle\tau$, i.e. to the large-time value of the integral in (14), while its tensorial properties reflect the rotational symmetries of the advecting velocity field. If the latter is isotropic, the tensor reduces to a diagonal form characterized by a single scalar value D_e. The main problem of turbulent diffusion is to obtain the effective diffusivity tensor given the velocity field $\mathbf{v}$ and the value of the molecular diffusivity κ. A huge amount of work has been devoted to it, both from the applied and the mathematical point of view, and exhaustive reviews of the problem are available in the literature [6–9].

The other general issue in turbulent diffusion is the condition on the velocity $\mathbf{v}(\mathbf{r}, t)$ ensuring that the Lagrangian correlation time τ is finite and an effective diffusion regime is taking place for large enough times. A sufficient condition valid for $\kappa \neq 0$ and both static and time-dependent flow, is a finite vector potential variance $\langle \mathbf{A}^2\rangle$, where the three-dimensional incompressible velocity $\mathbf{v} = \nabla \times \mathbf{A}$ [10–12]. The correlation time is experimentally known to be finite in developed turbulence whereas both subdiffusion (due to particle trapping) and superdiffusion (due to infinite Lagrangian correlation time) are possible in low-Reynolds-number flows.

B. Two-particle dispersion in a spatially smooth velocity

Even when velocity $\mathbf{v}(\mathbf{R}, t)$ is a smooth function of the coordinates, Lagrangian dynamics can be quite complicated. Indeed, d ordinary differential equations $\dot{\mathbf{R}} = \mathbf{v}(\mathbf{R}, t)$ generally produce chaotic dynamics (for $d \geq 3$ already for steady flows and for $d = 2$ for time-dependent flows). It is thus natural that the tools for the description of what is called chaotic advection [13] are similar to those of the theory of dynamical systems. The description in this Section consistently exploits two simple ideas: to single out the variables that can be represented by the sum of a large number of independent random quantities and to separate variables that fluctuate on different timescales.

We are interested here in the distance $\mathbf{R}_{12} = \mathbf{R}_1 - \mathbf{R}_2$ between two fluid particles with trajectories $\mathbf{R}_i(t) = \mathbf{R}(t; \mathbf{r}_i)$ passing at $t = 0$ through points $\mathbf{r}_i$. In the absence of noise, the distance satisfies the equation

$$\dot{\mathbf{R}}_{12} \;=\; \mathbf{v}(\mathbf{R}_1, t) - \mathbf{v}(\mathbf{R}_2, t). \tag{17}$$

If the distance R_{12} is smaller than the viscous scale of turbulence then the velocity field can be considered smooth on such a scale and we may expand: $\mathbf{v(R_1, t) - v(R_2, t)} = \sigma(\mathbf{t}, \mathbf{R_1})\mathbf{R_{12}}$ introducing the strain matrix σ which is traceless due to incompressibility. As a function of its spatial argument, σ changes on a scale that is supposed to be much larger than R_{12}. Then, σ can be treated as independent of $\mathbf{R}_{12}$ which thus satisfies locally a linear ordinary differential equation (we omit subscripts replacing $\mathbf{R}_{12}$ by $\mathbf{R}$)

$$\dot{\mathbf{R}}(t) = \sigma(t)\mathbf{R}(t) \; . \tag{18}$$

This equation, with $R(0) = r$ and the strain treated as a given function of time may be explicitly solved for arbitrary $\sigma(t)$ only in the 1D case

$$\ln[R(t)/r] = \ln W(t) = \int_0^t \sigma(s)\, ds \equiv X \, , \tag{19}$$

expressing $W(t)$ as the exponential of the time-integrated strain. When t is much larger than the correlation time τ of the strain, the variable X is a sum of N independent equally distributed random numbers with $N = t/\tau$. Using (13) we get

$$\mathcal{P}(r; R, t) \;\propto\; \exp\left\{-tH\left[t^{-1}\ln(R/r) - \lambda\right]\right\} \, , \tag{20}$$

Here we denoted $\lambda = \langle X\rangle/t$ which is called the Lyapunov exponent and is the growth (or decay) rate of the inter-particle distance $R(t)$. The moments $\langle [R(t)]^p\rangle$ behave exponentially as $\exp[E(p)t]$. The convexity of the entropy function leads to the convexity of $E(p)$. This implies, in particular, that even for $\lambda = E'(0) < 0$, high-order moments of R may grow exponentially in time (see Sect. II D below).

In the multidimensional case, the behavior of the vector $\mathbf{R}$ is determined by the product of random matrices rather than just random numbers. Still, the main properties of the propagator (sufficient for most physical applications) can be established for an arbitrary strain. The basic idea is coming back to Lyapunov [14] and it found further development in the Multiplicative Ergodic Theorem of Oseledec [15]. Introduce the evolution matrix W such that $\mathbf{R}(t) = W(t)\mathbf{R}(0)$. The modulus R is expressed via the positive symmetric matrix W^TW. The main result states that in almost every realization of the strain, the matrix $t^{-1}\ln W^T W$ stabilizes at $t \to \infty$, i.e. its eigenvectors tend to d fixed orthonormal eigenvectors $\mathbf{f}_i$. To understand that intuitively, consider some fluid volume, say a sphere, which evolves into an elongated ellipsoid at later times. As time increases, the ellipsoid is more and more elongated and it is less and less likely that the hierarchy of the ellipsoid axes will change. The limiting eigenvalues

$$\lambda_i = \lim_{t\to\infty} t^{-1} \ln |W\mathbf{f}_i| \tag{21}$$

are called Lyapunov exponents. The major property of the Lyapunov exponents is that they are realization-independent if the flow is ergodic (that is spatial and temporal averages coincide). We arrange the exponents in non-increasing order.

The relation (21) tells that two fluid particles separated initially by $\mathbf{r}$ pointing into the direction $\mathbf{f}_i$ will separate (or converge) asymptotically as $\exp(\lambda_i t)$. The incompressibility constraints $\det(W) = 1$ and $\sum \lambda_i = 0$ imply that a positive Lyapunov exponent will exist whenever at least one of the exponents is nonzero. Consider indeed

$$E(n) = \lim_{t\to\infty} t^{-1} \ln\langle [R(t)/r]^n \rangle \ , \tag{22}$$

whose derivative at the origin gives the largest Lyapunov exponent λ_1. The function $E(n)$ obviously vanishes at the origin. Furthermore, $E(-d) = 0$, i.e. incompressibility and isotropy make that $\langle R^{-d}\rangle$ is time-independent as $t \to \infty$ [16, 17]. Negative moments of orders $n < -1$ are indeed dominated by the contribution of directions $\mathbf{R}(0)$ almost aligned to the eigenvectors $\mathbf{f}_2, \ldots \mathbf{f}_d$. At $n < 1-d$ the main contribution comes from a small subset of directions in a solid angle $\propto \exp(d\lambda_d t)$ around $\mathbf{f}_d$. It follows immediately that $\langle R^n\rangle \propto \exp[\lambda_d(d+n)t]$ and that $\langle R^{-d}\rangle$ is a statistical integral of motion. Apart from $n = 0, -d$, the convex function $E(n)$ cannot have other zeroes if it does not vanish identically. It follows that dE/dn at $n = 0$, and thus λ_1, is positive. A simple way to appreciate intuitively the existence of a positive Lyapunov exponent is to consider the saddle-point 2D flow $v_x = \lambda x, v_y = -\lambda y$ with the axes randomly rotating after time interval T. A vector initially at the angle ϕ with the x-axis will be stretched after time T if $\cos\phi \geq [1 + \exp(2\lambda T)]^{-1/2}$, i.e. the measure of the stretching directions is larger than 1/2 [17].

A major consequence of the existence of a positive Lyapunov exponent for any random incompressible flow is the exponential growth of the inter-particle distance $R(t)$. In a smooth flow, it is also possible to analyze the statistics of the set of vectors $\mathbf{R}(t)$ and to establish a multidimensional analog of (13) for the general case of a non-degenerate Lyapunov exponent spectrum. The idea is to reduce the d-dimensional problem to a set of d scalar problems for slowly fluctuating stretching variables excluding the fast fluctuating angular degrees of freedom. Consider the matrix $I(t) = W(t)W^T(t)$, representing the tensor of inertia of a fluid element like the above mentioned ellipsoid. The matrix is obtained by averaging $R^i(t)R^j(t)d/\ell^2$ over the initial vectors of length ℓ and $I(0) = 1$. Introducing the variables that describe stretching as the lengths of the ellipsoid axis $e^{2\rho_1}, \ldots e^{2\rho_d}$ one can deduce similarly to (13,20) the asymptotic PDF [1, 18]:

$$\begin{aligned} \mathcal{P}(\rho_1, \ldots, \rho_d; t) \;\propto\; & \exp\left[-t\, H(\rho_1/t - \lambda_1, \ldots, \rho_{d-1}/t - \lambda_{d-1})\right] \\ & \times\; \theta(\rho_1 - \rho_2) \ldots \theta(\rho_{d-1} - \rho_d)\, \delta(\rho_1 + \ldots + \rho_d)\,. \end{aligned} \tag{23}$$

The entropy function H depends on the details of the statistics of σ and has the same general properties as above: it is non-negative, convex and it vanishes at zero. In the δ-correlated case, H is everywhere quadratic as in Sect. I B:

$$H(\mathbf{x}) \propto d^{-1} \sum_{i=1}^{d} x_i^2\,, \qquad \lambda_i \propto d(d - 2i + 1)\,. \tag{24}$$

For a generic initial vector $\mathbf{r}$, the long-time asymptotics of $\ln(R/r)$ coincides with that of ρ_1 whose PDF also takes the large-deviation form (23) at large times. The quadratic expansion of the entropy near its minimum corresponds to the log-normal distribution for the distance between two particles

$$\mathcal{P}(r; R, t) \;\propto\; \exp\left\{-\left[\ln(R/r) - \lambda_1 t\right]^2 / (2t\Delta)\right\}\,, \tag{25}$$

with $r = R(0)$ and $\Delta = C_{11}$.

Molecular diffusion is incorporated into the above picture by replacing the differential equation (18) by its noisy version [both independent noises of two particles contribute, hence the change in the noise coefficient comparing to (3)]:

$$d\mathbf{R}(t) = \sigma(t)\mathbf{R}(t)\, dt + \sqrt{2}\, d\mathbf{q}(t)\,, \langle q^i(t) q^j(t')\rangle = 2\kappa\delta^{ij} \min(t, t')\,. \tag{26}$$

This is an inhomogeneous linear stochastic equation whose solution is easy to express via the matrix $W(t)$. The tensor of inertia of a fluid element $I^{ij}(t) = R^i(t)R^j(t)d/\ell^2$ is now averaged both over the initial vectors of length ℓ and the noise, thus obtaining [1, 18]:

$$I(t) \;=\; W(t)W(t)^T \;+\; \tfrac{4\kappa}{\ell^2 d} \int_0^t W(t)\left[W(s)^T W(s)\right]^{-1} W(t)^T\, ds\,. \tag{27}$$

The last term in (27) is essential for the directions corresponding to negative λ_i. The molecular noise will indeed start to affect the motion of the marked fluid volume when the respective dimension gets sufficiently small. If ℓ is the initial size, the required condition $\rho_i < -\rho_i^* = -\ln(\ell^2|\lambda_i|/\kappa)$ is typically met for times $t \simeq \rho_i^*/|\lambda_i|$. For longer times, the Brownian motion does not allow the respective ρ_i to decrease much below $-\rho_i^*$, while the negative λ_i prevents it from increasing. As a result, the corresponding ρ_i becomes a stationary random process with a mean of the order $-\rho_i^*$. The relaxation times to the stationary distribution are determined by $\tilde{\sigma}$, which is diffusion independent, and they are thus much smaller than t. On the other hand, the components ρ_j corresponding to non-negative Lyapunov exponents are the integrals over the whole evolution time t. Their values at time t are thus not sensitive to the latest period of evolution lasting of the order of the relaxation times for the contracting ρ_i. Fixing the values of ρ_j at times $t \gg \rho_i^*/|\lambda_i|$ will not affect the distribution of the contracting ρ_i and the whole PDF is thus factorized [1, 18–20]. For example, there are two positive and one negative Lyapunov exponents in 3D developed Navier-Stokes turbulence [21]. For times $t \gg \rho_3^*/\lambda_3$ we have then

$$\mathcal{P}(\rho_1, \rho_2, \rho_3, t) \;\propto\; \exp\left[-t\, H\left(\rho_1/t - \lambda_1, \rho_2/t - \lambda_2\right)\right]\; \mathcal{P}_{\rm st}(\rho_3)\,, \tag{28}$$

with the same function H as in (23) since ρ_3 is independent of ρ_1 and ρ_2. The account of the molecular noise violates the condition $\sum \rho_i = 0$ as fluid elements at scales smaller than $\sqrt{\kappa/|\lambda_3|}$ cannot be distinguished. To avoid misunderstanding, note that (28) does not mean that the fluid is getting compressible: the simple statement is that if one tries to follow any marked volume, the molecular diffusion makes this volume growing.

Note that we have implicitly assumed ℓ to be smaller than the viscous length $\eta = \sqrt{\nu/|\lambda_3|}$ but larger than the diffusion scale $\sqrt{\kappa/|\lambda_3|}$. Even though ν and κ are both due to molecular motion, their ratio widely varies depending on the type of material. The theory of this section is applicable for the materials having the Schmidt number ν/κ large.

The universal forms (23) and (28) for the two-particle dispersion are basically everything we need for physical applications. In the Chapters III and IV, we show that the most negative Lyapunov exponent determines the small-scale statistics of a passively advected scalar in a smooth incompressible flow. For other problems, the whole spectrum of exponents and even the form of the entropy functions are relevant.

Generally, the Lyapunov spectrum and the entropy function cannot be derived from a given statistics of σ except for few limiting cases. The case of a short-correlated strain allows for a complete solution. As far as finite-correlated strain is concerned, one can express analytically λ_1 and Δ via the correlators of σ only in two dimensions for a long-correlated strain and at large space dimensionality [1].

C. Two-particle dispersion in a non-smooth incompressible flow

We now assume the Reynolds number sufficiently high and study the separation between two trajectories in the inertial interval of scales $\eta \ll r \ll L$, where L denotes the integral scale at which the flow is induced and η is a viscous scale.

Let us describe first the usual phenomenology of two-particle dispersion. In the inertial interval, the velocity differences exhibit an approximate scaling. Let us assume $\delta\mathbf{v}(\mathbf{r},t)| \propto r^\alpha$, rewriting then the equation (17) for the distance between two particles as $\dot{\mathbf{R}} = \delta\mathbf{v}(\mathbf{R},\mathbf{t})$, we infer that $dR^2/dt = 2\mathbf{R}\cdot\delta\mathbf{v}(\mathbf{R},t) \propto R^{1+\alpha}$. For $\alpha < 1$, this is solved (ignoring the proportionality constant) by

$$R(t)^{1-\alpha} - R(0)^{1-\alpha} \;\propto\; t \tag{29}$$

For large t, $R(t) \propto t^{1/(1-\alpha)}$ with the dependence of the initial separation quickly wiped out.

Of course, for the random process $\mathbf{R}(t)$, relation (29) is of the mean field type and should pertain (if true) to the large-time behavior of the averages:

$$\langle R(t)^p \rangle \;\propto\; t^{p/(1-\alpha)} \tag{30}$$

for $p > 0$ implying their super-diffusive growth, faster than the diffusive one $\propto t^{p/2}$. The power-law scaling (30) may be amplified to the scaling behavior of the PDF of the inter-particle distance:

$$\mathcal{P}(R,t) \;=\; \lambda\mathcal{P}(\lambda R, \lambda^{1-\alpha}t)\ . \tag{31}$$

Possible deviations from a linear behavior in the order p of the exponents in (30) should be interpreted as a signal of multiscaling of the Lagrangian velocity $\Delta\mathbf{v}(\mathbf{R}(t),t) \equiv \Delta\mathbf{V}(t)$. The power-law growth (30) for $p = 2$ and $\alpha = 1/3$, i.e. $\langle R(t)^2 \propto t^3$, is the celebrated Richardson dispersion relation stating that

$$\frac{d}{dt}\langle R(t)^2\rangle \;\propto\; \langle R(t)^2\rangle^{2/3}. \tag{32}$$

The Richardson relation was the first quantitative phenomenological prediction in developed turbulence. It seems to be confirmed by experimental data [22, 23] and by the numerical simulations [24, 25]. The more general property of self-similarity (31) (with $\alpha = 1/3$) has been observed in the inverse cascade of two-dimensional turbulence [23]. It is likely that (32) is exact within the inverse cascade of 2d turbulence while it may be only approximately correct in 3d. It is important to remark that, even assuming the validity of the Richardson relation, it is impossible to establish general large-time properties of the PDF $\mathcal{P}(R;t)$ such as those for the single particle PDF in Sect. II A or for the distance between two particles in Sect. II B . The physical reason becomes clear looking at the Lagrangian velocity difference correlation time

$$\tau_t = \int_0^t \langle \delta\mathbf{V}(t)\cdot\delta\mathbf{V}(s)\rangle \, ds / \langle (\delta\mathbf{V})^2 \rangle. \tag{33}$$

The numerator coincides with $d\langle R^2\rangle/dt$ and is thus proportional to $\langle R^2\rangle^{2/3}$, while the denominator $\propto \langle R^2\rangle^{1/3}$. It follows that τ_t grows as $\langle R^2\rangle^{1/3} \propto t$, i.e. the random process $\delta\mathbf{V}(t)$ has a correlation time comparable with its whole span. The absence of decorrelation explains why the Central Limit Theorem and the large deviation theory cannot be applied. There is in fact no *a priori* reason to expect $\mathcal{P}(R;t)$ to be Gaussian with respect to a power of R either, although we shall see that this happens to be the case in the Kraichnan ensemble.

It is instructive to contrast the exponential growth (22) of the distance between the trajectories within the viscous range with the power-law growth (30) in the inertial range. In the viscous regime, the closer two trajectories are initially the more time is needed to effectively separate them. As a result, the infinitesimally close trajectories never separate and trajectories in a fixed realization of the velocity field are continuously labelled by the initial conditions. They depend, however, in a sensitive way on the latter due to the exponential magnification of small deviation of the initial point. This sensitive dependence is usually considered as the defining feature of the dynamical chaos. On the other hand, in the inertial range the trajectories separate in a finite time independent of their initial distance $R(0)$, provided that the latter is also in the inertial range. For very high Reynolds numbers, the viscous scale η is negligibly small (a fraction of a millimeter in the turbulent atmosphere) and setting it to zero (or equivalently, setting the Reynolds number to infinity) is an appropriate abstraction if we want to concentrate on the behavior of the fluid trajectories in the inertial range. In such a limit, however, the power law separation extends down to infinitesimal distances between the trajectories: the infinitesimally close trajectories still separate to a finite distance in a finite time. This points to a marked difference in the behavior of trajectories in comparison to that in the chaotic regime: developed turbulence and chaos are clearly different phenomena. This explosive separation of trajectories results in a breakdown of the deterministic Lagrangian flow in the limit $Re \to \infty$, a rather dramatic effect [26–28]. Indeed, in this limit the trajectories cannot be labelled by the initial conditions. The sheer existence of the Lagrangian trajectories $\mathbf{R}(t;\mathbf{r})$ depending continuously on the initial position $\mathbf{r}$ would imply that $\lim_{\mathbf{r}_1 \to \mathbf{r}_2} \langle |\mathbf{R}(t;\mathbf{r}_1) - \mathbf{R}(t;\mathbf{r}_2)|^p \rangle = 0$ and contradict the persistence of a power law separation of the Richardson type for infinitesimally close trajectories. The breakdown of the deterministic Lagrangian flow at $Re \to \infty$ agrees with the fundamental theorem stating that the ordinary differential equation $\dot{\mathbf{R}} = \mathbf{v}(\mathbf{R}, t)$ has unique solution if $\mathbf{v}(\mathbf{r}, t)$ is Lipschitz in $\mathbf{r}$, i.e. if $|\delta \mathbf{v}(\mathbf{r}, t)| \leq \mathcal{O}(r)$. At $Re = \infty$, however, as first noticed by Onsager [29], the velocities are only Hölder continuous: $|\delta \mathbf{v}(\mathbf{r}, t)| \simeq \mathcal{O}(r^\alpha)$ with the exponent $\alpha < 1$ ($\alpha \simeq 1/3$ in Kolmogorov's phenomenology). As is shown by the example of the equation $\dot{x} = |x|^\alpha$ with two solutions $x = [(1-\alpha)t]^{\frac{1}{1-\alpha}}$ and $x = 0$ both starting at zero, one should expect multiple Lagrangian trajectories starting or ending at the same point for velocity fields with $\alpha < 1$. Does then the Lagrangian description of the fluid breaks down completely at $Re = \infty$?

Even though the deterministic Lagrangian description breaks down, a statistical description of the trajectories is still possible. As we have seen above, certain probabilistic questions concerning the flow, like the moments of the distance between initially close trajectories, should still have well defined answers in this limit. We expect that for typical velocity realization at $Re = \infty$, one can maintain a probabilistic description of Lagrangian trajectories and make sense of such objects as the propagator $P(\mathbf{r}, s; \mathbf{R}, t|\mathbf{v})$. The mathematical difference between the cases of smooth and rough velocities is that in the latter case the propagators are weak solutions of (8) rather than strong ones. What happens if we turn off molecular diffusion? If the velocity $\mathbf{v}(\mathbf{r}, t)$ is Lipschitz in $\mathbf{r}$ then $P(\mathbf{r}, s; \mathbf{R}, t|\mathbf{v})$ converges to (9) (we shall call this collapse property). It has been conjectured in [28] that for a generic $Re = \infty$ turbulent velocity field, $P(\mathbf{r}, s; \mathbf{R}, t|\mathbf{v})$ at $\kappa \to 0$ is a weak solution of the pure advection equation, $[\partial_t - \nabla_{\mathbf{R}} \cdot \mathbf{v}(\mathbf{R}, t)] P(\mathbf{r}, s; \mathbf{R}, t|\mathbf{v}) = 0$, that is a solution not concentrated at a single trajectory $\mathbf{R}(t; \mathbf{r}, s)$. This way the roughness of turbulent velocities resulting in the explosive separation of the Lagrangian trajectories would assure the persistence of stochasticity of the noisy trajectories in a fixed generic realization of the velocity field even in the limit $\kappa \to 0$. Let us stress again that, according to this claim, in the limit of large Reynolds numbers the Lagrangian trajectories behave stochastically already in a given velocity field and for negligible molecular diffusivity and not only due to a random noise or to random fluctuations of the velocities. This intrinsic stochasticity of fluid particles seems to constitute an important aspect of developed turbulence, an unescapable consequence of the Richardson dispersion law or of the Kolmogorov-like scaling of velocity differences in the limit $Re \to \infty$ and a natural mechanism assuring the energy flux constancy in the inertial interval of turbulence.

The general conjecture about the existence and diffuse nature of propagators is known to be true for the Kraichnan Gaussian ensemble (11) of velocities decorrelated in time. To model the non-smooth velocity field of turbulence, we choose $D^{ij}(\mathbf{r}) = D_0\delta^{ij} - (1/2)d^{ij}(\mathbf{r})$ with $D_0 = \mathcal{O}(L^\xi)$ and

$$d^{ij}(\mathbf{r}) \;=\; D_1[(d-1+\xi)\delta^{ij}r^\xi - \xi r^i r^j r^{\xi-2}]\;. \tag{34}$$

As we discussed in Sect. II A, D_0 gives the eddy diffusivity of a single fluid particle at long times. Notice that D_0 is dominated by the integral scale indicating that the effective diffusion of a single fluid particle is driven by the velocity fluctuations at the largest scales present. On the other hand, $d_{ij}(\mathbf{r})$ describes the statistics of the velocity differences: $\langle \delta v^i(\mathbf{r},t)\delta v^j(\mathbf{r},t')\rangle = 2\delta(t-t')d^{ij}(\mathbf{r})$. It picks up contributes of all scales.

The normalization constant D_1 has the dimensionality of $length^{2-\xi}time^{-1}$. For $0 < \xi < 2$, the Kraichnan ensemble is supported on the velocities that are Hölder continuous in space with a fixed exponent α arbitrarily close to $\xi/2$. It mimics this way the main property of the infinite Reynolds number turbulent velocities characterized by fractional Hölder exponents. The rough (distributional) behavior of Kraichnan velocities in time, although not very physical, is not expected to modify essentially the qualitative picture of the trajectory behavior (it is the spatial regularity, not the temporal one, of a vector field that is crucial for the uniqueness if its trajectories).

In the Kraichnan ensemble, one can directly calculate the Gaussian integral in (5) which gives the Gaussian single-point PDF that satisfies the heat equation $[\partial_t - (D_0+\kappa)\nabla^2_{\mathbf{r}}]\mathcal{P}(\mathbf{r},t) = 0$. That agrees with the all-time diffusive behavior of a single fluid particle in the Kraichnan ensemble characterized by the enhancement of the molecular diffusivity κ by the eddy diffusivity D_0 discussed at the end of Sect. II A.

In much the same way one can examine the joint PDF of the simultaneous values of the coordinates of two fluid particles averaged over the velocity ensemble:

$$\mathcal{P}_2(\mathbf{r}_1,\mathbf{r}_2,s;\mathbf{R}_1,\mathbf{R}_2,t) \;=\; \Big\langle P(\mathbf{r}_1,s;\mathbf{R}_1,t|\mathbf{v})P(\mathbf{r}_2,s;\mathbf{R}_2,t|\mathbf{v})\Big\rangle. \tag{35}$$

For the Kraichnan ensemble, it satisfies the equation

$$(\partial_t-\mathcal{M}_2)\mathcal{P}_2(\mathbf{r}_1,\mathbf{r}_2,s;\mathbf{R}_1,\mathbf{R}_2,t)=\delta(t-s)\delta(\mathbf{R}_1-\mathbf{r}_1)\delta(\mathbf{R}-\mathbf{r}_2)$$

with an explicit elliptic second-order differential operator

$$\mathcal{M}_2 \;=\; -\sum_{n,n'=1}^{2} D^{ij}(\mathbf{r}_n-\mathbf{r}_{n'})\nabla_{r_n^i}\nabla_{r_{n'}^j}\;, \tag{36}$$

a result which goes back to the original work of Kraichnan [4]. If we are interested only in the separation $\mathbf{R}=\mathbf{R}_1-\mathbf{R}_2$ of two fluid particles at time t, given their separation $\mathbf{r}$ at time s, then the relevant PDF $\mathcal{P}_2(\mathbf{r},s;\mathbf{R},t)$ is obtained by averaging over the simultaneous translations of the final (or initial) positions of the particle and is governed by the operator $\mathcal{M}_2$ restricted to the translationally invariant sector. The latter is equal to $-d^{ij}(\mathbf{r})\nabla_{r^i}\nabla_{r^j}$. Note that the eddy diffusivity D_0, dominated by the integral scale, drops out in the action on translation-invariant functions. The above result shows that the relative motion of two fluid particles in the Kraichnan ensemble of velocities is an effective diffusion with a distance-dependent diffusivity tensor scaling like r^ξ in the inertial range. This is a precise realization of the scenario for the turbulent diffusion put up by Richardson as far back as 1926 [30]. Similarly, the PDF $\mathcal{P}_2(r,s;R,t)$ of the distance R between two particles satisfies the equation

$$(\partial_t-M_2)\mathcal{P}_2(r,s;R,t)=\delta(t-s)\delta(r-R)\;, \tag{37}$$

where the restriction of $\mathcal{M}_2$ to the homogeneous and isotropic sector is $M_2=-D_1(d-1)r^{1-d}\partial_r r^{d-1+\xi}\partial_r$ and (37) can be readily solved [4, 31]. At $r\ll R$, the PDF has particularly simple form

$$\lim_{r\to 0}\;\mathcal{P}_2(r,s;R,t)\;\propto\;\frac{R^{d-1}}{|t-s|^{d/(2-\xi)}}\exp\left[-\mathrm{const.}\frac{R^{2-\xi}}{|t-s|}\right]\;. \tag{38}$$

That confirms the diffusive character of the limiting process describing the Lagrangian trajectories in fixed non-Lipschitz velocities: the endpoints of the process stay at finite distance when the initial points converge. If we set $\eta=0$ but maintain finite integral scale L, then the behavior (38) is modified for $R\gg L$ and crosses over to the simple diffusion with the diffusivity $2D_0$: at distances much larger than the integral scale two fluid particles undergo independent Brownian walks driven by the velocity fluctuations on scale L.

The PDF (38) changes from Gaussian to log-normal when ξ changes from zero to two. The PDF has the scaling form (31) for $\alpha = \xi - 1$ and implies the power law growth (30) of the averaged powers of the distance between trajectories. The Richardson dispersion $\langle R^2(t)\rangle \propto t^3$ is reproduced for $\xi = 4/3$ rather than for $\xi = 2/3$ when the spatial Hölder exponent of the typical Kraichnan ensemble velocities takes the Kolmogorov value 1/3. The reason is that the velocity temporal decorrelation cannot be ignored and we should replace the time t in the right hand side of (29) by the Brownian motion $\beta(t)$. That replacement indeed reproduces for $\alpha = \xi/2$ the large-time PDF (38) up to a geometric power-law prefactor.

Note the special case of the average $\langle R^{2-\xi-d}\rangle$ in the Kraichnan velocities. Since $M\, r^{2-\xi-d}$ is a contact term $\propto \delta(r)$ for $\kappa = 0$, one has $\partial_t \langle R^{2-\xi-d}\rangle \propto \mathcal{P}(r; 0; t)$. The latter is zero in the smooth case so that $\langle R^{-d}\rangle$ is a true integral of motion. In the non-smooth case, $\langle R^{2-\xi-d}\rangle \propto t^{1-d/(2-\xi)}$ and is not conserved due to a nonzero probability density to find two particles at the same place even when they started apart.

D. Two-particle dispersion in a compressible flow

Discussing the particle dispersion in incompressible fluids and exposing the different mechanisms of particle separation, we paid little attention to the detailed geometry of the flows, severely restricted by the incompressibility. The presence of compressibility allows for more flexible flow geometries with regions of compression trapping particles and counteracting their tendency to separate. To expose this effect and gauge its relative importance for smooth and non-smooth flows, we start from the simplest case of a time-independent $1d$ flow $\dot{x} = v(x)$. In $1d$, any velocity is potential: $v(x) = -\partial_x \phi(x)$, and the flow is the steepest descent in the landscape defined by the potential ϕ. The particles are trapped in the intervals where the velocity has a constant sign and they converge to the fixed points with lower value of ϕ at the ends of those intervals. In the regions where $\partial_x v$ is negative, nearby trajectories are compressed together. If the flow is smooth the trajectories take an infinite time to arrive at the fixed points (the particles might also escape to infinity in a finite time). Let us consider now a non-smooth version of the velocity, e.g. a Brownian path with Hölder exponent 1/2. At variance with the smooth case, the solutions will take a finite time to reach the fixed points at the ends of the trapping intervals and will stick to them at subsequent times, as in the example of the equation $\dot{x} = |x - x_0|^{1/2}$. The roughness of the velocity clearly amplifies the trapping effects leading to the convergence of the trajectories. A time-dependence of the velocity changes somewhat the picture. The trapping regions, as defined for the static case, start wandering and they do not enslave the solutions which may cross their boundaries. Still, the regions of ongoing compression effectively trap the fluid particles for long time intervals. Whether the tendency of the particles to separate or the trapping effects win is a matter of detailed characteristics of the flow.

In higher dimensions, the behavior of potential flows is very similar to the $1d$ case, with trapping totally dominating in the time-independent case, its effects being magnified by the velocity roughness and blurred by the time-dependence. The traps might of course have a more complicated geometry. Moreover, we might have both solenoidal and potential components in the velocity. The dominant tendency for the incompressible component is to separate the trajectories, as we discussed in the previous Sections. On the other hand, the potential component enhances trapping in the compressed regions. The net result of the interplay between the two components depends on their relative strength, spatial smoothness and temporal rate of change.

Let us consider first a smooth compressible flow with a homogeneous and stationary ergodic statistics. Similarly to the incompressible case discussed in Sect. II B, the stretching-contraction variables ρ_i, $i = 1, \ldots, d$, behave asymptotically as $t\lambda_i$ with the PDF of large deviations $x_i = \rho_i/t - \lambda_i$ determined by an entropy function $H(x_1, \ldots, x_d)$. The asymptotic growth rate of the fluid volume is given by the sum of the Lyapunov exponents $s = \sum\limits_{i=1}^{d} \lambda_i$. Note that density fluctuations do not grow in a statistically steady compressible flow because the pressure provides feedback from the density to the velocity field. That means that s vanishes even though the ρ_i variables fluctuate. However, to model the growth of density fluctuations in the intermediate regime, one can consider an idealized model with a steady velocity statistics having nonzero s. This quantity has the interpretation of the opposite of the entropy production rate, see Section III B below, and it is necessarily ≤ 0 [32, 33]. Indeed, in any statistically homogeneous flow, incompressible or compressible, the distribution of particle displacements is independent of their initial position and so is the distribution of the evolution matrix $W_{ij}(t; \mathbf{r}) = \partial R^i(t; \mathbf{r})/\partial r^j$. Since the total volume V (assumed finite in this argument) is conserved, the average $\langle \det W \rangle$ is equal to unity for all times and initial positions although the determinant fluctuates in the compressible case. The average of $\det W = \mathrm{e}^{\sum \rho_i}$ is dominated at long times by the saddle-point $\mathbf{x}^*$ giving the maximum of $\sum(\lambda_i + x_i) - H(\mathbf{x})$, which has to vanish to conform with the total volume conservation. Since $\sum x_i - H(\mathbf{x})$ is concave and vanishes at $x = 0$, its maximum value has to be non-negative. We conclude that the sum of the Lyapunov exponents is non-positive. The meaning of this result is transparent: there are more Lagrangian particles in the contracting regions which thus acquire higher weight, leading to negative average gradients in the Lagrangian frame. Let us stress the essential difference between the Eulerian and the Lagrangian averages in the compressible case: an Eulerian average is uniform over space, while in a Lagrangian average every trajectory comes with its own weight determined by the local rate of volume change.

For quantitative description we employ again the Kraichnan model. The compressible generalization of the Kraichnan ensemble for smooth velocities has the (non-constant part of the) pair correlation function defined as

$$d^{ij}(\mathbf{r}) \;=\; D_1 \left[(d+1-2\wp)\, \delta^{ij}\, r^2 \;+\; 2(\wp d - 1)\, r^i r^j\right] \;. \tag{39}$$

The degree of compressibility $\wp \equiv \langle (\nabla_i v^i)^2 \rangle / \langle (\nabla_i v^j)^2 \rangle$ is between 0 and 1 for the isotropic case at hand, with the the two extrema corresponding to the incompressible and the potential cases. The corresponding strain matrix $\sigma = \nabla \mathbf{v}$ has the Eulerian mean equal to zero and 2-point function

$$\langle\, \sigma_{ij}(t)\, \sigma_{k\ell}(t')\, \rangle = 2\, \delta(t-t')\, D_1 \left[(d+1-2\wp)\, \delta_{ik}\delta_{j\ell} + (\wp d - 1)(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{jk})\right] . \tag{40}$$

The volume growth rate $-\int_0^t \langle\, \sigma_{ii}(t)\, \sigma_{jj}(t') \rangle\, dt'$ is thus strictly negative, in agreement with the general discussion, and equal to $-\wp\, D_1\, d(d-1)(d+2)$ if we set $\int_0^\infty \delta(t)\, dt = 1/2$. The PDF $\mathcal{P}(\rho_1, \ldots, \rho_d; t)$ takes again the large deviation form (23), with the entropy function and the Lyapunov exponents given by [1, 36, 37]

$$H(\mathbf{x}) \;=\; \tfrac{1}{4D_1(d+\wp(d-2))} \Big[\sum_{i=1}^{d} x_i^2 \;+\; \tfrac{1-\wp d}{\wp(d-1)(d+2)} \Big(\sum_{i=1}^{d} x_i \Big)^2 \Big] , \tag{41}$$

$$\lambda_i \;=\; D_1 \left[d(d-2i+1) - 2\wp\, (d + (d-2)i) \right] \;. \tag{42}$$

Compare this expression to (24). Note how the form (41) of the entropy imposes the condition $\sum x_i \;=\; 0$ in the incompressible limit. The inter-particle distance $R(t)$ has the lognormal distribution (25) with $\bar{\lambda} = \lambda_1 = D_1(d-1)(d-4\wp)$ and $\Delta = 2D_1(d-1)(1+2\wp)$. Explicitly, $t^{-1}\ln\langle R^n\rangle \propto n[n+d+2\wp(n-2)]$ [37]. The quantity $R^{(4\wp-d)/(1+2\wp)}$ is thus statistically conserved. The highest Lyapunov exponent $\bar{\lambda}$ becomes negative when the degree of compressibility is larger than $d/4$ [36, 37]. Low-order moments of R, including its logarithm, would then decrease while high-order moments would grow with time. The decrease of the Lyapunov exponents when $\wp$ grows clearly signals the increase of trapping. The regime with $\wp > d/4$, with all the Lyapunov exponents becoming negative, is the one where trapping effects dominate. The dramatic consequences for the scalar fields advected by such flow will be discussed in Sect. IV A. Analysis of the Kraichnan model for a non-smooth case demonstrates even stronger effects of compressibility, with an increased tendency for the fluid particles to aggregate in a finite time [1, 38]. When the compressibility degree is large enough, even though the velocity is non-smooth, the Lagrangian trajectories in a fixed velocity field are determined by their initial positions. Moreover, trajectories starting at a finite distance collapse to zero distance and stay together with a positive probability growing with time.

As was mentioned, the aggregation of fluid particles can take place only as a transient process. The back reaction of the density on the flow eventually stops the growth of the density fluctuations. The transient trapping should, however, play a role in the creation of the shock structures observed in high Mach number compressible flows. That theory describes also the aggregation of real particles suspended in the fluid. Let us consider a small inertial particle of density ρ and radius a in a fluid of density ρ_0. Its movement may be approximated by that of a Lagrangian particle in an effective velocity field provided that a^2/ν is much smaller than the velocity time scale in the Lagrangian frame. The inertial difference between the effective velocity $\mathbf{v}$ of the particle and the fluid velocity $\mathbf{u}(\mathbf{r}, t)$ is proportional to the local acceleration: $\mathbf{v} = \mathbf{u} + (\beta - 1)\,\tau_s\, d\mathbf{u}/dt$, where $\beta = 3\rho/(\rho + 2\rho_0)$ and $\tau_s = a^2/3\nu\beta$ is the Stokes time. Considering such particles distributed in the volume, one may define the velocity field $\mathbf{v}(\mathbf{r}, t)$, whose divergence $\propto \nabla[(\mathbf{u}\cdot\nabla)\mathbf{u}]$ does not vanish even if the fluid flow is incompressible. As discussed above, this leads to a negative volume growth rate and the clustering of the particles [33–35].

E. Multi-particle configurations and zero modes

We describe here the time-dependent statistics of multi-particle configurations. Our main interest is in the long-time asymptotics of propagators when final distances far exceed initial ones. Particularly important question is what memory of initial configuration remain in the propagators in that limit. We shall see that to answer this question one must analyze the conservation laws of turbulent diffusion. As we have seen in the previous subsections, the two-particle statistics is characterized by the single separation vector. In non-smooth velocities, the length of the vector grows by a power law, while the initial separation is forgotten. Adding extra particles brings geometry into the game. Many-particle evolution in non-smooth velocities exhibits non-trivial statistical integrals of motion that are proportional to the positive powers of the distances. The integrals involve geometry in such a way that the distance growth is balanced by the decrease of the shape fluctuations. The existence of multi-particle conservation laws indicates the presence of a long-time memory and is a reflection of the coupling among the particles due to the simple fact that they are all in the same velocity field. The conserved quantities may be easily built for the limiting cases. Since the advection by a smooth velocity preserves straight lines, the d-volume $\epsilon_{i_1 i_2 \ldots i_d}\, R_{12}^{i_1} \ldots R_{1d}^{i_d}$ is conserved for $(d+1)$ Lagrangian trajectories. In particular, for any three trajectories in $d=2$, the area $\epsilon_{ij}\, \mathbf{R}_{12}^{i} \mathbf{R}_{13}^{j}$ of the triangle defined by the three particles remains constant, the growth of the sides being compensated by the decrease of the angle. In the opposite case of a very irregular velocity, the fluid particles undergo a Brownian motion. The distances between the Brownian particles grow according to $\langle R_{nm}^2(t) \rangle = R_{nm}^2(0) + Dt$. The statistical integrals of motion are $\langle R_{nm}^2 - R_{pr}^2 \rangle$, $\langle 2(d+2) R_{nm}^2 R_{pr}^2 - d(R_{nm}^4 + R_{pr}^4) \rangle$, and an infinity of similarly built polynomials (zero modes of Laplacian) where all powers of t cancel out. Another trivial case is the infinite-dimensional flow where the distances between particles do not fluctuate. The two-particle law $R_{nm}(t)^{1-\alpha} - R_{nm}(0)^{1-\alpha} \propto t$, implies then that the expectation of any function of $R_{nm}^{1-\alpha} - R_{pr}^{1-\alpha}$ does not change with time. Away from the degenerate limiting cases, the conserved quantities continue to exist yet they cannot be generally constructed so easily and they depend substantially on the number of particles. We thus see that the very existence of conserved quantities is natural. What is nontrivial in a general case is their precise form and their scaling. The intricate statistical conservation laws of multi-particle dynamics were first discovered for the Kraichnan velocities [39, 40].

The discovery has led to a new qualitative and quantitative understanding of intermittency of advected fields as will be described in Chapter IV. It has also revealed the aspects of the multi-particle evolution that seem both present and relevant in generic turbulent flows [1, 41].

As for many-body problems in other branches of physics (e.g. in kinetic theory or in quantum mechanics), the multi-particle dynamics may bring about new aspects due to the cooperative behavior of particles. In turbulence, such behavior is mediated by the velocity fluctuations correlated at large scales. If the velocities are statistically homogeneous, it is convenient to separate the absolute motion of particles from the relative one, as in the other many-body problems with spatial homogeneity. For N particles, we define the absolute motion as the one of the mean position $\overline{\mathbf{R}} = \sum \mathbf{R}_n/N$; as for any single particle, that motion is also expected to be diffusive on time scales longer that the Lagrangian correlation time (Sect. II A). Since for such time scales the particles may be considered as moving independently then the diffusivity of the absolute motion is N times smaller than that of a single particle. The statistics of the relative motion of N particles is described by the joint PDF averaged over rigid translations $\underline{\rho} = (\rho, \dots, \rho)$:

$$\mathcal{P}_{_N}^{\rm rel}(\underline{\mathbf{r}}, s; \underline{\mathbf{R}}, t) \;=\; \int \mathcal{P}_{_N}(s, \underline{\mathbf{r}}; \underline{\mathbf{R}} + \underline{\rho}, t) d\rho, \tag{43}$$

The PDF $\mathcal{P}_{_N}^{\rm rel}$ describes the distribution of the separations $\mathbf{R}_{nm} = \mathbf{R}_n - \mathbf{R}_m$ or the relative positions $\underline{\mathbf{R}}^{\rm rel} = (\mathbf{R}_{_1} - \overline{\mathbf{R}}, \dots, .\mathbf{R}_{_N} - \overline{\mathbf{R}})$.

The PDF $\mathcal{P}_{_N}$ are again expected to show a different short-distance behavior for smooth and non-smooth velocities. For smooth velocities, the existence of deterministic trajectories leads for $\kappa = 0$ to the collapse property

$$\lim_{\mathbf{r}_{_N} \to \mathbf{r}_{_{N-1}}} \mathcal{P}_{_N}(\underline{\mathbf{r}};\, \underline{\mathbf{R}};\, t) \;=\; \mathcal{P}_{_{N-1}}(\underline{\mathbf{r}}';\, \underline{\mathbf{R}}';\, t)\;\; \delta(\mathbf{R}_{_{N-1}} - \mathbf{R}_{_N}), \tag{44}$$

where $\underline{\mathbf{R}}' = (\mathbf{R}_{_1}, \dots, \mathbf{R}_{_{N-1}})$ and similarly for the relative PDF's. If all the distances between the particles are much less than the viscous length, one may consider velocity smooth and approximate the velocity field differences by linear expressions:

$$\mathcal{P}_{_N}^{\rm rel}(\underline{\mathbf{r}}, 0; \underline{\mathbf{R}}, t) \;=\; \int \Big\langle \prod_{n=1}^{N} \delta(\mathbf{R}_n + \rho - W(t)\mathbf{r}_n) \Big\rangle d\rho\ . \tag{45}$$

Clearly, the above PDF depend only on the statistics of the evolution matrix $W(t)$ that has been discussed in Sect. II B. Under the evolution governed by $W(t)$, all distances between points grow exponentially for large times while their ratios R_{nm}/R_{kl} tend to a constant. For whatever initial positions, asymptotically in time, the points tend to be situated on the line. This behavior and its dramatic consequences for passive scalar statistics are further discussed at the end of Sect. IV A.

The long-time asymptotics of the propagators in the non-smooth case can be found explicitly for the Kraichnan ensemble of velocities. The great simplification of the Kraichnan model consists in the Markov character of the effective N-trajectory processes which is due to the time decorrelation of the velocities. In other words, the PDF $\mathcal{P}_{_N}(s,\underline{\mathbf{r}};\underline{\mathbf{R}},t)$ and the relative version (43) satisfy the 2$^{\rm nd}$ order differential equations which can be derived by a straightforward generalization of the arguments employed for two particles, see (36):

$$(\partial_t - \mathcal{M}_{_N})\mathcal{P}_{_N}(\underline{\mathbf{r}},s;\underline{\mathbf{R}},t) = \delta(t-s)\delta(\underline{\mathbf{R}}-\underline{\mathbf{r}}), \tag{46}$$

$$\mathcal{M}_{_N} \;=\; -\sum_{n,m=1}^{N} D^{ij}(\mathbf{r}_{nm})\nabla_{r_n^i}\nabla_{r_m^j}\;, \tag{47}$$

where, $D^{ij}(\mathbf{r}) = D_0\delta^{ij} - \frac{1}{2}d^{ij}(\mathbf{r})$ is the spatial part of the velocity 2-point function. For the relative process,

$$(\partial_t + M_{_N})\mathcal{P}^{\rm rel}_{_N}(\underline{\mathbf{r}},s;\underline{\mathbf{R}},t) = \delta(t-s)\delta(\underline{\mathbf{R}}-\underline{\mathbf{r}}) \tag{48}$$

$$M_{_N} \;=\; \sum_{n<m} d^{ij}(\mathbf{r}_{nm})\nabla_{r_n^i}\nabla_{r_m^j}\;\;. \tag{49}$$

Note the multi-body structure of $\mathcal{M}_{_N}$ and $M_{_N}$.

Since $M_{_N}$ scales as $length^{\xi-2}$ then $time$ should scale as $length^{2-\xi}$ and

$$\mathcal{P}^{\rm rel}_{_N}(\underline{\mathbf{r}},0;\underline{\mathbf{R}},t) \;=\; \lambda^{(N-1)d}\mathcal{P}^{\rm rel}_{_N}(\lambda\underline{\mathbf{r}},0;\lambda\underline{\mathbf{R}},\lambda^{2-\xi}t). \tag{50}$$

Therefore, the asymptotics of the propagator is the same when initial points get close or final points get far apart and time gets large. We expect the multi-particle PDF to be factorized in that limit:

$$\lim_{\lambda\to 0}\mathcal{P}^{\rm rel}_{_N}(\lambda\underline{\mathbf{r}},0;\underline{\mathbf{R}},t) \;=\; \sum_\beta \lambda^{\zeta_\beta} f_\beta(\underline{\mathbf{r}})g_\beta(\underline{\mathbf{R}},t)\;. \tag{51}$$

Here we presume the functions to be scale invariant: $f_\beta(\lambda\underline{\mathbf{r}}) = \lambda^{\zeta_\beta}f_\beta(\underline{\mathbf{r}})$. To find f_β, g_β consider the composition of two PDFs [26]:

$$\int \mathcal{P}^{\rm rel}_{_N}(\lambda\underline{\mathbf{x}},t;\underline{\mathbf{y}},0)\mathcal{P}^{\rm rel}_{_N}(\underline{\mathbf{z}},0;\underline{\mathbf{x}},\tau)\,d\underline{\mathbf{x}} = \lambda^{d(1-N)}\int \mathcal{P}^{\rm rel}_{_N}(\underline{\mathbf{x}},t;\underline{\mathbf{y}},0)\mathcal{P}^{\rm rel}_{_N}(\underline{\mathbf{z}},0;\lambda\underline{\mathbf{x}},\tau)\,d\underline{\mathbf{x}} \tag{52}$$

$$= \int \mathcal{P}^{\rm rel}_{_N}(\underline{\mathbf{x}},t;\underline{\mathbf{y}},0)\mathcal{P}^{\rm rel}_{_N}(\lambda\underline{\mathbf{z}},0;\lambda\underline{\mathbf{x}},\tau\lambda^{2-\xi})\,d\underline{\mathbf{x}} = \mathcal{P}^{\rm rel}_{_N}(\underline{\mathbf{y}},t+\lambda^{2-\xi}\tau;\underline{\mathbf{z}},0)\,. \tag{53}$$

In deriving (53) we have used the scaling relation (50) and the composition property of propagators $\int \mathcal{P}(x,t_1;y,t_2)\mathcal{P}(y,t_2;z,t_3)\,dy = \mathcal{P}(x,t_1;z,t_3)$. Further, one makes Taylor expansion of (53) in τ and then applies the expansions (51) and compare it order-by-order with the straightforward expansion (51) of (52). As a result one can see that f_β must be taken as zero modes of $M^\dagger_{_N}$ and its powers while $\partial_t g_\beta = -M_{_N} g_\beta$.

The first term in the expansion is $\underline{\mathbf{r}}$-independent with the constant $f_0 = 1$ and $g_0(\underline{\mathbf{R}},t) = \mathcal{P}^{\rm rel}_{_N}(\underline{\mathbf{0}},0;\underline{\mathbf{R}},t)$ being the PDF of N initially overlapping particles. The zero mode of $M^\dagger_{_N}$ with the lowest positive scaling dimension ζ gives the first nonvanishing $\underline{\mathbf{r}}$-dependence in the propagator. The remarkable feature of the zero modes of $M^\dagger_{_N}$ can be appreciated by considering the Lagrangian average of an arbitrary translation-invariant functions F of the simultaneous positions of the particles (we assume that $\underline{\mathbf{R}}(0) = \underline{\mathbf{r}}$ and set $\underline{\mathbf{R}}' \equiv (\mathbf{R}_{_1},\dots \mathbf{R}_{_{N-1}})$):

$$\langle F(\underline{\mathbf{R}}(t))\rangle = \int F(\underline{\mathbf{R}})\ \mathcal{P}^{\rm rel}_{_N}(\underline{\mathbf{r}},0;\underline{\mathbf{R}},t)\ d\underline{\mathbf{R}}' \tag{54}$$

When F is taken as a zero mode of $M^\dagger_{_N}$ it is conserved in mean by the Lagrangian evolution. Indeed, the time derivative of $\langle f(t)\rangle$ vanishes since it brings down $M^\dagger_{_N}$ acting on f on the right hand side of (54):

$$\begin{aligned}\partial_t \langle f(\underline{\mathbf{R}}(t))\rangle &= \int f(\underline{\mathbf{R}}) M_{_N} \mathcal{P}^{\rm rel}_{_N}(\underline{\mathbf{r}},0;\underline{\mathbf{R}},t)\ d\underline{\mathbf{R}}' \\ &= \int \mathcal{P}^{\rm rel}_{_N}(\underline{\mathbf{r}},0;\underline{\mathbf{R}},t) M^\dagger_{_N} f(\underline{\mathbf{R}})\, d\underline{\mathbf{R}}'\end{aligned}$$

The importance of the scale-invariant conserved modes for the transport properties of short correlated velocities has been recognized independently in [39, 40, 42].

To understand how one can have conserved quantities in turbulent diffusion think about the evolution of N fluid particles as of that of a discrete cloud of marked points in the physical space. There are two elements in the relative evolution of the cloud: the growth of its size and the change of its shape. We shall define the overall size of the cloud as $R = [(2N)^{-1}\sum \mathbf{R}^2_{nm}]^{1/2}$ and its "shape" as $\widehat{\underline{\mathbf{R}}} = \underline{\mathbf{R}}^{\rm rel}/R$. For example, 3 particles form a triangle in the space, with labelled vertices, and the notion of shape that we are using includes the orientation of the triangle. To get convinced that zero modes do exist, let us first consider the limiting case $\xi \to 0$ of very rough velocity fields. In this limit, the operator $M_{_N}$ becomes proportional to $\underline{\nabla}^2$, the (Nd)-dimensional Laplacian restricted to the translation-invariant sector (note that for an incompressible flow, the operator is always self-adjoint: $M^\dagger_{_N} = M_{_N}$). The relative motion of particles becomes pure diffusion. With R

denoting the size-of-the-cloud variable,

$$\underline{\nabla}^2 \;=\; R^{-d_N+1}\,\partial_R\,R^{d_N-1}\,\partial_R \;+\; R^{-2}\,\widehat{\underline{\nabla}}^2\,, \tag{55}$$

where $d_N \equiv (N-1)d$ and $\widehat{\underline{\nabla}}^2$ is the angular Laplacian on the (d_N-1)-dimensional unit sphere of shapes $\widehat{\underline{\mathbf{R}}}$. The spectrum of the latter may be analyzed using the properties of the rotation group. Its eigenfunctions ϕ_ℓ have eigenvalues $-\ell(\ell+d_N-2)$ where $\ell=0,1,\dots$ is the angular momentum. The averages of the angular eigenfunctions decay as follows: $\langle\phi_\ell(\widehat{\underline{\mathbf{R}}})\rangle \propto t^{-\ell/2}$. To compensate for the decay, we introduce the functions $f_{\ell,0} = R^\ell\phi_\ell(\widehat{\underline{\mathbf{R}}})$ which are zero modes of the Laplacian with the scaling dimension ℓ — the contributions coming from the radial and the angular parts in (55) indeed cancel out. The averages $\langle f_{\ell,0}\rangle$ are thus conserved. All the scale-invariant zero modes of the Laplacian are of that form. The polynomials invariant under d-dimensional translations, rotations and reflections can be reexpressed as polynomials in R^2_{nm}. For even N, the irreducible $O(d)$-invariant zero mode with the lowest scaling dimension has then the form

$$f_N(\underline{\mathbf{R}}) \;=\; R^2_{12}R^2_{34}\dots R^2_{(N-1)N} \;+\; [\dots] \tag{56}$$

where $[\dots]$ denotes a combination of terms that depend on positions of $(N-1)$ or less particles. E.g. for four points, the zero mode is $R^2_{12}R^2_{34} - \frac{d}{2(d+2)}(R^4_{12}+R^4_{34})$, the example already mentioned before. The terms $[\dots]$ are not uniquely determined since we may add to them degree N zero modes for smaller number of points. Besides, the functions differing from f_N by a permutation of points are also zero modes so that we may symmetrize the above expressions and look only at the permutation invariant modes. Clearly, the scaling dimension $\zeta_N = N$. The linear in N growth of the dimension signals the absence of the extra attractive effect between the particles diffusing with a constant diffusivity (no particle binding in the shape evolution for $\xi = 0$). As we shall see in Sect. IV A, this leads to the disappearance of intermittency in the advected scalar which becomes a Gaussian field in the limit $\xi \to 0$. For small but positive ξ, the scaling dimension of the irreducible 4-point zero mode f_4 was first calculated to the linear order in ξ by Gawędzki and Kupiainen [40]. Parallelly, a similar calculation in the linear order in $1/d$ was performed by Chertkov, Falkovich, Kolokolov and Lebedev [39]. Those two papers present the first ever analytic calculations of the anomalous exponents in turbulence. A generalization for larger N has been achieved in [43, 44]:

$$\zeta_{_N} \;=\; \frac{N}{2}(2-\xi) \;-\; \frac{N(N-2)}{2(d+2)}\xi \tag{57}$$

giving the leading correction $\propto \xi$ to the scaling dimension of the lowest irreducible zero mode. Note that, to that order, the scaling dimension $\zeta_{_N}$ is a concave function of N. This can be interpreted as a result of particle interaction. From the form (49) of the generator of the process $\underline{\mathbf{R}}^{\rm rel}(t)$ we infer that, in the Kraichnan model, N fluid particles undergo an effective diffusion with the diffusivity depending on the inter-particle distances. In the inertial interval of distances $\eta \ll r \ll L$, where $d^{ij}(\mathbf{r}) \propto r^\xi$, the effective diffusivity grows as the power ξ of the distance. It should be intuitively clear that, in comparison to the standard diffusion with constant diffusivity, the particles will tend to spend longer time together when they are close and to separate faster when they become distant.

Let us stress that (51) *is not* a spectral decomposition of the resolvent $M_{_N}^{-1}$ (since $M_{_N}$ is positive with a continuous spectrum, such decomposition would be a continuous integral involving eigenfunctions). The scaling zero modes that govern the small-scale asymptotics are rather analogous to resonances in many-body systems with the scaling dimension ζ playing the role of energy. It is thus instructive to compare the shape-versus-size stochastic evolution of the Lagrangian cloud to the imaginary-time evolution of the quantum-mechanical many-particle systems governed by the Hamiltonians $H_{_N} = \sum_n p_n^2/2m + \sum_{n<k} V(\mathbf{r}_{nk})$. An attractive potential between the particles may lead to the creation of bound states at the bottom of the spectrum of $H_{_N}$. Those states determine the decomposition of the (Hermitian) imaginary-time evolution operators $\exp(-tH_{_N})$ in the translation-invariant sector with lowest levels determining the asymptotics at $t \to \infty$: $\sum_a \exp(-tE_{_{N,a}})|\psi_{_{N,a}}\rangle\langle\psi_{_{N,a}}|$. Breaking the system into subsystems of N_i particles by removing the potential coupling between them one raises the ground state energy: $E_{_N} < \sum\limits_i E_{_{N_i}}$. A very similar phenomenon occurs in the stochastic shape evolution in the Kraichnan model. For simplicity, let us only consider the case of even number of particles and of the isotropic sector. Let $\zeta_{_N}$ be the lowest value of the scaling dimension of the irreducible (i.e. dependent on positions of all N particles) zero mode invariant under d-dimensional translations, rotations and reflections. For $N=2$, the expansion starts from $r^{2-\xi}$ which is the zero mode of M^2 so that $\zeta_{_2} = 2-\xi$. Suppose now that we break the system into subsystems of N_i particles (with even N_i) by removing in $M_{_N}$ the derivative terms $d(\mathbf{r}_{nm})\nabla_{\mathbf{r}_n}\nabla_{\mathbf{r}_m}$ coupling the subsystems, see (49). The recalculation of the smallest dimension of the invariant irreducible zero modes gives now the value $\sum_i \zeta_{N_i}$. Indeed, if $N_i \geq 4$, the lowest irreducible zero mode for the broken system is the product of such modes for the subsystems. The crucial observation, confirmed by (57) is that the breaking of the system raises the minimal dimension of the irreducible zero modes: $\zeta_N < \sum_i \zeta_{N_i}$. In particular, $\zeta_N < \frac{N}{2}(2-\xi)$. Generally, one expects ζ_N to be a concave function of (even) N. Interesting that for $N \gg d$, the dependence of ζ_N on N saturates that is adding extra particles does not change the energy at all [31]. By analogy with the many-body quantum mechanics we may say that the irreducible zero modes are bound states of the shape evolution of the Lagrangian cloud. It is a cooperative phenomenon exhibiting a short-distance attraction of close Lagrangian trajectories diffusing with the diffusivity proportional to a power of the distance, superposed on the overall repulsion of the trajectories. The effective short-distance attraction slowing down the separation of close particles is a robust phenomenon that should be present also in time-correlated and non-Gaussian velocity fields. The effect is at the root of the anomalous scaling of the structure functions of the passive scalar advected by non-smooth Kraichnan velocities, as we shall see in Sect. IV A. We believe that it is responsible for intermittency in the transport of scalars by high Reynolds number flows.

To conclude, one is able to build statistically conserved quantities by compensating the growth of inter-particle distances by the decrease of the shape fluctuations of the particle configurations. The size of the cloud increases with time while the average of a generic functions of shape relaxes to a constant as a combination of negative powers (of R or t). The scaling exponents of the zero modes depend in a nontrivial way of the number N of the particles which is the manifestation of particle interaction.

III. UNFORCED EVOLUTION OF PASSIVE FIELDS

The qualification "passive" means that we disregard the back reaction of the advected fields on the advecting velocity. The first section of this chapter is devoted to the statistical initial value problem: how an initially created distribution of a passive scalar evolves in a statistically steady turbulent environment? The simplest question to address is which fields have their amplitudes decaying in time and which growing, assuming the velocity field to be statistically steady.

We consider the tracer (scalar density per unit mass) which satisfies the advection-diffusion equation:

$$\frac{\partial \theta}{\partial t} + (\mathbf{v} \cdot \nabla)\theta = \kappa \nabla^2 \theta \tag{58}$$

and the scalar density per unit volume, n (to be called concentration) whose evolution is governed by the continuity equation

$$\partial_t n + \nabla \cdot (n\, \mathbf{v}) = \kappa \nabla^2 n \ . \tag{59}$$

For incompressible flows, (58) and (59) obviously coincide. A tracer field always decays because of dissipative effects, with the law of decay depending on the velocity properties. The fluctuations of a passive density may grow in a compressible flow, with this growth saturated by diffusion after some time. We shall also briefly consider vector fields advected by the flow. A potential vector field can be considered as the gradient of a tracer $\omega = \nabla\theta$, obeying

$$\partial_t \omega + \nabla(\mathbf{v} \cdot \omega) = \kappa \nabla^2 \omega \ . \tag{60}$$

Solenoidal vector field (e.g. magnetic field) evolves in an incompressible flow according to

$$\partial_t \mathbf{B} + \mathbf{v} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{v} = \kappa \nabla^2 \mathbf{B} \ . \tag{61}$$

The fluctuations of both ω and $\mathbf{B}$ may grow exponentially as long as diffusion is unimportant. After diffusion comes into play, their destinies are different : ω decays, while the magnetic field continues to grow. This growth is known as dynamo process and it continues until saturated by the back-reaction of the magnetic field on the velocity.

Another important issue here is the presence or absence of a dynamic self-similarity : for example, is it possible to present the time-dependent PDF $\mathcal{P}(\theta; t)$ as a function of a single argument? In other words, does the form of the PDF remain invariant in time apart from a rescaling of the field? We shall show that for large times the scalar PDF tends to a self-similar limit when the advecting velocity is non-smooth, while self-similarity is broken in smooth velocities.

A. Decay of tracer fluctuations

For practical applications, e.g. in the diffusion of pollution, the most relevant quantity is the average $\langle \theta(\mathbf{r}, t) \rangle$. It follows from (2) that the average concentration is related to the single particle propagation discussed in Sect. II A. For times longer than the Lagrangian correlation time, the particle diffuses and $\langle\theta\rangle$ obeys the effective heat equation

$$\partial_t \Big\langle \theta(\mathbf{r},t) \Big\rangle = \left(D_e^{ij} + \kappa\delta_{ij} \right) \nabla_i \nabla_j \Big\langle \theta(\mathbf{r},t) \Big\rangle , \tag{62}$$

with the eddy diffusivity D_e^{ij} given by (16). The simplest decay problem is that of a uniform scalar spot of size ℓ released in the fluid. Its averaged spatial distribution at later times is given by the solution of (62) with the appropriate initial condition. On the other hand, the decay of the scalar in the spot is governed by the multi-point Lagrangian propagators. Another relevant situation is that where a homogeneous statistics with correlations decaying on the scale ℓ is initially prescribed. Taking the point of measurement inside the spot or averaging over space for a homogeneous statistics, consider the single-point moment $\langle\theta^N\rangle(t)$ described by (2):

$$\Big\langle \theta^N \Big\rangle(t) = \int \mathcal{P}_{_N}(\underline{\mathbf{0}}, t; \underline{\mathbf{R}}, 0) \theta(\mathbf{R}_{_1}, 0) \, \ldots \, \theta(\mathbf{R}_{_N}, 0) \; d\underline{\mathbf{R}} \, . \tag{63}$$

If there is no molecular diffusion and the trajectories are unique, particles that end at the same point remained together throughout the evolution and all the moments $\langle\theta^N\rangle(t)$ are preserved. From what we have learnt in Chapter II we expect the preservation at the limit $\kappa \to 0$ when velocity field either is smooth or has its non-smoothness overcame by compressibility. Note that the conservation laws are statistical: the moments are not dynamically conserved in every realization, but their averages over the velocity ensemble are. On the contrary, when velocity field is non-smooth and the propagator is diffusive we expect the decay of the tracer moments even at the limit $\kappa \to 0$. This is an example of the so-called dissipative anomaly which we shall discuss more below. One calls anomaly a finite effect of symmetry breaking even when the symmetry-breaking factor goes to zero. Here, the symmetry broken by molecular diffusion is time reversibility.

i) **Smooth velocity**. Let us start from the simplest problem: consider a small spherical spot of the tracer θ released in a spatially smooth incompressible 3d velocity field with $\lambda_1 > \lambda_2 > 0 > \lambda_3$. Physically, we imply the Schmidt number $Sc = \nu/\kappa$ to be large that is the viscous scale of the flow η is much larger than the diffusion scale of the scalar defined as $r_d = \sqrt{-\kappa/\lambda_3}$. The initial size of the spot L is assumed to satisfy $\eta \gg L \gg r_d$. The spot is stretched and contracted by the velocity field. As we have shown in Sect.II B, during the time less than $t_d = |\lambda_3|^{-1} \ln(L/r_d)$, diffusion is unimportant and θ inside the spot does not change. At larger time, the dimensions of the spot with negative Lyapunov exponents are frozen at r_d, while the rest keep growing exponentially, resulting in an exponential growth of the total volume $\exp(\rho_1+\rho_2)$. That leads to an exponential decay of scalar moments averaged over velocity statistics: $\langle[\theta(t)]^\alpha\rangle \propto \exp(-\gamma_\alpha t)$. The decay rates γ_α can be expressed via the PDF (28) of stretching variables ρ_i. Since θ decays as the inverse volume then

$$\langle[\theta(t)]^\alpha\rangle \propto \int d\rho_1 d\rho_2 \exp\left[-tH(\rho_1/t-\lambda_1,\rho_2/t-\lambda_2)-\alpha(\rho_1+\rho_2)\right] \ . \tag{64}$$

At large t, the integral is determined by the saddle point. At small α, the saddle-point lies within the parabolic domain of H so γ_α increases with α quadratically. At large α, the main contribution is due to the realization with smallest possible spot which has the volume L^3 so γ_α is independent of α [18, 19, 45, 46].

Let us consider now an initial random distribution of $\theta(0,\mathbf{r})$ statistically homogeneous in space. We pass to the reference frame which moves with the Lagrangian point $\mathbf{R}(t|T,\mathbf{r}_0)$ coming to $\mathbf{r}_0$ at T. Such $\theta(t,\mathbf{r})=\tilde\theta(t,\mathbf{r}-\mathbf{R}(t|T,\mathbf{r}_0))$ satisfies

$$\partial_t\tilde\theta+\tilde\sigma_{\alpha\beta}r_\beta\nabla_\alpha\tilde\theta=\kappa\nabla^2\tilde\theta \ . \tag{65}$$

Since the correlation functions of θ and $\tilde\theta$ coincide at the moment of observation we omit the tilde sign in what follows. One may treat diffusion in two equivalent ways: either by introducing Brownian motion or by making Fourier transform in (65). Here for a change we choose the second way defining the time-dependent wavevector $\mathbf{k}(t')=W^T(t,t')\mathbf{k}(t)$ and solving (65) as follows

$$\theta(t,\mathbf{k})=\theta_0\left(W^T(t)\mathbf{k}\right)\exp\left[-Q_{\mu\nu}k_\mu k_\nu\right] \ , \tag{66}$$

$$Q(t)=\kappa\int_0^t dt'\, W(t)W^{-1}(t')\left[W(t)W^{-1}(t')\right]^T \ . \tag{67}$$

The moments of $\theta(t,0)=\int d\mathbf{k}(2\pi)^{-3}\theta(t,\mathbf{k})$ are to be averaged both over velocity statistics and over the initial statistics of the scalar. As the long-time limit is independent of the statistics of $\theta(0,\mathbf{r})$ [18], we take it Gaussian with $\langle\theta(0,\mathbf{r})\theta(0,0)\rangle=\chi(r)=\chi_0\exp[-r^2/(8L^2)]$. Then the moments of θ are as follows

$$\langle[\theta(t)]^\alpha\rangle \propto \int d\rho_1 d\rho_2 \exp\left[-tH-\alpha(\rho_1+\rho_2)/2\right]. \tag{68}$$

Notice that in (68) the scalar amplitude is proportional to the square root of the volume factor as distinct from (64). This difference can be intuitively understood by imagining initially different blobs of size L with uncorrelated values of θ. At time t those blobs overlap. Mutual cancellations of θ from different blobs leads to the law of large numbers with initial statistics forgotten and the rms value of θ being proportional to the square root of the number of blobs. The number of blobs is inversely proportional to the volume $\exp(\rho_1+\rho_2)$. Similarly to (64), the same qualitative conclusions about the decay rates $\gamma_\alpha = \lim_{t\to\infty} t^{-1}\ln\{\langle[\theta(t)]^\alpha\rangle\}$ can be drawn from (68). In particular, for the Kraichnan model $\gamma_\alpha \propto \alpha(1-\alpha/8)$ for $\alpha < 4$ and $\gamma_\alpha = const$ for $\alpha > 4$ [18].

Note that in both cases (single spot and random homogeneous distribution) γ_α is not a linear function of α so that the scalar decay is not self-similar in a smooth velocity. Note also that, in the presence of large-scale cutoffs, the picture is modified at late times when blobs are stretched up to the size of the cutoffs (see iv below).

ii) **Non-smooth velocity**. For the decay in incompressible non-smooth flow, we shall specifically consider the case of a time reversible Kraichnan velocity field. The comments on the general case are reserved to the end of the subsection. The simplest objects to investigate are the single-point moments $\left\langle\theta^{2n}(t)\right\rangle$ and we are interested in their long-time behavior $t \gg \ell^{2-\xi}/D_1$. Here, ℓ is the correlation length of the random initial field and D_1 enters the velocity 2-point function as in (34). Using (2) and the scaling property (50) of the Green function we obtain

$$\left\langle\theta^{2n}(t)\right\rangle \;=\; \int \mathcal{P}_{2n}\left(\underline{\mathbf{0}};\,\underline{\mathbf{R}};-1\right)\, C_{2n}\left(t^{\frac{1}{2-\xi}}\underline{\mathbf{R}},0\right)\,d\underline{\mathbf{R}}. \tag{69}$$

There are two universality classes for this problem, corresponding to either non-zero or vanishing value of the so-called Corrsin integral $J_0 = \int C_2(\mathbf{r},t)\,d\mathbf{r}$. Note that the integral is generally preserved in time by the passive scalar dynamics.

We concentrate here on the case $J_0 \neq 0$ and refer the interested reader to the original paper [47] for more details. For $J_0 \neq 0$, the function $t^{\frac{d}{2-\xi}}\,C_2(t^{\frac{1}{2-\xi}}\mathbf{r},0)$ tends to $J_0\,\delta(\mathbf{r})$ in the long-time limit and (69) is reduced to

$$\langle\theta^{2n}(t)\rangle \;\approx\; (2n-1)!!\,J_0^n t^{\frac{nd}{\xi-2}} \int \mathcal{P}_{2n}\left(\underline{\mathbf{0}};\,\mathbf{R}_1,\mathbf{R}_1,\ldots\mathbf{R}_n,\mathbf{R}_n;-1\right)\,d\underline{\mathbf{R}}, \tag{70}$$

for a Gaussian initial condition. A few remarks are in order. First, the previous formula shows that the behavior in time is self-similar. In other words, the single point PDF $\mathcal{P}(t,\theta)$ takes the form $t^{\frac{d}{2(2-\xi)}}Q(t^{\frac{d}{2(2-\xi)}}\theta)$. That means that the PDF of $\theta/\sqrt{\bar\epsilon}$ is asymptotically time-independent with $\bar\epsilon(t) = \kappa\langle(\nabla\theta)^2\rangle$ being time-dependent (decreasing) dissipation rate. This should be contrasted with the lack of self-similarity found previously for the smooth case. Second, the result is asymptotically independent of the initial statistics (of course, within the universality class $J_0 \neq 0$). As in the previous subsection, this follows from the fact that the connected non-Gaussian part of C_{2n} depends on more than n separation vectors. Its contribution is therefore decaying faster than $t^{-\frac{nd}{2-\xi}}$. Third, it follows from (70) that the long-time PDF, although universal, is generally non-Gaussian. Its Gaussianity would indeed imply the factorization of the probability for the $2n$ particles to collapse in pairs at unit time. Due to the correlations existing among the particle trajectories, this is generally not the case, except for $\xi = 0$ where the particles are independent. The degree of non-Gaussianity is thus expected to increase with ξ [47]. And last but not least, notice the dissipative anomaly as the decay laws are independent of κ.

Other statistical quantities of interest are the structure functions $S_{2n}(\mathbf{r}, t) = \langle[\theta(\mathbf{r}, t) - \theta(\mathbf{0}, t)]^{2n}\rangle$ related to the correlation functions by

$$S_{2n}(\mathbf{r}, t) = \int_0^1 \dots \int_0^1 \partial_{\mu_1} \dots \partial_{\mu_{2n}} C_{2n}(\mu_1 \mathbf{r}, \dots, \mu_{2n}\mathbf{r}, t) \prod d\mu_i \;\equiv\; \Delta(\mathbf{r})\, C_{2n}(\cdot)\,. \tag{71}$$

To analyze their long-time behavior, we proceed similarly as in (69) and use the asymptotic expansion (51) to obtain

$$\begin{aligned} S_{2n}(\mathbf{r}, t) \;&=\; \int \Delta(t^{-\frac{1}{2-\xi}}\mathbf{r})\, \mathcal{P}_{2n}(\,\cdot\,; \mathbf{R}; -1)\;\; C_{2n}(t^{\frac{1}{2-\xi}}\underline{\mathbf{R}}, 0)\; d\underline{\mathbf{R}} \\ &\approx\; \Delta(\mathbf{r}) f_{2n}(\cdot) t^{\frac{\zeta_{2n}}{\xi-2}} \int g_{2n,0}(\underline{\mathbf{R}}, -1)\; C_{2n}(t^{\frac{1}{2-\xi}}\underline{\mathbf{R}}, 0)\; d\underline{\mathbf{R}} \;\;\propto\;\; \left(\frac{r}{\ell(t)}\right)^{\zeta_{2n}} \langle\theta^{2n}\rangle(t)\,. \end{aligned} \tag{72}$$

Here, f_{2n} is the irreducible zero mode in (51) with the lowest dimension and the scalar integral scale $\ell(t) \propto t^{\frac{1}{2-\xi}}$. As we shall see in Sect. IV, the scaling dimensions of the zero modes, ζ_{2n}, give also the scaling exponents of the structure function in the stationary state established in the forced case.

Let us briefly discuss the scalar decay for velocity fields having finite correlation times. The key ingredient for the self-similarity of the scalar PDF is the rescaling (50) of the propagator. Such property is generally expected to hold (at least for large enough times) for self-similar velocity fields regardless of their correlation times. This has been confirmed by the numerical simulations in [47]. For an intermittent velocity field the presence of various scaling exponents makes it unlikely that the propagator possesses a rescaling property like (50). The self-similarity in time of the scalar distribution might then be broken.

iii) **Scalar decay with both viscous and inertial interval of scales present**. Even when the Schmidt/Prandtl number is large and the initial scale of the scalar field l is smaller than the viscous scale η, separation of initially close particles brings their distance eventually into the inertial interval. Until the time of order $\lambda^{-1}\ln(\eta/l)$ the sizes of scalar blobs are contained within the viscous interval and the decay proceeds as described in the subsection i) above. After that time, however, large-scale structures of passive field are created with sizes in the inertial interval. The number of such structures overlapping after time t now grows as power of t as in the subsection ii) above. As a result, the structure function decays by a power law (that is slower than exponential) even in the viscous interval [48]:

$$S_2(r,t) \simeq t^{-2-d/(2-\xi)}\ln(r/r_d) \qquad \text{at} \qquad r_d \ll r \ll \eta\,, \qquad t \gg \lambda^{-1}\ln(\eta/l)\ . \tag{73}$$

Logarithmic r-dependence corresponds to a steady cascade of a scalar in a smooth velocity according to (86) below. One can interpret (73) as describing a cascade with a time-dependent flux, such interpretation is meaningful since the the flux changes (due to inertial-interval dynamics) much more slowly than the cascade proceeds below the viscous scale. At late times, the inertial interval thus serves as a reservoir of passive scalar. Note that the large-time law of decay of the single-point moments $\langle\theta^{2n}(t)\rangle$ is unknown in this case.

iv) **Scalar decay in a finite box**. Finiteness of the flow restricts the time when the above description (based on the separation of fluid particles) is valid. Consider the behavior of the average concentration $\langle\theta(\mathbf{r},t)\rangle$ in a spatially smooth chaotic flow in a finite box of size L. Until time of order $\lambda^{-1}\ln(L/r_d)$, scalar decay proceeds exponentially as described in the section i) above. After the average size of the scalar blob reaches the box size, scalar decay in the bulk is getting generally non-universal that is depending on the large-scale structure of the flow [49, 50]. According to Chertkov and Lebedev [48], the main remaining scalar field is however concentrated near solid boundaries where stretching is suppressed since flow incompressibility and no-slip condition require zero velocity gradient perpendicular to the boundary. At the distance q from the boundary, normal velocity is thus proportional to q^2 while tangential to q. Scalar diffusion is determined by the normal velocity while the correlation time by the (inverse) gradient of the normal velocity. That makes mixing short-correlated at $q \ll L$ with the eddy diffusivity (16) proportional to q^4. Considering the equation (62) with $D \propto q^4$ one can readily establish that the width of the boundary region where the scalar is preserved shrinks as $t^{-1/2}$. During that stage the leakage from the boundary regions makes the scalar concentration in the bulk decreasing by a power law $\langle\theta(\mathbf{r},t)\rangle \propto t^{-3/2}$. After the regions shrink down to the size of the diffusive boundary layer where $D \simeq \kappa$ (the width of the layer is proportional to $\kappa^{1/4}$), $\langle\theta\rangle$ decays exponentially with the rate proportional to $\kappa^{1/2}$ (this was also verified experimentally [51]). Recent work in [52] indicates that large-scale cutoffs strongly affect the decay of magnetic fields as well. It is in particular to be noted that the decay at late times is numerically found to be self-similar.

B. Growth of density fluctuations in compressible flow

The evolution of a passive density field $n(\mathbf{r},t)$ is governed by the equation (59). Consider smooth velocities and neglect diffusion. The density n changes along the trajectory as the inverse of the volume contraction factor. Let us introduce the matrix $\tilde{W}(t;\mathbf{r}) = W(t;\mathbf{R}(0;\mathbf{r},t))$, where $W(t;\mathbf{r})$ describes the forward evolution of small separations of the Lagrangian trajectories starting at time zero near $\mathbf{r}$. The volume contraction factor is $\det(\tilde{W}(t;\mathbf{r}))$ and

$$n(\mathbf{r},t) \;=\; [\det(\tilde{W}(t;\mathbf{r}))]^{-1}\, n(\mathbf{R}(0;\mathbf{r},t),0)\,. \tag{74}$$

Note that the matrix $\tilde{W}(t;\mathbf{r})$ is the inverse of the backward-in-time evolution matrix $W'(t;\mathbf{r})$ with the matrix elements $\partial R^i(0;\mathbf{r},t)/\partial r^j$. This is indeed implied by the identity $\mathbf{R}(t;\mathbf{R}(0;\mathbf{r},t),0) = \mathbf{r}$ and the chain rule for differentiation. We shall take the initial field on the right hand side of (74) to be uniform. This gives $n(\mathbf{r},t) = [\det(\tilde{W}(t;\mathbf{r}))]^{-1}$. Performing the velocity average and recalling the long-time asymptotics of the $\tilde{W}$ statistics, we obtain

$$\langle n^\alpha(t)\rangle \;\propto\; \int \exp\left[(1-\alpha)\sum_i \rho_i - tH(\rho_1/t-\lambda_1,\ldots,\rho_d/t-\lambda_d)\right]\,\prod d\rho_i\,. \tag{75}$$

The moments at long times may be calculated by the saddle-point method and they generally behave as $\propto \exp(\gamma_\alpha t)$. The growth rate function γ_α is convex, due to Hölder inequality, and vanishes both at the origin and for $\alpha = 1$ (by the total mass conservation). This leads to the conclusion that γ_α is negative for $0 < \alpha < 1$ and is otherwise positive: low-order moments decay, whereas high-order and negative moments grow. For a Kraichnan velocity field, the large deviations function H is given by (41) and the density field becomes lognormal with

$\gamma_\alpha \propto \alpha(\alpha-1)$ [34]. Note that the asymptotic rate $\langle n\ln n\rangle/t$ is given by the derivative at unity of γ_α and it is equal to $-\sum\tilde\lambda_i \le 0$. On the other hand, the derivative at zero of γ_α is negative and it determines the decay of the Eulerian average $\langle \ln n\rangle$. We thus conlcude that if the sum of the Lyapunov exponents is nonzero then density decays in almost any point in space and grows for almost any Lagrangian trajectory. The growth of high moments is due to density concentration in some (smaller and smaller) regions. The amplification of negative moments is due to the expansion of low density regions and density decay there. The positive quantity $-\sum\lambda_i$ has the interpretation of the mean (Gibbs) entropy production rate per unit volume. Indeed, if we define the Gibbs entropy $S(n)$ as $-\int(\ln n)n\,d\mathbf{r} = \int \ln\,\det(W(t;\mathbf{r}))\,d\mathbf{r}$ then the entropy transferred to the environment per unit time and unit volume is $-\ln\,\det(W)/t = -\sum\rho_i/t$ and it is asymptotically equal to $-\sum\lambda_i > 0$ [32].

The behavior of the density moments discussed above is the effect of a linear but random hyperbolic stretching and contracting evolution of the trajectory separations. In a finite volume, the linear evolution is eventually superposed with non-linear bending and folding effects. In order to capture the combined impact of the linear and the non-linear dynamics at long times, one may observe at fixed time t the density produced from an initially uniform distribution imposed at much earlier times t_0. When $t_0 \to -\infty$ and if $\lambda_1 > 0$, the density approaches weakly, i.e. in integrals against test functions, a realization-dependent fractal density $n_*(\mathbf{r},t)$ in almost all the realizations of the velocity. The resulting density field is the so-called SRB (Sinai-Ruelle-Bowen) measure. The fractal dimension of the SRB measures may be read from the values of the Lyapunov exponents. For the Kraichnan ensemble of smooth velocities, the SRB measures have a fractal dimension equal to $1+\frac{1-2\wp}{1+2\wp}$ if $0<\wp<\frac{1}{2}$ in $2d$. In $3d$, the dimension is $2+\frac{1-3\wp}{1+2\wp}$ if $0<\wp\le\frac{1}{3}$ and $1+\frac{3-4\wp}{5\wp}$ if $\frac{1}{3}\le\wp<\frac{3}{4}$, where $\wp$ is the compressibility degree [36].

The above considerations show that, as long as one can neglect diffusion, the passive density fluctuations grow in a random compressible flow. One particular case of the above phenomena is the clustering of inertial particles in an incompressible turbulent flow, see [33, 35] where the theory for a general flow and the account of the diffusion effects that eventually stops the density growth were presented.

C. Vector fields in a smooth velocity

i) **Gradients of the passive scalar** . For the passive scalar gradients $\omega = \nabla\theta$ in an unforced incompressible situation, we solve (60) by simply taking the gradient of the scalar expression (66). The initial distribution is assumed statistically homogeneous with a finite correlation length. The long-time limit is independent of the initial scalar statistics [18] and it is convenient to take it Gaussian with the 2-point function $\propto \exp[-\frac{1}{2d}(r/\ell)^2]$. The averaging over the initial statistics for the generating function $\mathcal{Z}(\mathbf{y}) = \langle\exp\left[i\mathbf{y}\cdot\omega\right]\rangle$ reduces then to Gaussian integrals involving the matrix $I(t)$ determined by (27). The inverse Fourier transform is given by another Gaussian integral over $\mathbf{y}$ and one finally obtains for the PDF of ω:

$$\mathcal{P}(\omega) \;\propto\; \Big\langle(\det I)^{d/4+1/2}\,\exp\Big[-\mathrm{const.}\sqrt{\det I}\,(\omega, I\omega)\Big]\Big\rangle\;. \tag{76}$$

As may be seen from (27), during the initial period $t < t_d = |\lambda_d^{-1}|\ln(\ell/r_d)$, the diffusion is unimportant, the contribution of the matrix Q to I is negligible, the determinant of the latter is unity and ω^2 grows as the trace of I^{-1}. In other words, the statistics of $\ln\omega$ and of $-\rho_d$ coincide in the absence of diffusion. The statistics of the gradients can therefore be immediately taken over from Section II B. The growth rate $(2t)^{-1}\langle\ln\omega^2\rangle$ approaches $|\lambda_d|$ while the gradient PDF depends on the entropy function. For the Kraichnan model, the PDF is lognormal with the average $\propto d(d-1)t$ and the variance $\propto 2(d-1)t$ read directly from (24). This result was obtained by Kraichnan [53] using the fact that, without diffusion, ω satisfies the same equation as the distance between two particles, whose PDF is (25).

As time increases, the wavenumbers (evolving as $\dot{\mathbf{k}} = \sigma^T\mathbf{k}$) reach the diffusive scale r_d^{-1} and the diffusive effects start to modify the PDF, propagating to lower and lower values of ω. High moments first and then lower ones will start to decrease. The law of decay at $t \gg t_d$ can be deduced from (76). Considering this expression in the eigenbasis of the matrix I, we observe that the dominant component of ω coincides with the largest eigendirection of the I^{-1} matrix, i.e. the one along the ρ_d axis. Recalling from the Section II B that the distribution of ρ_d is stationary, we infer that $\Big\langle|\omega|^\alpha(t)\Big\rangle \;\propto\; \Big\langle(\det I)^{-\alpha/4}\Big\rangle$. The comparison with (68) shows that the decay laws for the scalar and its gradients coincide [18, 45]. This is qualitatively understood by estimating $\omega \sim \theta/\ell_{\min}$, where $\ell_{\min}$ is the smallest size of the spot. Noting that θ and $\ell_{\min}$ are independent and that $\ell_{\min} \approx \mathrm{e}^{\rho_d}\,\ell$ at large times has a stationary statistics concentrated around r_d, it is quite clear that the decrease of ω is due to the decrease of θ.

ii)**Small-scale magnetic dynamo**

The magnetic fields of stars and galaxies are thought to have their origin in the turbulent dynamo action. In this problem, the magnetic field can be treated as passive. Furthermore, the viscosity-to-diffusivity ratio is often large enough for a sizable interval of scales between the viscous and the diffusive cut-offs to be present. That is the region of scales with the fastest growth rates of the magnetic fluctuations. In this Section, we consider the generation of inhomogeneous magnetic fluctuations below the viscous scale of incompressible turbulence.

The dynamo process is caused by the stretching of fluid elements already extensively discussed above and the major new point to be noted is the role of diffusion. In an ideal conductor, when the diffusion is absent, the magnetic field satisfies the same equation as the infinitesimal separation between two fluid particles (18): $d\mathbf{B}/dt = \sigma\,\mathbf{B}$. Any chaotic flow would then produce dynamo, with the growth rate

$$\bar{\gamma} = \lim_{t\to\infty}(2t)^{-1}\langle \ln B^2\rangle, \tag{77}$$

equal to the highest Lyapunov exponent λ_1. Recall that the gradients of a scalar grow with the growth rate $-\lambda_3$ during the diffusionless stage. If the initial scale of magnetic fluctuations is l then for time less than $t_d = |\lambda_3|\ln(l/r_d)$ the growth rate is insensitive to diffusion. The long-standing problem solved in [55] was whether the presence of a small, yet finite, diffusivity could stop the dynamo growth process at $t > t_d$ (as it is the case for the gradients of a scalar).

In a smooth flow, the magnetic field can be expressed in terms of the stretching matrix W and the backward Lagrangian trajectory: $\mathbf{B}(\mathbf{r},t) = W(t;\mathbf{r})\ \mathbf{B}(\mathbf{R}(0;\mathbf{r},t),0)$. The realizations contributing to the moments of $\mathbf{B}$ are those with the inter-particle separations almost orthogonal to the (backward) expanding direction ρ_3 of W^{-1}, the share of such realizations decreases as the angle $\propto \exp(\rho_3)$. As a result, moments of the magnetic field are to be obtained by averaging moments of $B^2 \propto \exp(2\rho_1+\rho_3)$ [1, 55]. In particular, the growth rate is now $\lambda_1+\lambda_3/2$ that is less than in a perfect conductor.

Note that the gradients of a scalar field are stretched by the same W^{-1} matrix that governs the growth of the Lagrangian separations. It is therefore impossible to increase the stretching factor of the gradient and keep the particle separation within the correlation length ℓ at the same time. That is why diffusion eventually kills all the gradients while the component B^i that points into the direction of stretching survives and grows with ∇B^i perpendicular to it. This simple picture also explains the absence of dynamo in $2d$ incompressible flow, where the stretching in one direction necessarily means the contraction in the other one. The consequences of these mechanisms for the curvature of magnetic fields lines have been explored in [54].

To conclude the Chapter III, note that an important lesson to learn is that the limits $\kappa \to 0$ and $t \to \infty$ do not commute for a smooth flow. Growth/decay rates of scalars and vectors are different before and after time $t_d \sim \lambda^{-1} \ln(l/r_d)$. Note that this difference is independent of diffusivity.

IV. CASCADES OF A PASSIVE TRACER

This Chapter describes forced turbulence of the passive scalar θ under the action of pumping which is statistically stationary in time and statistically homogeneous in space. To the advection-diffusion equation

$$\partial_t \theta + (\mathbf{v} \cdot \nabla)\theta = \kappa \nabla^{\mathbf{2}} \theta + \varphi \tag{78}$$

we added the pumping φ, characterized by the variance $\langle \varphi(t, \mathbf{r})\varphi(0, 0)\rangle = \Phi(r)\delta(t)$ with $\Phi(r)$ constant at $r < L$ and decaying fast at $r > L$. The below consideration is valid for a finite-correlated pumping too, as long as the pumping correlation time in Lagrangian frame is much smaller than the time of stretching from a given scale to the pumping correlation scale L. In most physical situations the sources do not move with the fluid so that the Lagrangian correlation time of the pumping is either its Eulerian correlation time or L/V, depending on which one is smaller, here V is the typical fluid velocity.

The scalar field along the Lagrangian trajectories $\mathbf{R}(t)$ changes as

$$\frac{d}{dt}\theta(\mathbf{R}(t), t) = \varphi(\mathbf{R}(t), t). \tag{79}$$

The N-th order scalar correlation function $\langle \theta(\mathbf{r}_1, t) \ldots \theta(\mathbf{r}_N, t)\rangle$ is therefore given by

$$\int_0^t \ldots \int_0^t \Big\langle \varphi(\mathbf{R}_1(s_1), s_1) \ldots \varphi(\mathbf{R}_N(s_N), s_N) \Big\rangle ds_1 \ldots ds_N, \tag{80}$$

with the Lagrangian trajectories satisfying the final conditions $\mathbf{R}_i(t) = \mathbf{r}_i$. For the sake of simplicity we have written down the expression for the case where the scalar field was absent at $t = 0$. Averaging (80) over the Gaussian pumping we get for $N = 2$:

$$\langle\theta(\mathbf{r}_1, t)\theta(\mathbf{r}_2, t)\rangle = \left\langle \int_0^t \Phi\left(\mathbf{R}_{12}(s)\right) ds \right\rangle \ . \tag{81}$$

Higher-order correlations are obtained similarly to (81) by using the Wick rule to average over the Gaussian forcing statistics and the remaining average is made over the ensemble of Lagrangian trajectories:

$$\begin{aligned}&\langle\theta(t, \mathbf{r}_1) \dots \theta(t, \mathbf{r}_{2n})\rangle = \int_0^t dt_1 \dots dt_n \\ &\times \langle\Phi(R(t_1|T, \mathbf{r}_{12})) \dots \Phi(R(t_n|T, \mathbf{r}_{2n-1,2n}))\rangle + \dots \ ,\end{aligned} \tag{82}$$

The functions Φ in (81,82) restrict integration to the time intervals where $R_{ij} < L$. If the Lagrangian trajectories separate, the correlation functions reach at long times the stationary form for all r_{ij}. Such steady states correspond to a direct cascade of the tracer (i.e. from large to small scales) and are considered in Sect. IV A. As we have seen in Section II D, particles cluster in flows with high enough compressibility. In this case, the correlation functions acquire parts which are independent of r and grow proportional to time: when Lagrangian particles cluster rather than separate, tracer fluctuations grow at larger and larger scales — phenomenon that can be loosely called an inverse cascade of a passive tracer [56, 57] and which is considered in Sect. IV B.

A. Direct cascade

Here we consider incompressible (and weakly compressible) flows where particles separate and the steady state exists. Let us first present the standard flux phenomenology. Assuming stationarity, one derives the flux relation of θ^2

$$\Big\langle(\mathbf{v}_1 \cdot \nabla_1 + \mathbf{v}_2 \cdot \nabla_2)\, \theta_1\theta_2\Big\rangle \ + \ 2\,\kappa \Big\langle \nabla_{_1}\theta_1 \cdot \nabla_{_2}\theta_2 \Big\rangle \ = \ \Phi(r_{12})\,, \tag{83}$$

where indices designate spatial points. The relative strength of the two terms on the left hand side depends on the distance. At some r_d (called the diffusion scale) the advection is comparable to diffusion. In the "diffusive interval" $r_{12} \ll r_d$, the diffusion term dominates in the left hand side of (83). Taking the limit of vanishing separations, we infer that the mean dissipation rate is equal to the mean injection rate $\bar{\epsilon} \equiv \langle\kappa(\nabla\theta)^2\rangle = \frac{1}{2}\Phi(0)$. This illustrates the aforementioned phenomenon of the "dissipative anomaly": the limit $\kappa \to 0$

of the mean dissipation rate is non-zero despite the explicit κ factor in its definition. In the "convective interval" $r_d \ll r_{12} \ll L$, one can drop the diffusive term in (83) while still neglect r-dependence in Φ:

$$\Big\langle (\mathbf{v}_1 \cdot \nabla_1 + \mathbf{v}_2 \cdot {}_2)\, \theta_1 \theta_2 \Big\rangle \approx \Phi(0) \ . \tag{84}$$

The relation (84) states that the mean flux of θ^2 stays constant within the convective interval and expresses analytically the downscale scalar cascade. The physical picture is that stretching and contraction by an inhomogeneous velocity provides for a cascade of a scalar from the pumping scale L (where it is generated) to the diffusion scale r_d (where it is dissipated). For velocity fields scaling as $\delta v \propto r^{\alpha}$, dimensional arguments suggest that [58, 59]

$$\delta\theta \propto r^{(1-\alpha)/2} \ . \tag{85}$$

This relation gives a proper qualitative understanding that the degrees of roughness of the scalar and the velocity are complementary, yet it suggests a wrong scaling for the scalar structure functions of order higher than two (see Sect. IV A below). For smooth velocity, (84) correctly suggests $\langle \theta_1 \theta_2 \rangle \propto ln\, r_{12}$ [60].

i) **Direct cascade in a smooth velocity**. In this subsection, all the scales are supposed to be much smaller than the viscous scale of turbulence so that the velocity field can be assumed spatially smooth and we may use the Lagrangian description developed in Sect. II B. We restrict ourselves by the incompressible case when $\lambda_3 < 0$ so that particles do separate and the steady state exists. We first treat the interval of scales between the diffusion scale r_d and the pumping scale L, which is called convective interval. Formula (81) simply tells us that the stationary pair correlation function of a tracer is twice the flux [i.e. $\Phi(0)$] times the average time that two particles spent in the past within the correlation scale of the pumping:

$$\begin{aligned} &\langle \theta(0,t)\theta(\mathbf{r},t)\rangle = -\lambda_3^{-1}\Phi(0)\ln(L/r)\,, \qquad \langle \theta^2 \rangle = -\lambda_3^{-1}\Phi(0)\ln(L/r_d) \\ &S_2(r) = -\lambda_3^{-1}\Phi(0)\ln(r/r_d) \ . \end{aligned} \tag{86}$$

Deep inside the convective interval when $r \ll L$, the statistics of passive scalar approaches Gaussian. Indeed, when we average (82) over $\mathcal{P}(\rho)$ and perform summation over all sets of the pairs of the points $\mathbf{r}_i$, the reducible part in

$$\langle \Phi[r_{12} e^{\rho_d(t_1)}] \ldots \Phi[r_{2n-1,2n} e^{\rho_d(t_n)}] \rangle$$

prevails for n less that the ratio between the transfer time $|\lambda_d|^{-1}\ln(L/r)$ and the correlation time τ_s of the stretching rate fluctuations. The reason is that the irreducible contributions have less large logarithmic factors than the reducible ones [61]. Therefore, for $n \ll n_{cr} \simeq (\lambda_d\tau_s)^{-1}\ln(L/r)$, the statistics of the passive tracer is Gaussian. Since $L \gg r$, then $n_{cr} \gg 1$. The single-point statistics is Gaussian up to $n_{cr} \simeq (\lambda_d\tau_s)^{-1}\ln(L/r_d)$. Larger n correspond to the exponential tails of tracer's pdf. The physics behind this is transparent and most likely valid also for a non-smooth velocity (even though the consistent derivation is absent in the non-smooth case). Indeed, large values of the scalar can be achieved only if during a large time the pumping works uninterupted by advection (which eventually brings diffusion into play). When the time in question is much larger than the typical stretching time from r_d to L then the stretching events can be considered as a Poisson process and the probability that no stretching occurs during time t is $\exp(-ct)$. Integrating that with a pumping-produced distribution we get: $\mathcal{P}(\theta) \propto \int dt \exp(-ct - \theta^2/2\Phi t) \propto \exp(-\theta\sqrt{2c/\Phi})$. The detailed derivation for a smooth case can be found in [18, 19, 26, 61].

From a general physical viewpoint, it is of interest to understand the properties of turbulence at scales larger than the pumping scale, i.e. at $r > L$. If only direct cascade exists, one may expect equilibrium equipartition at large scales with the effective temperature determined by small-scale turbulence [62, 63]. The peculiarity of our problem is that we consider scalar fluctuations at the scales that are larger than the scale of excitation yet smaller than the correlation scale of the velocity field, which provides for mixing of the scalar. In a smooth flow, the statistics at large scales lacks scale invariance and is very far from Gaussian [64]. The probability for two points separated by r_{12} to belong to the same blob of scalar originated from the pumping scale L is $(L/r_{12})^d$. Therefore, the pair correlation function is proportional to r_{12}^{-d}. Since an advection by a smooth velocity preserves straight lines then the same answer is true for the correlation function of arbitrary order if all the points lie on a line (when the largest distance between points was within L then all other distances were as well): $C_n \propto r^{-d}$. The fact that for collinear geometry $C_{2n}/C_2^n \sim (r/L)^{(n-1)d} \gg 1$ is due to strong correlation of the points along the line.

When we consider a non-collinear geometry, the opposite takes place, namely the stretching of different non-parallel vectors is generally anti-correlated because of incompressibility and volume conservation. Consider two-dimensional case and the contribution from $\int \mathrm{d}t_1 \mathrm{d}t_2 \langle \Phi[R_{12}(t_1)]\Phi[R_{34}(t_2)]\rangle$ into the fourth-order correlation function. Since the area

$|\mathbf{R}_{12}\times\mathbf{R}_{34}|$ is conserved, the answer is crucially dependent on the relation between $|\mathbf{r}_{34}\times\mathbf{r}_{12}|$ and L^2. When $|\mathbf{r}_{34}\times\mathbf{r}_{12}| \ll L^2$ we have a collinear answer $C_4 \propto r^{-2}$. Let us now consider the case of non-collinear geometry and find the probability of an event that during evolution R_{12} became of the order L, and then, at some other moment of time, R_{34} reached L (only such events will contribute into C_4). There is a reducible part in pumping, which makes C_4 nonzero (decaying as power of r_{ij}) even when $|\mathbf{r}_{34}\times\mathbf{r}_{12}| \gg L^2$. The probability that R_{12} came to L is L^2/r_{12}^2. Due to area conservation, there is an anti-correlation between R_{12} and R_{34}: if $R_{12}\sim L$, than $R_{34}\sim r_{12}r_{34}/L$. So probability for R_{34} to come back to L is $L^2/(r_{12}r_{34}/L)^2 = L^4/r_{12}^2r_{34}^2$. Therefore, the total probability can be estimated as L^6/r^6, which is much smaller than the naive Gaussian estimation L^4/r^4 while the collinear answer L^2/r^2 is much larger than Gaussian. Wee that the breakdown of scale invariance is related to the Lagrangian conservation laws. More details can be found in [1, 64].

ii) **Anomalies of tracer statistics in a non-smooth velocity**. In this subsection we shall analyze the steady cascade of a scalar in the inertial interval of scales where the velocities are effectively non-smooth. Here the main fundamental issue, as in any turbulence, is the degree of universality of scalar statistics [say, the PDF $\mathcal{P}(\delta\theta, r)$ of the scalar difference $\delta\theta$ measured at two points distance r apart] in the convective interval that is at $L \gg r \gg r_d$. One may ask, in particular, what symmetries exist in the convective interval. Finite-scale pumping breaks scale invariance while diffusion breaks time reversibility. Are those symmetries restored when $L\to\infty$ and $r_d \to 0$? As discussed in Sect. II C, in non-smooth velocities an explosive separation of trajectories separates however close particles in a finite time. That provides for the dissipation of the single-point moments of the scalar when $\kappa\to 0$. We called this phenomenon dissipative anomaly which tells that time-reversibility remains broken even when symmetry-breaking factor tends to zero. We shall see in this subsection that the same phenomenon of an explosive separation generally breaks the scale invariance of $\mathcal{P}(\delta\theta, r)$. Indeed, N-th moment of $\mathcal{P}(\delta\theta, r)$, the structure function $S_{N}(r)$, is expressed via the N-particle propagator which generally cannot be reduced to the two-particle propagator even though all particles end up in two points. We shall see that the structure functions are proportional to the respective zero modes which, as we learnt, have non-trivial scaling exponents ζ_{N}. When ζ_{N} is not a linear function of N it is called anomalous scaling since the scale invariance of $\mathcal{P}(\delta\theta, r)$ is not restored even at the limit $L\to\infty$. Explosive separation of trajectories is necessary but not sufficient condition of anomalous scaling; as we have seen in Sect. II E the turbulent diffusivity that govern interparticle separation must be scale-dependent which requires velocity field to have power correlation in space. Indeed, anomalous scaling disappears both for smooth and for extremely rough white-in-space velocity (respectively, cases $\xi = 2$ and $\xi = 0$ in the Kraichnan model).

In a non-smooth flow with $\delta v \propto r^\alpha$ the time to separate is proportional to $L^{1-\alpha} - r^{1-\alpha}$. Similarly, the pair structure function $S_2(r) = \langle(\theta_1 - \theta_2)^2\rangle$ is proportional to the time it takes for two coinciding particles to separate to a distance r. For $\delta v \propto r^\alpha$, one gets $S_2 \propto r^{1-\alpha}$ in agreement with (85). The analytic treatment of the multi-point correlation functions of the tracer is possible for the Kraichnan model (11, 34). Making a straightforward Gaussian averaging of (78) over the statistics of pumping and velocity one gets the following equation for the n-point simultaneous correlation function of the scalar $C_n(t, \mathbf{r}_1 \ldots \mathbf{r}_n) = \langle\theta(t, \mathbf{r}_1) \ldots \theta(t, \mathbf{r}_n)\rangle$ [19]

$$\partial_t C_n + M_n C_n = \sum_{k,l} \Phi(r_{kl}) C_{n-2} \ . \tag{87}$$

Here the operator M_n is given by (49) and, of course, (87) can be derived in a Lagrangian way by using the propagator (46) [26, 65]. The great simplification of scalar description in the Kraichnan model is due to the fact that the set of (87) for different n presents a recursive problem since the rhs is expressed in terms of lower-order correlation functions. There is no closure problem and any correlation function satisfies closed equation after the lower-order functions are found. We consider steady state and drop the time derivative.

One starts from the pair correlation function that depends on a single variable and satisfies an ordinary differential equation $M_2 C_2(r) = \Phi(r)$ [4] which is the Yaglom flux relation (84) for the Kraichnan model. This equation with two boundary conditions (zero at infinity and finiteness at zero) can be explicitly integrated

$$r^{1-d}\partial_r\Big[(d-1)D_1 r^{d-1+\xi} + 2\kappa r^{d-1}\Big]\partial_r C_2(r) = \Phi(r), \tag{88}$$

$$C_2(r) = \int_r^\infty \frac{x^{1-d}dx}{x^\xi + r_d^\xi} \int_0^x \Phi(y) y^{d-1} dy \ ,$$

where we introduced the diffusion scale $r_d^\xi = 2\kappa/D_1(d-1)$. Let us remind that we consider the pumping correlated on the scale L assumed to be much larger than r_d. There are thus three intervals of the distinct behavior. At $(D_0/D_1)^{1/\xi} \gg r \gg L$ the pair correlation function is given by the zero mode of $M_2(\kappa = 0)$: $C_2(r) = r^{2-\xi-d}\bar{\Phi}/d(d-1)(d+\xi-2)D_1$ which may be thought of as Rayleigh-Jeans equipartition $\langle\theta_k\theta_{k'}\rangle = \delta(k+k')\bar{\Phi}/\omega_k$ with the temperature $\bar{\Phi} = \int \Phi(x)x^{d-1}dx$ and $\omega_k = k^{2-\xi}d(d-1)(d+\xi-2)D_1$ being an inverse stretching rate. At the convective interval, $L \gg r \gg r_d$, C_2 is equal to a constant (another zero mode of M_2) plus the inhomogeneous part (zero mode of M_2^2): $S_2(r) = \langle[\theta(\mathbf{r})-\theta(0)]^2\rangle = 2C_2(0) - 2C_2(r) = r^{2-\xi}\Phi(0)/d(d-1)(d+\xi-2)(2-\xi)D_1$. Note that in the convective interval the degrees of roughness of the scalar and velocity are indeed complementary, a smooth velocity corresponds to a roughest scalar and vice versa. And finally, $S_2(r) \approx r^2\Phi(0)/4\kappa d$ at the diffusive interval. Note though that $S_2(r)$ is not analytic at zero since it expansion contains noninteger powers $r^{2n+\xi}$, $n = 1, 2, \ldots$. This is an artefact of extending velocity nonsmoothness to the smallest scales that is setting the viscous scale to zero (i.e. Schmidt/Prandtl number to infinity).

Consider now high-order correlation functions in the convective interval. Solving recursively the stationary version of (87) one finds that C_n generally contains powers from $r^{2-\xi}$ to $r^{n(2-\xi)}$ plus a constant and other zero modes of M_n [39, 40, 43, 44]. Note that one cannot satisfy the boundary conditions at large scales without the zero modes. In the structure function, $S_n(r) = \langle[\theta(\mathbf{r}) - \theta(0)]^n\rangle$, all the terms cancel except for the irreducible zero mode. We thus conclude that $S_n(r) = A_n r^{\zeta_n}$. Note that only S_2 is universal (that is determined by the flux only), all the other A_n depend on the pumping statistics [1]. As we have seen in Sect. II E, the anomalous exponents $\Delta_{2n} = n\zeta_2 - \zeta_{2n} = n(2-\xi) - \zeta_{2n}$ are positive for any $d < \infty$ and $\xi \neq 0, 2$. That means an anomalous scaling and small-scale intermittency of the scalar field: the ratio S_{2n}/S_2^n grows as r decreases. In the perturbative domain, $n\xi/d(2-\xi) \ll 1$, the scaling exponents are given by (57). At $n \gg d(2-\xi)/\xi$, the dependence $\zeta(n)$ saturates which means that the sharp fronts of the scalar determine high moments. The saturation value has been calculated for large d: $\zeta_n \to d(2-\xi)^2/8\xi$ [31].

It is instructive to discuss the limits $\xi = 0, 2$ and $d = \infty$ from the viewpoint of the scalar statistics. Since the scalar field at any point is the superposition of fields brought from d directions then it follows from a central limit theorem that scalar's statistics approaches Gaussian when space dimensionality d increases. In the case $\xi = 0$, an irregular velocity field acts like Brownian motion so that turbulent diffusion is much like linear diffusion: scalar statistics is Gaussian provided the input is Gaussian. What is general in both limits $d = \infty$ and $\xi = 0$ is that the degree of Gaussianity (say, flatness S_4/S_2^2) is independent of the ratio r/L. Quite contrary, we have seen in Sect. IV A that $\ln(L/r)$ is the parameter of Gaussianity in the Batchelor limit so that statistics is getting Gaussian at small scales whatever the input statistics. At $\xi = 2$ the mechanism of Gaussianity is temporal rather than spatial: since the stretching is exponential in a smooth velocity field then the cascade time grows logarithmically as the scale decreases. That leads to the essential difference: at small yet nonzero ξ/d, the degree of non-Gaussianity increases downscales while at small $(2-\xi)$ the degree of non-Gaussianity first decreases downscales until $\ln(L/r) \simeq 1/(2-\xi)$, and then starts to increase, the first region grows with ξ approaching 2. Already that simple reasoning shows that the perturbation theory is singular at the limit $\xi = 2$, which formally is manifested by the many-point correlation functions having singularity (smeared by molecular diffusion only) at the collinear geometry [66].

Note that the dependence $\Delta_n(\xi)$ has to be nonmonotonic since $\Delta_n(0) = \Delta_n(2) = 0$. There is a transparent physics behind the nonmonotonic dependence $\Delta(\xi)$ because the influence of velocity nonsmoothness (measured by ξ) on scalar intermittency is twofold: if one considers scalar fluctuation of some scale then velocity harmonics with comparable scales produce intermittency while small-scale harmonics act like diffusivity and smooth it out. At $\xi_* < \xi < 2$ the first mechanism is stronger while at $0 < \xi < \xi_*$ the second one takes over. Still, our understanding is only qualitative here, we don't know how the maximum position ξ_* depends on n and d.

The anomalous exponents determine also the moments of the dissipation field $\epsilon = \kappa|\nabla\theta|^2$. By a straightforward analysis of (87) one can show that $\langle\epsilon^n\rangle = c_n\langle\epsilon\rangle^n(L/r_d)^{\Delta_{2n}}$ [39, 43]. Here the mean dissipation $\langle\epsilon\rangle = \Phi(0)$ while the dimensionless constants c_n are determined by the fluctuations of dissipation scale, most likely they are of the form n^{qn} with yet unknown q. In the perturbative domain, $n \ll d(2-\xi)/\xi$, the main factor is $(L/r_d)^{\Delta_{2n}}$ and the dissipation PDF is close to lognormal since Δ_{2n} is a quadratic function of n [43], the form of the distant PDF tails are unknown.

The scalar correlation functions decay by power laws at scales r larger than that of the pumping. Remind that the time two particles spent within L is less than the time they spent within r by the small volume factor $(L/r)^d$. Therefore, the pair correlation function is proportional to $r_{12}^{1-\alpha-d}$ for $\delta v \propto r^\alpha$ or to $r^{2-\xi-d}$ in the Kraichnan model. Note that $\mathcal{M}_2 r^{2-\xi-d} \propto \delta(r)$. The analysis of higher-order correlation functions is simplified in a non-smooth case since straight lines are not preserved and no strong angular dependencies of the type encountered in the smooth case are thus expected. To determine the scaling behavior of the correlation functions, it is therefore enough to focus on a specific geometry. Consider for instance the equation $M_4 C_4 = \sum \chi(r_{ij}) C_2(r_{kl})$ for the fourth order correlation function. A convenient geometry to analyze is that with one distance among the points, say r_{12}, much smaller than the other distances which are of order R. At the dominant order in r_{12}/R, the solution of the equation is $C_4 \propto C_2(r_{12}) C_2(R) \sim (r_{12}\, R)^{2-\xi-d}$. Similar arguments apply to arbitrary orders. We conclude that the scalar statistics at $r \gg \ell$ is scale-invariant, i.e. $C_{2n}(\lambda \underline{\mathbf{r}}) = \lambda^{n(2-\xi-d)}\, C_{2n}(\underline{\mathbf{r}})$ as $\lambda \to \infty$. Note that the statistics is generally non-Gaussian when the distances between the points are comparable. As ξ increases from zero to two, the deviations from the Gaussianity starts from zero and reach their maximum for the smooth case described in Sect. IV A.

The results for spatially non-smooth flows have mostly been derived within the framework of the Kraichnan model with the white forcing. The conditions on the forcing are not crucial and may be easily relaxed since the scaling properties of the scalar correlation functions are universal with respect to the forcing, i.e. independent of its details, while the constant prefactors are not [1]. The situation with the velocity field is more interesting and nontrivial. Even though a short-correlated flow might in principle be produced by an appropriate forcing, all the cases of physical interest have a finite correlation time. The very existence of closed equations of motion for the particle propagators, which we heavily relied upon, is then lost. The existing numerical and experimental evidence is that the basic mechanisms for scalar intermittency are quite robust: anomalous scaling is still associated with statistically conserved quantities and the expansion (51) for the multiparticle propagator seems to carry over. The specific statistics of the advecting flow affects only quantitative details, such as the numerical values of the exponents [1].

B. Inverse cascade in a compressible flow

If the trajectories are unique, particles that start from the same point will remain together throughout the evolution. That means that advection preserves all the single-point moments $\langle \theta^N \rangle(t)$. Note that the conservation laws are statistical: the moments are not dynamically conserved in every realization, but their average over the velocity ensemble are. In the presence of pumping, the moments are the same as for the equation $\partial_t \theta = \varphi$ in the limit $\kappa \to 0$ (nonsingular now). It follows that the single-point statistics is Gaussian, with $\langle \theta^2 \rangle$ coinciding with the total injection $\Phi(0)t$ by the forcing. That growth is produced by the flux of scalar variance toward the large scales. As explained in Section IV A, correlation functions at very large scales are related to the probability for initially distant particles to come close. In a strongly compressible flow, the trajectories are typically contracting, the particles tend to approach and the distances will reduce to the forcing correlation length L (and smaller) for long enough times. On a particle language, the larger the time the large the distance starting from which particle come within L. As afar as the field θ is concerned, strong correlations at larger and larger scales are therefore established as time increases, signaling the inverse cascade process [37, 38].

The uniqueness of the trajectories greatly simplifies the analysis of the PDF $\mathcal{P}(\delta\theta, r)$. Indeed, the structure functions involve initial configurations with just two groups of particles separated by a distance r. The particles explosively separate in the incompressible case and we are immediately back to the full N-particle problem. Conversely, the particles that are initially in the same group remain together if the trajectories are unique. The only relevant degrees of freedom are then given by the intergroup separation and we are reduced to a two-particle dynamics. It is therefore not surprising that the scaling behaviors at the various orders are simply related in the inverse cascade regime [37, 38].

V. ACTIVE TRACERS

As we have have learnt in the previous Chapters, the most fundamental property of the propagators is whether they describe particles separating or clustering backwards in time. That property alone determines the direction of the cascade for the passive tracer. Another important distinction is whether the propagators possess the collapse property (44) at the limit $\kappa \to 0$. The absence of the property makes anomalies possible for the passive tracer. Here we consider the Lagrangian invariants (conserved along the fluid trajectories without pumping and diffusion) which are active that is related to the velocity that transports them. We shall see that the correlation between the Lagrangian tracer and velocity field makes it impossible to derive the direction of the cascade solely from the behavior of trajectories. In some situations, passive and active tracers cascade in opposite directions in the same velocity field. In Sect. V A we first describe Burgers turbulence which has clustering for the majority of trajectories (going to full measure in the inviscid limit) and collapse property for propagators, a passive tracer then undergoes inverse cascade in such velocity. On the contrary, powers of velocity (active tracers) have their dissipation determined by the minority of trajectories that separate. Velocity statistics thus corresponds to the direct cascade with both dissipation anomaly and anomalous scaling. We then consider 2d magnetohydrodynamics where velocity is non-smooth (separation of trajectories and no collapse) so that passive scalar must have direct cascade and dissipative anomaly. On the contrary, magnetic vector potential (active scalar) influences velocity field via Lorentz force in such a way that only those trajectories can come to the same point that carry the same value of the potential. As a result, the potential cascades upscales and there are no anomalies in the magnetic field statistics. In Sect. V B we consider 2d incompressible turbulence and describe the relations between passive tracer and active tracers (vorticity). We argue that in the domain of the direct vorticity cascade, both tracers cascade downscales in a very similar way. On the contrary, in the domain of the inverse energy cascade, passive scalar undergoes direct cascade while vorticity has some kind of equipartition and no flux.

A. Activity changing cascade direction

i) **Burgers turbulence**. We start from the simplest case of Burgers turbulence whose inviscid version describes a free propagation of fluid particles with velocity being Lagrangian invariant, while viscosity provides for a local interaction:

$$\partial_t v + v v_x - \nu v_{xx} = f \tag{89}$$

Without force, the evolution described by (89) conserves total momentum $\int v\,dx$. Burgers equation describes one-dimensional acoustics and many other systems. Under the action of a large-scale forcing (or in free decay of large-scale initial data) a cascade of kinetic energy towards the small scales takes place. The nonlinear term provides for steepening of negative gradients and the viscous term causes energy dissipation in the fronts that appear this way. In the limit of vanishing viscosity, the energy dissipation stays finite due to the appearance of velocity discontinuities called shocks. The Lagrangian statistics is peculiar in such an extremely non-smooth flow and can be closely analyzed even though it does not correspond to a Markov process. Forward and backward Lagrangian statistics are different, as it has to be in an irreversible flow. Lagrangian trajectories stick to the shocks. That provides for a strong interaction between the particles and results in an extreme anomalous scaling of the velocity field. A tracer field passively advected by such a flow undergoes an inverse cascade.

Here we briefly describe the picture of Burgers turbulence at the limit of small viscosity (see [1, 67] and the references therein for the details). At vanishing viscosity, the Burgers equation may be considered as describing a gas of particles moving in a force field. Indeed, in the Lagrangian frame defined for a regular velocity by $\dot{X} = v(X,t)$, relation (89) becomes the equation of motion of non-interacting unit-mass particles whose acceleration is determined by the force:

$$\ddot{X} = f(X,t)\ . \tag{90}$$

In order to find the Lagrangian trajectory $X(t;x)$ passing at time zero through x it is then enough to solve the second order equation (90) with the initial conditions $X(0) = x$ and $\dot{X}(0) = v(x,0)$. For sufficiently short times such trajectories do not cross and the Lagrangian map $x \mapsto X(t;x)$ is invertible. One may then reconstruct v at time t from the relation $v(X(t),t) = \dot{X}(t)$. At longer times, however, the particles collide creating velocity discontinuities, i.e. shocks. Once created shocks never disappear but they may merge so that they form a tree branching backward in time. The crucial question for the Lagrangian description of the Burgers velocities is what happens with the fluid particles after they reach shocks where their equation of motion $\dot{x} = v(x,t)$ becomes ambiguous. The question may be easily answered by considering the inviscid case as a limit of the viscous one where shocks become steep fronts with large negative velocity gradients. It is easy to see that the Lagrangian particles are trapped within such fronts and keep moving with them. In other words, the two particles arriving at the shock from the right and the left at a given moment aggregate upon the collision. Momentum is conserved so that their velocity after the collision is the mean of the incoming ones (recall that the particles have unit mass) and is equal to the velocity of the particles moving with the shock that have been absorbed at earlier times. The shock speed is thus the mean of the velocities on both sides of the shock. Note that in the presence of shocks the Lagrangian map becomes many-to-one, compressing whole space-intervals into the shock locations.

The Lagrangian picture of the Burgers velocities allows for a simple analysis of advection of scalar quantities carried by the flow. In the inviscid and diffusionless limit, the advected tracer satisfies the evolution equation

$$\partial_t\theta\,+\,\bar{v}\,\partial_x\theta\;=\;\varphi\,, \tag{91}$$

where φ represents an external source. As usual, the solution of the initial value problem is given in terms of the PDF $P(x,t;y,0\,|\,v)$ to find the backward Lagrangian trajectory at y at time 0, given that at later time t it passed by x. Except for the discrete set of time t shock locations, the backward trajectories are uniquely determined by x. As a result, a smooth initial scalar will develop discontinuities at shock locations but no stronger singularities. Since a given set of points $(x_{_1},\dots,x_{_N})\equiv \underline{x}$ avoids the shocks with probability 1, the joint backward PDF's of N trajectories $\mathcal{P}_{_N}(\underline{x};\underline{y};-t)$ should be regular for distinct x_n and should possess the collapse property (44). This leads to the conservation of $\langle\theta^2\rangle$ in the absence of scalar sources and to the linear pumping of the scalar variance when a stationary source is present. Such behavior corresponds to an inverse cascade of the passive scalar as in Sect. IV B.

As usual in compressible flows, the advected density n satisfies the continuity equation

$$\partial_t n + \partial_x(\bar{v}n) \;=\; \varphi \tag{92}$$

different from (91) for the tracer. The solution of the initial value problem is given by the forward Lagrangian PDF: $n(x,t)=\int p(y,0;x,t|v)n(y,0)dy$. Since the trajectories collapse, a smooth initial density will become singular under the evolution, with δ-function contributions concentrating all the mass from the regions compressed to shocks by the Lagrangian flow. Since the trajectories are determined by the initial point y, the joint forward PDF's $\mathcal{P}_{_N}(\underline{y};\underline{x};t)$ should have the collapse property (44) but they will also have contact terms in x_n's when the initial points y_n are distinct. Such terms signal a finite probability of the trajectories to aggregate in the forward evolution, the phenomenon that we have already met in the strongly compressible Kraichnan model discussed in Sect. (II D). The velocity gradient $\partial_x v$ is an example of an (active) density satisfying equation (92) with $\varphi=\partial_x f$. The behavior of the Lagrangian PDF's and the advected scalars summarized above have been established by a direct calculation in freely decaying Burgers velocities with random Gaussian finitely-correlated initial potentials ϕ [68].

The Burgers velocity itself and all its powers constitute an example of advected scalars. Indeed, the equation of motion (89) may be also rewritten as

$$(\partial_t + \bar{v}\,\partial_x - \lambda\,f)\,\mathrm{e}^{\lambda\,v} \;=\; 0 \tag{93}$$

from which the relation (91) for $\theta=v^n$ and $\varphi=nfv^{n-1}$ follows. Of course, v^n are active scalars so that in the random case their initial data, the source terms, and the Lagrangian trajectories are not independent, contrary to the case of passive scalars. That correlation makes the unlimited growth of $\langle v^2 \rangle$ impossible: the larger the value of local velocity, the faster it creates a shock and dissipates the energy. The difference between active and passive tracers is thus substantial enough to switch the direction of the energy cascade from inverse for the passive scalar to direct for the velocity. Indeed, in the presence of the force,

$$v(x,t) \;=\; v(x(0),0) + \int_0^t f(x(s),s)\, ds\,, \tag{94}$$

along the Lagrangian trajectories. The velocity is an active scalar and the Lagrangian trajectories are evidently dependent on the force that drives the velocity. One cannot write a formula like (81) obtained by two independent averages over the force and over the trajectories. Nevertheless, the main contribution to the distance-dependent part of the 2-point function $\langle v(x,t)\, v(x',t)\rangle$ is due, for small distances, to realizations with a shock in between the particles. It is insensitive to a large-scale force and hence approximately proportional to the time that the two Lagrangian trajectories ending at x and x' take to separate backwards to the injection scale L. With a shock in between x and x' at time t, the initial backward separation is linear so that the second order structure function becomes proportional to Δx. Other structure functions may be analyzed similarly and give the same linear dependence on the distance (all terms involve at most two trajectories):

$$\langle |\Delta v|^p \rangle \propto \Delta x\,, \qquad p \geq 1\ . \tag{95}$$

In particular, one can obtain the exact relations $\langle |\Delta v|^{2n+1} \rangle = -4(2n+1)\epsilon_n x/(2n-1)$ where ϵ_n are the mean dissipation rates of the inviscid integrals $\int v^{2n}\, dx/2$ which stay finite in the inviscid limit (consider, for instance, the shock-wave solution $v = 2u\{1+\exp[u(x-ut)/\nu]\}^{-1}$). We thus see that the same velocity field of forced Burgers equation gives no dissipative anomaly and inverse cascade for a passive tracer while provides for a dissipative anomaly and direct cascade of the active tracers (powers of velocity itself).

ii) **Two-dimensional magnetohydrodynamics**. Another example of an active tracer having its cascade opposite to that of a passive one is that of the magnetic vector potential in 2d MHD [69]. Magnetic vector potential a in 2d is related to the magnetic field as follows $\mathbf{B} = (-\partial_x a, \partial_y a)$. It satisfies the advection-diffusion equation with forcing:

$$\partial_t a + (\mathbf{v}\cdot\nabla) a = \kappa \Delta a + f_a\ . \tag{96}$$

Magnetic field acts on velocity by the Lorentz force:

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \nu\Delta\mathbf{v} - \nabla p - \Delta a \nabla a + f_v \ . \tag{97}$$

Numerics show that the velocity field is non-smooth so that the passive scalar must undergo a direct cascade and dissipation stays finite when diffusivity tends to zero. On the contrary, vector potential a undergoes inverse cascade [69]. The Lagrangian explanation for that remarkable fact is that even though different trajectories may come to the same point they must all bring the same value of a (otherwise Lorentz force would be infinite). That type of correlation between the trajectory and the value of active scalar that it carries provides for the absence of anomalies in a statistics even for a non-smooth velocity [70]. The PDF $\mathcal{P}(a,t)$ is Gaussian with a variance linearly growing with time, the PDF $\mathcal{P}(\delta a, r)$ of the increments is self-similar [69, 70].

B. Two-dimensional incompressible turbulence

Taking *curl* of two-dimensional Navier-Stokes equation and using incompressibility $\operatorname{div}\mathbf{v} = 0$ one obtains the advection-diffusion equation for the vorticity $\omega = curl\,\mathbf{v}$:

$$\partial_t \omega + (\mathbf{v}\cdot\nabla)\omega = \kappa\Delta a + \varphi \ . \tag{98}$$

The vorticity and all its powers are thus scalar Lagrangian invariants of the inviscid dynamics in two dimensions. In the presence of an external pumping φ injecting energy and enstrophy (squared vorticity), it is clear that both quantities may flow throughout the scales. If both cascades are present, they cannot go in the same direction: the different dependence of the energy and the enstrophy on the scale prevents their fluxes to be both constant in the same interval of scales. Since one cannot provide a turbulent cascade by a potential flow (completely determined by boundaries in 2d) then energy cannot flow to small scales where a finite energy dissipation would mean an infinite vorticity dissipation at the limit $\nu \to 0$. The natural conclusion is that, given a single pumping at some intermediate scale, the energy and the vorticity flow towards the large and the small scales, respectively [71–74].

i) **Direct vorticity cascade in 2D**. The basic knowledge of the Lagrangian dynamics presented in the Sections II B and IV is essentially everything one needs for understanding the direct cascade. The vorticity in 2D is a scalar and the analogy between the cascades of the vorticity and the passive scalar was noticed by Batchelor and Kraichnan already in the sixties. The vorticity is not passive though and such analogies may be very misleading as shown in the previous Section (another misleading analogy is that between the vorticity and the magnetic field in 3D which would wrongly suggest that dynamo is absent when viscosity equals magnetic diffusivity).

The basic flux relation for the enstrophy cascade is analogous to (84):

$$\langle (\mathbf{v}_1 \cdot \nabla_1 + \mathbf{v}_2 \cdot \nabla_2)\omega_1\omega_2 \rangle = \langle \varphi_1\omega_2 + \varphi_2\omega_1 \rangle = P_2 \ . \tag{99}$$

The subscripts indicate the spatial points $\mathbf{r}_1$ and $\mathbf{r}_2$ and the pumping is assumed to be Gaussian with $\langle \varphi(\mathbf{r},t)\phi(\mathbf{0},0)\rangle = \delta(t)\tilde{\Phi}(r/L)$ decaying rapidly for $r > L$. The constant $P_2 \equiv \tilde{\Phi}(0)$, of dimensionality $time^{-3}$, is the input rate of the enstrophy ω^2. Equation (99) states that the enstrophy flux is constant in the inertial range, that is for r_{12} much smaller than L and much larger than the viscous scale. A simple power counting suggests that the velocity difference scales as the first power of r_{12}. That roughly fits the idea of a scalar cascade in a spatially smooth velocity: as was discussed in Sect. IV, passive scalar correlation functions are logarithmic in that case and we expect the same of vorticity. Of course, logarithm means that velocity is actually (weakly) non-smooth which provides for a nonzero vorticity dissipation in the inviscid limit. Hypothetical power-law vorticity spectra [75–77] must be structurally unstable [78].

The physics of the enstrophy cascade is basically the same as that for a passive scalar: a fluid blob embedded into a larger-scale velocity shear is stretched along one direction and compressed along another; that provides for the vorticity flux toward the small scales, with the rate of transfer proportional to the strain. One can show that the vorticity correlation functions at a given scale are indeed solely determined by the influence of larger scales (that give exponential separation of the fluid particles) rather than smaller scales (that would lead to a diffusive growth as the square root of time). The subtle differences from the passive scalar case come from the active nature of the vorticity. The stretching of a blob depends on the vorticity it carries. Note that the relation is nontrivial here since the transfer rate is related to the strain (the symmetric part of the tensor of velocity derivatives) rather than to vorticity (the antisymmetric part). The analysis of the relations between the strain and the vorticity correlation functions shows however that in average the active nature of the vorticity accelerates the cascade as it goes downscales [78]. As far as the dominant logarithmic scaling of the correlation functions is concerned, the active nature of the vorticity simply amounts to the following: the field can be treated as a passive scalar, but the strain and the vorticity acting on it must be renormalized with the scale [71, 72, 78]. The law of renormalization is then established as follows: From (18), one has the dimensional relation that *time* behaves as $\omega^{-1}\ln(L/r)$ while the vorticity correlation function is $\langle\omega\,\omega\rangle \propto P_2 \times time$ according to (81). That gives the scaling $\omega \sim [P_2\ln(L/r)]^{1/3}$. The consequences are that the distance between two fluid particles satisfies: $\ln(R/r) \sim P_2^{1/2}t^{3/2}$, and that the pair correlation function $\langle\omega_1\omega_2\rangle \sim [P_2\ln(L/r_{12})]^{2/3}$. Experiments and numerics are compatible with that conclusion [79–81].

ii) **Inverse energy cascade in 2d**.

The energy is not a Lagrangian tracer and we cannot relate its inverse cascade to the behavior of trajectories. Still, we can get some important Lagrangian insight into the properties of the inverse energy cascade. If one assumes (after Kolmogorov) that $\bar{\epsilon}$ is the only pumping-related quantity that determines the statistics then the separation between the particles $\mathbf{R}_{12} = \mathbf{R}(t;\mathbf{r}_1) - \mathbf{R}(t;\mathbf{r}_2)$ has to obey the already mentioned Richardson law: $\langle R_{12}^2\rangle \propto \bar{\epsilon}t^3$. The equation for the separation follows from the Euler equation: $\partial_t^2\mathbf{R}_{12} = \mathbf{f}(\mathbf{R}(t;\mathbf{r}_1)) - \mathbf{f}(\mathbf{R}(t;\mathbf{r}_2)) - \nabla(P_1 - P_2)$. In the inertial range, R_{12} is much larger than the forcing correlation length. The forcing can therefore be considered short-correlated both in time and in space. Was the pressure term absent, one would get the separation growth: $\langle R_{12}^2\rangle/\bar{\epsilon}t^3 = 4/3$. The experimental data give a smaller numerical factor $\simeq 0.5$ [82], which is quite natural since the incompressibility constrains the motion. What is however important to note is that already the forcing term prescribes the law $\langle R_{12}^2\rangle \propto t^3$ consistent with the scaling of the energy cascade. Another amazing aspect of the $2d$ inverse energy cascade can be inferred if one considers it from the viewpoint of vorticity. First, there is no dissipative anomaly for enstrophy in the inertial interval of scales of the inverse cascade. Moreover, enstrophy is transferred toward the small scales and its flux at the large scales (where the inverse energy cascade is taking place) vanishes. By analogy with the passive scalar behavior at the large scales discussed in Sect. IV A ii, one may expect the behavior $\langle\omega_1\omega_2\rangle \propto r_{12}^{1-\alpha-d}$, where α is the scaling exponent of the velocity. The self-consistency of the argument dictated by the relation $\omega = \nabla\times\mathbf{v}$ requires $1-\alpha-d = 2\alpha-2$ which indeed gives the Kolmogorov scaling $\alpha = 1/3$ for $d = 2$. Experiments and numerical simulations indicate that the inverse energy cascade has a normal Kolmogorov scaling for all measured correlation functions [83–85]. No consistent theory is available yet, but the previous arguments based on the enstrophy equipartition might give an interesting clue. Since Kolmogorov scaling correspond to a non-smooth velocity (in the limit of pumping scale going to zero) then the passive scalar in such velocity field undergoes direct cascade with both dissipative anomaly and anomalous scaling while the active scalar (vorticity) has neither dissipative anomaly nor anomalous scaling.

From another perspective it is likely that the scale-invariance of inverse cascades is physically associated to the growth of the typical times with the scale. As the cascade proceeds, the fluctuations have indeed time to get smoothed out contrary to direct cascades with typical time decreasing in the direction of the cascade.

To conclude this Chapter, we note that what matters for the direction of the cascade of active tracers is the correlation between the tracer value and the type of trajectory it tracers.

VI. CONCLUSION

We hope that the reader have absorbed by now the two main lessons: the power of the Lagrangian approach to fluid turbulence and the importance of statistical integrals of motion for systems far from equilibrium.

The Lagrangian approach allows the analytical description of most important aspects of the statistics of particles and fields for velocity fields either spatially smooth or temporally decorrelated (or both). In a spatially smooth flow, the Lagrangian chaos with the ensuing exponentially separating trajectories is generally present. The respective statistics of passive scalar and vector fields is related to the statistics of the stretching and contraction rates in a way that is well understood. The theory finds a natural physical domain of application in the viscous range of scales. The most important open problem here seems to be the understanding of the back reaction of the advected field on the velocity. That would include an account of the buoyancy force in inhomogeneously heated fluids, the saturation of the small-scale magnetic dynamo and the polymer drag reduction. In non-smooth velocities, pertaining to the inertial interval of developed turbulence, the main Lagrangian phenomenon is the intrinsic stochasticity of the fluid particle trajectories that accounts for the dissipation at short distances. These phenomena are fully captured in the Kraichnan ensemble of non-smooth time-decorrelated velocities. It is an open problem to exhibit them for more realistic non-smooth velocities and to relate them to hydrodynamical evolution equations obeyed by the latter. The spontaneous stochasticity of Lagrangian trajectories enhances the interaction between fluid particles leading to intricate multi-particle stochastic conservation laws. There are open problems already in the framework of the Kraichnan model. First, there is the issue of whether one can build an operator product expansion, classifying the zero modes and revealing their possible underlying algebraic structure, both at large and small scales. The second class of problems is related to a consistent description of high-order moments of scalar, vector and tensor fields, especially in the situations where their amplitudes are growing, in a further attempt to describe feedback effects. Another major open problem is to identify the appropriate statistical integrals of motion for the active and the nonlocal cases. One sees there the potential direction of progress: coupling analytical, experimental and numerical studies to investigate the geometrical statistics of fluid turbulence with the primary aim to identify the underlying conservation laws.

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