TURBULENCE

Turbulence is a state of a nonlinear physical system that has energy distribution over many degrees of freedom strongly deviated from equilibrium. Turbulence is irregular both in time and in space. Turbulence can be maintained by some external influence or it can decay on the way to relaxation to equilibrium. The term first appeared in fluid mechanics and was later generalized to include far-from-equilibrium states in solids and plasmas.

If an obstacle of size L is placed in a fluid of viscosity ν that is moving with velocity V, a turbulent wake emerges for sufficiently large values of the Reynolds number

$$\mathrm{Re} \equiv VL/\nu.$$

At large Re, flow perturbations produced at scale L experience, a viscous dissipation that is small compared with nonlinear effects. Nonlinearity then induces motions at smaller and smaller scales until viscous dissipation terminates the process at a scale much smaller than L, leading to a wide (so-called inertial) interval of scales where viscosity is negligible and nonlinearity plays a dominant role.

Examples of this phenomenon include waves excited on a fluid surface by wind or moving bodies and waves in plasmas and solids that are excited by external electromagnetic fields. The state of such a system is called turbulent when the wavelength of the waves excited greatly differs from the wavelength of the waves that dissipate. Nonlinear interactions excite waves in the interval of wavelengths (called the transparency window or inertial interval as in fluid turbulence) between the injection and dissipation scales.

The ensuing complicated and irregular dynamics require a statistical description based on averaging over regions of space or intervals of time. Because nonlinearity dominates in the inertial interval, it is natural to ask to what extent the statistics are universal, in the sense of being independent of the details of excitation and dissipation. The answer to this question is far from evident for non-equilibrium systems. A fundamental physical problem is to establish which statistical properties are universal in the inertial interval of scales and which are features of different turbulent systems.

Constraints on dynamics are imposed by conservation laws, and therefore conserved quantities must play an essential role in turbulence. Although the conservation laws are broken by pumping and dissipation, these factors do not act in the inertial interval. Under incompressible turbulence, for example, the kinetic energy is pumped by external forcing and is dissipated by viscosity. As suggested by Lewis Fry Richardson in 1921, kinetic energy flows throughout the inertial interval of scales in a cascade-like process. The cascade idea explains the basic macroscopic manifestation of turbulence: the rate of dissipation of the dynamical integral of motion has a finite limit when the dissipation coefficient tends to zero. In other words, the mean rate of the viscous energy dissipation does not depend on viscosity at large Reynolds numbers. That means that symmetry of the inviscid equation (here, time-reversal invariance) is broken by the presence of the viscous term, even though the latter might have been expected to become negligible in the limit $\mathrm{Re} \to \infty$.

The cascade idea fixes only the mean flux of the respective integral of motion, requiring it to be constant across the inertial interval of scales. To describe an entire turbulence statistics, one has to solve problems on a case-by-case basis with most cases still unsolved.

Weak Wave Turbulence

From a theoretical point of view, the simplest case is the turbulence of weakly interacting waves. Examples include waves on the water surface, waves in plasma with and without a magnetic field, and spin waves in magnetics. We assume spatial homogeneity and denote by a_k the amplitude of the wave with the wave vector $\boldsymbol{k}$. When the amplitude is small, it satisfies the linear equation

$$\frac{\partial a_k}{\partial t} = -\mathrm{i}\omega_k a_k + f_k(t) - \gamma_k a_k. \tag{1}$$

Here, the dispersion law ω_k describes wave propagation, γ_k is the decrement of linear damping, and f_k describes pumping. For the linear system, a_k is different from zero only in the regions of $\boldsymbol{k}$-space where f_k is nonzero. To describe wave turbulence that involves wave numbers outside the pumping region, one must account for the interactions among different waves. Considering the wave system to be closed (no external pumping or dissipation) one can describe it as a Hamiltonian system using wave amplitudes as normal canonical variables (Zakharov et al., 1992). At small amplitudes, the Hamiltonian can be written as an expansion over a_k, where the second-order term describes non-interacting waves and high-order terms determine the interaction

$$H = \int \omega_k |a_k|^2 \, \mathrm{d}\boldsymbol{k} + \int \Big(V_{123} a_1 a_2^* a_3^* + \mathrm{c.c.} \Big)$$
$$\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) \, \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2 \, \mathrm{d}\boldsymbol{k}_3 + \mathrm{O}(a^4). \tag{2}$$

Here, $V_{123} = V(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ is the interaction vertex, and c.c. denotes complex conjugate. In this expansion, we presume every subsequent term smaller than the previous one, in particular, $\xi_k = |V_{kkk} a_k| k^d / \omega_k \ll 1$.

Wave turbulence that satisfies that condition is called weak turbulence. Also, space dimensionality d can be 1, 2, or 3.

A dynamic equation that accounts for pumping, damping, wave propagation, and interaction thus has the following form:

$$\frac{\partial a_k}{\partial t} = -\mathrm{i}\frac{\delta H}{\delta a_k^*} + f_k(t) - \gamma_k a_k. \qquad (3)$$

It is likely that the statistics of the weak turbulence at $k \gg k_f$ is close to Gaussian for wide classes of pumping statistics (this has not been shown rigorously). It is definitely the case for a random force with the statistics close to Gaussian. We consider here and below a pumping by a Gaussian random force statistically isotropic and homogeneous in space, and white in time. Thus,

$$\langle f_k(t) f_{k'}^*(t')\rangle = F(k)\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'), \qquad (4)$$

where angular brackets imply spatial averages, and $F(k)$ is assumed nonzero only around some k_f. For waves to be well defined, we assume $\gamma_k \ll \omega_k$.

Because the dynamic equation (3) contains a quadratic nonlinearity, the statistical description in terms of moments encounters the closure problem: the time derivative of the second moment is expressed via the third one, the time derivative of the third moment is expressed via the fourth one, and so on. Fortunately, weak turbulence in the inertial interval is expected to have the statistics close to Gaussian so one can express the fourth moment as the product of two second ones. As a result, one gets a closed kinetic equation for the single-time pair correlation function $\langle a_k a_{k'}\rangle = n_k \delta(\boldsymbol{k}+\boldsymbol{k}')$ (Zakharov et al., 1992):

$$\begin{aligned} \frac{\partial n_k}{\partial t} &= F_k - \gamma_k n_k + I_k^{(3)}, \\ I_k^{(3)} &= \int \left(U_{k12} - U_{1k2} - U_{2k1}\right) \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2 , \\ U_{123} &= \pi\left[n_2 n_3 - n_1(n_2+n_3)\right]|V_{123}|^2 \\ &\quad \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\delta(\omega_1-\omega_2-\omega_3). \end{aligned} \qquad (5)$$

This is called the kinetic equation for waves. The collision integral $I_k^{(3)}$ describes three-wave interactions: the first term in the integral corresponds to a decay of a given wave while the second and third terms correspond to a confluence with other waves.

One can estimate from (5) the inverse time of nonlinear interaction at a given k as $|V(k,k,k)|^2 n(k) k^d/\omega(k)$. We define k_d as the wave number where this inverse time is comparable to $\gamma(k)$ and assume nonlinearity to dominate over dissipation at $k \ll k_d$. As has been noted, wave turbulence appears when there is a wide (inertial) interval of scales where both pumping and damping are negligible, which requires $k_d \gg k_f$, the condition analogous to $\mathrm{Re} \gg 1$.

The presence of frequency delta-function in $I_k^{(3)}$ means that wave interaction conserves the quadratic part of the energy $E = \int \omega_k n_k\, \mathrm{d}\boldsymbol{k} = \int E_k\, \mathrm{d}k$. For the cascade picture to be valid, the collision integral has to converge in the inertial interval which means that energy exchange is small between motions of vastly different scales, a property called interaction locality in k-space. Consider now a statistical steady state established under the action of pumping and dissipation. Let us multiply (5) by ω_k and integrate it over either interior or exterior of the ball with radius k.

Taking $k_f \ll k \ll k_d$, one sees that the energy flux through any spherical surface (Ω is a solid angle),

$$P_k = \int_0^k k^{\mathrm{d}-1}\mathrm{d}k \int \mathrm{d}\Omega\, \omega_k I_k^{(3)},$$

is constant in the inertial interval and is equal to the energy production/dissipation rate:

$$P_k = \varepsilon = \int \omega_k F_k\, \mathrm{d}\boldsymbol{k} = \int \gamma_k E_k\, \mathrm{d}k. \qquad (6)$$

Let us assume now that the medium (characterized by ω_k and V_{123}) can be considered isotropic at the scales in the inertial interval. In addition, for scales much larger or much smaller than a typical scale in the medium (like the Debye radius in plasma or the depth of the water), the Hamiltonian coefficients are usually scale invariant: $\omega(k) = ck^\alpha$ and $|V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)|^2 = V_0^2 k^{2m} \chi(\boldsymbol{k}_1/k, \boldsymbol{k}_2/k)$ with $\chi \simeq 1$. Remember that we presumed statistically isotropic force. In this case, the pair correlation function that describes a steady cascade is also isotropic and scale invariant:

$$n_k \simeq \varepsilon^{1/2} V_0^{-1} k^{-m-d}. \qquad (7)$$

One can show that (7) reduces $I_k^{(3)}$ to zero (see Zakharov et al. 1992).

If the dispersion relation $\omega(k)$ does not allow for the resonance condition $\omega(k_1)+\omega(k_2) = \omega(|\boldsymbol{k}_1+\boldsymbol{k}_2|)$ then the three-wave collision integral is zero and one has to account for four-wave scattering which is always resonant, that is whatever $\omega(k)$ one can always find four wave vectors that satisfy $\omega(k_1)+\omega(k_2)=\omega(k_3)+\omega(k_4)$ and

$\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}_3+\boldsymbol{k}_4$. The collision integral that describes scattering,

$$I_k^{(4)} = \frac{\pi}{2}\int |T_{k123}|^2\big[n_2n_3(n_1+n_k) -n_1n_k(n_2+n_3)\big]\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3) \times\delta(\omega_k+\omega_1-\omega_2-\omega_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3, \tag{8}$$

conserves the energy and also the wave action $N=\int n_k\,\mathrm{d}\boldsymbol{k}$ (which can also be called the number of waves). Pumping generally provides for an input of both E and N. If there are two inertial intervals (at $k\gg k_f$ and $k\ll k_f$), then there should be two cascades. Indeed, if $\omega(k)$ grows with k then absorbing finite amount of E at $k_d\to\infty$ corresponds to an absorption of an infinitely small N. It is thus clear that the flux of N has to go in the opposite direction, that is to snall wave numbers. A so-called inverse cascade with a constant flux of N can thus be realized at $k\ll k_f$. A sink at small k can be provided by wall friction in the container or by long waves leaving the turbulent region in open spaces (as in sea storms).

The collision integral $I_k^{(3)}$ involves products of two n_k so that flux constancy requires $E_k\propto\varepsilon^{1/2}$ while for the four-wave case, one has $E_k\propto\varepsilon^{1/3}$. In many cases (when there is complete self-similarity), that knowledge is sufficient to obtain the scaling of E_k from a dimensional reasoning without actually calculating V and T. For example, short waves in deep water are characterized by the surface tension σ and density ρ so the dispersion relation must be $\omega_k\sim\sqrt{\sigma k^3/\rho}$, which allows for the three-wave resonance and thus $E_k\sim\varepsilon^{1/2}(\rho\sigma)^{1/4}k^{-7/4}$. For long waves in deep water, the surface-restoring force is dominated by gravity so that the gravitational acceleration g replaces σ as a defining parameter and $\omega_k\sim\sqrt{gk}$. Such a dispersion law does not allow for three-wave resonance so that the dominant interaction is four-wave scattering which permits two cascades. The direct energy cascade corresponds to $E_k\sim\varepsilon^{1/3}\rho^{2/3}g^{1/2}k^{-5/2}$. The inverse cascade carries the flux of N which we denote Q, it has the dimensionality $[Q]=[\varepsilon]/[\omega_k]$ and corresponds to $E_k\sim Q^{1/3}\rho^{2/3}g^{2/3}k^{-7/3}$.

Because the statistics of weak turbulence is near Gaussian, it is completely determined by the pair correlation function, which is in turn determined by the respective flux. We thus conclude that weak turbulence is universal in the inertial interval.

Strong Wave Turbulence

One cannot treat wave turbulence as a set of weakly interacting waves when the wave amplitudes are large ($\xi_k\geq 1$) and also in the particular case of linear (acoustic) dispersion where $\omega(k)=ck$ for arbitrarily small amplitudes. Indeed, there is no dispersion of wave velocity for acoustic waves, so waves moving in the same direction interact strongly and produce shock waves when viscosity is small. Formally, there is a singularity due to the coinciding arguments of delta-functions in (5) (and in the higher terms of perturbation expansion for $\partial n_k/\partial t$), which is thus invalid at however small amplitudes. Still, some features of the statistics of acoustic turbulence can be understood even without a closed description.

Consider a one-dimensional case which pertains, for instance, to sound propagating in long pipes. Because weak shocks are stable with respect to transversal perturbations (Landau & Lifshitz, 1987), quasi-one-dimensional perturbations may propagate in two and three dimensions as well. In a reference frame that moves with the sound velocity, weakly compressible 1-d flows ($u\ll c$) are described by the Burgers' equation (Landau & Lifshitz, 1987)

$$u_t+uu_x-\nu u_{xx}=0. \tag{9}$$

The Burgers' equation has a propagating shock-wave solution $u=2v\{1+\exp[v(x-vt)/\nu]\}^{-1}$ with the energy dissipation rate $\nu\int u_x^2\,\mathrm{d}x$ independent of ν. The shock width ν/v is a dissipative scale, and we consider acoustic turbulence produced by a pumping correlated on much larger scales (i.e., pumping a pipe from one end by frequencies much less than cv/ν). After some time, the system will develop shocks at random positions. Here we consider the single-time statistics of the Galilean invariant velocity difference $\delta u(x,t)=u(x,t)-u(0,t)$. The moments of δu are called structure functions $S_n(x,t)=\langle[u(x,t)-u(0,t)]^n\rangle$. Quadratic nonlinearity allows the time derivative of the second moment to be expressed via the third one:

$$\frac{\partial S_2}{\partial t}=-\frac{\partial S_3}{3\partial x}-4\varepsilon+\nu\frac{\partial^2 S_2}{\partial x^2}. \tag{10}$$

Here $\varepsilon=\nu\langle u_x^2\rangle$ is the mean energy dissipation rate. Equation (10) describes both a free decay (then ε depends on t) and the case of a permanently acting pumping which generates turbulence statistically steady at scales less than the pumping length.

In the first case, $\partial S_2/\partial t\simeq S_2u/L\ll\varepsilon\simeq u^3/L$ (where L is a typical distance between shocks) while in the second case, $\partial S_2/\partial t=0$ so that $S_3=12\varepsilon x+\nu\partial S_2/\partial x$. Consider now the limit $\nu\to 0$ at fixed x (and t for decaying turbulence). Shock dissipation provides for a finite limit of ε at $\nu\to 0$, then

$$S_3 = -12\varepsilon x. \qquad (11)$$

This formula is a direct analog of (6). Indeed, the Fourier transform of (10) describes the energy density $E_k = \langle |u_k|^2 \rangle /2$: $(\partial_t - \nu k^2)E_k = -\partial P_k/\partial k$ where the k-space flux

$$P_k = \int_0^k \mathrm{d}k' \int_{-\infty}^{\infty} \mathrm{d}x \, S_3(x) k' \sin(k'x)/24.$$

It is thus the flux constancy that fixes $S_3(x)$ which is universal (determined solely by ε) and depends neither on the initial statistics for decay nor on the pumping for steady turbulence. On the contrary, other structure functions $S_n(x)$ are not given by $(\varepsilon x)^{n/3}$. Indeed, the scaling of the structure functions can be readily understood for any dilute set of shocks (that is, when shocks do not cluster in space) which seems to be the case both for smooth initial, conditions and large-scale pumping in Burgers turbulence. In this case, $S_n(x) \sim C_n|x|^n + C_n'|x|$, where the first term comes from the regular (smooth) parts of the velocity while the second comes from $O(x)$ probability to have a shock in the interval x. The scaling exponents, $\xi_n = \mathrm{d}\ln S_n/\mathrm{d}\ln x$, thus behave as follows: $\xi_n = n$ for $n \le 1$ and $\xi_n = 1$ for $n > 1$. That means that the probability density function (PDF) of the velocity difference in the inertial interval $P(\delta u, x)$ is not scale-invariant, that is, the function of the rescaled velocity difference $\delta u/x^a$ cannot be made scale-independent for any a. As one goes to smaller scales, the lower-order moments decrease faster than the higher-order ones, that means that the smaller the scale the more probable are large fluctuations. In other words, the level of fluctuations increases with the resolution. When the scaling exponents ξ_n do not lie on a straight line, this is called an anomalous scaling since it is related again to the symmetry (scale invariance) of the PDF broken by pumping and not restored even when $x/L \to 0$. As an alternative to the description in terms of structures (shocks), one can relate the anomalous scaling in Burgers turbulence to the additional integrals of motion. Indeed, the integrals $E_n = \int u^{2n}\,\mathrm{d}x/2$ are all conserved by the inviscid Burgers' equation. Any shock dissipates the finite amount of E_n at the limit $\nu \to 0$ so that similar to (11) one denotes $\langle \dot{E}_n \rangle = \varepsilon_n$ and obtains $S_{2n+1} = -4(2n+1)\varepsilon_n x/(2n-1)$ for integer n.

Note that $S_2(x) \propto |x|$ corresponds to $E(k) \propto k^{-2}$, which is natural since every shock gives $u_k \propto 1/k$ at $k \ll v/\nu$, that is, the energy spectrum is determined by the type of structures (shocks) rather than by energy flux constancy. Similar ideas were suggested for other types of strong wave turbulence assuming them to be dominated by different structures. Weak wave turbulence, being a set of weakly interacting plane waves, can be studied uniformly for different systems (Zakharov et al., 1992). On the contrary, when nonlinearity is comparable to or exceeds dispersion, different structures appear in different systems. Identifying structures and the role they play in determining different statistical characteristics of strong wave turbulence remains to be investigated for most cases. Broadly, one distinguishes conservative structures (like solitons and vortices) from dissipative structures which usually appear as a result of finite-time singularity of the nondissipative equations (like shocks, light self-focusing, or wave collapse). For example, nonlinear wave packets are described by the nonlinear Schrödinger equation,

$$\mathrm{i}\Psi_t + \Delta\Psi + T|\Psi|^2\Psi = 0. \qquad (12)$$

Weak wave turbulence is determined by $|T|^2$ and is the same both for $T < 0$ (wave repulsion) and $T > 0$ (wave attraction). At high levels of nonlinearity, different signs of T correspond to dramatically different physics: At $T < 0$, one has a stable condensate, solitons and vortices, while at $T > 0$, instabilities dominate and wave collapse is possible at $d = 2, 3$. No analytic theory is yet available for strong turbulence described by (12).

Because the parameter of nonlinearity $\xi(k)$ generally depends on k then there may exist a weakly turbulent cascade until some k_* where $\xi(k_*) \sim 1$, and strong turbulence beyond this, wave number; thus weak and strong turbulence can coexist in the same system. Presuming that some mechanism (for instance, wave breaking) prevents the appearance of wave amplitudes that correspond to $\xi_k \gg 1$, one may hypothetize that some cases of strong turbulence correspond to the balance between dispersion and nonlinearity local in k-space so that $\xi(k) =$ constant throughout its domain in k-space. That would correspond to the spectrum $E_k \sim \omega_k^3 k^{-d}/|V_{kkk}|^2$ which is ultimately universal, that is independent even of the flux (only the boundary k_* depends on the flux). For gravity waves, this gives $E_k = \rho g k^{-3}$, the same spectrum one obtains presuming the wave profile to have cusps (another type of dissipative structure leading to whitecaps in stormy seas—see Phillips, 1977). It is unclear if such flux-independent spectra are realized.

Incompressible Turbulence

Incompressible fluid flow is described by the Navier–Stokes equation

$$\partial_t \boldsymbol{v}(\boldsymbol{r},t) + \boldsymbol{v}(\boldsymbol{r},t)\cdot\nabla\boldsymbol{v}(\boldsymbol{r},t) - \nu\nabla^2\boldsymbol{v}(\boldsymbol{r},t) = -\nabla p(\boldsymbol{r},t), \quad \operatorname{div}\boldsymbol{v} = 0.$$

We are again interested in the structure functions $S_n(\boldsymbol{r},t) = \langle\, [(\boldsymbol{v}(\boldsymbol{r},t) - \boldsymbol{v}(0,t))\cdot \boldsymbol{r}/r]^n\,\rangle$ and treat first the three-dimensional case. Similar to (10), one considers distance r smaller than the force correlation scale for a steady case and smaller than the size of the turbulent region for a decay case. For such r, one can derive the Karman–Howarth relation between S_2 and S_3 (see Landau & Lifshitz, 1987):

$$\frac{\partial S_2}{\partial t} = -\frac{1}{3r^4}\frac{\partial}{\partial r}(r^4 S_3) + \frac{4\varepsilon}{3} + \frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial S_2}{\partial r}\right). \quad (13)$$

Here $\varepsilon = \nu\,\langle\,(\nabla \boldsymbol{v})^2\,\rangle$ is the mean energy dissipation rate. Neglecting the time derivative (which is zero in a steady state and small compared to ε for decaying turbulence), one can multiply (13) by r^4 and integrate: $S_3(r) = -4\varepsilon r/5 + 6\nu\, \mathrm{d}S_2(r)/\mathrm{d}r$. Andrei Kolmogorov in 1941 considered the limit $\nu \to 0$ for fixed r and *assumed* nonzero limit for ε, which gives the so-called $\frac{4}{5}$ law (see Landau & Lifshitz, 1987; Frisch, 1995):

$$S_3 = -\tfrac{4}{5}\,\varepsilon\, r. \quad (14)$$

This relation is a direct analog of (6) and (11). It also means that the kinetic energy has a constant flux in the inertial interval of scales (the viscous scale η is defined by $\nu S_2(\eta) \simeq \varepsilon\eta^2$). Law (14) implies that the third-order moment is universal, that is, it does not depend on the details of the turbulence production but is determined solely by the mean energy dissipation rate. The rest of the structure functions have not yet been derived. Kolmogorov (and also Werner Heisenberg, Karl von Weizsacker, and Lars Onsager) presumed the pair correlation function to be determined only by ε and r which would give $S_2(r) \sim (\varepsilon r)^{2/3}$ and the energy spectrum $E_k \sim \varepsilon^{2/3}k^{-5/3}$. Experiments suggest that $\zeta_n = \mathrm{d}\ln S_n/\mathrm{d}\ln r$ lie on a smooth concave curve sketched in Figure 1. While ζ_2 is close to $2/3$, it has to be a bit larger because experiments show that the slope at zero $\mathrm{d}\zeta_n/\mathrm{d}n$ is larger than $\frac{1}{3}$ while, $\zeta(3) = 1$ in agreement with (14). As in Burgers turbulence, the PDF of velocity differences in the inertial interval is not scale-invariant in 3-d incompressible turbulence. No one has yet found an explicit relation between the anomalous scaling for 3-d Navier–Stokes turbulence and either structures or additional integrals of motion.

While not exact, the Kolomogorov approximation $S_2(\eta) \simeq (\varepsilon\eta)^{2/3}$ can be used to estimate the viscous scale: $\eta \simeq L\mathrm{Re}^{-3/4}$. The number of degrees of freedom involved in 3-d incompressible turbulence can thus be roughly estimated as $N \sim (L/\eta)^3 \sim \mathrm{Re}^{9/4}$. That means,

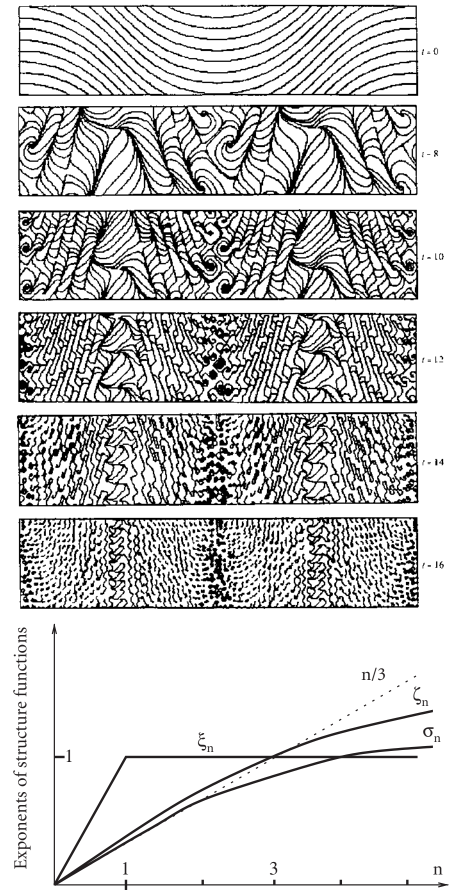

Figure 1. The scaling, exponents of the structure functions ξ_n for Burgers, ζ_n for Navier–Stokes, and σ_n for the passive scalar. The dotted straight line is the Kolmogorov hypothesis $n/3$.

in particular, that detailed computer simulation of water or oil pipe flows ($\mathrm{Re} \sim 10^4 - 10^7$) or turbulent clouds ($\mathrm{Re} \sim 10^6 - 10^9$) is out of question for the foreseeable future. To calculate correctly at least the large-scale part of the flow, it is desirable to have some theoretical model to parametrize the small-scale motions, the main obstacle being our lack of qualitative understanding and quantitative description of how turbulence statistics changes as one goes downscale.

Large-scale motions in a shallow fluid can be approximately considered two dimensional. When the velocities of such motions are much smaller than the velocities of the surface waves and the velocity of sound, such flows can be considered incompressible. Their description is important for understanding atmospheric and oceanic turbulence at the scales larger than atmosphere height and ocean depth.

Vorticity $\omega = \mathrm{curl}\, \boldsymbol{v}$ is a scalar in a two-dimensional flow. It is advected by the velocity field and dissipated by viscosity. Taking the curl of the Navier–Stokes equation, one gets

$$\partial_t \omega + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\omega = \nu \nabla^2 \omega. \tag{15}$$

Two-dimensional incompressible inviscid flow just transports vorticity from place to place and thus conserves spatial averages of any function of vorticity. In particular, we now have the second quadratic inviscid invariant (in addition to energy) which is called enstrophy: $\int \omega^2 \, d\boldsymbol{r}$. Since the spectral density of the energy is $|\boldsymbol{v}_k|^2/2$ while that of the enstrophy is $|\boldsymbol{k} \times \boldsymbol{v}_k|^2$, Robert Kraichnan suggested in 1967 that the direct cascade (towards large k) is that of enstrophy while the inverse cascade is that of energy. Again, for the inverse energy cascade, there is no consistent theory except for the flux relation that can be derived similar to (14):

$$S_3(r) = 4\varepsilon r/3. \tag{16}$$

The inverse cascade is observed in the atmosphere (at scales of 30–500 km) and in laboratory experiments. Experimental data suggest that there is no anomalous scaling; thus $S_n \propto r^{n/3}$. In particular, $S_2 \propto r^{2/3}$ which corresponds to $E_k \propto k^{-5/3}$. It is ironic that probably the most widely known statement on turbulence, the $\frac{5}{3}$ spectrum suggested by Kolmogorov for the 3-d case, is not correct in this case (even though the true scaling is close), while it is probably exact in Kraichnan's inverse 2-d cascade. Qualitatively, it is likely that the absence of anomalous scaling in the inverse cascade is associated with the growth of the typical turnover time (estimated, say, as $r/\sqrt{S_2}$) with the scale. As the inverse cascade proceeds, the fluctuations have enough time to get smoothed out as opposed to the direct cascade in three dimensions, where the turnover time decreases in the direction of the cascade.

Before discussing the direct (enstrophy) cascade, we describe a similar yet somewhat simpler problem of passive scalar turbulence, which allows one to introduce the necessary notions of Lagrangian description of the fluid flow. Consider a scalar quantity $\theta(\boldsymbol{r}, t)$ that is subject to molecular diffusion and advection by the fluid flow but has no back influence on the velocity (i.e., passive):

$$\partial_t \theta + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\theta = \kappa \nabla^2 \theta + \varphi. \tag{17}$$

Here κ is molecular diffusivity. In the same 2-d flow, ω and θ behave in the same way, but vorticity is related to velocity while the passive scalar is not. Examples of passive scalar are smoke in air, salinity in water, and temperature when one neglects thermal convection. If the source φ produces fluctuations of θ on some scale L then the inhomogeneous velocity field stretches, contracts, and folds the field θ producing progressively smaller and smaller scales. If the rms velocity gradient is Λ then molecular diffusion is substantial at scales less than the diffusion scale $r_d = \sqrt{\kappa/\Lambda}$. The ratio

$$\mathrm{Pe} = L/r_d$$

is called the Peclet number. It is an analog of the Reynolds number for passive scalar turbulence. When $\mathrm{Pe} \gg 1$, there is a long inertial interval where the flux constancy relation derived by A.M. Yaglom in 1949 holds:

$$\langle (\boldsymbol{v}_1 \cdot \boldsymbol{\nabla}_1 + \boldsymbol{v}_2 \cdot \boldsymbol{\nabla}_2)\theta_1\theta_2 \rangle = 2P, \tag{18}$$

where $P = \kappa \langle (\nabla\theta)^2 \rangle$ and subscripts denote the spatial points. In considering the passive scalar problem, the velocity statistics is presumed to be given. Still, the correlation function (18) mixes $\boldsymbol{v}$ and θ and does not generally allow one to make a statement on any correlation function of θ. The proper way to describe the correlation functions of the scalar at scales much larger than the diffusion scale is to employ the Lagrangian description, that is, to follow fluid trajectories. Indeed, if we neglect diffusion, then Equation (17) can be solved along the characteristics $\boldsymbol{R}(t)$ which are called Lagrangian trajectories and satisfy $d\boldsymbol{R}/dt = \boldsymbol{v}(\boldsymbol{R}, t)$. Presuming zero initial conditions at $t \to -\infty$, we write

$$\theta\Big(\boldsymbol{R}(t), t\Big) = \int_{-\infty}^{t} \varphi\Big(\boldsymbol{R}(t'), t'\Big)\, dt'. \tag{19}$$

In that way, the correlation functions of the scalar $F_n = \langle \theta(\boldsymbol{r}_1, t) \ldots \theta(\boldsymbol{r}_n, t) \rangle$ can be obtained by integrating the correlation functions of the pumping along the trajectories that satisfy the final conditions $\boldsymbol{R}_i(t) = \boldsymbol{r}_i$.

Consider first, the case of pumping which is Gaussian, statistically homogeneous, and isotropic in space and white in time: $\langle \varphi(\boldsymbol{r}_1, t_1)\varphi(\boldsymbol{r}_2, t_2) \rangle = \Phi(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)\, \delta(t_1 - t_2)$ where the function Φ is constant at $r \ll L$ and goes to zero at $r \gg L$. The pumping provides for symmetry $\theta \to -\theta$ which makes only even correlation functions F_{2n} nonzero. The pair correlation function is

$$F_2(r, t) = \int_{-\infty}^{t} \Phi\Big(R_{12}(t')\Big)\, dt'. \tag{20}$$

Here $R_{12}(t') = |\boldsymbol{R}_1(t') - \boldsymbol{R}_2(t')|$ is the distance between two trajectories and $R_{12}(t) = r$. The function Φ essentially restricts the integration to the time interval when the distance $R_{12}(t') \le L$. Simply speaking, the stationary pair correlation function of a tracer is $\Phi(0)$ (which is twice the injection rate of θ^2) times the average time $T_2(r, L)$ that two fluid particles spend within the correlation scale of the pumping. The larger r, the less time it takes for the particles to separate from r to L and the smaller is $F_2(r)$. Of course, $T_{12}(r, L)$ depends on the properties of the velocity field. A general theory is available only when the velocity field is spatially smooth at the scale of scalar pumping L. This so-called Batchelor regime happens, in particular, when the scalar cascade occurs at the scales less than the viscous scale of fluid turbulence. This requires the Schmidt number ν/κ (called the Prandtl number when θ is temperature) to be large, which is the case for very viscous liquids. In this case, one can approximate the velocity difference $\boldsymbol{v}(\boldsymbol{R}_1, t) - \boldsymbol{v}(\boldsymbol{R}_2, t) \approx \hat{\sigma}(t)\boldsymbol{R}_{12}(t)$ with the Lagrangian strain matrix $\sigma_{ij}(t) = \nabla_j v_i$. In this regime, the distance obeys the linear differential equation

$$\dot{\boldsymbol{R}}_{12}(t) = \hat{\sigma}(t)\boldsymbol{R}_{12}(t). \tag{21}$$

The theory of such equations is well developed and related to what is called Lagrangian chaos, as fluid trajectories separate exponentially as is typical for systems with dynamical chaos (see, e.g. Falkovich et al., 2001): At t much larger than the correlation time of the random process $\hat{\sigma}(t)$, all moments of R_{12} grow exponentially with time and $\langle\, \ln[R_{12}(t)R_{12}(0)]\,\rangle = \lambda t$, where λ is called a senior Lyapunov exponent of the flow (note that for the description of the scalar we need the flow taken backwards in time which is different from that taken forward because turbulence is irreversible). Dimensionally, $\lambda = \Lambda f(\mathrm{Re})$ where the limit of the function f at $\mathrm{Re} \to \infty$ is unknown. We thus obtain

$$F_2(r) = \Phi(0)\lambda^{-1}\ln(L/r) = 2P\lambda^{-1}\ln(L/r). \tag{22}$$

In a similar way, one shows that for $n \ll \ln(L/r)$, all F_n are expressed via F_2 and the structure functions $S_{2n} = \langle\, [\theta(\boldsymbol{r}, t) - \theta(0, t)]^{2n}\,\rangle \propto \ln^n(r/r_d)$ for $n \ll \ln(r/r_d)$. This can be generalized for an arbitrary statistics of pumping as long as it is finite-correlated in time (Falkovich et al., 2001).

One can use the analogy between passive scalar and vorticity in two dimensions as has been shown by Falkovich and Lebedev in 1994 following the line suggested by Kraichnan in 1967. For the enstrophy cascade, one derives the flux relation analogous to (18):

$$\langle(\boldsymbol{v}_1 \cdot \nabla_1 + \boldsymbol{v}_2 \cdot \nabla_2)\omega_1\omega_2\rangle = 2D, \tag{23}$$

where $D = \langle\, \nu(\nabla\omega)^2\,\rangle$. The flux relation along with $\omega = \mathrm{curl}\,\boldsymbol{v}$ suggests the scaling $\delta v(r) \propto r$, that is, velocity being close to spatially smooth (of course, it cannot be perfectly smooth to provide for a nonzero vorticity dissipation in the inviscid limit, but the possible singularitites are indeed shown to be no stronger than logarithmic). That makes the vorticity cascade similar to the Batchelor regime of passive scalar cascade with a notable change in that the rate of stretching λ acting on a given scale is not a constant but is logarithmically growing when the scale decreases. Since λ scales as vorticity, the law of renormalization can be established from dimensional reasoning and one gets $\langle\omega(\boldsymbol{r}, t)\omega(0, t)\,\rangle \sim [D\ln(L/r)]^{2/3}$ which corresponds to the energy spectrum $E_k \propto D^{2/3}k^{-3}\ln^{-1/3}(kL)$. Higher-order correlation functions of vorticity are also logarithmic, for instance, $\langle\,\omega^n(\boldsymbol{r}, t)\omega^n(0, t)\,\rangle \sim [D\ln(L/r)]^{2n/3}$. Note that both passive scalar in the Batchelor regime and vorticity cascade in two dimensions are universal, that is, determined by the single flux (P and D, respectively) despite the existence of higher-order conserved quantities. Experimental data and numeric simulations support these conclusions.

Zero Modes and Anomalous Scaling

Let us now return to the Lagrangian description and discuss it when velocity is not spatially smooth, for example, that of the energy cascades in the inertial interval. One can assume that it is Lagrangian statistics that are determined by the energy flux when the distances between fluid trajectories are in the inertial interval. That assumption leads, in particular, to the Richardson law for the asymptotic growth of the interparticle distance:

$$\langle R_{12}^2(t)\rangle \sim \varepsilon t^3, \tag{24}$$

which was first established from atmospheric observations (in 1926) and later confirmed experimentally for energy cascades both in 3-d and in 2-d. There is no consistent theoretical derivation of (24) and it is unclear whether it is exact (likely to be in 2-d) or just approximate (possible in 3-d). The semi-heuristic argument usually presented in textbooks is based on the mean-field estimate: $\dot{\boldsymbol{R}}_{12} = \delta\boldsymbol{v}(\boldsymbol{R}_{12}, t) \sim (\varepsilon R_{12})^{1/3}$ which upon integration gives: $R_{12}^{2/3}(t) - R_{12}^{2/3}(0) \sim \varepsilon^{1/3}t$. For the passive scalar it gives, by virtue of (20), $F_2(r) \sim \Phi(0)\varepsilon^{-1/3}[L^{2/3} - r^{2/3}]$ which was suggested by S. Corrsin and A.M. Oboukhov. The structure function is then $S_2(r) \sim \Phi(0)\varepsilon^{-1/3}r^{2/3}$. Experiments measuring the scaling exponents $\sigma_n = \mathrm{d}\ln S_n(r)/\mathrm{d}\ln r$ generally give σ_2 close to $2/3$ but higher exponents deviating from the straight lineane even stronger than the exponents of the velocity in 3-d. Moreover, the scalar exponents σ_n are anomalous even when advecting velocity has a normal scaling like in the 2-d energy cascade.

To better understand the Lagrangian dynamics (and passive scalar statistics) in a spatially nonsmooth velocity, Kraichnan suggested considering the model of a velocity field having the simplest statistical and temporal properties, namely Gaussian velocity which is white in time:

$$\langle v^i(\boldsymbol{r},t)v^j(0,0)\rangle = \delta(t)\Big[D_0\delta_{ij} - d_{ij}(\boldsymbol{r})\Big],$$
$$d_{ij} = D_1 r^{2-\gamma}\Big[(d+1-\gamma)\,\delta^{ij} + (\gamma-2)r^i r^j r^{-2}\Big]. \quad (25)$$

Here the exponent $\gamma \in [0,2]$ is a measure of the velocity nonsmoothness with $\gamma = 0$ corresponding to a smooth velocity and $\gamma = 2$ corresponding to a velocity very rough in space (distributional). Richardson–Kolmogorov scaling of the energy cascade corresponds to $\gamma = 2/3$. Lagrangian flow is a Markov random process for the Kraichnan ensemble (25). Every fluid particle undergoes a Brownian random walk with the so-called eddy diffusivity D_0. The PDF for two particles to be separated by r after time t satisfies the diffusion equation (see, e.g., Falkovich et al., 2001)

$$\partial_t P(r,t) = L_2 P(r,t)\,,$$
$$L_2 = d_{ij}(\boldsymbol{r})\nabla^i\nabla^j = D_1(d-1)r^{1-d}\partial_r r^{d+1-\gamma}\partial_r\,, \quad (26)$$

with the scale-dependent diffusivity $D_1(d-1)r^{2-\gamma}$. The asymptotic solution of (26) is lognormal for the Batchelor case while for $\gamma > 0$

$$P(r,t) = r^{d-1}t^{d/\gamma}\exp\Big(-\text{const}\, r^\gamma/t\Big). \quad (27)$$

For $\gamma = 2/3$, it reproduces, in particular, the Richardson law. Multiparticle probability distributions also satisfy diffusion equations in the Kraichnan model as well as all the correlation functions of θ. Multiplying equation (17) by $\theta_2 \ldots \theta_{2n}$ and averaging over the Gaussian statistics of $\boldsymbol{v}$ and φ, one derives

$$\partial_t F_{2n} = L_{2n}F_{2n} + \sum_{l,m} F_{2n-2}\Phi(\boldsymbol{r}_{lm})\,,$$
$$L_{2n} = \sum d_{ij}(\boldsymbol{r}_{lm})\nabla_l^i\nabla_m^j. \quad (28)$$

This equation enables one, in principle, to derive inductively all steady-state F_{2n} starting from F_2. The equation $\partial_t F_2(r,t) = L_2F_2(r,t) + \Phi(r)$ has a steady solution $F_2(r) = 2[\Phi(0)/\gamma d(d-1)D_1][dL^\gamma/(d-\gamma) - r^\gamma]$, which has the Corrsin–Oboukhov form for $\gamma = 2/3$. Further, F_4 contains the so-called forced solution having the normal scaling 2γ but also, remarkably, a zero mode Z_4 of the operator L_4: $L_4Z_4 = 0$. Such zero modes necessarily appear (to satisfy the boundary conditions at $r \simeq L$) for all $n > 1$ and the scaling exponents of Z_{2n} are generally different from $n\gamma$ that is anomalous. In calculating the scalar structure functions, all terms cancel out except a single zero mode (called irreducible because it involves all distances between $2n$ points). Calculation of Z_n and their scaling exponents σ_n were carried out analytically at $\gamma \ll 1$, $2-\gamma \ll 1$ and $d \gg 1$, and numerically for all γ and $d = 2, 3$ (Falkovich et al., 2001).

That gives σ_n lying on a convex curve (as in Figure 1) which saturates to a constant at large n. Such saturation (confirmed by experiments) is a signature that most singular structures in a scalar field are shocks (as in Burgers' turbulence), the value σ_n at $n \to \infty$ is the fractal codimension of fronts in space. Interestingly, the Kraichnan model enables one to establish the relation between the anomalous scaling and conservation laws of a new type. Thus, the combinations of distances between points that constitute zero modes are the statistical integrals of Lagrangian evolution. To give a simple example, in a Brownian walk, the mean distance between every two particles grows with time, $\langle R_{lm}^2(t)\rangle = R_{lm}^2(0) + \kappa t$, while $\langle R_{lm}^2 - R_{pq}^2\rangle$ and $\langle 2(d+2)R_{lm}^2R_{pq}^2 - d(R_{lm}^4 + R_{pq}^4)\rangle$ (and an infinity of similarly built harmonic polynomials) are conserved. Note that the integrals are not dynamical, they are conserved only in average. In a turbulent flow, the form of such conserved quantities is more complicated but the essence is the same: the increase of averaged distances between fluid particles is compensated by the decrease in shape fluctuations. The existence of statistical conserved quantities breaks the scale invariance of scalar statistics in the inertial interval and explains why scalar turbulence knows more about pumping than just the value of the flux. Note that both symmetries, one broken by pumping (scale invariance) and another by damping (time reversibility) are not restored even when $r/L \to 0$ and $r_d/r \to 0$.

For the vector field (like velocity or magnetic field in magnetohydrodynamics), the Lagrangian statistical integrals of motion may involve both the coordinate of the fluid particle and the vector it carries. Such integrals of motion were built explicitly and related to the anomalous scaling for the passively advected magnetic field in the Kraichnan ensemble of velocities (Falkovich et al., 2001). Doing that for velocity that satisfies the Navier–Stokes equation remains a task for the future.

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See also **Chaos vs. turbulence; Development of singularities; Intermittency; Kolmogorov cascade; Lagrangian chaos; Magnetohydrodynamics; Mixing;**

Navier–Stokes equation; Nonlinear Schrödinger equations; Water waves; Wave packets, linear and nonlinear

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