Brownian motion and the Central Limit Theorem

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1 Introduction

In this tutorial we shall review Brownian motion as emerging from the collective effect of random independent forces. This point of view explains the ubiquity of Brownian-like (and diffusion like) motions in many different scenarios. We start by reviewing the general central limit theorem (CLT) for a sum of independent random variables and then apply the CLT for the case of a pollen in a fluid, much like was done in class.

Just as a historical remark, the importance of Brownian motion in physics comes from its role in proving the atomic picture of matter (by Einstein and Smoluchowski (1905-6) from the theoretical perspective and by Chaudesaigues and Perrin (1908-9) from the experimental perspective). It is also a prototypical example for stochastic process which has great usage in many areas of science from physics through chemistry and biology to economics and social sciences.

2 Central Limit Theorem (CLT)

Consider a set of N independent variables $\{X_i\}$ with finite variance $\sigma_i^2 = \langle X_i^2 \rangle$. Let's start by assuming $\langle X_i \rangle = 0$. Define another random variable

$$Y = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} X_j \tag{1}$$

To find the distribution of Y, $\rho(y)dy = P(y < Y < y + dy)$, in the limit of large N. Define the generating function of Y to be

$$G_Y(k) = \langle e^{ikY} \rangle = \int e^{iky} \rho(y) dy \tag{2}$$

As this is simply proportional to the Fourier transform of its distribution function, one can obtain $\rho(y)$ from $G_Y(k)$ by the inverse transform. Now

$$\begin{aligned} G_Y(k) &= \left\langle \exp\left[\frac{ik}{\sqrt{N}}\sum_{j=1}^{N} X_j\right]\right\rangle \\ &= \left\langle \prod_{j=1}^{N} \exp\left[\frac{ik}{\sqrt{N}} X_j\right]\right\rangle \end{aligned}$$

From independence of X_j we get

$$G_Y(k) = \prod_{j=1}^{N}\left\langle \exp\left[\frac{ik}{\sqrt{N}} X_j\right]\right\rangle \equiv \exp\left[\sum_{j=1}^{N} A_j\left(\frac{k}{\sqrt{N}}\right)\right] \tag{3}$$

$$A_j\left(\frac{k}{\sqrt{N}}\right) \equiv \ln\left\langle \exp\left(\frac{ik}{\sqrt{N}} X_j\right)\right\rangle \tag{4}$$

As we want to probe the large N behavior, we assume $k/\sqrt{N} \ll 1$ and expand

$$\begin{aligned} A_j\left(\frac{k}{\sqrt{N}}\right) &= \ln\left\langle 1 + \frac{ik}{\sqrt{N}} X_j - \frac{k^2}{2N} X_j^2 + O\left(N^{-3/2}\right)\right\rangle \\ &= \ln\left(1 + \langle X_j\rangle \frac{ik}{\sqrt{N}} - \left\langle X_j^2\right\rangle \frac{k^2}{2N} + O\left(N^{-3/2}\right)\right) \end{aligned}$$

Using $\langle X_j\rangle = 0$, $\left\langle X_j^2\right\rangle = \sigma_j^2$, and expanding the ln function yields

$$A_j\left(\frac{k}{\sqrt{N}}\right) = -\sigma_j^2\frac{k^2}{2N} + O\left(N^{-3/2}\right)$$

Hence to leading order in N^{-1} we find

$$G_Y(k) = \exp\left[-\frac{1}{2}\sigma^2 k^2\right] \tag{5}$$

$$\sigma^2 \equiv \frac{1}{N}\sum_{j=1}^{N}\sigma_j^2 \tag{6}$$

The inverse Fourier transform yields

$$\begin{aligned} \rho(y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iky}\exp\left[-\frac{1}{2}\sigma^2 k^2\right] dk \\ &= \frac{1}{2\pi\sigma}\exp\left[-\frac{y^2}{2\sigma^2}\right]\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}(k - iy)^2\right] dk \\ &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{y^2}{2\sigma^2}\right] \end{aligned} \tag{7}$$

where in the second line we changed the variable $\sigma k \to k$ and in the third $k - iy \to k$ and used $\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}x^2\right] dx = \sqrt{2\pi}$. Hence we find that Y is a Gaussian variable.

Hence we finally got that to leading order in N^{-1}, Y is a Gaussian variable. If $\langle X_i \rangle \neq 0$ we can define

$$\tilde{Y} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} (X_j - \langle X_j \rangle)$$

which will be a Gaussian variable (to leading order in N^{-1}) and $Y = \tilde{Y} + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \langle X_j \rangle$.

3 Brownian Motion

Brownian motion is the motion of a small macroscopic object in a liquid medium. It is named after Robert Brown, who observed the movement of pollen grains under the microscope and identified their irregular motion. It is now known that Brownian motion is caused by the thermal (equilibrium) movement of atoms which pass momentum to the macroscopic object upon collision, but at the beginning of the 20th century, this understanding actually helped proving that atoms actually exist.

A possible starting point for analyzing Brownian motion is the following Langevin equation

$$\frac{d\mathbf{p}}{dt} = -\lambda \mathbf{p} + \mathbf{f} \tag{8}$$

where $\mathbf{f}$ is a random force with $\langle \mathbf{f} \rangle = 0$. This equation has a simple physical interpretation: The force acting on a body moving in a fluid is on average in opposite direction to the direction of motion (as, for instance, more particles hit the front of the body than its back). The average of the force should vanish when $\mathbf{p} = 0$ (equilibrium, symmetry) and hence for small $\mathbf{p}$ it can be assumed to be linear, which yields the resistance term $-\lambda \mathbf{p}$. Subtracting the average, what is left is a noisy term with zero average, which we denote by $\mathbf{f}$.

3.1 Momentum

The solution of Eq.(8) is

$$\mathbf{p} = \int_{-\infty}^{t} f(t') e^{\lambda(t'-t)} dt' \tag{9}$$

$$= \int_{0}^{\infty} f(t-t') e^{-\lambda t'} dt' \tag{10}$$

as can be verified by taking a derivative of (9) with respect to t. The correlation function of the random force is defined as

$$\begin{aligned} C(t) &\equiv \frac{1}{3}\langle f(t')f(t'+t)\rangle \\ &= \frac{1}{3W}\int_0^W f(t')f(t'+t)dt' \end{aligned}$$

where W is a large enough window of time. Here we defined the average as a time average, but from ergodicity it is the same as ensemble average, hence the distinction between the two is irrelevant in this case. The correlations are assumed to decay with a characteristic time scale τ - the *correlation time* - which means that for $t \gg \tau$ the correlation $C(t)$ is essentially zero. We also assume that $\lambda\tau \ll 1$ and hence the integration in (9) can be regarded as sum of many independent random variables. Formally this can be done by dividing the integral to segments of length $\tau \ll t'' \ll \lambda^{-1}$ and noting that the correlation between every two segments is bounded by $\frac{\tau}{t''} \ll 1$. Hence, from the CLT we find that the momentum is a Gaussian variable

$$\rho(\mathbf{p}) = \frac{1}{(2\pi\sigma^2)^{3/2}}\exp\left[-\frac{p^2}{2\sigma^2}\right]$$

From symmetry $\langle p_x^2\rangle = \langle p_y^2\rangle = \langle p_z^2\rangle = \sigma^2$ and hence

$$\begin{aligned} \sigma^2 &= \frac{1}{3}\left\langle\left(\int_{-\infty}^{t} f(t_1)e^{\lambda(t_1-t)}dt_1\right)^2\right\rangle = \frac{1}{3}\left\langle\left(-\int_0^{\infty} f(t-t_1)e^{-\lambda t_1}dt_1\right)^2\right\rangle \\ &= \frac{1}{3}\left\langle\int_0^\infty dt_1\int_0^\infty dt_2 f(t-t_1)f(t-t_2)e^{-\lambda(t_1+t_2)}\right\rangle \\ &= \int_0^\infty dt_1\int_\infty^0 dt_2 e^{-\lambda(t_1+t_2)}C(t_1-t_2) \end{aligned}$$

The change of variables $s = t_1+t_2$, $t' = t_1-t_2$ has Jacobian $\frac{1}{2}$, hence

$$\begin{aligned} \sigma^2 &= \frac{1}{2}\int_0^\infty ds e^{-\lambda s}\int_{-s}^{s} dt' C(t') \\ &= \frac{1}{2\lambda}\int_{-\infty}^{\infty} dt' C(t') \end{aligned}$$

where we used the fact that the correlation time is much shorter than λ^{-1}.

From equipartition theorem $\langle p_x^2/2m\rangle = T/2$, hence $\langle p_x^2\rangle = mT$. From this we find the relation

$$\lambda = \frac{1}{2mT}\int_{-\infty}^{\infty} dt' C(t')$$

which is a relation between the friction coefficient (non-equilibrium quantity) and the fluctuations of the force (an equilibrium quantity). This is a particular example of the fluctuation dissipation theorem.

3.2 Displacement

The displacement is give by

$$\Delta r \equiv r(t+t') - r(t) = \int_0^{t'} v(t'')dt''$$

where $v = \dot{r} = p/m$. We found that the momentum is a random variable, and we shall see below that its correlation time is of order λ^{-1} which is quite intuitive. Hence, the same reasoning used above implies that for $t \gg \lambda^{-1}$, $\Delta r(t)$ is also a Gaussian random variable (actually as the momentum is a Gaussian random variable, even for times $t < \lambda^{-1}$ their sum is Gaussian).

To find the correlation time of the velocity - or the momentum - it is more convenient to write the solution of (8) at $t+t'$ with an initial condition $v(t)$ at t

$$v(t+t') = v(t)e^{-\lambda t'} + \frac{1}{m}\int_0^{t'} f(t+t'-t'')e^{-\lambda t''}dt'$$

Then

$$\langle v_x(t+t')v_x(t)\rangle = \left\langle v_x^2 e^{-\lambda t'} + \frac{v_x(t)}{m}\int_0^{t'} f(t+t'-t'')e^{-\lambda t''}dt' \right\rangle \approx \left\langle v_x^2 \right\rangle e^{-\lambda t'}$$

where we assumed that $v(t)$ is uncorrelated with $f(t+t')$ for $t' > 0$ (there is actually a small correlation, but it is negligible). According to the equipartition theorem $\left\langle v_x^2 \right\rangle = \frac{T}{m}$, hence

$$\langle v(t+t')v(t)\rangle = \frac{3T}{m}e^{-\lambda t'}$$

Hence we find that the correlation time of the velocity is λ^{-1} as stated above, much longer than τ. For times $t \gg \lambda^{-1}$ we thus find that $\Delta r(t)$ is a Gaussian random variable. The second moment of Δr is given by

$$\begin{aligned}
\left\langle (\Delta r(t))^2 \right\rangle &= \int_0^t dt_1 \int_0^t dt_2 \left\langle v(t_1)v(t_2)\right\rangle \\
&= \int_0^t dt_1 \int_0^t dt_2 \frac{3T}{m} e^{-\lambda|t_1-t_2|} \\
&= \frac{3T}{2m}\int_0^{2t} ds \int_{-s}^{s} dt' e^{-\lambda|t'|} \\
&= \frac{3T}{2m}\int_0^{2t} ds \frac{2}{\lambda}\left(1-e^{-\lambda s}\right) \\
&= \frac{3T}{m\lambda}\int_0^{2t} ds \left(1-e^{-\lambda s}\right) \\
&= \frac{3T}{m\lambda^2}\left(2\lambda t + e^{-2\lambda t} - 1\right)
\end{aligned}$$

which has a ballistic behavior for $t \ll \lambda^{-1}$ and diffusive behavior for $t \gg \lambda^{-1}$,

$$\left\langle (\Delta r_x(t))^2 \right\rangle = \frac{1}{3} \left\langle (\Delta r(t))^2 \right\rangle \approx \begin{cases} \frac{2T}{m} t^2 & t \ll \lambda^{-1} \\ \frac{2T}{m\lambda} t & t \gg \lambda^{-1} \end{cases}$$

where we implicitly assumed isotropy so $\left\langle (\Delta r_y(t))^2 \right\rangle = \left\langle (\Delta r_y(t))^2 \right\rangle = \left\langle (\Delta r_y(t))^2 \right\rangle$. Defining the diffusion coefficient by

$$\left\langle (\Delta r_x(t))^2 \right\rangle \equiv 2Dt$$

we find that in $d = 3$

$$D = \frac{T}{m\lambda}$$

which is known as *the Einstein relation*. So finally we get

$$\rho(\Delta r, t) = \frac{1}{(4\pi Dt)^{3/2}} \exp\left[-\frac{(\Delta r)^2}{4Dt} \right]$$