Renormalization Group analysis of 2D Ising model

Amir Bar

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1 Introduction

In this tutorial we will see explicitly how RG can be used to probe the phase diagram of $d > 1$ systems, focusing as usual on the Ising model in $d = 2$. Remember that the "bare" Ising model partition function reads

$$Z(K_1) = \sum_{\{\sigma\}} exp\left[K_1 \sum_{\langle (i,j),(k,l)\rangle} s_{ij}s_{kl}\right] \tag{1}$$

We shall first try to follow the same decimation procedure as done for the $d = 1$ case, and after we will understand why it is not working as well as in the $d = 1$ case, we shall introduce a couple of approximation schemes to overcome the difficulties.

2 Straight forward decimation

As illustrated in Fig.1, we split the lattice to two sublattices A and B so that spins of type A (which we'll denote by A_{ij}) have neighbors of type B (which we will denote accordingly). We shall decimate now, summing over the B spins

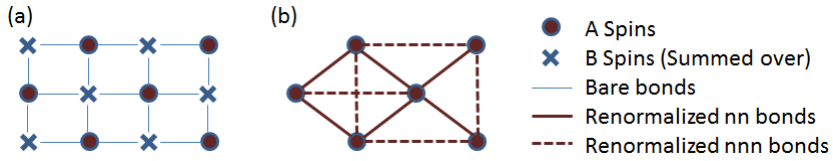

Figure 1: The $b = \sqrt{2}$ renormalization scheme: (a) The original lattice with the A and B spin types marked, and (b) The renormalized lattice with the emergent next nearest neighbors (nnn) links appear as dashed lines

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\begin{aligned}
Z &= \sum_{\{A\}}\sum_{\{B\}} exp\left[K_1 \sum_{\langle (i,j),(k,l)\rangle} A_{ij}B_{kl}\right] \\
&= 2\sum_{\{A\}}\prod_{\{ij\}} cosh\left[K_1\tilde{A}_{ij}\right] \qquad (2) \\
\tilde{A}_{ij} &\equiv A_{i+1,j} + A_{i-1,j} + A_{i,j+1} + A_{i,j-1}
\end{aligned}
$$

where the ij indices are in the B sub-lattice. From the symmetry of the expression it is clear that on top of the nearest-neighbor interactions $A_{i+1,j}A_{i,j+1} + ...$ interactions that will be formed, there will be also diagonal interactions of the form $A_{i+1,j}A_{i-1,j}$ and $A_{i,j+1}A_{i,j-1}$. In addition, a fourth order term will be generated $A_{i+1,j}A_{i-1,j}A_{i,j+1}A_{i,j-1}$ (the third order will not appear due to symmetry). As we shall proceed in the RG flow (with these new terms) higher order terms will be generated as well, and hence we cannot write a closed set of equations in terms of only a small number of coupling constants. So how can we proceed?

We know (from the exact solution) that in the Ising model, for $H = 0$, we need to tune only one parameter (the temperature for instance) to get criticality. This implies that in the exact RG flow (with all of the possible coupling constants) the unstable fixed point (corresponding to the critical point) has only a single unstable direction and all other are irrelevant parameters. Hence, it can be hoped that we still get the qualitative behavior of the model even if we truncate the order of the coupling in some way. Below we present two (approximate!) methods to do so:

3 K expansion

We can assume that K_1 is small and expand (2) in orders of K_1 to second order (as the first order will vanish due to symmetry)

$$
\begin{aligned}
Z &\approx 2\sum_{\{A\}}\prod_{\{ij\}}\left(1 + \frac{1}{2}K_1^2\tilde{A}_{ij}^2\right) \qquad (3) \\
&\approx 2\sum_{\{A\}} exp\left(\frac{1}{2}K_1^2\sum_{\{ij\}}\tilde{A}_{ij}^2\right) \qquad (4) \\
\tilde{A}_{ij}^2 &= 4 + 2\underbrace{(A_{i+1,j}A_{i,j+1} + ...)}_{nearest\ neighbours} + 2\underbrace{\ (A_{i+1,j}A_{i-1,j} + ...)}_{next\ nearest\ neighbours} \qquad (5)
\end{aligned}
$$

We see that even in this crude approximation next nearest neighbors (nnn) interactions appear. The consistent thing to do is to acount for these from the beginning in order to see their effect on the nearest neighbors (nn) interactions,

i.e. start with

$$Z\left(K_1,K_2\right) = \sum_{\{\sigma\}} exp\left[K_1 \sum_{\langle(i,j),(k,l)\rangle} s_{ij}s_{kl} + K_2 \sum_{\langle\langle(i,j),(k,l)\rangle\rangle} s_{ij}s_{kl}\right] \quad (6)$$

where $\langle\langle(i,j),(k,l)\rangle\rangle$ stands for nnn. Noting that the nnn of A spins are A spins and similarly for B we find

$$Z = \sum_{\{A\}} exp\left[K_2 \sum_{\langle\langle(i,j),(k,l)\rangle\rangle_A} A_{ij}A_{kl}\right] Z_B \quad (7)$$

$$Z_B = \sum_{\{B\}} exp\left[K_1 \sum_{\langle(i,j),(k,l)\rangle} A_{ij}B_{kl} + K_2 \sum_{\langle\langle(i,j),(k,l)\rangle\rangle_B} B_{ij}B_{kl}\right] \quad (8)$$

Expanding in K_1, K_2 to leading order yields

$$\begin{aligned} Z_B &= \sum_{\{B\}} \left[1 + \frac{1}{2}K_1^2 \left[\sum_{\langle(i,j),(k,l)\rangle} A_{ij}B_{kl}\right]^2 + K_2 \sum_{\langle\langle(i,j),(k,l)\rangle\rangle_B} B_{ij}B_{kl}\right] \\ &= \left[1 + g\left(K_2\right) + \frac{1}{2}K_1^2 \sum_{kl}\left(\tilde{A}_{kl}\right)^2\right] \\ &\approx exp\left[g\left(K_2\right) + \frac{1}{2}K_1^2 \sum_{kl}\left(\tilde{A}_{kl}\right)^2\right] \end{aligned} \quad (9)$$

The term $g\left(K_2\right)$ comes from the sum over the only-B's term, while the K_1 expansion follows as before (and similarly the linear term vanish due to symmetry). Now we rescale the lattice by $b = \sqrt{2}$ and relabel vertices, use the previous calculation of $\tilde{A}_{kl}^2$ (5), and finally note that each $A's$ nn interaction in (5) appears twice in the sum of (9) so that

$$Z\left(K_1',K_2'\right) = \sum_{\{A\}} exp\left[g\left(K_1,K_2\right) + \left(2K_1^2 + K_2\right) \sum_{\langle(i,j),(k,l)\rangle} A_{ij}A_{kl} + K_1^2 \sum_{\langle\langle(i,j),(k,l)\rangle\rangle} A_{ij}A_{kl}\right] \quad (10)$$

Again $g\left(K_1,K_2\right)$ absorb any A-independent terms. Comparing (10) with (6) we find the RG equations

$$\begin{aligned} K_1' &= 2K_1^2 + K_2 \\ K_2' &= K_1^2 \end{aligned}$$

The fixed points are

$$\left(K_1,K_2\right) = \left(0,0\right), \left(\infty,\infty\right), \left(\frac{1}{3},\frac{1}{9}\right)$$

The first two are the high temperature and low temperature fixed points, while the third one is the critical point. Analyzing stability:

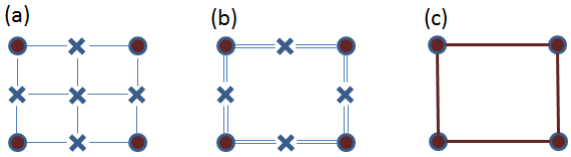

Figure 2: The Migdal Kadanoff ($b = 2$) rescaling scheme: (a) The original lattice; (b) The lattice after the bonds moving step; (c) the renormalized lattice, with only nearest neighbors bonds.

- Near $(0,0)$: If $K_2 \ll K_1 \ll 1$ then obviously $K_1' \ll K_1 \Rightarrow$ stable fixed point
- Near (∞,∞): If $K_1, K_2 \gg 1$ then obviously $K_1' \gg K_1 \Rightarrow$ stable fixed point
- Near $\left(\frac{1}{3},\frac{1}{9}\right)$: linearizing $K_1 = \frac{1}{3}+\alpha$, $K_2 = \frac{1}{9}+\beta$:

$$\begin{aligned} \alpha' &= \frac{4}{3}\alpha + \beta \\ \beta' &= \frac{2}{3}\alpha \end{aligned}$$

The eigenvalues are given by

$$\left(\lambda - \frac{4}{3}\right)\lambda - \frac{2}{3} = 0 \quad \Rightarrow \quad \lambda_\pm = \frac{1}{2}\left(\frac{4}{3} \pm \sqrt{\left(\frac{4}{3}\right)^2 + \frac{8}{3}}\right) = \frac{1}{3}\left(2 \pm \sqrt{10}\right)$$

So we see that there is one positive (relevant) eigenvalue and one negative (irrelevant) eigenvalue $\Rightarrow$ unstable fixed point, with one control parameter (as we wished!)

Summarizing, we got indeed a phase transition for finite (K_1, K_2) with a single unstable direction (temperature). However, K_1 at the critical fixed point is not really small, hence there is no obvious reason why the truncation of the expansion at second order in K_1 is legitimate (and obviously no *a priori* reason to believe the $K_1 = \infty$ fixed point). Indeed the critical point found in this method is not accurate and also critical exponents that are derived from it are not the correct ones. In principle, by including higher powers of K - and respectively higher order terms in coupling - we can become more accurate.

4 Migdal-Kadanoff approximation

A different approximation that was used by Kadanoff is driven by a somewhat more physical motivation, though it is less controlled than the previous one. The starting point is the nn Ising model (1), and the rescaling factor is $b = 2$

instead of the previous $b = \sqrt{2}$. This is done by keeping in each iteration only the 4 corner spins of a 3×3 block of spins, as illustrated in Fig.2. In order to avoid generating new interactions when summing over the middle spin, it is just removed and the bonds attached to it are "moved" to the sides so as to double the bonds between the other spins of the block. This move, besides having a somewhat physical excuse of keeping the energetics of the model, is mandatory as without it the procedure would be identical to the 1D procedure, and hence no transition would be expected.

The RG equation in 1D and $H = 0$ is $K' = \frac{1}{2} ln\left(cosh\left(2K\right)\right)$, so in our case due to the bond doubling the recursion relation is

$$K' \quad = \quad \frac{1}{2} ln\left(cosh\left(4K\right)\right)$$

Again we find $K = 0$ and $K = \infty$ fixed points, but unlike the 1D case for $K \gg 1$, $K' \approx \frac{1}{2} ln\left(\frac{1}{2} exp(4K)\right) \approx 2K$ and hence the low temperature fixed point is also stable. (for $K \ll 1$, $K' \approx \frac{1}{2} ln\left(1 + 16K^2\right) \approx 8K^2$ so the high temperature fixed point is stable as it is in 1D). Hence in between we must have another unstable fixed point, which is given by

$$e^{2K^*} = \frac{e^{4K^*} + e^{-4K^*}}{2} \Rightarrow K^* \approx 0.305$$

Comparing to the previous approximation, the Migdal-Kadanoff approximation gives a result further away from the true answer of $K_c \approx 0.441$, and in addition it is an uncontrollable approximation in the sense that there is no small parameter in which it was expanded. But on the other hand it is in a sense more straightforward as the procedure did not generate any additional interactions.