Statistical Mechanics 2012/2013   Problem Set 2

Submission date: 3.12.12

1.1 Paramagnetic cooling (25 points)

Consider N noninteracting spins $1/2$, each having a magnetic moment $\pm\mu$. Derive the magnetization M and the entropy S as functions of the magnetic field H and the temperature T. Obtain S also in the microcanonical ensemble by counting states and confirm that S is a unique function of M.

(b) Plot schematically the equilibrium $S(T)$ for two different magnetic fields H_1 and H_2, where $H_1 < H_2$. (Look also at the curves for $M(T)$ at these two fields and make sure you understand why one lies above the other).

(c) Imagine that the system is started at H_1 and at some temperature T_1, and H is increased isothermally to H_2. Has the entropy of the system has decreased? Are the spins less or more ordered as a result of this process?

(d) Now, the magnetic field is reduced adiabatically back to H_1. Note that the system is now again on the curve $S(T)$ of H_1, but at a lower temperature. This is the principle of "Nuclear Paramagnetic Cooling". Plot a sketch of this cooling process on top of your $S(T)$ curve.

(e) We first want to check whether the model described above is a good approximation of the physical setting. Suppose the system is first cooled by other means to below $1K$. Estimate the nuclear magneton times a field of a few Tesla compared to k_BT at $T = 1K$. What does it imply about the average magnetization of the nuclear magnetic moments? Consider a metal at $T = 0.1K$ and compare the entropy of the electrons, phonons and nuclear spins at $H \simeq 1T$, where T here denotes Tesla. Assume that the electron are not affected by the magnetic field. Which one is the largest? Explain why you had to check these details.

(f) Estimate how much cooling can be obtained using 10^{22} nuclear spins, starting at $0.1K$ and $1/2$ Tesla and going to 5 Tesla.

1.2 Model for electrons in a metal (30 points)

Consider a simple model for electrons in a metal, in which the metal is characterized as a three-dimensional potential well of depth U and linear size L. We denote the electron density in this well as $n = N/L^3$. The minimum energy needed to remove an electron from the metal, ϕ, is thus given by $\phi = eU - \epsilon_F$, where ϵ_F is the Fermi energy of the electrons. ϕ is called the work function of the metal (see Figure 1). Throughout the problem all Coulomb interactions will be neglected. We consider here the limit where $eU \gg \epsilon_F$ in which the energy levels of the electrons in the metal can be approximated by those of a particle in an infinite potential well.

(a) Obtain an expression for ϵ_F as a function of n, and evaluate it for $n = 10^{22}$ cm^{-3}.

(b) Consider an electron gas outside the metal in thermal equilibrium with the electrons in the metal at temperature T. Since typically at room temperature $k_BT \ll \phi$, the electron density outside the metal can be assumed to be small. By equating chemical potentials, find the mean electron density outside the metal, n_g. Evaluate n_g for n of (a) and $\phi = 2$ eV at room temperature, and verify that in these conditions the free electron gas can safely be approximated as a classical gas.

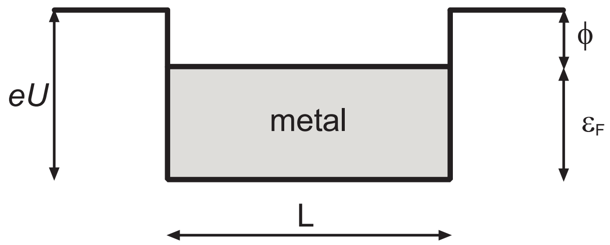

Figure 1: Terminology and notation for problem 1.2

(c) Calculate the pressure of the electron gas outside the metal and that of the electrons in the metal.

(d) Discuss what happens when L is decreased at constant N. Does n_g increase or decrease? why? can you identify a value of L beyond which your results from the previous paragraphs are no longer valid? which assumptions will no longer be valid close to this value of L?

1.3 Idea Bose gas in two dimensions (25 points)

(a) Consider the relation that determines the chemical potential: the number of atoms equal to the sum of the Bose distributions over all momenta - see first paragraph in section 3.2.4 in the lecture notes. Is there a condensation at finite temperature in two dimensions ?

(b) Now consider the example of N Bosons in a harmonic anisotropic potential which you saw in class, for an arbitrary dimension d:

$$V = \frac{1}{2} \sum_{i=1}^{d} m\omega_i x_i^2 \tag{1}$$

Generalize the results from class for the transition temperature and the occupation number of the condensate as a function of temperature for this case. Is there a condensation for $d = 2$? Does this conform with the result in the previous section? Explain why.

1.4 Intermediate statistics (20 points)

Consider a hypothetical system where each quantum state can be occupied by no more than p particles. Find the mean occupation number of the state with the energy ϵ when the chemical potential of the system is μ (system is considered within the grand-canonical ensemble). Check how the resulting formula goes into the Fermi or Bose distributions within the proper limits.