Statistical Mechanics 2011-12 — Problem Set 3

due: December 22, 2011

3.1 Fermi gas in two dimensions (30 points)

In class you have seen that the Sommerfeld expansion is a useful tool for calculating the low-T behavior of ideal Fermi gases. The following problem will examine a case in which some complications with the expansion occur. To this end, an ideal gas of electrons in two dimensions will be studied.

(a) In two dimensions, what is the relation between the Fermi wave number, k_F, and the density of electrons, $n \equiv N/V$ (where V is the volume in two dimension, i.e., the area)?

(b) Let $g(\epsilon)$ be the free-electron density of an energy level (that is, $g(\epsilon)d\epsilon$ is the number of single-electron levels per unit volume in the energy range from ϵ to $\epsilon + d\epsilon$). Prove that in two dimensions $g(\epsilon)$ is a constant for $\epsilon > 0$ and is 0 for $\epsilon < 0$. What is the constant?

(c) Show that because $g(\epsilon)$ is constant, every term in the Sommerfeld expansion for n vanishes, except for the $T = 0$ term. Deduce that $\mu = \epsilon_F$ at any temperature.

(d) In the two-dimensional case, the integral in $n = \int_{\infty}^{\infty} d\epsilon g(\epsilon) f(\epsilon)$ (where $f(\epsilon)$ is the Fermi distribution) can be solved exactly at all temperatures, without the use of the Sommerfeld expansion. Solve this integral and show that when $g(\epsilon)$ is as in (b), the exact expression for $\mu(T, n)$ satisfies

$$\mu + k_B T \ln\left(1 + e^{-\mu/k_B T}\right) = \epsilon_F(n) \ . \qquad (1)$$

What is the origin of the difference between the results of (c) and (d) ?

3.2 Debye model of lattice (20 points)

Using Debye model of lattice vibrations, find the mean squared fluctuation of the atom deviation from the equilibrium position, $\langle |\vec{r} - \langle \vec{r} \rangle|^2 \rangle$.

3.3 Lyapunov exponent (20 points)

Consider the infinitesimal distance between two trajectories in d-dimensional phase space $\mathbf{r}(t)$. Following lectures, assume that the evolution is statistically isotropic and can be characterized by the evolution matrix $\hat{W}$ so that $\mathbf{r}(t) = \hat{W}(t)\mathbf{r}(0)$. Remind that at $t \to \infty$ there exists d directions $\mathbf{f}_i$ such that vectors pointed initially at these directions are getting stretched by the factor $\exp(\lambda_i t)$, where λ_i are Lyapunov exponents. Using the Liouville theorem show that $\langle r^{-d} \rangle$ is time-independent as $t \to \infty$.

3.4 Large Deviation theory (30 points)

In class we considered the large deviation principle for the average one N independent identically distributed variables, y_i. We now consider the case where each variable can assume M possible values with probablity p_m,

$$P(y_i = v_m) = p_m \qquad \forall m = 1, \ldots, M \tag{2}$$

(a) We define the empirical vector as the average number of time the value m appears in the set of N varaibles,

$$L_m = \sum_{i=1}^{N} \delta_{y_i - v_m}. \tag{3}$$

Show that probability distribution of the empirical vector, $P_N(\vec{L})$ where $\vec{L} = \{L_1, L_2, \ldots L_M\}$, obeys a large deviation principle in the limit $N \gg 1$ and find its rate function. One way to compute it is to follow the lines of the derivation of Cramer's theorem presented in section 4.3 of the lecture notes. How is the resulting rate function related to the concept of relative entropy (also known as the Kullback-Leibler divergence) ? Can you explain it intuitively ?

(b) Consider the result above for the case of an unbiased dice with M possible outcomes. In this case $v_m = m$ and $m = 1, \ldots, M$. Compute using $P_N(\vec{L})$ the large deviation principle for the sum of N tosses, $X = \sum_{m=1}^{M} mL_m$ in the limit of $1 \ll M \ll N$. Compute the same physical quantity using Carmer's theorem, where you calculate the large deviation principle for $X = \sum_{i=1}^{N} y_i$. Do you obtain the same results?