Statistical Mechanics 2011-12 - Problem Set 4

due: January 5, 2012

4.1 Coin weighing

Suppose one has n coins, among which there may or may not be one counterfeit coin. If there is a counterfeit coin, it may be either heavier or lighter than the other coins. The coins are to be weighed by a balance.

(a) We denote by W_i the outcome of the i'th weighing. What is the maximal entropy of this variable, $S(W_i)$?

(b) Find the maximal information gain (often referred to as 'mutual information') from the first weighing defined as

$$I(Q, W_1) = S(Q) - S(Q|W_1) \tag{1}$$

where Q is a random variable which denotes the state of the n-coin system. The conditional entropy is defined for two general random variables as

$$\begin{aligned} S(X|Y) &= -\sum_y P(Y=y) \sum_x P(X=x|Y=y) \log_2 P(X=x|Y=y) \\ &= \sum_y P(Y=y) S(X|Y=y). \end{aligned} \tag{2}$$

Hint : Assume in this weighing that we put m coins at each side of the balance and find the maximizing m.

(c) Using the symmetry of the information gain, $I(X, Y) = I(Y, X)$, find an upper bound for $I(Q, W_1)$. Does your result in (b) agrees with this bound ?

(d) Generalize the bound in (c) to the case of the i'th weighing, and use it to set a lower bound on $S(Q|W_1, \ldots, W_k)$. Find the value of k for which this lower bound reaches zero. The result is the minimal number of weighings we need in order to reduce the uncertainty of the system of n coins to zero. Zero uncertainty means that we have found the counterfeit coin (if any) and have correctly declared it to be heavier or lighter. However, only one or sometimes several optimal strategies of weighings could yield an answer within k weighings.

(f) **5 points bonus** : What is the coin weighing strategy for $k = 3$ weighings and $n = 12$ coins ?

4.2 Alphabet

A source produces a character x from the alphabet $x \in \{0, 1, \dots, 9, a, b, \dots, z\}$. We divide characters into three sets, numeral, vowels (a,e,i,o,u) and the 21 consonants. With probability $1/3$, x is a numeral, with probability $1/3$ x is a vowel and with probability $1/3$ it is a consonant. Within each set x is uniformly distributed.

(a) What is the entropy of per character in bits of a message composed of several such characters, $\{x_1, \dots, x_N\}$?

(b) Repeat (a) for the case where x is uniformly distributed over all possible characters. Do you obtain a large or smaller entropy per character ? Could you have guessed the answer without computing explicitly ?

4.3 Fluctuation of the number of particles and of the wave intensity

(a) Find the mean squared fluctuation of the number of particles in the state number k of the ideal Fermi and Bose gases.

(b) Consider a gauge measuring the amplitude of light coming from a noncoherent source. This can be described by a set of classical light waves of a specific frequency, ω, with random phases and amplitudes. Calculate the fluctuation in the mean intensity of the radiation, $\langle(\Delta I)^2\rangle$, by averaging over long time and over the random phases. Express your result in term of the mean intensity of the radiation, $\langle I\rangle$. How this result related to the quantum result you have computed in (a) for Bose gas.

4.4 Ornstein-Zernicke approximation

Derive within the Ornshtein-Zernicke approximation the expression for the two point correlation function found in the lecture notes,

$$\langle \Delta n(0) \Delta n(r) \rangle = \frac{T}{8\pi g r} e^{-r/r_c}. \tag{3}$$

Use the identity

$$(\kappa^2 - \Delta) \frac{e^{-\kappa r}}{r} = 4\pi \delta(\mathbf{r}). \tag{4}$$