Extra:

0.1 The energy of $N$ interacting oscillators with frequencies $\omega_i$ is

$$E = \sum_{i=1}^{N} \epsilon_i n_i + \lambda/2 \sum_{i=1}^{N} \sum_{j=1}^{N} V_{ij} n_i n_j \ . \tag{1}$$

Here $\epsilon_i = \hbar n_i$ are the energy levels, $n_i = 0, 1, \ldots$ -occupation number for $i$-th oscillator, $V_{ij} = V_{ji}$ and $V_{ii} = 0$. Assuming $\lambda$ small find the partition function of the system to first order in $\lambda$. Can it be approximated as the partitions function of a non-interacting system with different energy levels?

0.2 Below critical temperature, there are two equilibrium values of the order parameter $\eta_0$ and $-\eta_0$. Consider a flat boundary between two regions in space, one with $\eta_0$, another with $-\eta_0$. Find how the order parameter depends on the distance to the boundary. Find the surface tension i.e. the energy of a unit area of the boundary. Use Landau theory.

1. Consider a classical ideal gas of atoms whose mass is $m$ moving in an attractive potential of an impenetrable wall: $V(x) = Ax^2/2$ with $A > 0$ for $x > 0$ and $V(x) = \infty$ for $x < 0$. Atoms move freely along $y$ and $z$. Let $T$ be the temperature and $n$ be the number of atoms per unit area of the wall. Consider the thermodynamic limit.

a) Calculate the concentration $n(x)$ — the number of atoms per unit volume as a function of the distance $x$ from the wall. Note that $n = \int_0^\infty n(x)\, dx$.

b) Calculate the energy and specific heat per unit area of the wall.

c) Find the free energy and chemical potential.

d) Find the pressure the atoms exert on the wall (i.e. at $x = 0$).

2. A cavity containing a gas of electrons has a small hole of area A through which electrons can escape. External electrodes are so arranged that voltage is V between inside and outside of the cavity. Assume that

i) a constant number density of electrons is maintained inside (for example, by thermionic emission);

ii) electrons are in thermal equilibrium with temperature T and chemical potential $\mu$ such that $kT \ll V - \mu$;

iii) electrons moving towards the hole escape if they have an energy greater than V.

Estimate the total current carried by escaping electrons.

3. A $d$-dimensional container is divided into two regions A and B by a fixed wall. The two regions contain identical Fermi gases of spin $1/2$ particles which have a magnetic moment $\tau$. In region A there is a magnetic field of strength $H$, but there is no field in region B. Initially, the entire system is at zero temperature, and the numbers of particles per unit volume are the same in both regions. If the wall is now removed, particles may flow from one region to the other. Determine the direction in which particles begin to flow, and how the answer depends on the space dimensionality $d$.

4. One may think of certain networks, such as the internet, as a directed graph $G$ of $N$ vertices. Every pair of vertices, say $i$ and $j$, can be connected by multiple edges (e.g. hyperlinks) and loops may connect vertices to themselves. The graph can therefore be described in terms of an adjacency matrix $A_{ij}$ with $N^2$ elements; each $A_{ij}$ counts the number of edges connecting $i$ and $j$ and can be any non-negative integer, 0, 1, 2...

The entropy of the ensemble of all possible graphs is $S = -\sum_G p_G \ln p_G = -\sum_{\{A_{ij}\}} p(A_{ij}) \ln p(A_{ij})$. Consider such an ensemble with the fixed average number of edges per vertex $\langle k \rangle$.

(i) Write an expression for the number of edges per vertex $k$ for a given graph $A_{ij}$. Use the maximum entropy principle to calculate $p_G(A_{ij})$ and the partition function $Z$ (denote the Lagrange multiplier that accounts for fixed $\langle k \rangle$ by $\tau$). What is the equivalent of the Hamiltonian? What are the degrees of freedom? What kind of "particles" are they? Are they interacting?

(ii) Calculate the free energy $F = -\ln Z$, and express it in terms of $\tau$. Is it extensive with respect to any number?

(iii) Write down an expression for the mean occupation number $\langle A_{ij} \rangle$ as a function of $\tau$. What is the name of this statistics? What is the "chemical potential" and why?

(iv) Express $F$ via $N$ and $\langle k \rangle$. Express $p_G$ for a given graph as a function of $k$, $\langle k \rangle$ and $N$.

Good luck!

Solutions

Problem 1.

a) $n(x) = n(0)\exp(-\beta A x^2/2) = 2n(A\beta/2\pi)^{1/2}\exp(-\beta A x^2/2)$.

b) For $x_i > 0$,

$$\mathcal{H} = \sum_i \left(\frac{p_i^2}{2m} + \frac{Ax_i^2}{2}\right) .$$

Equipartition gives $E = 4(T/2)N = 2TN$ and $C_v = 2N$.

c)

$$\begin{aligned}
Z_N &= \frac{h^{3N}}{N!}\int \pi dp dx \exp\left[-\beta\sum_i\left(\frac{p_i^2}{2m} + \frac{Ax_i^2}{2}\right)\right] \\
&= \frac{h^{3N}L^{2N}}{N!}(\sqrt{2\pi m T})^{3N}\left(\frac{\sqrt{2\pi T/A}}{2}\right)^N , \\
F &= -T\left[3N\ln h - N\ln N + N + 2N\ln L + N\ln\left(2\pi^2 T^2 m^{3/2}/A^{1/2}\right)\right] , \\
\mu &= \frac{\partial F}{\partial N} = -T\left[3\ln h - \ln N + 2\ln L + \ln\left(2\pi^2 T^2 m^{3/2}/A^{1/2}\right)\right] \\
&= -T\left[3\ln h - \ln n + \ln\left(2\pi^2 T^2 m^{3/2}/A^{1/2}\right)\right] .
\end{aligned}$$

d) $P(x) = n(x)T$. Note that the pressure is inhomogeneous so it is not $\partial F/\partial l$ where $l$ is the length of the system in the x-direction — the free energy $F$ is not extensive in $l$ and at the limit $l \to \infty$ we have $\partial F/\partial l = 0$.

Problem 2

Our main task is to find the number of particles per unit time escaping from the cavity. To this end, we first suppose that the cavity is a cube of side $L$ and that single-particle wavefunctions satisfy periodic boundary conditions at its walls. Then the allowed momentum eigenvalues are $p_i = hn_i/L$ ($i = 1, ..., 3$), where the $n_i$ are positive or negative integers. Then (allowing for two spin polarizations) the number of states with momentum in the range $d^3p$ is $(2V/h^3)d^3p$, where $V = L^3$ is the volume. (To an excellent approximation, this result is independent of the shape of the cavity.) Multiplying by the grand canonical occupation numbers for these states, we find that the number of particles per unit volume with momentum in the range $d^3p$ near $\mathbf{p}$ is $n(p)d^3p$, where

$$n(p) = \frac{2}{h^3}\frac{1}{\exp[\beta(\epsilon(p) - \eta)] + 1} \tag{2}$$

with $\epsilon(p) = |\mathbf{p}|^2/2m$. Now we consider, in particular, a small volume enclosing the hole in the cavity wall, and adopt a polar coordinate system with the usual angles $\theta$ and $\phi$, such that the axis $\theta = 0$ (the $z$ axis, say) is the outward normal to the hole (see corresponding figure).

The number of electrons per unit volume whose volume has a magnitude between $p$ and $p + dp$ and is directed into a solid angle $d\Omega = sin\theta d\theta d\phi$ surrounding the direction $(\theta, \phi)$ is

$$n(p)p^2\sin\theta dp d\theta d\phi \tag{3}$$

and, since these electrons have a speed $p/m$, the number per unit time crossing a unit area normal to the $(\theta, \phi)$ direction is

$$\frac{1}{m}n(p)p^3\sin\theta dp d\theta d\phi . \tag{4}$$

The hole subtends an area $\delta A\cos\theta$ normal to this direction, so the number of electrons per unit time passing through the hole with momentum between $p$ and $p + dp$ into the solid angle $d\Omega$ is

$$\frac{\delta A}{m}n(p)p^3\sin\theta\cos\theta dp d\theta d\phi . \tag{5}$$

It is useful to check this for the case of a classical gas, with $n(p) = n(2\pi mk_BT)^{-3/2}e^{-p^2/2mk_BT}$, where $n$ is the total number density, and with $V = 0$. Bearing in mind that only those particles escape for which $0 \le \theta < \pi/2$, we then find that the total number of particles escaping per unit time is

$$\frac{\delta An}{m(2\pi mk_BT)^{3/2}} \int_0^\infty dp \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi p^3 \exp\left[-\frac{p^2}{2mk_BT}\right] \sin\theta\cos\theta = \frac{1}{4}n\bar{c}\delta A\,, \tag{6}$$

where $\bar{c} = \sqrt{8k_BT/\pi m}$ is the mean speed. This standard result can be obtained by several methods in elementary kinetic theory. For the case in hand, the number of electrons escaping per unit time is

$$\frac{dN}{dt} = \frac{\delta A}{m} \int_{\sqrt{2mV}}^\infty dp \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi p^3 n(p) \sin\theta\cos\theta \tag{7}$$

$$= \tfrac{2\pi\delta A}{mh^3} \int_{\sqrt{2mV}}^\infty \tfrac{p^3}{e^{\beta(\epsilon(p)-\mu)}+1} dp\,. \tag{8}$$

If $V - \mu \gg k_BT$, then $\beta(\epsilon(p) - \mu)$ is large for all values of $p$ in the range of integration, and we can use the approximation

$$\frac{dN}{dt} \simeq \frac{2\pi\delta A}{mh^3} \int_{\sqrt{2mV}}^\infty p^3 e^{-(\epsilon(p)-\mu)\beta} dp \simeq \frac{\pi\delta A}{mh^3}(2mk_BT)^2 e^{-(V-\mu)\beta}\left(1 + \frac{V}{k_BT}\right)\,. \tag{9}$$

Since $\mu$ is positive for a Fermi gas at low temperatures, we also have $V \gg k_BT$, so the 1 and $1 + V/k_BT$ can be neglected. Finally, therefore, on multiplying by the charge $-e$ each electron, we estimate the current as

$$I = -\left(\frac{4\pi me}{h^3}\right)\delta AVk_BT \exp\left[-\beta(V - \mu)\right]\,. \tag{10}$$

If the temperature is low enough that $k_BT \ll \epsilon_F$, then $\mu$ can be replaced by the Fermi energy $\epsilon_F = (h^2/8m)(3n/\pi)^{2/3}$, where $n$ is the total number of particles per unit volume. Of course, the current, that we have calculated is the charge per unit time emerging from the hole. The number of electrons per unit solid angle at an angle $\theta$ to the normal ($z$ direction) is proportional to $\cos\theta$, so, although no electrons emerge tangentially to the wall ($\theta = \pi/2$), not all of them travel in the $z$ direction.

Problem 3

In general, particles flow from a region of higher chemical potential to a region of lower chemical potential. We therefore need to find out in which region the chemical potential is higher, and we do this by considering the grand canonical expression for the number of particles per unit volume. In the presence of a magnetic field, the single- particle energy is $\epsilon \pm \tau H$, where $\epsilon$ is the kinetice energy, depending on whether the magnetic moment is parallel or antiparallel to the field. The total number of particles is then given by

$$N = \int_0^\infty d\epsilon g(\epsilon)\frac{1}{\exp[\beta(\epsilon - \eta - \tau H)] + 1} + \int_0^\infty d\epsilon g(\epsilon)\frac{1}{\exp[\beta(\epsilon - \eta + \tau H)] + 1}\,. \tag{11}$$

For non-relativistic particles in a $d-$dimensional volume $V$, the density of states is $g(\epsilon) = \gamma V\epsilon^{d/2-1}$, where $\gamma$ is a constant. At $T = 0$, the Fermi distribution function is

$$\lim_{\beta\to\infty}\left(\frac{1}{e^{\beta(\epsilon-\mu\pm\tau H)} + 1}\right) = \theta(\mu \mp \tau H - \epsilon) \tag{12}$$

where $\theta(\cdot)$ is the step function, so the integrals are easily evaluated with the result

$$\frac{N}{V} = \frac{2\gamma}{d}\left[(\mu + \tau H)^{d/2} + (\mu - \tau H)^{d/2}\right]\,. \tag{13}$$

At the moment that the wall is removed, $N/V$ is the same in regions $A$ and $B$; so (with $H = 0$ is the region $B$) we have

$$(\mu_A + \tau H)^{d/2} + (\mu_A - \tau H)^{d/2} = 2\mu_B^{d/2}\,. \tag{14}$$

For small fields, we can make use of the Taylor expansions

$$(1 \pm x)^{d/2} = 1 \pm \frac{d}{2}x + \frac{d}{4}\left(\frac{d}{2} - 1\right)x^2 + ... \tag{15}$$

to obtain

$$\left(\frac{\mu_B}{\mu_A}\right)^{d/2} = 1 + \frac{d(d-2)}{8}\left(\frac{\tau H}{\mu_A}\right)^2 + ... \tag{16}$$

We see that, for $d = 2$, the chemical potentials are equal, so there is no flow of particles. For $d > 2$, we have $\mu_B > \mu_A$ so particles flow towards the magnetic field in region $A$ while, for $d < 2$, the opposite is true. We can prove that the same result holds for any magnetic field strength as follows. For compactness, we write $\lambda = \tau H$. Since our basic equation $(\mu_A + \lambda)^{d/2} + (\mu_A - \lambda)^{d/2} = 2\mu_B^{d/2}$ is unchanged if we change $\lambda$ to $-\lambda$, we can take $\lambda > 0$ without loss of generality. Bearing in mind the $\mu_B$ is fixed, we calculate $d\mu_A/d\lambda$ as

$$\frac{d\mu_A}{d\lambda} = \frac{(\mu_A - \lambda)^{d/2-1} - (\mu_A + \lambda)^{d/2-1}}{(\mu_A - \lambda)^{d/2-1} + (\mu_A + \lambda)^{d/2-1}}\,. \tag{17}$$

Since $\mu_A + \lambda > \mu_A - \lambda$, we have $(\mu_A + \lambda)^{d/2-1} > (\mu_A - \lambda)^{d/2-1}$ if $d > 2$ and vice versa. Therefore, if $d > 2$, then $d\mu_A/d\lambda$ is negative and, as the field is increased, $\mu_A$ decreased from its zero-field value $\mu_B$ and is always smaller than $\mu_B$. Conversely, if $d < 2$, then $\mu_A$ is always greater than $\mu_B$. For $d = 2$, we have $\mu_A = \mu_B$ independent of the field.

Problem 4

(i) $k = N^{-1}\sum_{i,j} A_{ij}$. The thermodynamic potential (Lagrangian in mechanical terms) to be minimized is therefore

$$\mathcal{L} = -\sum_{\{A_{ij}\}} p(A_{ij}) \ln p(A_{ij}) + \lambda_0 \sum_{\{A_{ij}\}} p(A_{ij}) - \tau N^{-1} \sum_{\{A_{ij}\}} p(A_{ij}) \sum_{i,j} A_{ij}.$$

Minimizing $\mathcal{L}$, that is $\partial\mathcal{L}/\partial p(A_{ij}) = 0$, one finds the distribution and the partition function

$$p_G(A_{ij}) = \frac{\exp\left[-(\tau/N)\sum_{i,j} A_{ij}\right]}{Z}, \quad Z = \sum_{\{A_{ij}\}} \exp\left[-(\tau/N)\sum_{i,j} A_{ij}\right].$$

The equivalent of the Hamiltonian is $k$. The degrees of freedom are the $N^2$ entries of $A_{ij}$. Each d.o.f. is a “boson” since it takes all non-negative integers as possible values. The bosons are non-interacting in the “Hamiltonian” .

(ii)

$$F = -\ln Z = -\ln \sum_{\{A_{ij}\}} \exp\left(-\tau/N \sum_{i,j} A_{ij}\right) = -\ln \sum_{\{A_{ij}\}} \prod_{i,j} \exp\left(-\tau/N A_{ij}\right) =$$
$$-\ln \sum_{\{A_{ij}\}} \prod_{i,j} \exp\left(-\tau/N A_{ij}\right) = -\ln \prod_{i,j} \sum_{A_{ij}} \exp\left(-\tau/N A_{ij}\right) = N^2 \ln\left(1 - e^{-\tau/N}\right).$$

$F$ is extensive in $N^2$.

(iii) $\langle A_{ij}\rangle = \left(e^{\tau/N} - 1\right)^{-1}$ , that is a Bose-Einstein statistics with zero chemical potential since additional edges are free to form.

(iv) $\langle k\rangle = N^{-1}\sum_{i,j}\langle A_{ij}\rangle = N\left(e^{\tau/N} - 1\right)^{-1} \Rightarrow \tau/N = \ln(N/\langle k\rangle + 1)$. It follows that the free energy is $F = N^2 \ln\left(1 - e^{-\tau/N}\right) = -N^2 \ln(1 + \langle k\rangle/N)$. Similarly $p_G(A_{ij}) = (N/\langle k\rangle + 1)^{-Nk}(1 + \langle k\rangle/N)^{-N^2}$.