15.2.2012

Statistical Physics — Final Exam

Duration: 4 hours

Open material during the exam: class notes only!

1. One way to create Bose-Einstein condensation is to put atoms in a three-dimensional optical lattice which is an atom trap created by standing light waves. There is no more than one atom per site cooled to the lowest vibrational state. The ratio between the number of atoms and number of sites is $\kappa \leq 1$. The lattice is then adiabatically removed (the intensity of light is lowered) so that the entropy of the gas is preserved. If $\kappa = 1$ then after lattice removal the atomic wavefunctions delocalize and overlap so that the atoms become a zero-temperature Bose-Einstein condensate (BEC).

   Now consider $\kappa < 1$.

   (a) Calculate the entropy of a partially filled lattice and compare it with the entropy of the Bose gas.

   (b) Find the condition that determines the smallest κ for which the formation of BEC is still possible.

   (c) Use the approximate value $\frac{5\zeta(5/2)}{2\zeta(3/2)} \approx 1.3$ and estimate the smallest κ numerically.

2. Consider an over-damped Brownian particle in a potential $V(q)$ so that the respective equation of motion is

$$\dot{q} = -\frac{dV}{dq} + \eta \, . \tag{1}$$

   Here the noise is white Gaussian with $\langle \eta(0)\eta(t) \rangle = 2\delta(t)$. The potential has the asymptotic $V(q \to \pm\infty) \to q^5$, that is $V(q \to +\infty) \to +\infty$ and $V(q \to -\infty) \to -\infty$. The space q has the topology of a circle i.e. $q = \infty$ and $q = -\infty$ is the same point.

(a) Can such a potential support a Boltzmann-Gibbs steady state (which has zero probability current)?

(b) Find the stationary probability distribution $\rho(q)$.

(c) Describe the form of the asymptotic of $\rho(q)$ at large $|q|$.

3. Consider the 3-state Potts model with spins $s_i = 1, 2, 3$, on sites i of a d-dimensional lattice, interacting with their nearest neighbors via a Hamiltonian

$$-\beta\mathcal{H} = K \sum_{\langle ij \rangle} \delta_{s_i,s_j} \ . \qquad (2)$$

(a) In $d = 1$ find the exact recursion relations by a $b = 2$ renormalization/decimation process. Identify all fixed points and note their stability.

(b) write down the recursion relation $K'(K)$ for $d = 2$ square lattice, using the Migdal-Kadanoff bond moving scheme with $b = 2$.

(c) By considering the stability of the fixed points at zero and infinite coupling, prove the existence of a non-trivial fixed point at a finite K^*.

(d) Assuming the value of K^* is known (there is no need to calculate it), write down the expression for the critical exponent ν.

Good Luck!

Solutions

Solution 1: Suppose we have $P \gg 1$ sites of which κP sites are occupied. The entropy is

$$S = \ln[P!/(\kappa P)!(P-\kappa P)!] \approx P[(\kappa-1)\ln(1-\kappa) - \kappa \ln \kappa] \; . \tag{3}$$

The entropy of the κP atoms of a Bose gas below T_c is given by the formula (91) from the lecture notes

$$S = \frac{5g_{5/2}}{2g_{3/2}} \kappa P \left(\frac{T}{T_c}\right)^{3/2} \approx 1.3 \cdot \kappa P \left(\frac{T}{T_c}\right)^{3/2} \; . \tag{4}$$

The condition that the entropies (3) and (4) are equal gives the temperature of the Bose gas which appears after the trap is adiabatically removed. Requiring that this temperature is equal to $T = T_c$ gives the $\kappa \approx 0.5$.

Solution 2: It is straightforward to check that the Boltzmann-Gibbs distribution $e^{-V(q)}$ is non-normalizable, i.e. its integral over q diverges. Yet the Fokker-Planck equation is of the second order, so it must have two solutions. Therefore, we need another solution:

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial q}\left(\frac{\partial \rho}{\partial q} + \frac{dV}{dq}\rho\right) = -\frac{\partial J}{\partial q} \; . \tag{5}$$

Apart from the Boltzmann-Gibbs distribution, which has zero current, a solution with a constant current is steady as well. Solving equation

$$\frac{\partial \rho}{\partial q} + \frac{dV}{dq}\rho = -J = \text{const} \; , \tag{6}$$

we find the true steady-state solution,

$$\rho(q) = -Je^{-V(q)} \int_{-\infty}^{q} e^{V(q')} dq' \; , \tag{7}$$

which is normalizable. Note that the current must be negative i.e. directed towards $-\infty$. For this solution to have a physical meaning, the phase space must be a circle i.e. the current that flows to $-\infty$ returns from $+\infty$. Asymptotic at large q is $\rho(q) \propto q^{-4}$, which corresponds to a fast (finite-time) escape to infinity according to the law $\dot{q} = -q^4$.

Solution 3: (a) Tracing over all 3 values of the spin s_2 yields

$$\sum_{s_2=1}^{3} e^{K(\delta_{s_1,s_2}+\delta_{s_2,s_3})} = \begin{cases} e^{2K}+2 & \text{if } s_1 = s_3 \\ 2e^{K}+1 & \text{if } s_1 \neq s_3 \end{cases} . \tag{8}$$

The recursion relations are therefore $e^{K'+a'} = e^{2K}+2$ and $e^{a'} = 2e^{K}+1$, from which we find

$$K' = \log \frac{e^{2K}+2}{2e^{K}+1} . \tag{9}$$

To find the fixed points of this equation denote $x = e^{K^*}$, which yields $x^2+x-2=0$. The only physical solution is $x=1$, which corresponds to $K^*=0$. This is a stable fixed point, because for $K \ll 1$ we have

$$K' \approx \ln \frac{3+2K+2K^2}{3+2K+K^2} \approx \ln(1+K^2/3) \approx \frac{K^2}{3} < K . \tag{10}$$

In addition, $K^* = \infty$ is also a fixed point. For $K \gg 1$,

$$K' \approx \ln \frac{e^K}{2} = K - \ln 2 < K, \tag{11}$$

and therefore this fixed point is unstable.

(b) In the Migdal-Kadanoff approximation, moving the bonds strengthens each bond by a factor of 2. By replacing $K \to 2K$ in equation (9) we find that the recursion relation for K is

$$K' = \log \frac{e^{4K}+2}{2e^{2K}+1} . \tag{12}$$

(c) once again, $K^* = 0, \infty$ are fixed points. For $K \ll 1$, by replacing $K \to 2K$ in equation (10), we find that $K' \approx 4K^2/3 < K$, and therefore $K^*=0$ is stable. Similarly, for $K^* = \infty$, by replacing $K \to 2K$ in equation (11) we find that $K' \approx 2K > K$, and so $K^* = \infty$ is also stable. Therefore, a non-trivial unstable fixed point must exist between them. This fixed point can be found explicitly: denoting as before $x = e^K$, we have $x = (x^4+2)/(2x^2+1)$, or $x^4 - 2x^3 - x + 2 = (x-2)(x^3-1) = 0$. Therefore, there is a fixed point at $K^* = \ln 2$.

(d) The thermal exponent y_t at the non-trivial fixed point is found from

$$2^{y_t} = \left.\frac{\partial K'}{\partial K}\right|_{K^*} = 4\left[\frac{e^{4K^*}}{e^{4K^*}+2} - \frac{e^{2K^*}}{2e^{2K^*}+1}\right] = \frac{16}{9} , \tag{13}$$

and $\nu = 1/y_t \approx 1.2$.