Exam 2013

1. To see how much of the exam grade spread is pure luck, consider the exam with ten identically difficult problems of ten points each. The exam is multiple-choice that is the correct choice of the answer brings 10 points and a wrong choice brings zero. Assume that all students are identical and they choose answers at random with the probability $p = 0.8$ to get it right. What are the expected mean grade $\bar{g}$ and the standard deviation $\sqrt{\langle (g - \bar{g})^2 \rangle}$?

Solution. For one question the mean grade is $\bar{g}_1 = 10p = 8$ and the standard deviation is $\Delta_1 = \sqrt{\langle g_1^2 \rangle - \bar{g}_1^2} = 10\sqrt{p(1-p)} = \sqrt{0.8 \times 100 - 64} = \sqrt{16}$. Since the problems are independent they can be treated as a random walk with ten steps. For a random walk, $\Delta_N = \Delta_1 \sqrt{N} = \sqrt{100Np(1-p)}$ and $\Delta_{10} = \sqrt{160} \approx 13$. Alternatively, you can notice that the probability to have $k = g/10$ right answers is $P(k, N) = C_N^k p^k (1-p)^{N-k}$, called binomial distribution. For binomial distribution, the mean value $\langle k \rangle = p$ and the standard deviation is $\sqrt{Np(1-p)}$. One can also treat N as a large number and obtain the standard deviation by expanding $\ln P(k, N)$ around its maximum $k_* \approx pN$ where the binomial distribution is close to Gaussian.

2. Consider a cubic sample of solid dielectric with the volume $1\,cm^3$. The speed of sound in this material is $u = 1$ km/sec. Planck constant and Boltzmann constants are respectively $\hbar = 10^{-34}$ Jsec and $k = 1.4 \cdot 10^{-23}$ J/K. Estimate at what temperature the relative fluctuation of the thermal energy (temperature-dependent part of the energy) is of order unity.

Solution: The squared energy fluctuation $\langle (\Delta E)^2 \rangle = kT^2 C_v$ is expressed via the specific heat. We expect the relative fluctuation to be of order unity at low temperatures where we can use the Debye formula $C_v \simeq kN(kT/\theta)^3$ see (86) from Lecture Notes. The thermal energy is $E \simeq NkT(kT/\theta)^3$ so that $\langle (\Delta E)^2 \rangle / E^2 \simeq (\theta/kT)^3/N \simeq (hu/kT)^3 V^{-1}$. Therefore, the relative fluctuations are of order unity when $kT \simeq hu/L$ which corresponds to $T \simeq 10^{-6}$K.

3. Gas molecules interact with the wall. The energy of interaction $U(x)$ depends on the distance to the wall x and changes from ∞ at $x = 0$ to 0 at $x \to -\infty$. Find the pressure on the wall if the temperature is T and the concentration of molecules far from the wall is n_0.

Solution: The molecules at x are under the action of the force $-dU/dx$ and act on the force with the opposite force. The total force acting on the unit area of the wall is

$$\int_{-\infty}^{0} \frac{dU}{dx} n(x) dx = \int_{-\infty}^{0} \frac{dU}{dx} n_0 e^{-U(x)/T} dx = n_0 \int_0^{\infty} e^{-U(x)/T} dU = n_0 T \,,$$

i.e. ideal gas pressure (it is important that there is interaction between molecules themselves).

If one is lazy but smart, one can obtain the answer without any computation: imagine a wall at $x = -\infty$, where the pressure must be $n_0 T$ - the same pressure must be on the wall at $x = 0$.

4. Renormalization group and central limit theorem. Consider the space of random variables x having the probability distribution $\rho(x)$ with zero mean and variance $\sigma = \int x^2 \rho(x) dx$. Consider the renormalization-group transformation which consists of two steps:

1) Take two random variables and add them. The new distribution of sums is $\rho'(z) = \int \rho(x)\rho(y)\delta(x + y - z) dx dy$.

2) Since the step 1) increases variance, make the re-scaling such that returns the variance back to σ but keeps the distribution normalized: $\rho''(z) = \lambda \rho'(\lambda z)$.

a) Determine λ.
b) Which equation $\rho(x)$ must satisfy to be a fixed point of the procedure 1) + 2)?
c) Find the solution of this equation (hint: use Fourier representation).

Solution: The step 1) increases the variance by the factor 2, so we need $\lambda = \sqrt{2}$. The equation for the fixed point,

$$\rho(z) = \sqrt{2} \int \rho(\sqrt{2}z - y)\rho(y)dy,$$

is a convolution equation, so it is easiest to solve it in Fourier representation, where $\rho(k) = \int \rho(z)e^{ikz}dz$ satisfies the equation $\rho(\sqrt{2}k) = \rho^2(k)$. The solution of this equation is Gaussian whose Fourier transform is Gaussian too: $\rho(z) \propto \exp(-z^2/2\sigma^2)$. The hint was given in the title of the problem by words "central limit theorem", which tells us that the Gaussian distribution appears after adding random variables.

Bonus question:
B. A raw chicken egg, when put into a large pot with boiling water usually cooks in about five minutes. An ostrich egg has about the same shape as a chicken egg, but its linear size is three times larger. Approximately how long does it take to cook an ostrich egg?
B. Solution: Let us recall what happens when we put an egg into boiling water. We assume that initially the entire egg is at some uniform room temperature. Once in the boiling water, the surface layer of the egg quickly heats up to the temperature of the water (we assume rapid heat exchange in the water outside the egg). Then, due to the temperature gradient, heat flows from the outside of the egg towards the yoke, eventually raising its temperature to the point when it coagulates, i.e., becomes solid. At this point, we proclaim the egg cooked. The heat conduction is governed by the diffusion equation, which suggests the scaling "time=distance squared". Therefore, cooking an ostrich egg must take about 45 minutes.