Diffusion with Stochastic Resetting

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Introduction: Search Problems

Search Problems are ubiquitous in nature and occur in a variety of contexts

- from foraging of animals to target location on DNA
- from internet searches to the mundane task of finding one's misplaced possessions

How does one search for lost keys?

after a while go back to where they should be and start looking again i.e. reset the search

Plan: Diffusion with Stochastic Resetting

Plan

- I Stationary state
- II Survival probability and mean absorption time
- III Many searchers

References:

M. R. Evans and S. N. Majumdar, Phys. Rev. Lett. 106, 160601 (2011)

I Diffusion with resetting

Consider resetting diffusive particle to the initial position x_0 with rate r:

Forward equation for $p(x, t|x_0)$ is modified from diffusion

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i.e. loss rate r from all $x \neq x_0$ provides source at x_0

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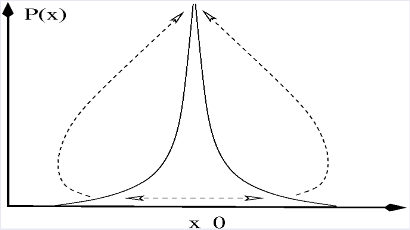
For $t \to \infty$ the stationary state probability density is

$$p_{\rm st}(x|x_0) = \frac{\alpha_0}{2}\exp(-\alpha_0|x-x_0|) \quad \text{where} \quad \alpha_0 = \sqrt{r/D}$$

Nonequilibrium stationary state



II Survival probability with resetting

The Backward equation for the survival probability $q(t|z)$ when there is an absorbing target at the origin

$$\frac{\partial q(t|z)}{\partial t} = D\frac{\partial^2 q(t|z)}{\partial z^2}$$

with boundary/initial conditions $q(t|0) = 0$ and $q(0|z) = 1$ for $z \neq 0$

z is the starting position (variable)

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Laplace transform satisfies

$$D\frac{\mathrm{d}^2\widetilde{q}(s|z)}{\mathrm{d}z^2} - (s+r)\widetilde{q}(s|z) = -1 - r\widetilde{q}(s|x_0)$$

Solution which fits boundary/initial conditions

$$\widetilde{q}(s|z) = \left[1 + r\widetilde{q}(s|x_0)\right]\frac{1 - \mathrm{e}^{-\alpha z}}{s+r}$$

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$$\widetilde{q}(s|z) = \left[1 + r\widetilde{q}(s|x_0)\right]\frac{1-\mathrm{e}^{-\alpha z}}{s+r}$$

Then solve self-consistently for

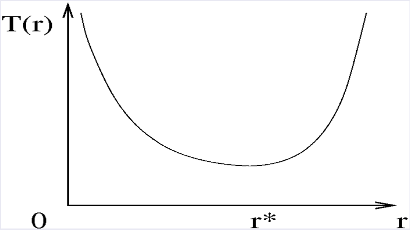
$$\boxed{\widetilde{q}(s|x_0) = \frac{1-\mathrm{e}^{-\alpha x_0}}{s+r\mathrm{e}^{-\alpha x_0}}} \quad \text{where} \quad \alpha = \left(\frac{s+r}{D}\right)^{1/2}$$

II Mean first passage time (MFPT)

MFPT $T = -\int_0^\infty \mathrm{d}t\, t \frac{\partial q(t|x_0)}{\partial t} = \widetilde{q}(0|x_0)$ is now finite for $0 < r < \infty$

$$\boxed{T = \frac{\mathrm{e}^{x_0 (r/D)^{1/2}} - 1}{r}}$$

$T(r)$ has a minimum at r^*



$$\frac{\mathrm{d}T}{\mathrm{d}r} = 0$$

$$\Rightarrow \frac{y}{2} = 1 - \mathrm{e}^{-y}$$

where $y = x_0 (r/D)^{1/2}$

y= distance from target : typical distance diffused between resets

Optimal $y^* = 1.5936...$

II Survival probability

The long-time behaviour of the $q(t|x_0)$ is now controlled by

simple pole of $\widetilde{q}(s|x_0) = \dfrac{1 - \mathrm{e}^{-\alpha x_0}}{s + r\mathrm{e}^{-\alpha x_0}}$ at

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For $y = x_0(r/D)^{1/2} \gg 1 \quad s_0 \simeq -r \exp -y$ and

$$q(t|x_0) \simeq \exp(-rt\,\mathrm{e}^{-y})$$

which has the form of a Gumbel distribution which gives the cumulative distribution for the maximum of independent r.v.s

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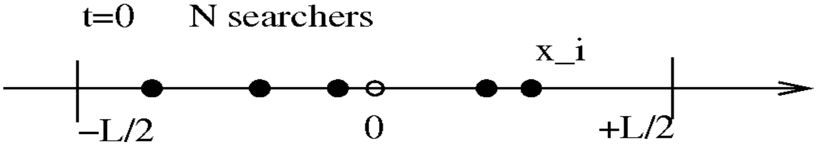
Explanation

On average there are rt resets. For each reset the process is "renewed" and the particle trajectory is independent. The particle must not reach the origin in any reset to survive.

So survival is probability that max excursion to left, out of $\simeq rt$ resets, is less than x_0

III Many Searchers

Consider the survival probability of a target at the origin in the presence of many particles (searchers/traps).



N searchers beginning at x_i $\quad i = 1 \ldots N$ uniformly distributed with density $\rho = \dfrac{N}{L}$

Survival probability of target $Q(t|\{x_i\}) = \displaystyle\prod_{i=1}^{N} q(t|x_i)$

Average (annealed)

$$Q^{av}(t) = \langle q(t|x) \rangle_x^N$$

Typical (quenched)

$$Q^{typ}(t) = \exp N \langle \ln q(t|x) \rangle_x$$

III Many Searchers continued

For diffusive particles

$$Q^{av,typ}(t) = \exp(-\lambda_{av,typ}\rho\sqrt{Dt})$$

where $\lambda_{av,typ}$ is a constant.

With Resetting

$rt \ll 1$ recovers diffusive results

$rt \gg 1$

$$Q^{av}(t) \simeq \text{constant}\, t^{-2\rho\sqrt{D/r}}$$

$$Q^{typ}(t) \simeq \exp\left[-t\,\rho\sqrt{Dr}\,8(1-\ln 2) + O(t^{1/2})\right]$$

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Explanation of different behaviours

$Q^{av}(t) \gg Q^{typ}(t)$ since average behaviour dominated by rare realisations of $\{x_i\}$ far from target

$\rightarrow$ memory of initial conditions

Summary and Outlook

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- Resetting gives finite mean first passage time
- Survival probability for single searcher decays exponentially
- Connection to statistics of extremes and a renewal process
- For many searchers survival probability of target *typically* decays exponentially but rare events make *average* decay more slowly

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Outlook

- Higher spatial dimensions can be studied
- $r(x)$ can be made position dependent
- A target distribution can be considered
- A resetting distribution $\mathcal{P}(x_0)$ can be considered
- Other optimisation problems e.g. cost for resetting