Bose–Einstein Condensation in the Large Deviations Regime with Applications to Information System Models

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The Problem

We are interested in the large deviations behavior of a system whose steady–state obeys

$$P(n_0, n_1, \ldots, n_{M-1}) = \frac{\prod_{i=0}^{M-1} p_i^{n_i}}{Z},$$

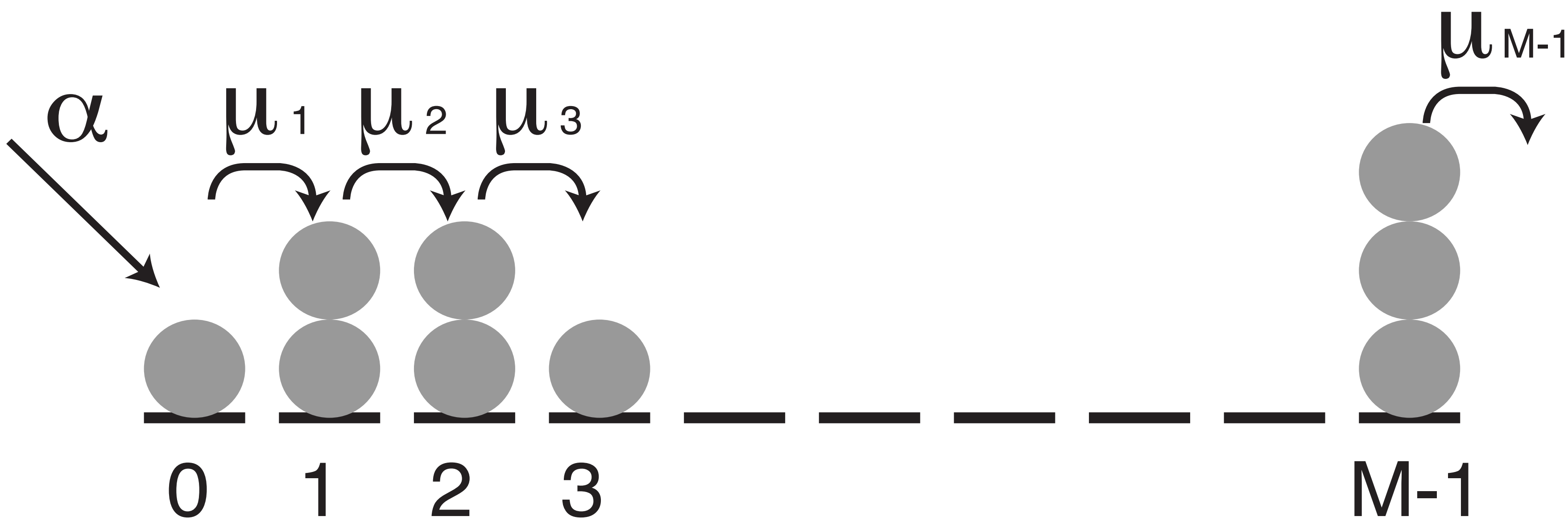
where $n_i =$ number of elements (particles/agents,...) in 'site' no. i.

E.g., assess $\Pr\{\sum_i n_i \geq M \cdot U\} \sim e^{-M \cdot I(U)}$ for $M \to \infty$.

Examples of Relevance:

- Grand–canonical ideal Boson gas: $p_i = ze^{-\beta\epsilon_i}$ (black body radiation).
- One–way Markov chain: $i \to i$ w.p. p_i; $\quad i \to i+1$ w.p. $1 - p_i$.
- Jackson network of M queues: $p_i = \lambda_i / \mu_i$.
- Hopping model for transport in a disordered medium: $p_i = \alpha / \mu_i$ (ZRP's).

Hopping Model (Or A Cascade of Queues)



$$P(n_0, n_1, \ldots, n_{M-1}) = \frac{\prod_{i=0}^{M-1} (\alpha/\mu_i)^{n_i}}{Z},$$

The Problem (Cont'd)

The large deviations rate function

$$I(U) = \lim_{M\to\infty}\left[-\frac{1}{M}\log \textsf{Pr}\left\{\sum_i n_i \geq MU\right\}\right]$$

may exhibit phase transitions – Bose–Einstein condensation (BEC): Given the event, a macroscopic fraction – jammed in one 'site' for $U > U_c$.

This is the case also with general linear combinations of $\{n_i\}$, e.g.,

$$I(U) = \lim_{M\to\infty}\frac{-1}{M}\log \textsf{Pr}\{\underbrace{\sum_i n_i/\mu_i}_{\text{est. time}} \geq MU\}.$$

More interestingly, how about $I(U,V)$ of Pr$\{\sum_i n_i \geq MU,\ \sum_i n_i/\mu_i \geq MV\}$?

Answer gives rise to a notion of 2D BEC: A very rich phase diagram.

A Single Constraint

$$\begin{aligned}
\Pr\left\{\sum_i n_i \geq MU\right\} &\leq \left\langle z^{\sum_i n_i - MU}\right\rangle \qquad z \geq 1 \\
&= z^{-MU}\prod_i \frac{1-p_i}{1-p_i z} \\
&= \exp\left\{-M\left[U\ln z - \frac{1}{M}\sum_i \ln\left(\frac{1-p_i}{1-zp_i}\right)\right]\right\}
\end{aligned}$$

$$I(U) = \sup_{z\geq 1}\left[U\ln z - \lim_{M\to\infty}\frac{1}{M}\sum_{i=0}^{M-1}\ln\left(\frac{1-p_i}{1-p_i z}\right)\right].$$

Assume that $\{p_i\}$ have a density $g(p)$:

fraction of $\{p_i\} \in [p, p+dp] \to g(p)dp.$

A Single Constraint (Cont'd)

Resulting equation in z (saddle point):

$$U = \lim_{M\to\infty} \frac{1}{M} \sum_{i=0}^{M-1} \frac{z p_i}{1 - p_i z} \equiv z \cdot \int_0^{p_{\max}} \frac{p \cdot g(p)\mathrm{d}p}{1 - pz} \triangleq \boldsymbol{U}(z)$$

A solution z^* is sought in $[1, 1/p_{\max})$.

In analogy to the BEC, if

$$\lim_{p\uparrow p_{\max}} \frac{g(p)}{(p_{\max} - p)^{\xi}} < \infty \quad \text{for some } \xi > 0,$$

then

$$U_c = \boldsymbol{U}(1/p_{\max}) \equiv \int_0^{p_{\max}} \frac{p g(p)\mathrm{d}p}{p_{\max} - p} < \infty$$

Condensation for $U > U_c$.

A Single Constraint (Cont'd)

The rate function exhibits two phase transitions

$$I(U) = \begin{cases} 0 & U < U_{\min} \equiv \boldsymbol{U}(1) \\ \sup_z \left[U \ln z - \int_0^{p_{\max}} \mathrm{d}p g(p) \ln\left(\frac{1-p}{1-pz}\right)\right] & U_{\min} \le U < U_c \\ U \ln\left(\frac{1}{p_{\max}}\right) - \int_0^{p_{\max}} \mathrm{d}p g(p) \ln\left(\frac{1-p}{1-p/p_{\max}}\right) & U \ge U_c \end{cases}$$

- $U < U_{\min}$ – not a rare event.
- $U_{\min} \le U < U_c$ – rare event; no condensation. $I(U)$ is convex.
- $U > U_c$ – BEC: $n_i/N \sim 1 - U_c/U$ for that i of $p_i = p_{\max}$. $I(U)$ is affine.

Second order phase transition: $I(U)$ and $I'(U)$ are continuous, $I''(U)$ is not.

A More General Constraint

Consider now the event $\{\sum_i n_i u_i \geq MU\}$ for some deterministic sequence $u_1, u_2, \ldots$ which has some density, or more simply, let $u_i = u(p_i)$ for some function $u(\cdot)$.

Examples:

- Disordered transport model: $u(p) = p/\alpha \;\rightarrow\; \sum_i n_i/\mu_i$ (estimated time).
- Ideal Boson gas: $u(p) = -\frac{1}{\beta}\ln p \;\rightarrow\; \sum_i n_i \epsilon_i$ (energy).

A similar derivation yields the saddle–point equation

$$U = \boldsymbol{U}(z) \equiv \int_0^{p_{\max}} \frac{p u(p) z^{u(p)} g(p) \mathrm{d}p}{1 - p z^{u(p)}},$$

whose solution z is sought in $[1, Z)$, where $Z \equiv \inf_{p\in[0,p_{\max}]}\{p^{-1/u(p)}\}$.

Again, condensation takes place if $U_c = \boldsymbol{U}(Z) < \infty$ and $U > U_c$.

Two Constraints

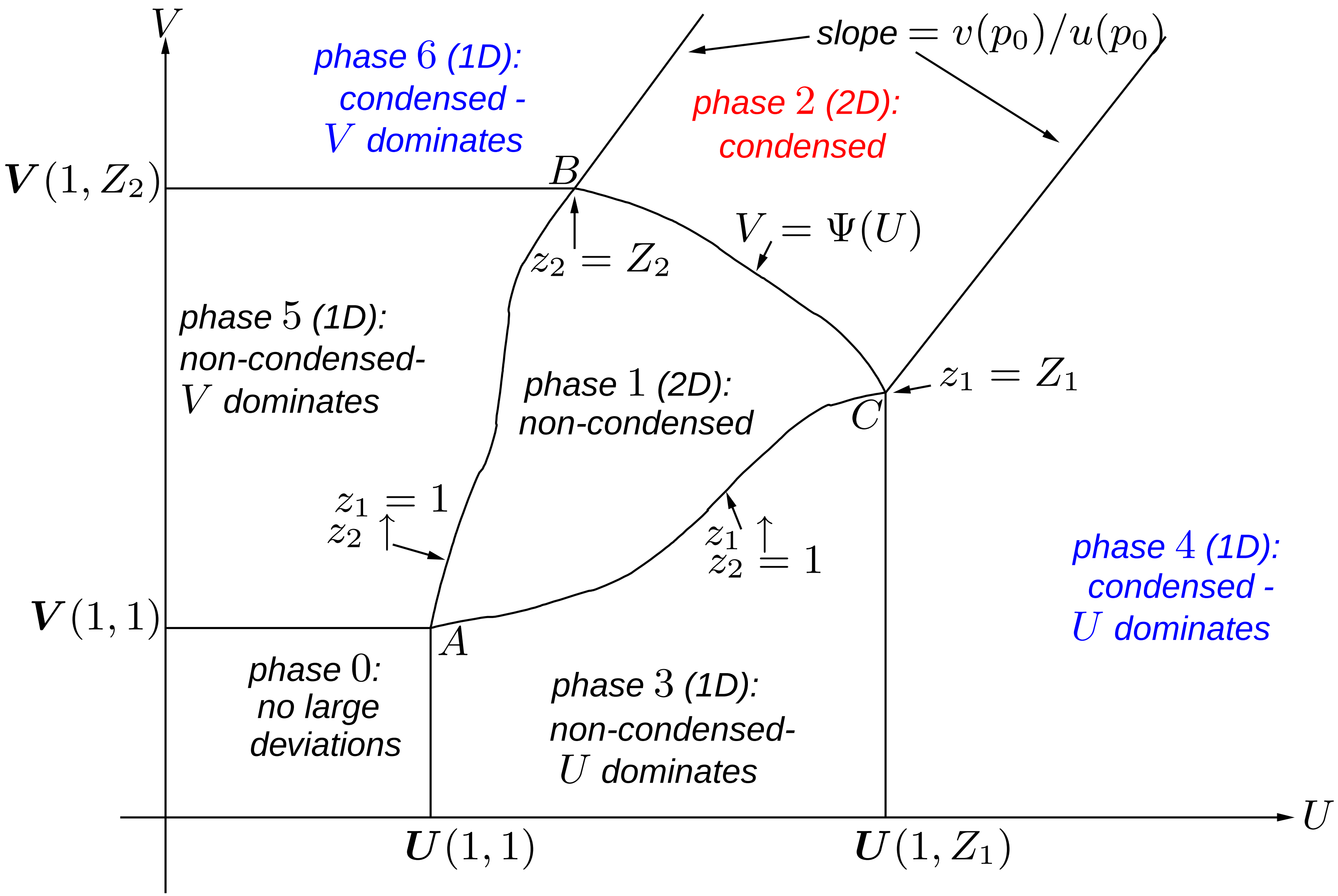
Consider now the event $\{\sum_i n_i u_i \geq MU,\ \sum_i n_i v_i \geq MV\}$ for $u_i = u(p_i)$ and $v_i = v(p_i)$. Here the large deviations analysis yields two equations with two unknowns:

$$
\begin{aligned}
U &= \boldsymbol{U}(z_1, z_2) \equiv \int_0^{p_{\max}} \frac{p u(p) z_1^{u(p)} z_2^{v(p)} g(p) \mathsf{d}p}{1 - p z_1^{u(p)} z_2^{v(p)}} \\
V &= \boldsymbol{V}(z_1, z_2) \equiv \int_0^{p_{\max}} \frac{p v(p) z_1^{u(p)} z_2^{v(p)} g(p) \mathsf{d}p}{1 - p z_1^{u(p)} z_2^{v(p)}}
\end{aligned}
$$

The rate function is

$$
I(U, V | z_1, z_2) = U \ln z_1 + V \ln z_2 - \int \mathsf{d}p g(p) \ln \left[\frac{1 - p}{1 - p z_1^{u(p)} z_2^{v(p)}} \right].
$$

Here, z_1 and z_2 are jointly limited by the inequality:
$f(z_1, z_2) \stackrel{\Delta}{=} \sup_p [p z_1^{u(p)} z_2^{v(p)}] < 1$. Solutions are sought in the region $\mathcal{A} = \{(z_1, z_2):\ z_1 \geq 1,\ z_2 \geq 1,\ f(z_1, z_2) < 1\}$.


V
$\boldsymbol{V}(1, Z_2)$
$\boldsymbol{V}(1,1)$
U
$\boldsymbol{U}(1,1)$
$\boldsymbol{U}(1, Z_1)$
slope $= v(p_0)/u(p_0)$
phase 6 (1D): condensed - V dominates
phase 2 (2D): condensed
phase 5 (1D): non-condensed- V dominates
phase 1 (2D): non-condensed
phase 4 (1D): condensed - U dominates
phase 0: no large deviations
phase 3 (1D): non-condensed- U dominates
B
A
C
$V = \Psi(U)$
$z_2 = Z_2$
$z_1 = Z_1$
$z_1 = 1$
$z_2 \uparrow$
$z_1 \uparrow$
$z_2 = 1$

There Can Be Even More Phases ...

We assumed that along the curve $f(z_1, z_2) = 1$,

$$\max_p [p z_1^{u(p)} z_2^{v(p)}]$$

is achieved by $p = p_0$ that is independent of z_1.

What happens when this is not the case?

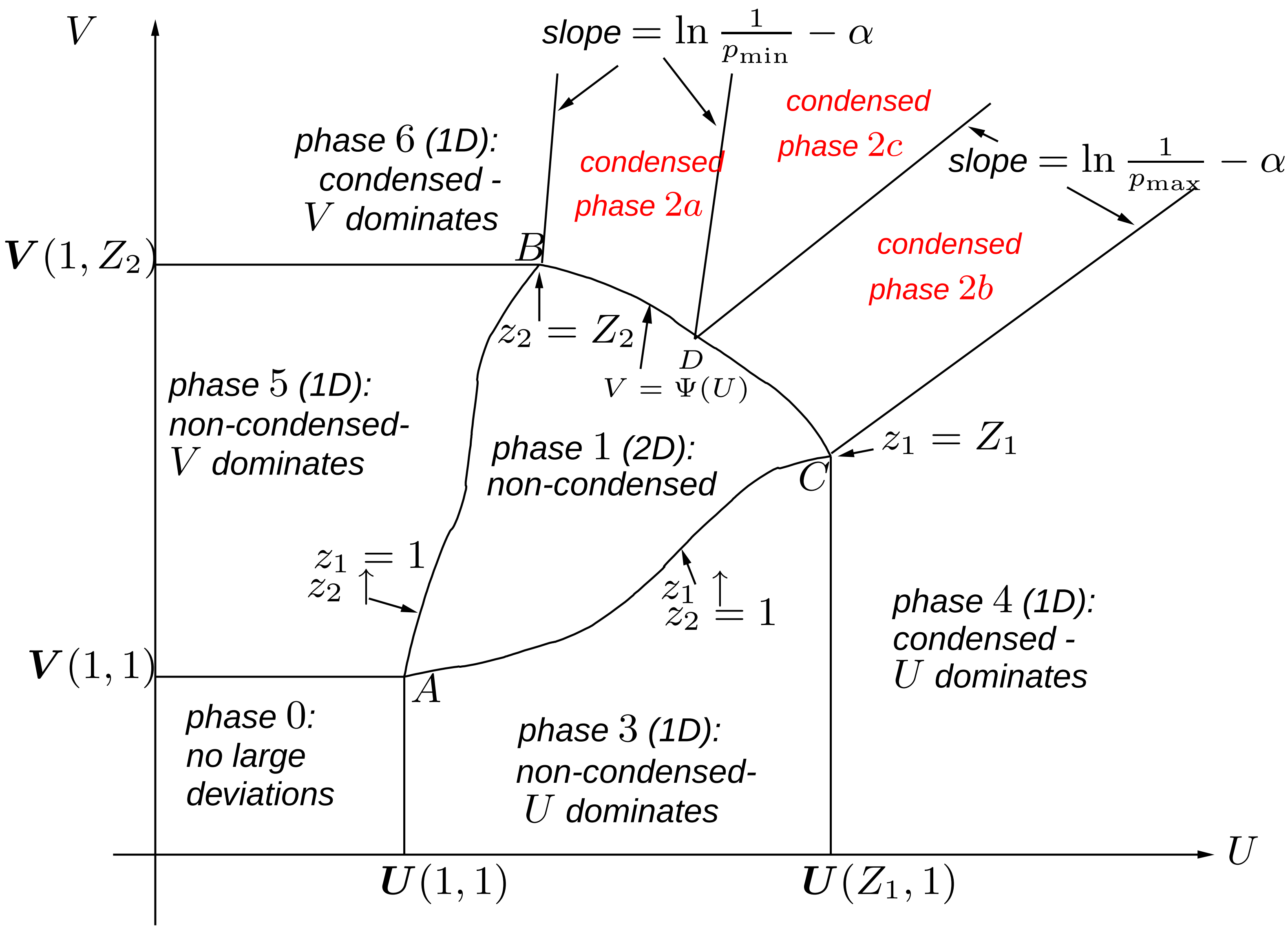
For example,

$$u(p) \equiv 1; \quad v(p) = -\alpha - \ln p.$$

Here,

$$\text{argmax}_{f(z_1,z_2)=1} [p z_1^{u(p)} z_2^{v(p)}] = \begin{cases} p_{\min} & z_1 < e^{\alpha} \\ p_{\max} & z_1 > e^{\alpha} \end{cases}$$

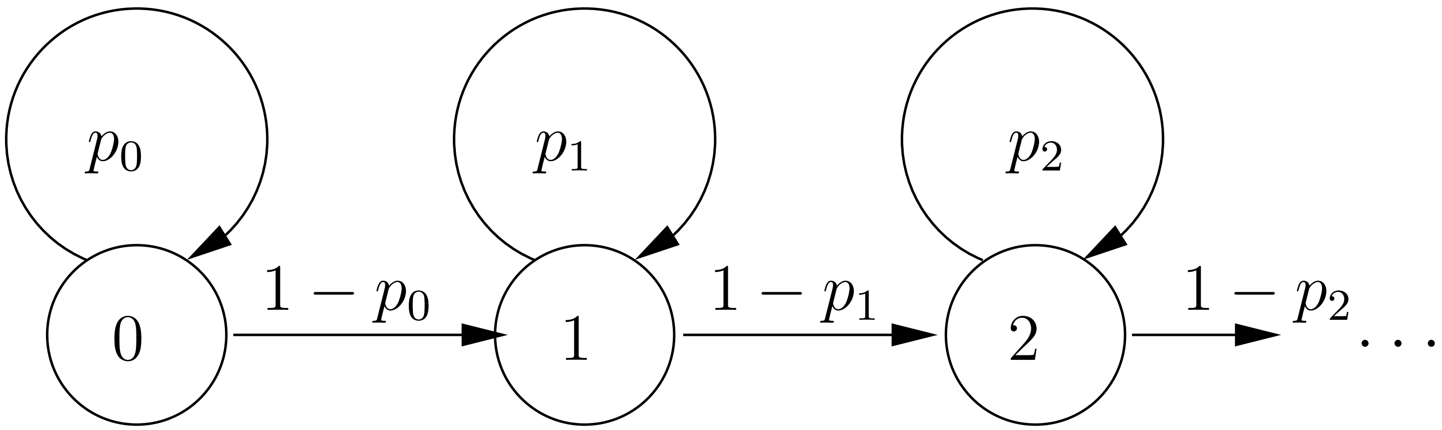
Here, the 2D condensed phase, splits into three sub–phases.



V
U
slope $= \ln \frac{1}{p_{\min}} - \alpha$
slope $= \ln \frac{1}{p_{\max}} - \alpha$
phase 6 (1D): condensed - V dominates
condensed phase $2a$
condensed phase $2c$
condensed phase $2b$
$\boldsymbol{V}(1, Z_2)$
B
$z_2 = Z_2$
D
$V = \Psi(U)$
phase 5 (1D): non-condensed- V dominates
phase 1 (2D): non-condensed
$z_1 = Z_1$
C
$z_1 = 1$
$z_2 \uparrow$
$z_1 \uparrow$
$z_2 = 1$
phase 4 (1D): condensed - U dominates
$\boldsymbol{V}(1, 1)$
A
phase 0: no large deviations
phase 3 (1D): non-condensed- U dominates
$\boldsymbol{U}(1, 1)$
$\boldsymbol{U}(Z_1, 1)$

Applications

One–Way Markov Chains



$$P(n_0, n_1, \ldots) = \prod_i [p_i^{n_i}(1-p_i)].$$

Optimum data compression: Encode n_i using

$$\ell_i(n_i) = -\log P(n_i) = n_i \log(1/p_i) - \log(1-p_i) \text{ bits}$$

Here $u(p) = -\log p$. The large deviations event $=$ buffer overlfow.
Condensation – beyond a certain buffer size for certain densities.

Queueing Networks

A natrual application is a Jackson network with M queues, with

$$p_i = \lambda_i/\mu_i \quad \text{utilization}$$

where $\lambda_i =$ arrival rate to queue no. i, and $\mu_i =$ service rate of queue no. i.

Examples of relevant (undesirable) events:

- Excess of $\sum_i n_i =$ total number of customers.
- Excess of $\sum_i n_i/\mu_i =$ estimation of total waiting time in all queues.

Condensation: queue with the worst utilization is jammed.

Gordon–Newell network: fixed number of customers – canonical Bose–Einstein distribution. Related to zero–range processes (ZRP's) in stat mech with conservation of particles.

Jackson (1963) extended his results to allow state–dependent service times: seems to include results on ZRP's as special cases.

Thank You!